A NONLINEAR PARTIAL DIFFERENTIAL EQUATION FOR THE VOLUME PRESERVING MEAN CURVATURE FLOW

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Dedicated to Professor Hiroshi Matano on the occasion of his 60th birthday

Abstract. We analyze the evolution of multi-dimensional normal graphs over the unit sphere under volume preserving mean curvature flow and derive a non-linear partial differential equation in polar coordinates. Furthermore, we construct finite difference numerical schemes and present numerical results for the evolution of non-convex closed plane curves under this flow, to observe that they become convex very fast.

1. **Introduction.** In this paper, we study the evolution of normal graphs over the unit sphere under volume preserving mean curvature flow (VPMCF). We use a proper diffeomorphism and a general parametrization of the unit sphere. Our main goal is to express evolution as an initial and boundary value problem with periodic conditions for a second order non-linear partial differential equation with non-local integral terms. In addition for the two-dimensional case, we solve the problem numerically by applying finite difference schemes.

Given an initial simple and closed hypersurface S_0 in $\mathbb{R}^{n+1}$, we seek a family

$$S = \{S_t; \ t \geq 0\}$$

of smooth closed hypersurfaces in $\mathbb{R}^{n+1}$ on which the following equation is satisfied, [4]

$$V = h - H \quad \text{on} \ \ S_t, \ \ t \geq 0. \tag{1}$$

Here $V = V(\sigma, t)$ and $H = H(\sigma, t)$ denote respectively the normal velocity and mean curvature of a point σ on S_t. The mean curvature on S_t is defined as an average of principal curvatures or equivalently as the trace of the second fundamental form. The function $h = h(t)$ is defined as the average of the mean curvature on S_t

$$h(t) := \frac{\int_{S_t} H(\sigma, t) \, d\sigma}{\int_{S_t} d\sigma}, \ \ t \geq 0. \tag{2}$$

A basic property of the averaged mean curvature flow (1) is that it defines a volume preserving and area shrinking hypersurfaces family $\{S_t; \ t \geq 0\}$. Any Euclidean sphere in $\mathbb{R}^{n+1}$ is a static solution of (1). Existence and uniqueness of solution of (1) for smooth initial hypersurfaces is proved in [9] and [5]. Huisken [9] (and

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Gage [5] in the case of curves) proved that the solution of (1) exists globally and converges to a sphere, if the initial surface S_0 is convex and smooth, while for any $t \geq 0$ S_t remains convex. Extending the previous results, Escher and Simonett in [4] proved that convexity is not necessary for global existence, i.e. there are non-convex initial conditions S_0 such that the solution of (1) exists globally and converges exponentially fast to a sphere. Gage and Hamilton analyzed the heat equation on convex plane curves in [6].

Alikakos and Freire [1] have shown neckpinching of certain class of rotationally symmetric surfaces under volume preserving mean curvature flow. Later, Gang and Sigal analyzed the motion of surfaces of revolution under mean curvature flow, [7]. Escher and Simonett in [4] proved for (1) by means of a center manifold analysis the asymptotic stability of spheres under Hölder norm. In [2] a proof of this result is given in Sobolev norms by orthogonal decomposition of the solutions near the manifold of Euclidean spheres and by making use of certain properties of Lyapunov functionals.

In Section 2, introducing a suitable diffeomorphism we present S_t as a normal graph over the unit sphere in polar coordinates. In the next section we use this co-ordinate system and write (1) in an equivalent formulation. The resulting equation presented in Theorem 3.2 for general n and Remark 5 for the two-dimensional case is an evolutionary in time non-linear partial differential equation (p.d.e.). Struwe, in [12], derived in a more geometric manner in cartesian coordinates the evolution equation for the mean curvature flow considering the multidimensional case. In this paper, we analyze the *volume preserving mean curvature flow* and propose the use of a polar coordinates system in order to present evolution as an initial and boundary value problem.

Finally, in Section 4 we solve the problem numerically for non-convex initial conditions by applying explicit finite difference schemes. The numerical results agree to the theoretical result of asymptotic convergence to spheres; a general experimental observation is that non-convex curves evolving under volume preserving mean curvature flow become convex very fast. Our numerical experiments verify for the VPMCF the elegant theoretical result of Grayson [8] proved for smooth embedded curves in $\mathbb{R}^2$ evolving under mean curvature flow. The result of Grayson completed the proof of the conjecture that curve shortening shrinks embedded plane curves smoothly to points, with round limiting shape, while these curves become convex without developing singularities. In our case, the flow of evolution is different since we refer to the *volume preserving mean curvature flow* (VPMCF), while the curve under this flow does not shrink to a point but converges asymtotically to a sphere always keeping the initial enclosed volume.

2. **Normal graphs.** Let Γ be a smooth hypersurface in $\mathbb{R}^{n+1}$. A hypersurface S in $\mathbb{R}^{n+1}$ is a graph in the normal direction over Γ if there exists a function

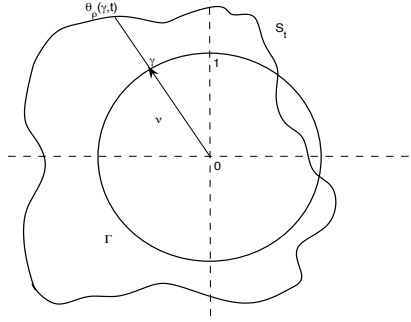$$\varrho : \Gamma \to \mathbb{R}$$

such that the function

$$\theta_\varrho : \Gamma \to S$$

defined by

$$\theta_\varrho := id + \varrho\nu$$

is a diffeomorphism from Γ to S, i.e. is a one-to-one smooth function onto S and θ_ϱ^{-1} is smooth too. Here ν is the unit outward normal vector field in Γ and id is the

identity function on Γ. Moreover, S is said to be in the class $H^s(\Gamma)$ if ϱ is in the class H^s.

Figure 1: Normal graph over the unit sphere in $\mathbb{R}^2$.

