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Singular Limit of a Spatially Inhomogeneous Lotka–Volterra Competition–Diffusion System

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We discuss the generation and the motion of internal layers for a Lotka-Volterra competition-diffusion system with spatially inhomogeneous coefficients. We assume that the corresponding ODE system has two stable equilibria $(\bar{u}, 0)$ and $(0, \bar{v})$ with equal strength of attraction in the sense to be specified later. The equation involves a small parameter ε, which reflects the fact that the diffusion is very small compared with the reaction terms. When the parameter ε is very small, the solution develops a clear transition layer between the region where the u species is dominant and the one where the v species is dominant. As ε tends to zero, the transition layer becomes a sharp interface, whose motion is subject to a certain law of motion, which is called the "interface equation". A formal asymptotic analysis suggests that the interface equation is the motion by mean curvature coupled with a drift term.

We will establish a rigorous mathematical theory both for the formation of internal layers at the initial stage and for the motion of those layers in the later stage. More precisely, we will show that, given virtually arbitrary smooth initial data, the solution develops an internal layer within the time scale of $O(\varepsilon^2 \log \varepsilon)$ and that the width of the layer is roughly of $O(\varepsilon)$. We will then prove that the motion of the layer converges to the formal interface equation as $\varepsilon \to 0$. Our results also give an optimal convergence rate, which has not been known even for spatially homogeneous problems.

Keywords Competition–diffusion; Interface motion; Matched asymptotic expansion; Nonlinear diffusion; Reaction–diffusion system; Singular perturbation.

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1. Introduction

It is well-known that some classes of nonlinear diffusion equations give rise to rather sharp transition layers (or interfaces) when the diffusion coefficient is very small or the reaction term is very large. And the motion of such interfaces is often driven by their curvature. In this paper we consider a Lotka–Volterra competition–diffusion system with spatially inhomogeneous coefficients of the form

$$\begin{cases} \varepsilon u_t = \varepsilon D_1 \nabla \cdot (k(x)\nabla u) + \dfrac{1}{\varepsilon} h(x)(R_1 - a_1 u - b_1 v)u & (x \in \Omega, t > 0), \\ \varepsilon v_t = \varepsilon D_2 \nabla \cdot (k(x)\nabla v) + \dfrac{1}{\varepsilon} h(x)(R_2 - a_2 u - b_2 v)v & (x \in \Omega, t > 0), \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 & (x \in \partial\Omega, t > 0), \\ u(x,0) = u_0(x), v(x,0) = v_0(x) & (x \in \Omega), \end{cases} \tag{1.1}$$

under the Neumann boundary conditions, where ε is a small parameter, Ω is a bounded domain in $\mathbf{R}^n$, $\partial/\partial\nu$ is the outward normal derivative on $\partial\Omega$, D_j, R_j, a_j, b_j $(j = 1, 2)$ are positive constants and $k(x), h(x)$ are positive C^2 functions representing spatially heterogeneous diffusion and reaction rates, respectively.

By suitable transformation, we can reduce equation (1.1) to

$$\begin{cases} \varepsilon u_t = \varepsilon \nabla \cdot (k(x)\nabla u) + \dfrac{1}{\varepsilon} h(x)(R_1 - u - bv)u, \\ \varepsilon v_t = \varepsilon D \nabla \cdot (k(x)\nabla v) + \dfrac{1}{\varepsilon} h(x)(R_2 - au - v)v, \end{cases} \quad (x \in \Omega, t > 0), \tag{1.2}$$

where a, b, D are positive constants. In what follows we assume

$$a > \frac{R_2}{R_1} > \frac{1}{b}, \tag{1.3}$$

which implies that the ODE system

$$\begin{cases} \dfrac{dp}{d\tau} = f(p,q), \\ \dfrac{dq}{d\tau} = g(p,q), \\ p(\xi,\eta,0) = \xi, \quad q(\xi,\eta,0) = \eta, \end{cases} \tag{1.4}$$

where

$$f(p,q) = (R_1 - p - bq)p, \quad g(p,q) = (R_2 - ap - q)q,$$

has two stable equilibria $(R_1, 0)$, $(0, R_2)$. In addition to the above two stable equilibria, the ODE system (1.4) has two other equilibria $(0, 0)$ and (u^*, v^*), where

$$u^* = \frac{bR_2 - R_1}{ab - 1}, \quad v^* = \frac{aR_1 - R_2}{ab - 1}. \tag{1.5}$$

The point (u^*, v^*) is a saddle, while $(0, 0)$ is an unstable node.

We also assume the following:

Assumption 1. The following system has a solution.

$$\begin{cases} U_{zz} + (R_1 - U - bV)U = 0 \quad (-\infty < z < +\infty), \\ DV_{zz} + (R_2 - aU - V)V = 0 \quad (-\infty < z < +\infty), \\ (U(-\infty), V(-\infty)) = (R_1, 0), \\ (U(+\infty), V(+\infty)) = (0, R_2). \end{cases} \tag{1.6}$$

Assumption 1 means that the rescaled diffusion system

$$\begin{cases} U_t = U_{zz} + (R_1 - U - bV)U \\ V_t = DV_{zz} + (R_2 - aU - V)V \end{cases} \quad (-\infty < z < +\infty) \tag{1.7}$$

has a stationary solution whose values at $z = -\infty$ and $z = +\infty$ are $(R_1, 0)$ and $(0, R_2)$; in other words, the speed of the travelling wave connecting $(R_1, 0)$ and $(0, R_2)$ is zero. Kan-on (1995) shows that under the bistability assumption (1.3), the system (1.7) has a unique travelling wave connecting $(R_1, 0)$ and $(0, R_2)$, and that the travelling wave speed depends continuously on the coefficients. He further studies the parameter range for which (1.6) has a solution. The fact that the travelling wave speed is zero can be interpreted that the two stable constant steady-states have "equal strength of attraction". Such a condition is quite standard in the context of a single Allen-Cahn type equation (Allen and Cahn, 1979)

$$U_t = U_{zz} + f(U) \quad \text{with } f(0) = f(1) = 0,$$

where it simply reduces to $\int_0^1 f(u)du = 0$.

When $\varepsilon > 0$ is very small, the domain Ω is loosely divided into two regions – the region where u is dominant and the one where v is dominant –, and the two regions are separated by rather a clear transition layer. As $\varepsilon \to 0$, a formal asymptotic analysis shows that this transition layer converges to a hypersurface $\Gamma(t)$ and that (u, v) takes the value $(R_1, 0)$ on one side of $\Gamma(t)$, while it takes the value $(0, R_2)$ on the other side. This hypersurface $\Gamma(t)$, which we may call a "sharp interface" moves according to the following law of motion:

$$V = -(N-1)k(x)\kappa - \frac{\partial}{\partial n}k(x) - \frac{2k(x)(C+1)}{K(x)}\frac{\partial}{\partial n}K(x). \tag{1.8}$$

Here V is the normal velocity of the interface $\Gamma(t)$, κ the mean curvature, n the unit normal vector and C is a constant determined by the coefficients D, R_j, a, b $(j = 1, 2)$ (see (2.22) below), $K(x)$ is defined by

$$K(x) = \sqrt{\frac{h(x)}{k(x)}}. \tag{1.9}$$

We call (1.8) the interface equation. It can be seen as a (formal) singular limit of the diffusion equation (1.2). Note that the signs of V, κ and $\partial/\partial n$ are all correlated, therefore equation (1.8) is well-defined irrespective of where n denotes the outer normal or inner normal.

Interestingly, the interface motion arising from the above equation involves not only a curvature term but also a drift term, despite the fact that no drift term is present in the original diffusion equation. The same thing has been observed earlier for the case of a single equation, (Ei et al., 1997; Hilhorst et al., in preparation; Nakamura et al., 1999), but the analysis is far more involved in the case of systems of equations.

Before stating our main results, we have to clarify the concept of transition layers for the system (1.2). To do so, let us introduce some notations. First, let the stable manifold of (u^*, v^*) be denoted by

$$S := \{(\xi, \eta) \in \mathbf{R}_+ \times \mathbf{R}_+ \mid (p(\tau; \xi, \eta), q(\tau; \xi, \eta)) \to (u^*, v^*) \text{ as } \tau \to \infty\}.$$

Here $\mathbf{R}_+ = (0, \infty)$ and $(p(\tau; \xi, \eta), q(\tau; \xi, \eta))$ is a solution of (1.4) with initial data (ξ, η). S is called a *separatrix* and it divides the first quadrant of the pq-plane into two parts, namely

$$\begin{aligned} \Delta_1 &:= \{(\xi, \eta) \in \mathbf{R}_+ \times \mathbf{R}_+ \mid (p(\tau; \xi, \eta), q(\tau; \xi, \eta)) \to (R_1, 0) \text{ as } \tau \to \infty\}, \\ \Delta_2 &:= \{(\xi, \eta) \in \mathbf{R}_+ \times \mathbf{R}_+ \mid (p(\tau; \xi, \eta), q(\tau; \xi, \eta)) \to (0, R_2) \text{ as } \tau \to \infty\}. \end{aligned} \tag{1.10}$$

This is due to the fact that every solution of (1.4) in the first quadrant $\{p > 0, q > 0\}$ converges to one of the four equilibria as $\tau \to \infty$ (see Lemma 3.1) and that no solution with $\xi > 0, \eta > 0$ converges to $(0, 0)$. This result is well-known, so we omit the proof; see for example, Chapter 12 of Hirsch and Smale (1974).

We will see later that, when ε is very small, the value of (u, v) is very close to either $(R_1, 0)$ or $(0, R_2)$ in most part of Ω. The domain Ω is divided into two regions: the one where the value of (u, v) is very close to $(R_1, 0)$ and the one where is nearly $(0, R_2)$, and steep transition layers appear between the two regions. The location of these layers is called the 'interface', and it is precisely where the value of (u, v) lies on S.

In view of this, we define the initial interface Γ_0 as follows:

$$\Gamma_0 = \{x \in \Omega \mid (u_0(x), v_0(x)) \in S\}.$$

Assumption 2 (Initial Data)**.** u_0, v_0 are continuous on $\overline{\Omega}$ and satisfy $|u_0(x)| + |v_0(x)| > 0$ on $\overline{\Omega}$.