Let Γ be the unit sphere in $\mathbb{R}^{n+1}$ of zero center and consider a family $\{S_t,\ t \geq 0\}$ of closed hypersurfaces in $\mathbb{R}^{n+1}$ where for any $t \geq 0$ S_t is a graph in the normal direction over Γ (see Fig. 1). More specifically, for $\Gamma = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$, we assume that there exists function $\rho^* : \Gamma \times \mathbb{R} \to \mathbb{R}$ defining for t fixed a diffeomorphism $\theta_{\rho^*}(\cdot, t)$ onto S_t:

$$\theta_{\rho^*} : \Gamma \times t \to S_t : \quad \theta_{\rho^*}(\gamma, t) := \gamma + \rho^*(\gamma, t)\nu(\gamma), \quad \gamma \in \Gamma, \quad t \geq 0.$$

Since $\theta_{\rho^*}(\Gamma, t) = S_t$ we deduce that in cartesian coordinates $x_1, \cdots, x_{n+1}$, S_t is represented by

$$S_t = \left\{ x \in \mathbb{R}_*^{n+1} : \quad |x| - 1 - \rho^*\left(\frac{x}{|x|}, t\right) = 0 \right\}, \tag{3}$$

where $\mathbb{R}_*^{n+1} := \mathbb{R}^{n+1} - \{0\}$. By setting $\widehat{\rho} := 1 + \rho^*$, [2], we define the diffeomorphism

$$\theta_{\widehat{\rho}}(\gamma, t) := \widehat{\rho}(\gamma, t)\gamma = \theta_{\rho^*}(\gamma, t) - id(\gamma).$$

We represent S_t by using the diffeomorphism $\theta_{\widehat{\rho}}$. S_t is identified by the function $\widehat{\rho}(\cdot, t) : \Gamma \to \mathbb{R}$.

Let $x = (x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1}$ in cartesian coordinates and consider the change of variables in polar coordinates $u = (u_1, \cdots, u_{n+1})$

$$x = x(u) = \Big(x_1(u_1, \cdots, u_{n+1}), \cdots, x_{n+1}(u_1, \cdots, u_{n+1}) \Big), \tag{4}$$
$$x_1 = u_{n+1} y_1(u_1, \cdots, u_n), \cdots, x_{n+1} = u_{n+1} y_{n+1}(u_1, \cdots, u_n),$$
$$\text{where} \quad u_{n+1} = |x| \in [0, +\infty).$$

The function $y = (y_1, \cdots, y_{n+1})$ is on Γ and $y_i\ i = 1, \cdots, n+1$ in polar coordinates may be expressed by the following formulas for $n = 1, 2$, ([10])

$$(n = 1) \quad y_1 = \cos(u_1), \quad y_2 = \sin(u_1), \tag{5}$$
$$(n = 2) \quad y_1 = \cos(u_1)\cos(u_2), \quad y_2 = \sin(u_1)\cos(u_2), \quad y_3 = \sin(u_2),$$

where $u_1, u_2 \in [0, 2\pi] \times [0, \pi)$. If $n \geq 3$ then y can be defined by

$$y_1 = \cos(u_1), \ y_k = \Big(\prod_{j=1}^{k-1} \sin(u_j) \Big) \cos(u_k), \ k = 2, \cdots, n-1, \tag{6}$$

$$y_n = \Big(\prod_{j=1}^{n-1} \sin(u_j) \Big) \cos(u_n), \ y_{n+1} = \Big(\prod_{j=1}^{n-1} \sin(u_j) \Big) \sin(u_n),$$

([11]), where $0 \leq u_j < \pi$ for $j = 1, \cdots, n-1$, $0 \leq u_n < 2\pi$.

Note that the geometrical properties of S_t are independent of the choice of parametrization y of Γ. We may write

$$u = u(x) = (u_1(x_1, \cdots, x_{n+1}), \cdots, u_{n+1}(x_1, \cdots, x_{n+1}))$$

as the change of variables is invertible.

Remark 1. Obviously, $\rho^* \Big(\frac{x}{|x|}, t \Big)$ is a function of $x = (x_1, \cdots, x_{n+1})$ and t for any $x \in \mathbb{R}_*^{n+1}$, therefore, for t fixed we define

$$\tilde{\rho}(\cdot, t) : \mathbb{R}_*^{n+1} \to \mathbb{R}$$

by

$$\tilde{\rho}(x_1, \cdots, x_{n+1}, t) := 1 + \rho^* \Big(\frac{x}{|x|}, t \Big) =: \widehat{\rho} \Big(\frac{x}{|x|}, t \Big).$$

Note that the above gives that $\tilde{\rho}$ is independent from $|x|$ while it depends only on the directional angles, and thus any change of variables of $\mathbb{R}^{n+1}$ from cartesians to polar coordinates will give for $x \in \mathbb{R}_*^{n+1}$

$$\tilde{\rho}(x_1, \cdots, x_{n+1}, t) = \rho(u_1, \cdots, u_n, u_{n+1}, t) = \rho(u_1, \cdots, u_n, t),$$

since $u_{n+1} := |x|$. In this paper we compute ρ as a solution of an initial and boundary value problem. Then we may use this ρ to construct S_t as follows: If Γ is represented by

$$\Gamma := \{y \in \mathbb{R}^{n+1} : \ y = (y_1(u_1, \cdots, u_n), \cdots, y_{n+1}(u_1, \cdots, u_n))\},$$

where y is given for example by (5), (6) then

$$S_t := \Big\{ x \in \mathbb{R}^{n+1} : \ x = \rho(u_1, \cdots, u_n, t) \Big(y_1(u_1, \cdots, u_n), \cdots, y_{n+1}(u_1, \cdots, u_n) \Big) \Big\}. \tag{7}$$

3. **The evolution equation.** In this Section we consider S_t to be a normal graph over the unit sphere Γ defined by (3) and prove an equivalent formulation for (1) presented as an evolution equation in time for $\rho = \rho(u_1, \cdots, u_n, t)$ in polar coordinates. Then S_t may be constructed in $\mathbb{R}^{n+1}$ by utilizing (7).

We prove the next lemma.

Lemma 3.1. *If S_t satisfies the VPMCF (1) then $\widetilde{\rho}$ satisfies*

$$\partial_t \widetilde{\rho} = h \sqrt{1 + |\nabla_x \widetilde{\rho}|^2} + \frac{1}{n} \Big\{ -\frac{n}{|x|} + \Delta_x \widetilde{\rho} - \frac{|\nabla_x \widetilde{\rho}|^2}{|x|(1 + |\nabla_x \widetilde{\rho}|^2)} - \frac{\nabla_x \widetilde{\rho} \operatorname{Hess}_x(\widetilde{\rho}) \nabla_x \widetilde{\rho}^T}{(1 + |\nabla_x \widetilde{\rho}|^2)} \Big\} \Big|_{x \in S_t}. \tag{8}$$

Proof. By definition it holds that

$$\tilde{\rho}(x_1, \cdots, x_{n+1}, t) = 1 + \rho^*\Big(\frac{x}{|x|}, t\Big),$$

thus (3) gives that

$$S_t = \{x \in \mathbb{R}_*^{n+1} : \phi(x, t) = 0\}, \tag{9}$$

where for t fixed, $\phi(\cdot, t) : \mathbb{R}_*^{n+1} \to \mathbb{R}$ is given by

$$\phi(x, t) := |x| - \widetilde{\rho}(x, t). \tag{10}$$

Obviously any $x \in S_t$ is a root of $\phi(x, t)$. In what follows, we use that $\tilde{\rho}$ is a function defined on any $x \in \mathbb{R}_*^{n+1}$, independent of $|x|$. The derivatives in formulae are applied in the space $\mathbb{R}^{n+1}$ in cartesians and at the end we consider $x \in S_t \subset \mathbb{R}^{n+1}$ in order to compute the exact values on the hypersurface S_t.

For the velocity and mean curvature of S_t we have respectively the formulae, [4]

$$V = -\frac{\partial_t \phi}{|\nabla_x \phi|}\Big|_{x \in S_t}, \quad H = \frac{1}{n}\mathrm{div}_x\Big(\frac{\nabla_x \phi}{|\nabla_x \phi|}\Big)\Big|_{x \in S_t}.$$

Using now the velocity formula in (1) we arrive at

$$\partial_t \phi = (H - h)|\nabla_x \phi| \quad \text{on} \quad S_t. \tag{11}$$

From (10) we compute

$$\partial_t \phi(x, t) = -\partial_t \widetilde{\rho}(x, t), \quad \nabla_x \phi = \Big(\frac{\partial|x|}{\partial x_1} - \widetilde{\rho}_{x_1}, \cdots, \frac{\partial|x|}{\partial x_{n+1}} - \widetilde{\rho}_{x_{n+1}}\Big), \tag{12}$$

while for any $0 \le i \le n+1$ it follows that

$$\frac{\partial|x|}{\partial x_i} = \frac{1}{2}\Big(\sum_{j=1}^{n+1} x_j^2\Big)^{-1/2} 2x_i = \frac{x_i}{|x|}.$$

Therefore, the second equality of (12) gives

$$\nabla_x \phi = \frac{x}{|x|} - \nabla_x \widetilde{\rho}. \tag{13}$$

In (11) we replace $\partial_t \phi$ by (12) and use (13) to obtain finally

$$\partial_t \widetilde{\rho} = (h - H)\Big|\frac{x}{|x|} - \nabla_x \widetilde{\rho}\Big|. \tag{14}$$

Let us consider $\lambda > 0$, then the next equality easily follows

$$\widetilde{\rho}(\lambda x, t) = 1 + \rho^*\Big(\frac{\lambda x}{|\lambda x|}, t\Big) = 1 + \rho^*\Big(\frac{x}{|x|}, t\Big) = \widetilde{\rho}(x, t).$$

Hence, for any $\lambda > 0$, we obtain

$$0 = \partial_\lambda(\widetilde{\rho}(x, t)) = \partial_\lambda(\widetilde{\rho}(\lambda x, t)) = \sum_{i=1}^{n+1} \widetilde{\rho}_{y_i}(y, t)\frac{\partial y_i}{\partial \lambda},$$

where

$$\widetilde{\rho}(\lambda x, t) = \widetilde{\rho}(y, t),$$

and $y := \lambda x$. So, $x\nabla_y \widetilde{\rho}(y, t) = 0$ and therefore $\lambda > 0$ yields

$$y\nabla_y \widetilde{\rho}(y, t) = \lambda x \nabla_y \widetilde{\rho}(y, t) = 0.$$

Setting $\lambda := 1$ we get that

$$\frac{x}{|x|} \perp \nabla_x \widetilde{\rho}(x, t),$$

and thus

$$\left|\frac{x}{|x|} - \nabla_x\widetilde{\rho}\right| = \left(\left|\frac{x}{|x|}\right|^2 + |\nabla_x\widetilde{\rho}|^2\right)^{1/2} = \sqrt{1 + |\nabla_x\widetilde{\rho}|^2}. \tag{15}$$

Replacing (15) in (14) we obtain

$$\partial_t\widetilde{\rho} = (h - H)\sqrt{1 + |\nabla_x\widetilde{\rho}|^2}. \tag{16}$$

The next step is to calculate the mean curvature H in terms of $\widetilde{\rho}$. By (13), (15) and the definition (10) of ϕ it follows that

$$nH := \text{div}_x\left(\frac{\nabla_x\phi}{|\nabla_x\phi|}\right) = \quad \left(1 + |\nabla_x\widetilde{\rho}|^2\right)^{-\frac{1}{2}}\left(\frac{n}{|x|} - \Delta_x\widetilde{\rho}\right) \tag{17}$$

$$-\left(1 + |\nabla_x\widetilde{\rho}|^2\right)^{-\frac{3}{2}}|\nabla_x\widetilde{\rho}|\sum_{j=1}^{n+1}(\frac{x_j}{|x|} - \widetilde{\rho}_{x_j})\frac{\partial}{\partial_{x_j}}(|\nabla_x\widetilde{\rho}|).$$

Further,

$$\frac{\partial}{\partial_{x_j}}(|\nabla_x\widetilde{\rho}|) = \frac{\partial}{\partial x_j}\left(\left(\sum_{i=1}^{n+1}\widetilde{\rho}_{x_i}^2\right)^{\frac{1}{2}}\right) = |\nabla_x\widetilde{\rho}|^{-1}\sum_{i=1}^{n+1}\widetilde{\rho}_{x_i}\widetilde{\rho}_{x_ix_j},$$

so replacing in (17) we get

$$nH = \mathcal{A} - \left(1 + |\nabla_x\widetilde{\rho}|^2\right)^{-\frac{3}{2}}\mathcal{B}, \tag{18}$$

for

$$\mathcal{A} := \left(1 + |\nabla_x\widetilde{\rho}|^2\right)^{-\frac{1}{2}}\left(\frac{n}{|x|} - \Delta_x\widetilde{\rho}\right)$$

and

$$\mathcal{B} := \sum_{j=1}^{n+1}(\frac{x_j}{|x|} - \widetilde{\rho}_{x_j})\left(\sum_{i=1}^{n+1}\widetilde{\rho}_{x_i}\widetilde{\rho}_{x_ix_j}\right).$$