Assumption 3 (Initial Interface)**.** Γ_0 is a C^2 closed hypersurface in Ω and satisfies $\partial\Omega \cap \Gamma_0 = \emptyset$.

Assumption 4 (Solution of the Interface Equation)**.** The classical solution $\Gamma(t)$ of (1.8) with initial data $\Gamma(0) = \Gamma_0$ exists on an interval $0 \le t \le T$ and is a smooth closed hypersurface in Ω for every $t \in [0, T]$.

Assumption 5 (Nondegeneracy Condition)**.** There exists a constant $A_0 > 0$ such that

$$\mathrm{dist}_{\mathbf{R}^2}((u_0(x), v_0(x)), S) \ge A_0 \mathrm{dist}(x, \Gamma_0) \quad x \in \Omega,$$

where $\mathrm{dist}_{\mathbf{R}^2}$ denotes the Euclidian distance on the uv-plane and dist denotes the Euclidian distance in $\mathbf{R}^n$.

Remark. Note that the existence of a local classical solution of (1.8) for any smooth initial hypersurface Γ_0 is well-known, as (1.8) can be reduced to a quasilinear uniformly parabolic equation on the manifold Γ_0, at least for small $t > 0$. See, for example, Hilhorst et al. (in preparation) for details. Assumption 4 simply requires that this local solution can be continued up to $t = T$ without developing singularity nor touching the boundary $\partial\Omega$.

The closed hypersurface $\Gamma(t)$ divides Ω into two parts:

$$\Omega_{in}(t) = \text{subregion of } \Omega \text{ inside } \Gamma(t), \quad \Omega_{out}(t) = \text{subregion of } \Omega \text{ outside } \Gamma(t).$$

Without loss of generality we may assume that $(u_0(x), v_0(x))$ satisfies

$$\begin{aligned} \Omega_{in}(0) &= \{x \in \Omega \mid (u_0(x), v_0(x)) \in \Delta_1\}, \\ \Omega_{out}(0) &= \{x \in \Omega \mid (u_0(x), v_0(x)) \in \Delta_2\}, \end{aligned} \tag{1.11}$$

since the same argument holds if we exchange Δ_1 and Δ_2.

Let the solution of (1.2) be denoted by $(u^\varepsilon, v^\varepsilon)$, and define $\Gamma^\varepsilon(t)$, $\Omega^\varepsilon_{in}(t)$, $\Omega^\varepsilon_{out}(t)$ as follows:

$$\begin{aligned} \Gamma^\varepsilon(t) &= \{x \in \Omega \mid (u^\varepsilon(x,t), v^\varepsilon(x,t)) \in S\}, \\ \Omega^\varepsilon_{in}(t) &= \{x \in \Omega \mid (u^\varepsilon(x,t), v^\varepsilon(x,t)) \in \Delta_1\}, \\ \Omega^\varepsilon_{out}(t) &= \{x \in \Omega \mid (u^\varepsilon(x,t), v^\varepsilon(x,t)) \in \Delta_2\}. \end{aligned}$$

In what follows we will call $\Gamma^\varepsilon(t)$ the interface at time t and set

$$\Gamma^\varepsilon = \bigcup_{0 \le t \le T} (\Gamma^\varepsilon(t) \times \{t\}). \tag{1.12}$$

We are now ready to state our main theorems:

Our first main theorem (Theorem 1) states that after a very short time of order $\varepsilon^2 \log(1/\varepsilon)$, the value of (u, v) comes close to $(R_1, 0)$ on one side of $\Gamma^\varepsilon(t)$ and to $(0, R_2)$ on the other side.

Theorem 1 (Generation of Interface)**.** *For any $\sigma > 0$, there exist $\widetilde{C} > 0$ and t^ε with $t^\varepsilon = O(\varepsilon^2 \log(1/\varepsilon))$ such that for all $t \in [t^\varepsilon, T]$,*

$$\begin{aligned} (u^\varepsilon(x,t), v^\varepsilon(x,t)) &\in \mathscr{B}_\sigma(R_1, 0) \quad \text{for } \Omega_{in}(t) \setminus \mathscr{N}_{C\varepsilon}(\Gamma(t)), \\ (u^\varepsilon(x,t), v^\varepsilon(x,t)) &\in \mathscr{B}_\sigma(0, R_2) \quad \text{for } \Omega_{out}(t) \setminus \mathscr{N}_{C\varepsilon}(\Gamma(t)), \end{aligned}$$

where $\mathscr{B}_\sigma(p_0.q_0) := \{(p,q) \in \mathbf{R}^2 \mid |(p,q) - (p_0, q_0)| < \sigma\}$.

Theorem 2 (Location of Interface)**.** *There exists a constant $C > 0$ such that*

$$d_{\mathscr{H}}(\Gamma^\varepsilon(t), \Gamma(t)) \le C\varepsilon \quad \text{for } 0 \le t \le T,$$

where $d_{\mathscr{H}}$ denotes the Hausdorff distance between compact sets. In other words, the following inclusion relations hold:

$$\Gamma^\varepsilon(t) \subset \mathscr{N}_{C\varepsilon}(\Gamma(t)), \quad \Gamma(t) \subset \mathscr{N}_{C\varepsilon}(\Gamma^\varepsilon(t)).$$

Here $\mathscr{N}_\delta(A)$ denotes the δ-neighborhood of a set A.

Corollary 1.1 (Convergence of Interface)**.** *The following hold for $0 \le t \le T$*:

$$\lim_{\varepsilon \to 0} \Gamma^\varepsilon(t) = \Gamma(t), \quad \lim_{\varepsilon \to 0} \Omega^\varepsilon_{in}(t) = \Omega_{in}(t), \quad \lim_{\varepsilon \to 0} \Omega^\varepsilon_{out}(t) = \Omega_{out}(t).$$

Here the convergence is in the sense of Hausdorff distance.

We also have the following convergence result which can be derived from Theorem 1 easily.

Theorem 3 (Convergence Away from Interface)**.** *The following holds for any $0 < t \le T$*:

$$\lim_{\varepsilon \to 0} (u^\varepsilon(x,t), v^\varepsilon(x,t)) = \begin{cases} (R_1, 0), & x \in \Omega_{in}(t) \\ (0, R_2), & x \in \Omega_{out}(t). \end{cases}$$

Moreover, the convergence is uniform on every compact set in $\overline{\Omega}\backslash\Gamma(t)$.

Our main tool for deriving the above results is the method of upper and lower solutions. We will use two different pairs of upper and lower solutions, namely $(u^\pm, v^\pm)$ and $(U^\pm, V^\pm)$. The first pair—(u^+, v^+) and (u^-, v^-)—is used to analyze the rapid formation of internal layers that takes place in a very fast time scale ("generation of interface"). The second pair—(U^+, V^+) and (U^-, V^-)—is used to study the motion of the internal layer in a relatively slow time scale ("motion of interface"). The transition from the initial stage to the second stage occurs within a time scale of $\varepsilon^2 \log(1/\varepsilon)$, but its precise timing of transition varies from place to place due to the inhomogeneity of the coefficients $k(x)$, $h(x)$. Since the behaviors of solutions are so different between the two stages, it is important to know the right timing to switch from $(u^\pm, v^\pm)$ to $(U^\pm, V^\pm)$ at each place. In this paper we will combine $(u^\pm, v^\pm)$ and $(U^\pm, V^\pm)$ with a variable switching time which depends on the location. A similar idea has been used in Hilhorst et al. (in preparation), which deals with a single Allen-Cahn equation in spatially inhomogeneous media.

The organization of this paper is as follows. In Section 2, we present a formal derivation of the interface equations corresponding to the system (1.2). In Section 3, we study the dynamics of the ODE system (1.4) in detail, particularly the behavior of orbits near the saddle point (u^*, v^*). This information will play a crucial role in Section 4, where we derive a result on the generation of interface, namely we prove that the solution develops internal layers within a time span of order $\varepsilon^2 \log(1/\varepsilon)$. As mentioned above, this will be done by constructing a suitable pair of upper and lower solutions, namely (u^+, v^+) and (u^-, v^-). These upper and lower solutions are constructed by using the solution of the ODE system (1.4). In Section 5 we construct another pair of upper and lower solutions, namely (U^+, V^+) and (U^-, V^-), that will be used for studying the motion of the interface. These upper and lower solutions are constructed by using the terms in the formal asymptotic expansion

(2.5). Finally, in Sections 6 and 7, we prove the main theorems. The key idea is to combine $(u^{\pm}, v^{\pm})$ and $(U^{\pm}, V^{\pm})$ to form a single pair of upper and lower solutions $(\hat{u}^{+}, \hat{v}^{+}), (\hat{u}^{-}, \hat{v}^{-})$.

2. A Formal Derivation of the Interface Equation

In this section we present a formal derivation of the interface equation for

$$\begin{cases} \varepsilon u_t = \varepsilon D_1 \nabla \cdot (k(x)\nabla u) + \dfrac{1}{\varepsilon} h(x)(R_1 - u - bv)u, & (x \in \mathbf{R}^N, t > 0), \\ \varepsilon v_t = \varepsilon D_2 \nabla \cdot (k(x)\nabla v) + \dfrac{1}{\varepsilon} h(x)(R_2 - au - v)v, & (x \in \mathbf{R}^N, t > 0). \end{cases} \tag{2.1}$$

Using the multiple-time scaling method, one finds that the evolution of solutions of (2.1) consists of two stages. To explain what these stages are, let us first consider the special case where $h(x) \equiv k(x) \equiv const.$, in which case system (2.1) reduces to

$$\begin{cases} \varepsilon u_t = \varepsilon \Delta u + \dfrac{1}{\varepsilon}(R_1 - u - bv)u, & x \in \mathbf{R}^N, \ \ t > 0, \\ \varepsilon v_t = \varepsilon D \Delta v + \dfrac{1}{\varepsilon}(R_2 - au - v)v, & x \in \mathbf{R}^N, \ \ t > 0. \end{cases}$$

In what follows $(u^{\varepsilon}, v^{\varepsilon})$ will denote a solution of the above equation.