But we note that

$$\nabla_x\widetilde{\rho}\,\text{Hess}_x(\widetilde{\rho})\,\nabla_x\widetilde{\rho}^T = \sum_{j=1}^{n+1}\sum_{i=1}^{n+1}\widetilde{\rho}_{x_i}\widetilde{\rho}_{x_ix_j}\widetilde{\rho}_{x_j} = \sum_{j=1}^{n+1}\widetilde{\rho}_{x_j}\sum_{i=1}^{n+1}\widetilde{\rho}_{x_i}\widetilde{\rho}_{x_ix_j}, \tag{19}$$

thus replacing in $\mathcal{B}$ we get

$$\mathcal{B} = \sum_{j=1}^{n+1}\frac{x_j}{|x|}\left(\sum_{i=1}^{n+1}\widetilde{\rho}_{x_i}\widetilde{\rho}_{x_ix_j}\right) - \nabla_x\widetilde{\rho}\,\text{Hess}_x(\widetilde{\rho})\,\nabla_x\widetilde{\rho}^T, \tag{20}$$

where $\nabla_x\widetilde{\rho}^T$ is the transpose of $\nabla_x\widetilde{\rho}$. Since $x\perp\nabla_x\widetilde{\rho}$ we deduce that

$$\sum_{j=1}^{n+1}x_j\widetilde{\rho}_{x_j} = 0.$$

By differentiation with respect to x_i we obtain

$$\sum_{j=1}^{n+1}x_j\widetilde{\rho}_{x_jx_i} + \widetilde{\rho}_{x_i} = 0$$

and consequently

$$\sum_{j=1}^{n+1}\frac{x_j}{|x|}\left(\sum_{i=1}^{n+1}\widetilde{\rho}_{x_i}\widetilde{\rho}_{x_ix_j}\right) = -\frac{1}{|x|}\sum_{i=1}^{n+1}\widetilde{\rho}_{x_i}\widetilde{\rho}_{x_i} = -\frac{|\nabla_x\widetilde{\rho}|^2}{|x|}. \tag{21}$$

By (20) combined with (21) the next relation follows

$$\mathcal{B} = -\frac{|\nabla_x \widetilde{\rho}|^2}{|x|} - \nabla_x \widetilde{\rho} \operatorname{Hess}_x(\widetilde{\rho}) \nabla_x \widetilde{\rho}^T.$$

We replace $\mathcal{B}$ and $\mathcal{A}$ in (18) to arrive at

$$nH$$

$$= \frac{n}{|x|\sqrt{1 + |\nabla_x \widetilde{\rho}|^2}} - \frac{\Delta_x \widetilde{\rho}}{\sqrt{1 + |\nabla_x \widetilde{\rho}|^2}} + \frac{|\nabla_x \widetilde{\rho}|^2}{|x|(1 + |\nabla_x \widetilde{\rho}|^2)^{\frac{3}{2}}} + \frac{\nabla_x \widetilde{\rho} \operatorname{Hess}_x(\widetilde{\rho}) \nabla_x \widetilde{\rho}^T}{(1 + |\nabla_x \widetilde{\rho}|^2)^{\frac{3}{2}}}\Big|_{x \in S_t}. \tag{22}$$

Plugging the above in (16) we obtain equation (8). $\qquad\square$

The non-linear evolution equation for the VPMCF in polar coordinates is presented in the next theorem.

Theorem 3.2. *Let S_t be a graph in normal direction over Γ determined by the function $\rho(\cdot, t) : \Gamma \to \mathbb{R}$. If S_t satisfies the VPMCF (1) then ρ satisfies the evolution equation*

$$\partial_t \rho = G(\rho), \quad t \geq 0, \tag{23}$$

where

$$G(\rho) := J(\rho) + \frac{h}{\rho}\sqrt{\rho^2 + \mathcal{R}_2(\rho)}.$$

Here $J(\rho)$ is defined as

$$J(\rho) := \frac{1}{n}\Big\{ -\frac{n}{\rho} + \frac{\mathcal{R}_1(\rho)}{\rho^2} - \frac{\mathcal{R}_2(\rho)}{\rho(\rho^2 + \mathcal{R}_2(\rho))} - \frac{\mathcal{R}_3(\rho)}{\rho^2(\rho^2 + \mathcal{R}_2(\rho))} \Big\}, \tag{24}$$

with

$$\mathcal{R}_1(\rho) := \Big[\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 \rho}{\partial u_i \partial u_j} \sum_{m=1}^{n+1} \frac{\partial u_j}{\partial y_m} \frac{\partial u_i}{\partial y_m} + \sum_{q=1}^{n} \frac{\partial \rho}{\partial u_q} \sum_{m=1}^{n+1} \frac{\partial^2 u_q}{\partial y_m^2} \Big], \tag{25}$$

$$\mathcal{R}_2(\rho) := \sum_{k=1}^{n+1}\sum_{m=1}^{n}\sum_{i=1}^{n} \frac{\partial \rho}{\partial u_m} \frac{\partial u_m}{\partial y_k} \frac{\partial \rho}{\partial u_i} \frac{\partial u_i}{\partial y_k}, \tag{26}$$

$$\mathcal{R}_3(\rho) := \sum_{j=1}^{n+1}\sum_{i=1}^{n+1} \Big(\Big[\sum_{s=1}^{n} \frac{\partial \rho}{\partial u_s} \frac{\partial u_s}{\partial y_i} \Big] \tag{27}$$

$$\Big[\sum_{l=1}^{n}\sum_{k=1}^{n} \frac{\partial^2 \rho}{\partial u_l \partial u_k} \frac{\partial u_k}{\partial y_i} \frac{\partial u_l}{\partial y_j} + \sum_{q=1}^{n} \frac{\partial \rho}{\partial u_q} \frac{\partial^2 u_q}{\partial y_i \partial y_j} \Big] \Big[\sum_{q=1}^{n} \frac{\partial \rho}{\partial u_q} \frac{\partial u_q}{\partial y_j} \Big] \Big),$$

while

$$h := \frac{\int_{S_t} H \, d\sigma}{\int_{S_t} d\sigma} = -\Big(\int_{\Gamma} \rho J(\rho)(\rho^2 + \mathcal{R}_2(\rho))^{-\frac{1}{2}} \mu_\rho \Big)\Big(\int_{\Gamma} \mu_\rho \Big)^{-1},$$

where μ_ρ is the Jacobian in polar coordinates.

Proof. By Lemma 3.1 the VPMCF (1) is transformed to (8). We express (8) in terms of ρ. First, we calculate $\Delta_x \widetilde{\rho}(x)|_{x \in S_t}$. Let $x \in \mathbb{R}_*^{n+1}$, then for $\widetilde{\rho}(x, t) = \rho(u_1, \cdots, u_n, t)$ (polar coordinates in space), we apply the chain rule and use that $x = u_{n+1} y = |x| y$.

In details, for any $x \in \mathbb{R}_*^{n+1}$, we consider

$$\widetilde{\rho}(x, t) = \rho(u_1, \cdots, u_n, t)$$

and compute

$$
\begin{aligned}
\widetilde{\rho}_{x_l x_m} \quad &= \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2\rho}{\partial u_i \partial u_j}\frac{\partial u_j}{\partial x_l}\frac{\partial u_i}{\partial x_m} + \sum_{q=1}^{n}\frac{\partial\rho}{\partial u_q}\frac{\partial^2 u_q}{\partial x_l \partial x_m} \\
&= \frac{1}{|x|^2}\Big[\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2\rho}{\partial u_i \partial u_j}\frac{\partial u_j}{\partial y_l}\frac{\partial u_i}{\partial y_m} + \sum_{q=1}^{n}\frac{\partial\rho}{\partial u_q}\frac{\partial^2 u_q}{\partial y_l \partial y_m}\Big],
\end{aligned}
\tag{28}
$$

for any $l, m \le n+1$. Here we used that

$$
\frac{\partial u_i}{\partial x_m} = \frac{1}{|x|}\frac{\partial u_i}{\partial y_m},
$$

and that

$$
\frac{\partial^2 u_q}{\partial x_l \partial x_m} = \frac{1}{|x|^2}\frac{\partial^2 u_q}{\partial y_l \partial y_m},
$$

since for any $x \in \mathbb{R}_*^{n+1}$ there exists $y \in \Gamma$ such that $x = |x|y$. Note that y is defined as a parametrization of Γ, for example by (5), (6). Hence we obtain

$$
\begin{aligned}
\Delta_x \widetilde{\rho} = \quad &\sum_{m=1}^{n+1}\widetilde{\rho}_{x_m x_m} = \frac{1}{|x|^2}\sum_{m=1}^{n+1}\Big[\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2\rho}{\partial u_i \partial u_j}\frac{\partial u_j}{\partial y_m}\frac{\partial u_i}{\partial y_m} + \sum_{q=1}^{n}\frac{\partial\rho}{\partial u_q}\frac{\partial^2 u_q}{\partial y_m^2}\Big] \\
&= \quad \frac{1}{|x|^2}\Big[\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2\rho}{\partial u_i \partial u_j}\sum_{m=1}^{n+1}\frac{\partial u_j}{\partial y_m}\frac{\partial u_i}{\partial y_m} + \sum_{q=1}^{n}\frac{\partial\rho}{\partial u_q}\sum_{m=1}^{n+1}\frac{\partial^2 u_q}{\partial y_m^2}\Big].
\end{aligned}
\tag{29}
$$

If $x \in S_t$ then (9), (10) give that $|x| = 1 + \rho^*(\frac{x}{|x|}, t) = \rho(u_1, \cdots, u_n, t)$. Thus (29) yields

$$
\Delta_x \widetilde{\rho}(x)|_{x \in S_t} = \frac{1}{\rho^2}\mathcal{R}_1(\rho),
\tag{30}
$$

for $\mathcal{R}_1(\rho)$ defined by (25). We note that the terms appearing in (25) can be computed because y is a known function given for example by (5) or (6).

Applying the chain rule at $\widetilde{\rho}(x,t) = \rho(u_1, \cdots, u_n, t)$ we arrive at

$$
\frac{\partial\widetilde{\rho}}{\partial x_k} = \sum_{l=1}^{n}\frac{\partial\rho}{\partial u_l}\frac{\partial u_l}{\partial x_k},
$$

for any $k \le n+1$, consequently

$$
|\nabla_x \widetilde{\rho}|^2 = \sum_{k=1}^{n+1}\Big(\sum_{l=1}^{n}\frac{\partial\rho}{\partial u_l}\frac{\partial u_l}{\partial x_k}\Big)^2.
\tag{31}
$$

By (31) and by making use of the identity

$$
\Big(\sum_{l=1}^{n}\varepsilon_l\Big)^2 = \sum_{m=1}^{n}\sum_{i=1}^{n}\varepsilon_m \varepsilon_i
$$

for

$$
\varepsilon_l := \frac{\partial\rho}{\partial u_l}\frac{\partial u_l}{\partial x_k},
$$

we obtain

$$|\nabla_x \widetilde{\rho}|^2 = \sum_{k=1}^{n+1}\sum_{m=1}^{n}\sum_{i=1}^{n} \frac{\partial \rho}{\partial u_m}\frac{\partial u_m}{\partial x_k}\frac{\partial \rho}{\partial u_i}\frac{\partial u_i}{\partial x_k}$$

$$= \frac{1}{|x|^2}\sum_{k=1}^{n+1}\sum_{m=1}^{n}\sum_{i=1}^{n} \frac{\partial \rho}{\partial u_m}\frac{\partial u_m}{\partial y_k}\frac{\partial \rho}{\partial u_i}\frac{\partial u_i}{\partial y_k}, \quad x \in \mathbb{R}^{n+1}.$$

Thus we get

$$|\nabla_x \widetilde{\rho}|^2|_{x\in S_t} = \frac{1}{\rho^2}\mathcal{R}_2(\rho), \tag{32}$$

for $\mathcal{R}_2(\rho)$ defined by (26).