In the first stage, which takes place in a very fast time scale of order $\tau = t/\varepsilon^2$, the effect of diffusion is negligible and $(u^{\varepsilon}, v^{\varepsilon})$ evolves according to the ordinary differential equation

$$\varepsilon u_t = (R_1 - u - bv)u, \quad \varepsilon v_t = (R_2 - au - v)v.$$

Thus the value of $(u^{\varepsilon}, v^{\varepsilon})$ quickly approaches $(R_1, 0)$ if $(u^{\varepsilon}(x, 0), v^{\varepsilon}(x, 0)) \in \Delta_1$ and approaches $(0, R_2)$ if $(u^{\varepsilon}(x, 0), v^{\varepsilon}(x, 0)) \in \Delta_2$. Accordingly, a steep transition layer develops between the two regions $\{(u^{\varepsilon}, v^{\varepsilon}) \approx (R_1, 0)\}$ and $\{(u^{\varepsilon}, v^{\varepsilon}) \approx (0, R_2)\}$, or, in other words, near the area where $\{x \in \mathbf{R}^N \mid (u^{\varepsilon}(x, 0), v^{\varepsilon}(x, 0)) \in S\}$, which coincides with Γ_0 in the previous notation.

In the second stage which takes place in a slower but still relatively fast time scale of order $\tilde{\tau} = t/\varepsilon$, the diffusion term Δu^{ε}, Δv^{ε} near the interface becomes large enough to balance the reaction term. Here the interface starts to move with normal velocity equal to its mean curvature (Ei and Yanagida, 1994).

When $h(x)$ or $k(x)$ is not a constant function, the above-mentioned scenario remains the same up to the first stage. An intriguing difference appears in the second stage. Namely that the normal velocity of the interface now depends not only on the curvature but also on the gradient of $h(x)$ and $k(x)$, thus the spatial inhomogeneity of the coefficient of the reaction term gives rise to a drift effect. In what follows we shall derive this law of motion by using the so-called matched asymptotic expansions along the same line as is done in Nakamura et al. (1999) and Hilhorst et al. (in preparation) for single equations.

2.1. *Matched Asymptotic Expansions*

Let $d(x, t)$ be the signed distance function with respect to the interface $\Gamma(t)$, namely,

$$d(x,t) = \begin{cases} -\text{dist}(x, \Gamma(t)) & x \in \overline{\Omega_{in}(t)}, \\ \text{dist}(x, \Gamma(t)) & x \in \Omega_{out}(t). \end{cases} \tag{2.2}$$

Here $\text{dist}(x, \Gamma(t))$ is the distance from x to the hypersurface $\Gamma(t)$ in $\mathbf{R}^N$. We also define

$$\begin{aligned} \Gamma &= \bigcup_{t\geq 0}(\Gamma(t) \times \{t\}), \\ Q_{in} &= \bigcup_{t\geq 0}(\Omega_{in}(t) \times \{t\}), \\ Q_{out} &= \bigcup_{t\geq 0}(\Omega_{out}(t) \times \{t\}). \end{aligned} \tag{2.3}$$

We assume that the solution $(u^\varepsilon, v^\varepsilon)$ has the expansions

$$\begin{aligned} u^\varepsilon(x,t) &= \widetilde{U}_0(x,t) + \varepsilon \widetilde{U}_1(x,t) + \varepsilon^2 \widetilde{U}_2(x,t) + \cdots \\ v^\varepsilon(x,t) &= \widetilde{V}_0(x,t) + \varepsilon \widetilde{V}_1(x,t) + \varepsilon^2 \widetilde{V}_2(x,t) + \cdots \end{aligned} \tag{2.4}$$

away from the interface $\Gamma^\varepsilon(t)$ (the outer expansion) and

$$\begin{aligned} u^\varepsilon(x,t) &= U_0(\zeta,x,t) + \varepsilon U_1(\zeta,x,t) + \varepsilon^2 U_2(\zeta,x,t) + \cdots \\ v^\varepsilon(x,t) &= V_0(\zeta,x,t) + \varepsilon V_1(\zeta,x,t) + \varepsilon^2 V_2(\zeta,x,t) + \cdots \end{aligned} \tag{2.5}$$

near $\Gamma^\varepsilon(t)$ (the inner expansion), where $\zeta = d(x,t)/\varepsilon$. The stretched space variable ζ gives exactly the right spatial scaling to describe the rapid transition between the regions $\{(u, v) \approx (R_1, 0)\}$ and $\{(u, v) \approx (0, R_2)\}$.

To make the inner and outer expansions consistent, we require that

$$(U_k(-\infty,x,t), V_k(-\infty,x,t)) = (U_k^{in}(x,t), V_k^{in}(x,t)) \quad \text{if } x \in \Omega_{in}(t) \cup \Gamma(t), \tag{2.6}$$

$$(U_k(+\infty,x,t), V_k(+\infty,x,t)) = (U_k^{out}(x,t), V_k^{out}(x,t)) \quad \text{if } x \in \Omega_{out}(t) \cup \Gamma(t), \tag{2.7}$$

for all (x, t) near Γ and all $k \geq 0$ (matching conditions), where (U_k^{in}, V_k^{in}) and (U_k^{out}, V_k^{out}) respectively denote the terms of outer expansion (2.4) in the region Q_{in} and the region Q_{out}. In particular, if $x \in \Gamma_t$, then one has to take into account both of the conditions (2.7), (2.6).

2.2. *Motion of the Interface for Equation* (1.2)

Substituting the outer expansion (2.4) into (1.2) and collecting the ε^{-1} and ε^0 terms respectively, we get

$$f(\widetilde{U}_0(x,t), \widetilde{V}_0(x,t)) = 0, \quad g(\widetilde{U}_0(x,t), \widetilde{V}_0(x,t)) = 0, \tag{2.8}$$

$$\begin{pmatrix} f_u(\widetilde{U}_0(x,t),\widetilde{V}_0(x,t)) & f_v(\widetilde{U}_0(x,t),\widetilde{V}_0(x,t)) \\ g_u(\widetilde{U}_0(x,t),\widetilde{V}_0(x,t)) & g_v(\widetilde{U}_0(x,t),\widetilde{V}_0(x,t)) \end{pmatrix} \begin{pmatrix} \widetilde{U}_1(x,t) \\ \widetilde{V}_1(x,t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{2.9}$$

in $Q_{in} \cup Q_{out}$. Equation (2.8) implies $(\widetilde{U}_0, \widetilde{V}_0) = (0, R_2)$, $(\widetilde{U}_0, \widetilde{V}_0) = (u^*, v^*)$ or $(\widetilde{U}_0, \widetilde{V}_0) = (R_1, 0)$. Since we are studying interfaces between the regions $\{(u,v) \approx (R_1, 0)\}$ and $\{(u,v) \approx (0, R_2)\}$, and since (1.11) holds, we have

$$(\widetilde{U}_0, \widetilde{V}_0) = (R_1, 0) \quad \text{in } Q_{in}, \quad \text{and} \quad (\widetilde{U}_0, \widetilde{V}_0) = (0, R_2) \quad \text{in } Q_{out}. \tag{2.10}$$

As for (2.9), note that

$$\begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix} = \begin{pmatrix} R_1 - u - bv & -bu \\ -cv & R_2 - cu - v \end{pmatrix}. \tag{2.11}$$

Consequently, we get $(\widetilde{U}_1(x,t), \widetilde{V}_1(x,t)) = (0,0)$ in $Q_{in} \cup Q_{out}$, since the matrix in (2.9) is equal to

$$\begin{pmatrix} -R_1 & -bR_1 \\ 0 & R_2 - aR_1 \end{pmatrix} \text{ in } Q_{in}, \quad \begin{pmatrix} R_1 - bR_2 & 0 \\ -aR_2 & -R_2 \end{pmatrix} \text{ in } Q_{out},$$

both of which by (1.3) are regular matrices. Next, substituting the inner expansion (2.5) into (1.2) and collecting the ε^{-1} and ε^0 terms, we obtain

$$k(x)\begin{pmatrix} U_{0\zeta\zeta} \\ DV_{0\zeta\zeta} \end{pmatrix} + h(x)\begin{pmatrix} f(U_0, V_0) \\ g(U_0, V_0) \end{pmatrix} = 0, \tag{2.12}$$

$$\begin{aligned} k(x)\begin{pmatrix} U_{1\zeta\zeta} \\ DV_{1\zeta\zeta} \end{pmatrix} &+ h(x)\begin{pmatrix} f_u(U_0, V_0) & f_v(U_0, V_0) \\ g_u(U_0, V_0) & g_v(U_0, V_0) \end{pmatrix}\begin{pmatrix} U_1 \\ V_1 \end{pmatrix} \\ &= \begin{pmatrix} U_{0\zeta} \\ DV_{0\zeta} \end{pmatrix}(d_t - \nabla \cdot (k(x)\nabla d)) - 2k(x)\begin{pmatrix} \nabla U_{0\zeta} \cdot \nabla d \\ D\nabla V_{0\zeta} \cdot \nabla d \end{pmatrix}. \end{aligned} \tag{2.13}$$

Both (2.12) and (2.13) are ordinary differential equations, with x, t acting the role of parameters. From (2.12) together with the matching conditions (2.6), (2.7), and (2.10), we find that

$$U_0(\zeta, x) = \phi_0(K(x)\zeta), \quad V_0(\zeta, x) = \psi_0(K(x)\zeta), \tag{2.14}$$

for all $\zeta \in \mathbf{R}$ and all (x,t) near Γ, where $K(x)$ is defined by (1.9) and $(\phi_0(z), \psi_0(z))$ is a solution of the stationary problem

$$\begin{cases} \phi_{zz} + f(\phi, \psi) = 0, & (-\infty < z < +\infty) \\ D\psi_{zz} + g(\phi, \psi) = 0, & (-\infty < z < +\infty) \\ (\phi(-\infty), \psi(-\infty)) = (R_1, 0), & (\phi(+\infty), \psi(+\infty)) = (0, R_2), \end{cases} \tag{2.15}$$

which is equivalent to (1.6).

By shifting the ζ coordinates, we may assume without loss of generality that

$$(\phi(0), \psi(0)) \in S, \tag{2.16}$$

which we call the normalization condition for (ϕ, ψ). It is shown in Kan-on (1995) that (2.16) determines the solution of (2.15) uniquely.

The following lemma gives estimates of ϕ_0, ψ_0 and their derivatives:

Lemma 2.1. *There exist constants* $C > 0$, $M > 0$ *such that*

$$0 < \phi_0(z) < C\exp(-M|z|), \quad 0 < R_2 - \psi_0(z) < C\exp(-M|z|) \quad \text{for } z \geq 0,$$
$$0 < R_1 - \phi_0(z) < C\exp(-M|z|), \quad 0 < \psi_0(z) < C\exp(-M|z|), \quad \text{for } z \leq 0$$
$$\phi_0'(z) > 0, \quad \psi_0'(z) < 0 \quad \text{for } z \in \mathbf{R}$$
$$|D^j\phi_0(z)| < C\exp(-M|z|), \quad |D^j\psi_0(z)| < C\exp(-M|z|) \quad \text{for } z \in \mathbf{R}, \ j = 1, 2.$$

Proof. The strict monotonicity of ϕ_0, ψ_0 is proved in Kan-on (1995). The exponential decay estimates follow from the fact that, in the 4-dimensional phase space corresponding to the system (2.15), the equilibria $(\phi, \psi, \phi_z, \psi_z) = (R_1, 0, 0, 0)$ and $(\phi, \psi, \phi_z, \psi_z) = (0, R_2, 0, 0)$ are non-degenerate saddle points. Details are omitted.

Substituting (2.14) into (2.13), we get

$$k(x)\begin{pmatrix} U_{1\zeta\zeta} \\ DV_{1\zeta\zeta} \end{pmatrix} + h(x)\begin{pmatrix} f_u(U_0, V_0) & f_v(U_0, V_0) \\ g_u(U_0, V_0) & g_v(U_0, V_0) \end{pmatrix}\begin{pmatrix} U_1 \\ V_1 \end{pmatrix}$$
$$= K(x)\begin{pmatrix} \phi_0' \\ D\psi_0' \end{pmatrix}(d_t - \nabla\cdot(k(x)\nabla d)) - 2k(x)\nabla d\cdot\nabla(K(x))\begin{pmatrix} \phi_0''\zeta K(x) + \phi_0' \\ D\psi_0''\zeta K(x) + D\psi_0' \end{pmatrix}, \tag{2.17}$$

with the normalization condition $U_1(0, x, t) = 0$. To give a solvability condition for (2.17), we need the following two lemmas.

Lemma 2.2. *There exists a solution* $(\phi^*(z, x, t), \psi^*(z, x, t))$ *of the equation*

$$\begin{pmatrix} \phi^*_{zz} \\ D\psi^*_{zz} \end{pmatrix} + \begin{pmatrix} f_u(U_0, V_0) & g_u(U_0, V_0) \\ f_v(U_0, V_0) & g_v(U_0, V_0) \end{pmatrix}\begin{pmatrix} \phi^* \\ \psi^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (-\infty < z < +\infty), \tag{2.18}$$

satisfying $\phi^* > 0$, $\psi^* < 0$. *Moreover, the solution of* (2.18) *is unique up to multiplication of a constant.*

Remark. (ϕ^*, ψ^*) spans the kernel of the adjoint operator of

$$\frac{d^2}{dz^2} + \begin{pmatrix} f_u(U_0, V_0) & f_v(U_0, V_0) \\ g_u(U_0, V_0) & g_v(U_0, V_0) \end{pmatrix}.$$

With the notation in (4.5), the condition $\phi^* > 0$, $\psi^* < 0$ can be written as $(\phi^*, \psi^*) \succ (0, 0)$. Therefore Lemma 2.2 states that the adjoint operator has a positive eigenfunction corresponding to the eigenvalue 0.

Lemma 2.2 is a consequence of the fact that: (a) the above operator and its adjoint have the same spectra; (b) only the principal eigenvalue has an eigenfunction satisfying $(\phi, \psi) \succ 0$; (c) the principal eigenvalue is simple. For details see Volpert et al. (1994, Proposition 1.3, pp. 155–156), and Alexander et al. (1990).

Lemma 2.3. *Let* $(A_1(z, x, t), A_2(z, x, t))$ *be given and assume that* $A_i(z, x, t) = O(e^{-\delta|z|})$ *as* $|z| \to \infty$ *for some* $\delta > 0$ *for* $i = 1, 2$. *Then for each fixed* (x, t), *the following equation*

$$\begin{pmatrix} \phi_{zz} \\ D\psi_{zz} \end{pmatrix} + \begin{pmatrix} f_u(U_0, V_0) & f_v(U_0, V_0) \\ g_u(U_0, V_0) & g_v(U_0, V_0) \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} A_1(z, x, t) \\ A_2(z, x, t) \end{pmatrix} \tag{2.19}$$

has a solution if and only if

$$\int_{\mathbf{R}} \{\phi^*(z, x, t)A_1(z, x, t) + \psi^*(z, x, t)A_2(z, x, t)\}dz = 0.$$

In addition, the solution, if it exists, is unique under the normalization condition $\phi_1(0, x, t) = 0$ *and satisfies*

$$\phi(z, x, t) = O\big(e^{-\hat{\delta}|z|}\big), \quad \psi(z, x, t) = O\big(e^{-\hat{\delta}|z|}\big) \tag{2.20}$$

for some $\hat{\delta} \in (0, \delta]$ *as* $|z| \to \infty$.

Proof. Since (ϕ^*, ψ^*) is the kernel of

$$\frac{d^2}{dx^2} + \begin{pmatrix} f_u(U_0, V_0) & f_v(U_0, V_0) \\ g_u(U_0, V_0) & g_v(U_0, V_0) \end{pmatrix},$$

by Volpert et al. (1994), one can use the method of variation of constants to find a solution of (2.19) explicitly. Then the lemma follows from direct calculation.

By Lemma 2.3, the solvability condition for (2.17) is written as

$$[\{d_t - \nabla \cdot (k(x)\nabla d)\}K(x) - 2k\nabla d \cdot \nabla K] \int_{\mathbf{R}} \{\phi^*\phi' + D\psi^*\psi'\}dz$$
$$- 2k\nabla d \cdot \nabla K \int_{\mathbf{R}} \{z\phi^*\phi'' + zD\psi^*\psi''\}dz = 0.$$

Lemma 2.2 assures that $\int_{\mathbf{R}} \{\phi^*\phi' + D\psi^*\psi'\}dz < 0$, which implies

$$d_t - \nabla(k(x)\nabla d) = \frac{2(C + 1)k\nabla d\nabla K}{K}, \tag{2.21}$$

$$C = \frac{\int_{\mathbf{R}} \{z\phi^*\phi_0'' + zD\psi^*\psi_0''\}dz}{\int_{\mathbf{R}} \{\phi^*\phi_0' + D\psi^*\psi_0'\}dz}. \tag{2.22}$$

Incidentally, we have

$$U_1(\zeta, x, t) = \frac{\nabla K\nabla d}{K^2}\phi_1(K(x)\zeta, x, t), \quad V_1(\zeta, x, t) = \frac{\nabla K\nabla d}{K^2}\psi_1(K(x)\zeta, x, t), \tag{2.23}$$

where (ϕ_1, ψ_1) is a solution of

$$\begin{cases} \begin{pmatrix} \phi_{zz} \\ D\psi_{zz} \end{pmatrix} + \begin{pmatrix} f_u(U_0, V_0) & f_v(U_0, V_0) \\ g_u(U_0, V_0) & g_v(U_0, V_0) \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix} = -2z \begin{pmatrix} \phi_{0zz} \\ D\psi_{0zz} \end{pmatrix} + 2C \begin{pmatrix} \phi_{0z} \\ D\psi_{0z} \end{pmatrix}, \\ \begin{pmatrix} \phi(-\infty) \\ \psi(-\infty) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \phi(+\infty) \\ \psi(+\infty) \end{pmatrix} = \begin{pmatrix} 0 \\ 0, \end{pmatrix} \end{cases} \tag{2.24}$$

and where C is defined in (2.22).

Let us derive the equation of interface motion from (2.21). Since ∇d $(=\nabla_x d(x, t))$ coincides with the outward normal unit vector to the hypersurface Γ_t, one easily sees that $-d_t(x, t) = V$, where V is the normal velocity of the interface Γ_t. It is also known that the mean curvature κ of the interface is equal to $\nabla^2 d/(N-1)$. Thus the equation (2.21) is equivalent to

$$V = -(N-1)k(x)\kappa - \frac{\partial}{\partial n}(k(x)) - \frac{2(C+1)k}{K}\frac{\partial}{\partial n}(K(x)) \quad \text{on } \Gamma_t, \tag{2.25}$$

where C is given in (2.22).

3. Basic Properties of the ODE

In this section we discuss dynamics of the ODE (1.4) that will be needed later. We first start with some basic properties that hold not only for the Lotka–Volterra system (1.4) but for any two-species competition system of the form

$$\begin{cases} \dfrac{dp}{d\tau} = F(p, q), \\ \dfrac{dq}{d\tau} = G(p, q). \end{cases} \tag{3.1}$$

The ODE system (3.1) is called a "competition system" in the region $\Delta := \{(p, q) \in \mathbf{R}^2 \mid p \geq 0, q \geq 0\}$ if

$$\frac{\partial F}{\partial q} \leq 0, \quad \frac{\partial G}{\partial p} \leq 0 \tag{3.2}$$

holds for any $(p, q) \in \Delta$. The following property is well known:

Lemma 3.1 (Comparison Principle). *Let* (3.1) *be a competition system in* Δ. *Suppose that* $(p_1(\tau), q_1(\tau))$ *and* $(p_2(\tau), q_2(\tau))$ *satisfy* $(p_i(\tau), q_i(\tau)) \in \Delta$ $(i = 1, 2)$,

$$\frac{dp_1}{d\tau} \geq f(p_1, q_1), \quad \frac{dq_1}{d\tau} \leq g(p_1, q_1) \tag{3.3}$$

$$\frac{dp_2}{d\tau} \leq f(p_2, q_2), \quad \frac{dq_2}{d\tau} \geq g(p_2, q_2), \tag{3.4}$$

for $\tau \geq 0$ *and that*

$$p_1(0) \geq p_2(0), \quad q_1(0) \leq q_2(0).$$

Then

$$p_1(\tau) \geq p_2(\tau), \quad q_1(\tau) \leq q_2(\tau) \text{ for } \tau \geq 0.$$

For the proof see, for example, Hirsch and Smale (1974). We say $(p_1(\tau), p_2(\tau))$ is *an upper solution* if it satisfies (3.3), while any function $(p_2(\tau), q_2(\tau))$ satisfying (3.4) is called *a lower solution*. A function $(p(\tau), q(\tau))$ is a solution of (3.1) if and only if it is an upper solution and a lower solution at the same time. The following lemma is also well known:

Lemma 3.2. *Let* $(p(\tau), q(\tau))$ *be a bounded solution of the competition system* (3.1) *satisfying* $(p(\tau), q(\tau)) \in \Delta$ $(i = 1, 2)$ *for* $\tau \geq 0$. *Then* $(p(\tau), q(\tau))$ *converges to an equilibrium point as* $\tau \to \infty$.