Next, we express in terms of ρ the operator $\nabla_x \widetilde{\rho}\,\mathrm{Hess}_x(\widetilde{\rho})\,\nabla_x \widetilde{\rho}^T$ appearing in (8). The equality $\widetilde{\rho}(x_1,\cdots,x_{n+1},t) = \rho(u_1,\cdots,u_n,t)$ yields

$$\widetilde{\rho}_{x_j} = \sum_{l=1}^{n}\frac{\partial \rho}{\partial u_l}\frac{\partial u_l}{\partial x_j}, \quad \widetilde{\rho}_{x_i x_j} = \sum_{l=1}^{n}\sum_{k=1}^{n}\frac{\partial^2 \rho}{\partial u_l \partial u_k}\frac{\partial u_k}{\partial x_i}\frac{\partial u_l}{\partial x_j} + \sum_{q=1}^{n}\frac{\partial \rho}{\partial u_q}\frac{\partial^2 u_q}{\partial x_i \partial x_j}.$$

By (19) and the above, we obtain for any $x \in \mathbb{R}^{n+1}$

$$\nabla_x \widetilde{\rho}\,\mathrm{Hess}_x(\widetilde{\rho})\,\nabla_x \widetilde{\rho}^T = \sum_{j=1}^{n+1}\sum_{i=1}^{n+1}\widetilde{\rho}_{x_i}\widetilde{\rho}_{x_i x_j}\widetilde{\rho}_{x_j} =$$

$$\sum_{j=1}^{n+1}\sum_{i=1}^{n+1}\left(\Big[\sum_{s=1}^{n}\frac{\partial \rho}{\partial u_s}\frac{\partial u_s}{\partial x_i}\Big]\Big[\sum_{l=1}^{n}\sum_{k=1}^{n}\frac{\partial^2 \rho}{\partial u_l \partial u_k}\frac{\partial u_k}{\partial x_i}\frac{\partial u_l}{\partial x_j} + \sum_{q=1}^{n}\frac{\partial \rho}{\partial u_q}\frac{\partial^2 u_q}{\partial x_i \partial x_j}\Big]\right. \tag{33}$$

$$\left.\Big[\sum_{q=1}^{n}\frac{\partial \rho}{\partial u_q}\frac{\partial u_q}{\partial x_j}\Big]\right) = \frac{1}{|x|^4}\sum_{j=1}^{n+1}\sum_{i=1}^{n+1}\left(\Big[\sum_{s=1}^{n}\frac{\partial \rho}{\partial u_s}\frac{\partial u_s}{\partial y_i}\Big]\right.$$

$$\left.\Big[\sum_{l=1}^{n}\sum_{k=1}^{n}\frac{\partial^2 \rho}{\partial u_l \partial u_k}\frac{\partial u_k}{\partial y_i}\frac{\partial u_l}{\partial y_j} + \sum_{q=1}^{n}\frac{\partial \rho}{\partial u_q}\frac{\partial^2 u_q}{\partial y_i \partial y_j}\Big]\Big[\sum_{q=1}^{n}\frac{\partial \rho}{\partial u_q}\frac{\partial u_q}{\partial y_j}\Big]\right).$$

Hence, we compute

$$\nabla_x \widetilde{\rho}\,\mathrm{Hess}_x(\widetilde{\rho})\,\nabla_x \widetilde{\rho}^T|_{x\in S_t} = \frac{1}{\rho^4}\mathcal{R}_3(\rho), \tag{34}$$

for $\mathcal{R}_3(\rho)$ given by (27).

Utilizing the definition of $J(\rho)$ in (24), the expression of H in (22) and the definitions of $\mathcal{R}_1(\rho), \mathcal{R}_2(\rho), \mathcal{R}_3(\rho)$, we obtain that

$$J(\rho) = -H\sqrt{1 + |\nabla_x \widetilde{\rho}|^2},$$

hence (32) yields

$$H = \frac{-J(\rho)\rho}{\sqrt{\rho^2 + \mathcal{R}_2(\rho)}}.$$

Using the previous expression in $h = \frac{\int_{S_t} H\,d\sigma}{\int_{S_t} d\sigma}$ and the values given by (30), (32), (34) in (8) we get finally (23) since $\partial_t \widetilde{\rho} = \partial_t \rho$. $\square$

Remark 2. The operators $\mathcal{R}_1, \mathcal{R}_2$ and $\mathcal{R}_3$, e.g. (25)-(27), may be expressed in terms of the Beltrami differential parameters of first and second order. Considering the first fundamental form $G = (G)_{ij}$, $i,j = 1,\cdots,n$ of the surface ρ we define in cartesians

$$\partial \rho := (\partial^1 \rho, \cdots, \partial^n \rho),$$

where

$$\partial^k \rho := \sum_{m=1}^{n} g^{km} \frac{\partial \rho}{\partial u_m}, \quad k = 1, \cdots, n,$$

while g^{km} are the elements of G^{-1}. Further let

$$|\partial \rho|_{\mathcal{G}}^2 := \partial \rho G \partial \rho^T.$$

This expression is equal to the first differential parameter of Beltrami which is invariant with respect to allowable transformations of coordinates, [10], [2]. We also define $\Delta_\Gamma \rho$ as the Laplace-Beltrami operator on the unit sphere which is invariant too and also called as second differential parameter of Beltrami, [10]. Finally, let the $n \times n$ matrix Hessρ be for t fixed the second covariant derivative of the scalar function $\rho(u_1, \cdots, u_n, t)$, [10], given by

$$(\text{Hess}\rho)_{rs} := \frac{\partial^2 \rho}{\partial u_r \partial u_s} - \sum_{p=1}^{n} \Gamma_{rs}^p \frac{\partial \rho}{\partial u_p},$$

where Γ_{rs}^p are the Christoffel symbols of second kind, then it follows that

$$\mathcal{R}_1(\rho) = \Delta_\Gamma \rho, \quad \mathcal{R}_2(\rho) = |\partial \rho|_{\mathcal{G}}^2, \quad \mathcal{R}_3(\rho) = \partial \rho \, \text{Hess}\rho \, \partial \rho^T,$$

and the VPMCF (1) admits an elegant representation in terms of Beltrami operators and of covariant Hessian, [2].