Proof. Set $\varphi = \frac{dp}{d\tau}$, $\psi = \frac{dq}{d\tau}$. Then $\varphi(\tau), \psi(\tau)$ satisfy

$$\frac{d\varphi}{d\tau} = a(\tau)\varphi + \beta(\tau)\psi, \quad \frac{d\psi}{d\tau} = \gamma(\tau)\varphi + \delta(\tau)\psi,$$

where

$$a(\tau) = \frac{\partial F}{\partial p}(p(\tau), q(\tau)), \quad \beta(\tau) = \frac{\partial F}{\partial q}(p(\tau), q(\tau)),$$
$$\gamma(\tau) = \frac{\partial G}{\partial p}(p(\tau), q(\tau)), \quad \delta(\tau) = \frac{\partial G}{\partial q}(p(\tau), q(\tau)).$$

By the assumption, we have $\beta(\tau) \leq 0$, $\gamma(\tau) \leq 0$. Therefore, if $\varphi(\tau_0) \geq 0$, $\psi(\tau_0) \leq 0$ for some $\tau_0 \geq 0$, then we have

$$\varphi(\tau) \geq 0, \quad \psi(\tau) \leq 0 \text{ for any } \tau \geq \tau_0.$$

This means both $p(\tau)$ and $q(\tau)$ are monotone in $\tau \geq \tau_0$, therefore they converge as $\tau \to \infty$. The same conclusion holds if $\varphi(\tau_0) \leq 0$, $\psi(\tau_0) \geq 0$ for some $\tau_0 \geq 0$. It remains to consider the case where neither of these inequalities hold for any $\tau_0 \geq 0$. In other words we have either $\varphi(\tau) \geq 0$, $\psi(\tau) \geq 0$ for all $\tau \geq 0$ or $\varphi(\tau) \leq 0$, $\psi(\tau) \leq 0$ for all $\tau \geq 0$. In either case, $p(\tau)$ and $q(\tau)$ are monotone, hence the convergence of $p(\tau)$, $q(\tau)$ follows. The lemma is proved.

Now we turn to our original system (1.4). As we mentioned earlier, $(R_1, 0)$, $(0, R_2)$ are stable nodes, $(0, 0)$ is an unstable node and (u^*, v^*) is a saddle point. Also recall that the separatrix

$$S := \{(\xi, \eta) \in \mathbf{R}_+ \times \mathbf{R}_+ \mid (p(\tau; \xi, \eta), q(\tau; \xi, \eta)) \to (u^*, v^*) \text{ as } \tau \to \infty\}$$

divides the first quadrant into the two regions Δ_1, Δ_2 defined in (1.10).

Lemma 3.3. *The separatrix* S *is expressed as the graph of a strictly monotone increasing function* $v = W(u)$ *satisfying* $W(0) = 0$.

Proof. We first show that every orbit $(p(\tau), q(\tau))$ on $S\backslash\{(u^*, v^*)\}$ satisfies either

$$p'(\tau) > 0, q'(\tau) > 0 \quad \text{or} \quad p'(\tau) < 0, q'(\tau) < 0. \tag{3.5}$$

Suppose the contrary. Then there exists $\tau_0 \in \mathbf{R}^2$ such that $p'(\tau_0) \geq 0$, $q'(\tau_0) \leq 0$ or $p'(\tau_0) \leq 0$, $q'(\tau_0) \geq 0$. Then as we have seen in the proof of the previous lemma, we have

$$p'(\tau) \geq 0, q'(\tau) \leq 0 \text{ for } \tau \geq \tau_0 \quad \text{or} \quad p'(\tau) \leq 0, q'(\tau) \geq 0 \text{ for } \tau \geq \tau_0.$$

However, since $(p(\tau), q(\tau))$ converges to (u^*, v^*) as $\tau \to \infty$, a local phase plane analysis near the saddle point (u^*, v^*) shows that either $p'(\tau) > 0$, $q'(\tau) > 0$ or $p'(\tau) < 0$, $q'(\tau) < 0$ for all large τ. This contradiction proves (3.5). The conclusion of the lemma easily follows from this observation and the fact that $(0, 0)$ lies on the boundary of both Δ_1 and Δ_2.

Our next task is to investigate the behavior of orbits near the separatrix S in more detail. Before doing so, we choose a large constant $\widetilde{M} > 0$ and small constants $\sigma_0, \sigma_1 > 0$ such that

$$\sigma_0 \leq u_0(x), v_0(x) \leq \widetilde{M} \quad \text{for } x \in \overline{\Omega}. \tag{3.6}$$

and that $\mathscr{B}_{\sigma_0}(0, 0), \mathscr{B}_{\sigma_1}(R_1, 0), \mathscr{B}_{\sigma_1}(0, R_2), \mathscr{B}_{\sigma_1}(u^*, v^*)$ are all mutually disjoint, where $\mathscr{B}_\sigma(u, v)$ denotes the σ-neighborhood of the point (u, v) in the uv-plane. We also assume that σ_1 is chosen sufficiently small so that the constant $\mu := -\beta^* - C_4\sigma_1$, which appears in (3.19), is positive. Hereafter we fix these constants $\widetilde{M}, \sigma_0, \sigma_1 > 0$ and define

$$\Theta = [0, \widetilde{M}] \times [0, \widetilde{M}]\backslash\big(\mathscr{B}_{\sigma_0}(0, 0) \cup \mathscr{B}_{\sigma_1}(R_1, 0) \cup \mathscr{B}_{\sigma_1}(0, R_2) \cup \mathscr{B}_{\sigma_1}(u^*, v^*)\big),$$

where $\mathscr{B}_\sigma(u, v)$ denotes the σ-neighborhood of the point (u, v) in the uv-plane.

The next lemma shows that the solution (p, q) cannot stay in Θ for a very long time:

Lemma 3.4. *For each $(\xi, \eta) \in \Theta$, define*

$$\tilde{\tau}(\xi, \eta) = \sup\{\tau \geq 0 \,|\, (p(s; \xi, \eta), q(s; \xi, \eta)) \in \Theta \textit{ for every } s \in [0, \tau]\},$$

then there exists $C_1' > 0$ such that

$$\tilde{\tau}(\xi, \eta) \leq C_1' \quad \textit{for every } (\xi, \eta) \in \Theta.$$

Proof. Since (p, q) converges to an equilibrium point as $\tau \to \infty$, we have $\tilde{\tau}(\xi, \eta) < \infty$ for every $(\xi, \eta) \in \Theta$. In order to prove that $\sup_{(\xi,\eta)\in\Theta} \tilde{\tau}(\xi, \eta) < \infty$, suppose the contrary.

Then there exists a sequence $(\xi_n, \eta_n) \in \Theta$ such that $\tilde{\tau}(\xi_n, \eta_n) \to \infty$ as $n \to \infty$. Since Θ is compact, we may assume without loss of generality that (ξ_n, η_n) converges to some $(\xi_\infty, \eta_\infty) \in \Theta$ as $n \to \infty$. By the definition of $\tilde{\tau}(\xi, \eta)$, we have

$$(p(\tau; \xi_n, \eta_n), q(\tau; \xi_n, \eta_n)) \in \Theta \quad \text{for } \tau \in [0, \tilde{\tau}(\xi_n, \eta_n)].$$

Letting $n \to \infty$ and recalling that Θ is a closed set, we obtain

$$(p(\tau; \xi_\infty, \eta_\infty), q(\tau; \xi_\infty, \eta_\infty)) \in \Theta \quad \text{for all } \tau \geq 0.$$

This, however, is impossible, since $(p(\tau; \xi_\infty, \eta_\infty), q(\tau; \xi_\infty, \eta_\infty))$ must converge to an equilibrium point as $\tau \to \infty$ but there is no equilibrium point in Θ. The lemma is proved.

The following two lemmas show that the solution $(p(\tau; \xi, \eta), q(\tau; \xi, \eta))$ cannot come close to S if the initial data (ξ, η) is not close to S.

Lemma 3.5. *There exists $C_2' > 0$ such that for any initial data $(\xi, \eta) \in \Theta$ it holds that*

$$\operatorname{dist}((p(\tau; \xi, \eta), q(\tau; \xi, \eta)), S) \geq C_2' \ \operatorname{dist}((\xi, \eta), S) \quad \textit{for } 0 \leq \tau \leq \tilde{\tau}(\xi, \eta).$$

Proof. For any $(\xi, \eta) \in \Theta$, we can choose a point $(\xi^s, \eta^s) \in S$ such that

$$\sqrt{(\xi - \xi^s)^2 + (\eta - \eta^s)^2} = dist((\xi, \eta), S).$$

Since S has a positive slope by Lemma 3.2 and since the vector $(\xi - \xi^s, \eta - \eta^s)$ is orthogonal to the tangent line of S at (ξ^s, η^s), $\xi - \xi^s$, and $\eta - \eta^s$ have opposite signs with respect to each other. Without loss of generality, we may assume that

$$\xi - \xi^s \geq 0, \quad \eta - \eta^s \leq 0. \tag{3.7}$$

We write $(p(\tau), q(\tau)) := (p(\tau; \xi, \eta), q(\tau; \xi, \eta))$ and $(p^s(\tau), q^s(\tau)) := (p(\tau; \xi^s, \eta^s), q(\tau; \xi^s, \eta^s))$ for simplicity and set $P(\tau) = p(\tau) - p^s(\tau)$, $Q(\tau) = q(\tau) - q^s(\tau)$. Then since both of $(p(\tau), q(\tau))$ and $(p^s(\tau), q^s(\tau))$ are solutions of (1.4), we have

$$\frac{d}{d\tau}\begin{pmatrix} P \\ Q \end{pmatrix} = \begin{pmatrix} f_u(s_1, r_1) & f_v(s_2, r_2) \\ g_u(s_3, r_3) & g_v(s_4, r_4) \end{pmatrix}\begin{pmatrix} P \\ Q \end{pmatrix}, \quad \begin{pmatrix} P(0) \\ Q(0) \end{pmatrix} = \begin{pmatrix} \xi - \xi^s \\ \eta - \eta^s \end{pmatrix}. \tag{3.8}$$

Here $s_i := s_i(\tau)$ is some value between p and p^s, $r_i := r_i(\tau)$ is some value between q and q^s, for $i = 1, 2, 3, 4$. It is easily seen that $P(\tau) \geq 0$, $Q(\tau) \leq 0$ by considering the sign of f_v, g_u and the initial condition (3.7). Note that the solution of the following ODE

$$\frac{d}{d\tau}\widehat{P} = h_1\widehat{P}, \quad \frac{d}{d\tau}\widehat{Q} = h_2\widehat{Q}$$
$$\widehat{P}(0) = \xi - \xi^s, \quad \widehat{Q}(0) = \eta - \eta^s,$$

with

$$h_1 = \min_{(s,r)\in[0,\widetilde{M}]\times[0,\widetilde{M}]} f_u(s, r), \quad h_2 = \min_{(s,r)\in[0,\widetilde{M}]\times[0,\widetilde{M}]} g_v(s, r)$$

is a lower solution of (3.8). Consequently by Lemma 3.1, we have

$$\begin{aligned} P(\tau) &\geq \widehat{P}(\tau) = (\xi - \xi^s)\exp(h_1\tau), \\ Q(\tau) &\leq \widehat{Q}(\tau) = (\eta - \eta^s)\exp(h_2\tau). \end{aligned} \tag{3.9}$$

This and Lemma 3.4 imply that

$$P(\tau) \geq e^{-h_1 C_1'} P(0), \quad Q(\tau) \leq e^{-h_2 C_1'} Q(0) \tag{3.10}$$

for all $0 \leq \tau \leq \hat{\tau}(\xi, \eta)$. This implies, in particular, that $P(\tau) > 0$, $Q(\tau) < 0$, therefore the vector $(p(\tau) - p^s(\tau), q(\tau) - q^s(\tau))$ always forms a strictly positive angle with the separatrix S, since the latter has a strictly positive slope. Consequently, there exist positive constants θ_1, θ_2 such that

$$\theta_1\sqrt{(\xi - \xi^s)^2 + (\eta - \eta^s)^2} \leq \text{dist}((p(\tau), q(\tau)), S) \leq \theta_2\sqrt{(\xi - \xi^s) + (\eta - \eta^s)^2}.$$

This and (3.10) prove the lemma.

Next, we discuss the behavior of (p, q) in a neighborhood of the saddle point (u^*, v^*). The linearization of (1.4) around the saddle point (u^*, v^*) is given by the matrix

$$\begin{pmatrix} -u^* & -bu^* \\ -cv^* & -v^* \end{pmatrix} \tag{3.11}$$

We denote by α^*, β^* the eigenvalues of this matrix where $\alpha^* > 0 > \beta^*$.

Lemma 3.6. *Let σ_1 be sufficiently small. There exist $C_3', C_4' > 0$ such that for any initial data $(\xi, \eta) \in \mathcal{B}_{\sigma_1}(u^*, v^*)$, the following inequalities hold as long as $(p(\tau; \xi, \eta), q(\tau; \xi, \eta))$ remains in $\mathcal{B}_{\sigma_1}(u^*, v^*)$:*

$$\begin{aligned} C_3' \exp(\alpha^*\tau)\text{dist}((\xi, \eta), S) &\leq \text{dist}\left(\begin{pmatrix} p(\tau, \xi, \eta) \\ q(\tau, \xi, \eta) \end{pmatrix}, S\right) \\ &\leq C_4' \exp(\alpha^*\tau)\text{dist}((\xi, \eta), S). \end{aligned} \tag{3.12}$$

Proof. By a suitable affine transformation, the ODE (1.4) can be written as

$$\frac{dX}{d\tau} = \alpha^* X + F(X, Y), \quad \frac{dY}{d\tau} = \beta^* Y + G(X, Y), \tag{3.13}$$

where F, G are homogeneous quadratic forms of X and Y with $F(0,0) = F_X(0,0) = F_Y(0,0) = 0$ and $G(0,0) = G_X(0,0) = G_Y(0,0) = 0$. The stable and unstable manifold of $(X, Y) = (0, 0)$ are expressed as $\{X = \Psi(Y)\}$ and $\{Y = \Phi(X)\}$, respectively, where $\Phi(0) = \Phi'(0) = 0$, $\Psi(0) = \Psi'(0) = 0$. Set

$$\widetilde{Y} = Y - \Phi(X), \quad \widetilde{X} = X - \Psi(Y).$$

Then it follows from (3.13) that

$$\frac{d\widetilde{X}}{d\tau} = \frac{dX}{d\tau} - \Psi'(Y)\frac{dY}{d\tau} = \alpha^* X + F(X, Y) - \Psi'(Y)(\beta^* Y + G(X, Y)). \tag{3.14}$$

Note that $\frac{dX}{d\tau} = \Psi'(Y)\frac{dY}{d\tau}$ holds everywhere on the stable manifold $\{X = \Psi(Y)\}$, hence

$$\alpha^*\Psi(Y) + F(\Psi(Y), Y) - \Psi'(Y)(\beta^* Y + G(\Psi(Y), Y) = 0. \tag{3.15}$$

Subtracting (3.15) from (3.14) yields

$$\frac{d\widetilde{X}}{d\tau} = \alpha^*\widetilde{X} + (F(\widetilde{X} + \Psi(Y), Y) - F(\Psi(Y), Y)) - \Psi'(Y)(G(\widetilde{X} + \Psi(Y), Y) - G(\Psi(Y), Y)).$$

Considering that

$$F(0,0) = F_X(0,0) = F_Y(0,0) = 0, \quad G(0,0) = G_X(0,0) = G_Y(0,0) = 0,$$

we can rewrite the equation as follows:

$$\frac{d\widetilde{X}}{d\tau} = \alpha^*\widetilde{X} + \widetilde{F}(\widetilde{X}, \widetilde{Y})\widetilde{X}, \tag{3.16}$$

where $\widetilde{F}$ satisfies the following estimate near the origin

$$\widetilde{F}(\widetilde{X}, \widetilde{Y}) \le C(|\widetilde{X}| + |\widetilde{Y}|).$$

Here and in what follows, C_i $(i = 1, 2, \dots)$ will denote positive constants that are independent of τ. Similarly

$$\frac{d\widetilde{Y}}{d\tau} = \beta^*\widetilde{Y} + \widetilde{G}(\widetilde{X}, \widetilde{Y})\widetilde{Y} \quad \text{with } \widetilde{G}(\widetilde{X}, \widetilde{Y}) \le C_2(|\widetilde{X}| + |\widetilde{Y}|). \tag{3.17}$$

Since $(p, q) \in \mathscr{B}_{\sigma_1}(u^*, v^*)$ implies

$$|\widetilde{G}(\widetilde{X}, \widetilde{Y})| \le C_4\sigma_1.$$

This and (3.17) yield

$$\left|\frac{d\widetilde{Y}}{d\tau}\right| \le (\beta^* + C_4\sigma_1)|\widetilde{Y}|.$$

Therefore

$$|\widetilde{Y}(\tau)| \le C_5 \exp\big((\beta^* + C_4\sigma_1)\tau\big). \tag{3.18}$$

It follows from this and the inequality in (3.16) that

$$|\widetilde{F}(\widetilde{X}, \widetilde{Y})| \le C_1(|\widetilde{X}| + C_5 e^{-\mu\tau}). \tag{3.19}$$

Here $\mu = -\beta^* - C_4\sigma_1$. If we set $\sigma_1 > 0$ sufficiently small, then $\mu > 0$. Without loss of generality we may assume that $\widetilde{X}(0) > 0$. It follows from (3.16) and (3.19) that

$$\dot{\widetilde{X}} \ge \alpha^*\widetilde{X} - C_1(\widetilde{X} + C_5 e^{-\mu\tau})\widetilde{X} = \alpha^*\widetilde{X} - C_1\widetilde{X}^2 - C_1 C_5 \widetilde{X} e^{-\mu\tau}.$$

Dividing this inequality by $\alpha^*\widetilde{X} - C_1\widetilde{X}^2$ and using the fact that $C_1|\widetilde{X}(\tau)| \le C_1 C_3 \sigma_1 \le \frac{\alpha^*}{2}$, we obtain

$$\frac{\dot{\widetilde{X}}}{(\alpha^* - C_1\widetilde{X})\widetilde{X}} \ge 1 - C^* e^{-\mu\tau} \quad \text{where } C^* = \frac{2C_1C_5}{\alpha^*}.$$

Integrating the inequality with respect to τ, we have

$$\widetilde{X}(\tau) \ge \frac{\alpha^* - C_1\widetilde{X}(\tau)}{\alpha^* - C_1\widetilde{X}(0)} \exp\left(-\frac{\alpha^* C^*}{\mu}\right) \exp(\alpha^*\tau)\widetilde{X}(0) \ge \frac{1}{2}\exp\left(-\frac{\alpha^* C^*}{\mu}\right)\exp(\alpha^*\tau)\widetilde{X}(0).$$

Here, we have used again the estimate $\alpha^* - C_1\widetilde{X}(\tau) \ge \frac{\alpha^*}{2}$ to show the second inequality. Finally we remark that there exist $\theta_1, \theta_2 \ge 0$ such that

$$\theta_1\widetilde{X}(\tau) \le \operatorname{dist}\left(\begin{pmatrix} p(\tau;\xi,\eta) \\ q(\tau;\xi,\eta)\end{pmatrix}, S\right) \le \theta_2\widetilde{X}(\tau).$$

The lemma is proved.