Remark 3. Considering the standard parametrization (6), we note that

$$u_i = \text{arccot}\Big(y_i (y_i^2 + y_{i+1}^2 + \cdots + y_{n+1}^2)^{-\frac{1}{2}} \Big), \quad i = 1, \cdots, n.$$

Remark 4. In order to compute an explicit formula for (23), we may use the standard parametrization given by (5), or (6). We supplement the non-linear p.d.e. (23) by an initial periodic condition $\rho(\cdot, 0)$ given for any

$$(u_1, \cdots, u_n) \in (0, \pi) \times \cdots \times (0, \pi) \times [0, 2\pi].$$

We also impose periodic and Dirichlet boundary conditions and derive an initial and boundary value problem. More specifically, if $n \geq 2$ we consider the p.d.e. (23) at any $t > 0$ for $(u_1, \cdots, u_{n-1})$ in the open set $\mathcal{A} := (0, \pi) \times \cdots \times (0, \pi)$, and for any $u_n \in (0, 2\pi)$. Since S_t is closed we impose a periodic boundary condition on the azimuth u_n by

$$\rho(u_1, \cdots, u_{n-1}, 0, t) = \rho(u_1, \cdots, u_{n-1}, 2\pi, t) \text{ for any } (u_1, \cdots, u_{n-1}) \in \mathcal{A}, \ t \geq 0.$$

In addition, we assign boundary values at the south and north poles $0, \pi$. Since the coordinate system is polar, these values at the poles must be independent of the azimuth u_n (u_n is measured along the equator), so we impose Dirichlet conditions on the azimuthal derivatives for any $u_n \in [0, 2\pi]$ and any $t > 0$ as follows

$$\frac{\partial \rho}{\partial u_n}(0, u_2, \cdots, u_n, t) = 0, \ \ \frac{\partial \rho}{\partial u_n}(\pi, u_2, \cdots, u_n, t) = 0, \ \ u_i \in (0, \pi), \ \ i \neq 1, n,$$

$$\frac{\partial \rho}{\partial u_n}(u_1, 0, \cdots, u_n, t) = 0, \ \ \frac{\partial \rho}{\partial u_n}(u_1, \pi, \cdots, u_n, t) = 0, \ \ u_i \in (0, \pi), \ \ i \neq 2, n,$$

....

$$\frac{\partial \rho}{\partial u_n}(u_1, u_2, \cdots, 0, u_n, t) = 0, \ \ \frac{\partial \rho}{\partial u_n}(u_1, u_2, \cdots, \pi, u_n, t) = 0,$$

$$u_i \in (0, \pi), \ \ i \neq n - 1, n.$$

For the case $n = 1$ we only consider periodicity on azimuth.

Remark 5. For the 2-dimensional VPMCF (23), i.e. in the case of curves in $\mathbb{R}^2$, let $n = 1$. We use the symbol $u_1 =: \theta$, to obtain $\rho = \rho(\theta, t)$, $y_1 = \cos\theta$, $y_2 = \sin\theta$, $\theta = \arctan(\frac{y_2}{y_1})$, $0 \leq \theta \leq 2\pi$, and we further compute $\frac{\partial\theta}{\partial y_1} = -\sin\theta$, $\frac{\partial\theta}{\partial y_2} = \cos\theta$, $\frac{\partial^2\theta}{\partial y_1^2} = -2\cos\theta\sin\theta$, $\frac{\partial^2\theta}{\partial y_1 \partial y_2} = -1 + 2\cos^2\theta$, $\frac{\partial^2\theta}{\partial y_2^2} = 2\cos\theta\sin\theta$. We replace in $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_3$ e.g. (25)-(27), to obtain after straightforward calculations that

$$\mathcal{R}_1(\rho) = \rho_{\theta\theta}, \ \ \mathcal{R}_2(\rho) = \rho_\theta^2, \ \ \mathcal{R}_3(\rho) = \rho_\theta^2 \rho_{\theta\theta}.$$

Thus by (24) we get

$$J(\rho) = \frac{\rho_{\theta\theta} - \frac{\rho_\theta^2}{\rho}}{\rho^2 + \rho_\theta^2} - \frac{1}{\rho}.$$

In order to calculate h we write

$$S_t = \Big\{ z \in \mathbb{R}^2 : \ z = (z_1(\theta, t), z_2(\theta, t)) = \rho(\theta, t)(\cos\theta, \sin\theta), \ \ \theta \in [0, 2\pi] \Big\},$$

thus $\int_{S_t} d\sigma = \int_0^{2\pi} \sqrt{z_{1\theta}^2 + z_{2\theta}^2} d\theta$. We compute $z_{1\theta}^2 + z_{2\theta}^2 = \rho_\theta^2 + \rho^2$ to get

$$\int_{S_t} d\sigma = \int_0^{2\pi} \sqrt{\rho_\theta^2 + \rho^2} d\theta, \ \ \ \ h = -\frac{\int_0^{2\pi} \rho J(\rho) d\theta}{\int_0^{2\pi} \sqrt{\rho_\theta^2 + \rho^2} d\theta}.$$

We replace in (23) and obtain the final equation

$$\partial_t \rho = \frac{\rho_{\theta\theta} - \frac{\rho_\theta^2}{\rho}}{\rho^2 + \rho_\theta^2} - \frac{1}{\rho} - \frac{\sqrt{\rho^2 + \rho_\theta^2} \int_0^{2\pi} \rho J(\rho) d\theta}{\rho \int_0^{2\pi} \sqrt{\rho_\theta^2 + \rho^2} d\theta}. \tag{35}$$

Remark 6. In three dimensions ($n = 2$) using (5) we write

$$S_t = \Big\{ z \in \mathbb{R}^3 : \ z = \rho(\theta, \phi, t)(\cos\theta\cos\phi, \sin\theta\cos\phi, \sin\phi), \ \theta \in [0, 2\pi], \ \phi \in [0, \pi] \Big\}.$$

In this case we may compute H by using the first and second fundamental forms for hypersurfaces.

4. Numerical experiments for the 2-dimensional VPMCF.

4.1. **Finite difference schemes.** We consider the case $n = 1$. The VPMCV can be presented (Remark 5, eqn. (35)) as the following non-linear initial and boundary value problem for ρ

$$\partial_t \rho = \frac{\rho_{\theta\theta} - \frac{\rho_\theta^2}{\rho}}{\rho^2 + \rho_\theta^2} - \frac{1}{\rho} + \sqrt{\rho^2 + \rho_\theta^2} \frac{\int_0^{2\pi} \left(\frac{-\rho_{\theta\theta} + \frac{\rho_\theta^2}{\rho}}{\rho^2 + \rho_\theta^2} + \frac{1}{\rho} \right) \rho \, d\theta}{\rho \int_0^{2\pi} \sqrt{\rho^2 + \rho_\theta^2} \, d\theta}, \ \ 0 < \theta < 2\pi, \ t > 0,$$

$$\rho(\theta, 0) = \rho_0(\theta), \ \ 0 \leq \theta \leq 2\pi, \ \ \ \ \rho(0, t) = \rho(2\pi, t), \ \ t \geq 0, \tag{36}$$

with periodic conditions and smooth periodic initial data ρ_0. We will consider the case when ρ_0 is non-convex. We approximate numerically (36) by explicit finite difference schemes using the trapezoid rule for the non-local integral terms.

More specifically, let define the uniform partition $0 = \theta_0 < \theta_1 < \theta_2 < \cdots < \theta_J = 2\pi$, with $\theta_j := jh$, $j = 0, \cdots J$, for $h := \frac{2\pi}{100}$, $J := 100$. We approximate the terms

$$\rho(\theta_j, t^n), \ \ \ \partial_t \rho(\theta_j, t^{n+1}), \ \ \ \rho_\theta(\theta_j, t^n), \ \ \ \rho_{\theta\theta}(\theta_j, t^n)$$

for $j = 1, \cdots, J - 1$, by

$$\rho_j^n, \ \ \ \frac{\rho_j^{n+1} - \rho_j^n}{k}, \ \ \ \frac{\rho_{j+1}^n - \rho_{j-1}^n}{2h} \ \text{ and } \ \frac{\rho_{j+1}^n - 2\rho_j^n + \rho_{j-1}^n}{h^2}$$

respectively, for $k = \frac{1}{100}$, and $t^n := nk$, $n = 0, \cdots, N$. We also approximate the values $\rho(\theta_0, t^{n+1}) = \rho(\theta_J, t^{n+1})$ by ρ_1^{n+1} for $n = 0, \cdots, N$, and use the initial condition $\rho_j^0 := \rho_0(\theta_j)$, $j = 1, \cdots, J$. Obviously

$$S_0 = \{z \in \mathbb{R}^2 : \ z = \rho_0(\theta)(\cos\theta, \sin\theta), \ \theta \in [0, 2\pi]\},$$

while for any $t > 0$

$$S_t = \{z \in \mathbb{R}^2 : \ z = \rho(\theta, t)(\cos\theta, \sin\theta), \ \theta \in [0, 2\pi]\}.$$

In details, let $\rho_j^0 := \rho_0(\theta_j)$, $j = 1, \cdots, J$. For any $n = 1, \cdots, N$ we solve the $J - 1 \times J - 1$ diagonal system

$$\frac{\rho_j^{n+1} - \rho_j^n}{k} = \frac{\frac{\rho_{j+1}^n - 2\rho_j^n + \rho_{j-1}^n}{h^2} - \frac{\left(\frac{\rho_{j+1}^n - \rho_{j-1}^n}{2h}\right)^2}{\rho_j^n}}{(\rho_j^n)^2 + \left(\frac{\rho_{j+1}^n - \rho_{j-1}^n}{2h}\right)^2} - \frac{1}{\rho_j^n} + \frac{\sqrt{(\rho_j^n)^2 + \left(\frac{\rho_{j+1}^n - \rho_{j-1}^n}{2h}\right)^2}}{\rho_j^n} \frac{A_n}{B_n},$$

where $j = 1, \cdots, J - 1$, while A_n, B_n are the approximations of the non-local integral terms of (36) at $t := t^n$ calculated by the trapezoid rule. Further using the periodic conditions for $j = J$ and any $n = 0, \cdots, N$ we set $\rho_0^{n+1} := \rho_1^{n+1}$, $\rho_J^{n+1} := \rho_1^{n+1}$.

4.2. **Numerical results.** For the first experiment (Case 1), we use as initial condition the following non-convex smooth and periodic function

$$\rho_0(\theta) = (4 + \cos^3\theta)(2 + \sin^3\theta).$$

Figure 2 presents the evolution of the closed initial curve S_0 for various times $t_0 = 0$, $t_1 = 1$, $t_2 = 10$, $t_3 = 20$. In this case the asymptotic convergence to a sphere is observable.
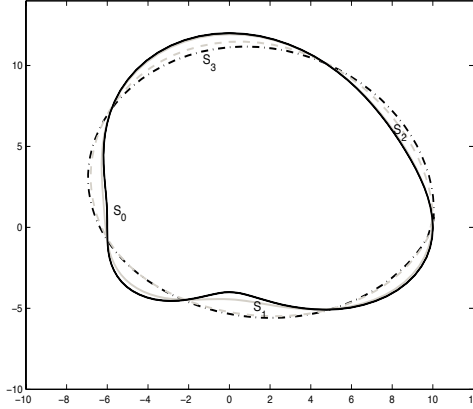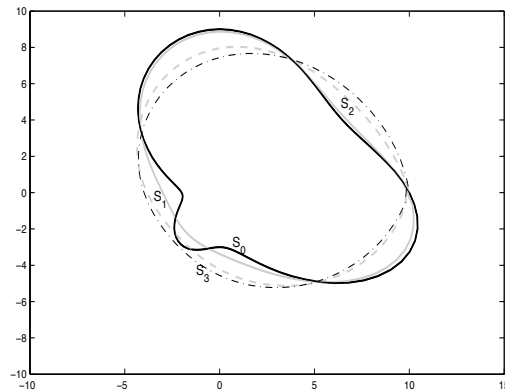

Figure 2: Case 1.

We next consider for the second experiment (Case 2)

$$\rho_0(\theta) = (1.5 + \cos^3\theta)(2 + \sin^3\theta)(2 - \cos^3\theta\sin\theta).$$

The above function creates a closed curve which is locally intensively non-convex. In Figure 3 we present S_t for $t_0 = 0$, $t_1 = 1$, $t_2 = 10$, $t_3 = 20$. A general observation stemming from these experiments is that the VPMCF converges first rapidly to a convex curve and after asymtotically to a sphere.

Figure 3: Case 2.

5. **Conclusions.** The (VPMCF) acting on normal graphs over the unit sphere is presented as a non-linear initial and boundary value evolutionary problem for the radial function in polar coordinates. The resulting equation is a second order partial differential equation containing some non-local integral terms. The (VPMCF) is an optimization procedure that drives hypersurfaces to spheres (i.e. to minimal area surfaces) under the constraint of constant enclosed volume. Our numerical results for the 2-dimensional case indicate that convexity is a local minimizer and arise the question if this is indeed true in two or higher dimensions.

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