Next we fix $\sigma_1 > 0$ and define a function $\hat{\tau}(\xi,\eta)$ on $[0,\widetilde{M}] \times [0,\widetilde{M}]$ as follows:

$$\hat{\tau}(\xi,\eta) = \min\{\tau \ge 0 \mid (p(s;\xi,\eta), q(s;\xi,\eta)) \in \mathscr{B}_{\sigma_1}(R_1,0) \cup \mathscr{B}_{\sigma_1}(0,R_2) \text{ for all } s > \tau\}.$$

Here we understand that $\hat{\tau} = \infty$ if there is no such τ as above. Since every solution of (3.1) with initial data in $[0,\widetilde{M}] \times [0,\widetilde{M}] \setminus (S \cup (0,0))$ converges to either $(R_1, 0)$ or $(0, R_2)$ as $\tau \to \infty$ by virtue of Lemma 3.2, $\hat{\tau}(\xi,\eta)$ is finite on this set. Moreover, the following estimate holds:

Lemma 3.7. *There exists a constant $\widehat{C}_0 \ge 0$ such that*

$$\frac{1}{\alpha^*}\log\frac{1}{\operatorname{dist}((\xi,\eta),S)} - \widehat{C}_0 \le \hat{\tau}(\xi,\eta) \le \frac{1}{\alpha^*}\log\frac{1}{\operatorname{dist}((\xi,\eta),S)} + \widehat{C}_0 \qquad (3.20)$$

for every $(\xi,\eta) \in [0,\widetilde{M}] \times [0,\widetilde{M}] \setminus (S \cup \mathscr{B}_{\sigma_0}(0,0))$, where α^* is the positive eigenvalue of the matrix (3.11).

Proof. By Lemma 3.4, every solution (p,q) with initial data $(\xi,\eta) \in [0,\widetilde{M}] \times [0,\widetilde{M}]$ enters—at least temporarily—the set $\mathscr{B}_{\sigma_0}(0,0) \cup \mathscr{B}_{\sigma_1}(u^*,v^*) \cup \mathscr{B}_{\sigma_1}(R_1,0) \cup \mathscr{B}_{\sigma_1}(0,R_2)$ within the time period $0 \le \tau \le C_1'$. Considering that $(R_1,0)$ and $(0,R_2)$ are asymptotically stable equilibria, we see that every solution that does not enter $\mathscr{B}_{\sigma_1}(u^*,v^*)$ will remain in $\mathscr{B}_{\sigma_1}(R_1,0) \cup \mathscr{B}_{\sigma_1}(0,R_2)$ for all $\tau \ge C_1''$, where C_1'' is a constant independent of the choice of initial data. To prove (3.20), it suffices to consider the case where the solution enters $\mathscr{B}_{\sigma_1}(u^*,v^*)$ temporarily. The estimate then follows easily from Lemmas 3.5 and 3.6. The proof is complete.

Using Lemmas 3.4–3.6, we obtain the following lemma.

Lemma 3.8. *Given any $\sigma_1 > 0$, there exists $\widehat{C}_0 > 0$ such that for any (ξ, η) satisfying $dist((\xi, \eta), S) \geq \varepsilon\ell$, either* (i) *or* (ii) *holds*:

$$\text{(i)}\quad (p(\tau; \xi, \eta), q(\tau; \xi, \eta)) \in \mathscr{B}_{\sigma_1}(R_1, 0) \quad \text{for } \tau > \frac{1}{\alpha^*}\log\frac{1}{dist((\xi, \eta), S)} + \widehat{C}_0;$$

$$\text{(ii)}\quad (p(\tau; \xi, \eta), q(\tau; \xi, \eta)) \in \mathscr{B}_{\sigma_1}(0, R_2) \quad \text{for } \tau > \frac{1}{\alpha^*}\log\frac{1}{dist((\xi, \eta), S)} + \widehat{C}_0.$$

At the end of this section we present a lemma concerning the derivatives of p, q with respect to the parameters ξ, η. Since the proof is rather long, we give it in the Appendix.

Lemma 3.9. *For any $C > 0$, there exist constants $B_i > 0$, $i = 1, \ldots, 7$, such that the following estimates hold for $0 \leq \tau < \frac{1}{\alpha^*}\log\frac{1}{dist((\xi,\eta),S)} + \widehat{C}_0$, where $\widehat{C}_0$ is the same constant as in Lemma* 3.8 *and α^* is the positive eigenvalue of* (3.11).

i) $B_1\exp(\alpha^*\tau) \leq p_\xi(\tau; \xi, \eta) \leq B_2\exp(\alpha^*\tau), -B_3\exp(\alpha^*\tau) \leq q_\xi(\tau; \xi, \eta) \leq 0$
ii) $-B_4\exp(\alpha^*\tau) \leq p_\eta(\tau; \xi, \eta) \leq 0, B_5\exp(\alpha^*\tau) \leq q_\xi(\tau; \xi, \eta) \leq B_6\exp(\alpha^*\tau)$
iii) $|p_{\xi\xi}| + |p_{\xi\eta}| + |p_{\eta\eta}| + |q_{\xi\xi}| + |q_{\xi\eta}| + |q_{\eta\eta}| \leq B_7\exp(2\alpha^*\tau)$.

4. Generation of Interfaces

In this section we study the formation of internal layers that takes place in a fast time scale at the early stage. The basic tool for analyzing this phenomenon is again the upper-lower solution method. However, the upper and lower solutions we use in this section completely differ from what we will use in Section 5 to analyze the motion of interface that takes place in the later stage.

The upper and lower solutions for the early stage are constructed by modifying the solution of the following problem:

$$\begin{cases} \tilde{u}_t = \dfrac{h(x)}{\varepsilon^2} f(\tilde{u}, \tilde{v}) \\ \tilde{v}_t = \dfrac{h(x)}{\varepsilon^2} g(\tilde{u}, \tilde{v}) \\ \tilde{u}(x, 0) = u_0(x), \quad \tilde{v}(x, 0) = v_0(x). \end{cases} \tag{4.1}$$

The solution of (4.1) has the form

$$\tilde{u}(x, t) = p\left(\frac{h(x)}{\varepsilon^2}t, u_0(x), v_0(x)\right), \quad \tilde{v}(x, t) = q\left(\frac{h(x)}{\varepsilon^2}t, u_0(x), v_0(x)\right), \tag{4.2}$$

where $(p(\tau; \xi, \eta), q(\tau; \xi, \eta))$ is a solution of (1.4) with initial data (ξ, η). The time scale $\tau = \frac{h(x)}{\varepsilon^2}t$, which appears in the above expression of $\tilde{u}, \tilde{v}$ will play an important role in the analysis of interface generation at the initial stage.

Remark. Problem (4.1) is obtained by dropping the diffusion terms in (1.2). Therefore, the behavior of the solution of (1.2) can be well approximated by that of $(\tilde{u}, \tilde{v})$ during the very early stage, where the diffusion terms are relatively small compared with the reaction terms $\frac{h(x)}{\varepsilon^2}f$, $\frac{h(x)}{\varepsilon^2}g$. In particular, $(\tilde{u}, \tilde{v})$ roughly describes how the internal layers are developed at the very beginning stage. To perform a more rigorous study, we will construct an upper and a lower solution $(u^{\pm}, v^{\pm})$ by modifying $(\tilde{u}, \tilde{v})$ appropriately.

4.1. *Definition of Upper and Lower Solutions*

Let $(u^+(x,t), v^+(x,t))$ and $(u^-(x,t), v^-(x,t))$ be smooth functions defined on $\overline{\Omega} \times [t_1, t_2]$. We say (u^+, v^+) is an *upper solution* for equation (1.2) (in the time interval $t_0 \le t \le t_1$) if it satisfies

$$\begin{cases} \varepsilon u_t^+ - \varepsilon \nabla \cdot (k(x)\nabla u^+) - \dfrac{1}{\varepsilon} h(x)(R_1 - u^+ - bv^+)u^+ \ge 0, \\ \varepsilon v_t^+ - \varepsilon D \nabla \cdot (k(x)\nabla v^+) - \dfrac{1}{\varepsilon} h(x)(R_2 - au^+ - v^+)v^+ \le 0, \end{cases} \tag{4.3}$$

for $x \in \Omega$, $t_1 \le t \le t_2$ along with the boundary condition

$$\frac{\partial u^+}{\partial n} \ge 0, \quad \frac{\partial v^+}{\partial n} \le 0 \quad (x \in \partial\Omega,\, t_0 \le t \le t_1).$$

We say (u^-, v^-) is a *lower solution* for equation (1.2) if it satisfies

$$\begin{cases} \varepsilon u_t^- - \varepsilon \nabla \cdot (k(x)\nabla u^-) - \dfrac{1}{\varepsilon} h(x)(R_1 - u^- - bv^-)u^- \le 0, \\ \varepsilon v_t^- - \varepsilon D \nabla \cdot (k(x)\nabla v^-) - \dfrac{1}{\varepsilon} h(x)(R_2 - au^- - v^-)v^- \ge 0, \end{cases} \tag{4.4}$$

for $x \in \Omega$, $t_1 \le t \le t_2$ along with the boundary condition

$$\frac{\partial u^-}{\partial n} \le 0, \quad \frac{\partial v^-}{\partial n} \ge 0 \quad (x \in \partial\Omega,\, t_0 \le t \le t_1).$$

The following is a consequence of the maximum principle and the fact that (1.2) is a competition system. The proof is omitted.

Proposition 4.1. *Let (u^+, v^+) be an upper solution and (u^-, v^-) be a lower solution of* (1.2) *for $t_0 \le t \le t_1$. Suppose that a solution (u, v) of* (1.2) *satisfies $u^-(x, t_0) \le u(x, t_0) \le u^+(x, t_0)$ and $v^-(x, t_0) \ge v(x, t_0) \ge v^+(x, t_0)$ for $x \in \overline{\Omega}$. Then the solution (u, v) satisfies $u^- \le u \le u^+$ and $v^- \ge v \ge v^+$ for $t \in [t_0, t_1]$ and $x \in \overline{\Omega}$.*

Now let us introduce the following order relation:

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \succeq \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \overset{def}{\Longleftrightarrow} u_1(x) \ge u_2(x) \quad \text{and} \quad v_1(x) \le v_2(x) \quad \text{on } \overline{\Omega}. \tag{4.5}$$

With this notation the above proposition can be restated as follows:

If (u^+, v^+) and (u^-, v^-) are an upper and a lower solution respectively and if

$$\begin{pmatrix} u^+ \\ v^+ \end{pmatrix}_{t=t_0} \succeq \begin{pmatrix} u \\ v \end{pmatrix}_{t=t_0} \succeq \begin{pmatrix} u^- \\ v^- \end{pmatrix}_{t=t_0},$$

then

$$\begin{pmatrix} u^+ \\ v^+ \end{pmatrix} \succeq \begin{pmatrix} u \\ v \end{pmatrix} \succeq \begin{pmatrix} u^- \\ v^- \end{pmatrix} \quad \text{for } t \in [t_0, t_1].$$

The following is also an immediate consequence of the above proposition:

Corollary 4.2 (Comparison Principle**).** *If* (u, v) *and* $(\tilde{u}, \tilde{v})$ *are two solutions of* (1.2) *and if* $\binom{u}{v}_{t=t_0} \preceq \binom{\tilde{u}}{\tilde{v}}_{t=t_0}$ *then* $\binom{u}{v} \preceq \binom{\tilde{u}}{\tilde{v}}$ *for* $t \geq t_0$.

Remark. This comparison principle reduces to Lemma 3.1 in the case of the ODE system (1.4). More precisely

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} \succeq \begin{pmatrix} \tilde{\xi} \\ \tilde{\eta} \end{pmatrix} \quad \text{implies} \begin{pmatrix} p(\tau; \xi, \eta) \\ q(\tau; \xi, \eta) \end{pmatrix} \succeq \begin{pmatrix} p(\tau; \tilde{\xi}, \tilde{\eta}) \\ q(\tau; \tilde{\xi}, \tilde{\eta}) \end{pmatrix} \quad \text{for } \tau \geq 0. \tag{4.6}$$

4.2. *Construction of Upper and Lower Solutions*

We will first consider the case where $(u_0(x), v_0(x))$ satisfies the Neumann zero boundary conditions on $\partial\Omega$. The general case will be considered in Remark 4.7.

Definition 4.3. For every $(x, t) \in \overline{\Omega} \times [0, \infty)$, we set

$$\begin{aligned} u^+(x, t) &= p(\tau; u_0(x) + \varepsilon^2 c_1(\exp(\alpha^*\tau) - 1), v_0(x) - \varepsilon^2 c_2(\exp(\alpha^*\tau) - 1)), \\ v^+(x, t) &= q(\tau; u_0(x) + \varepsilon^2 c_1(\exp(\alpha^*\tau) - 1), v_0(x) - \varepsilon^2 c_2(\exp(\alpha^*\tau) - 1)), \end{aligned} \tag{4.7}$$

$$\begin{aligned} u^-(x, t) &= p(\tau; u_0(x) - \varepsilon^2 c_1(\exp(\alpha^*\tau) - 1), v_0(x) + \varepsilon^2 c_2(\exp(\alpha^*\tau) - 1)), \\ v^-(x, t) &= q(\tau; u_0(x) - \varepsilon^2 c_1(\exp(\alpha^*\tau) - 1), v_0(x) + \varepsilon^2 c_2(\exp(\alpha^*\tau) - 1)), \end{aligned} \tag{4.8}$$

where

$$\tau = \frac{h(x)}{\varepsilon^2} t,$$

and the constants c_1, c_2 are to be chosen large enough and α^* is the positive eigenvalue of (3.11). Note that

$$u^{\pm}(x, 0) = u_0(x), \quad v^{\pm}(x, 0) = v_0(x).$$

We will show that (u^+, v^+) (resp. (u^-, v^-)) is an upper (resp. a lower) solution in the short time period $0 \leq t \leq t^*(x)$, where $t^*(x)$ indicates the timing for transfer from the early stage of interface formation to the later stage of interface propagation. In order to define $t^*(x)$, we choose $\delta_0 > 0$ sufficiently small so that the function $(u, v) \mapsto dist((u, v); S)$ is smooth in the region $\mathcal{N}_{\delta_0}(S) \backslash S$, where $\mathcal{N}_{\delta_0}(S) = \{(u, v) \in \mathbf{R}^2 \mid dist((u, v); S) \leq \delta_0\}$. We define $\delta^*(u, v)$ to be a smooth function in $\mathbf{R}^2$ such that

$$\delta^*(u, v) = \begin{cases} \varepsilon\ell & \text{if } 0 \leq dist((u, v); S) \leq \varepsilon\ell, \\ dist((u, v); S) & \text{if } 2\varepsilon\ell \leq dist((u, v); S) \leq \delta_0, \\ 2\delta_0 & \text{if } dist((u, v); S) \geq 2\delta_0, \end{cases} \tag{4.9}$$

$$\varepsilon\ell \le \delta^*(u,v) \le 2\varepsilon\ell \quad \text{if } \varepsilon\ell \le dist((u,v);S) \le 2\varepsilon\ell,$$
$$\delta_0 \le \delta^*(u,v) \le 2\delta_0 \quad \text{if } \delta_0 \le dist((u,v);S) \le 2\delta_0,$$

where $\ell > 1$ is a positive constant (see (5.49)). Now we define

$$\tau^*(x) = \frac{1}{\alpha^*}\log\frac{1}{\delta^*(u_0(x), v_0(x))}$$

and

$$t^*(x) := \frac{\varepsilon^2}{h(x)}\tau^*(x). \tag{4.10}$$

Lemma 4.4. *For any $B > 0$, there exists $\varepsilon_0 > 0$ and $c_1 > 0$, $c_2 > 0$ such that for $0 < \varepsilon < \varepsilon_0$, the functions $(u^\pm, v^\pm)$ are a pair of upper and lower solutions of (1.2) in the space-time region $\{(x,t) \mid x \in \overline{\Omega}, 0 \le t \le t^*(x) + B\varepsilon^2\}$.*

Set $B_0 = \widehat{C}_0/\min_{x\in\overline{\Omega}} h(x)$, where $\widehat{C}_0$ is the same constant as in Lemma 3.8. We also recall that $\mathscr{B}_\sigma(u,v)$ denotes the σ-neighborhood of the point in the (u,v) plane.

Lemma 4.5. *For any $\sigma_1 > 0$ and any $B > B_0$, there exist $\widetilde{C} > 0$ and $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$ and any $x \in \overline{\Omega}$ satisfying $dist(x, \Gamma_0) \ge \varepsilon\widetilde{C}$, either of the following holds:*

(i) $(u^+(x,t), v^+(x,t)), (u^-(x,t), v^-(x,t)) \in \mathscr{B}_{\sigma_1}(0, R_2)$,

(ii) $(u^+(x,t), v^+(x,t)), (u^-(x,t), v^-(x,t)) \in \mathscr{B}_{\sigma_1}(R_1, 0)$,

for every $x \in \overline{\Omega}$ and $t \in [t^*(x) + B_0\varepsilon^2, t^*(x) + B\varepsilon^2]$.

Proof of Lemma 4.5. For any large $B > B_0$, we can choose $\widetilde{C}_1 > 0$ such that

$$\varepsilon^2\left(\exp\left(\frac{\alpha^* h(x)t}{\varepsilon^2}\right) - 1\right) < 2\widetilde{C}_1\varepsilon \quad \text{for } t \in [0, t^*(x) + B\varepsilon^2]. \tag{4.11}$$

For $\varepsilon > 0$ sufficiently small, we set

$$P^+(x,t) = p\left(\frac{h(x)}{\varepsilon^2}t; \xi_\varepsilon^+(x), \eta_\varepsilon^+(x)\right),$$
$$P^-(x,t) = p\left(\frac{h(x)}{\varepsilon^2}t; \xi_\varepsilon^-(x), \eta_\varepsilon^-(x)\right),$$
$$Q^+(x,t) = q\left(\frac{h(x)}{\varepsilon^2}t; \xi_\varepsilon^+(x), \eta_\varepsilon^+(x)\right),$$
$$Q^-(x,t) = q\left(\frac{h(x)}{\varepsilon^2}t; \xi_\varepsilon^-(x), \eta_\varepsilon^-(x)\right),$$

where

$$\xi_\varepsilon^+(x) = u_0(x) + 2c_1\widetilde{C}_1\varepsilon, \quad \eta_\varepsilon^+(x) = v_0(x) - 2c_2\widetilde{C}_1\varepsilon,$$
$$\xi_\varepsilon^-(x) = u_0(x) - 2c_1\widetilde{C}_1\varepsilon, \quad \eta_\varepsilon^-(x) = v_0(x) + 2c_2\widetilde{C}_1\varepsilon,$$

with c_1, c_2 being the same constants as in Definition 4.3.

Next

$$\tilde{\xi}_\varepsilon^+(x) := u_0(x) + \varepsilon^2 c_1\left(\exp\left(\frac{\alpha^* h(x)}{\varepsilon^2}t - 1\right)\right),$$
$$\tilde{\xi}_\varepsilon^-(x) := u_0(x) - \varepsilon^2 c_1\left(\exp\left(\frac{\alpha^* h(x)}{\varepsilon^2}t - 1\right)\right),$$
$$\tilde{\eta}_\varepsilon^+(x) := v_0(x) - \varepsilon^2 c_2\left(\exp\left(\frac{\alpha^* h(x)}{\varepsilon^2}t - 1\right)\right),$$
$$\tilde{\eta}_\varepsilon^-(x) := v_0(x) + \varepsilon^2 c_2\left(\exp\left(\frac{\alpha^* h(x)}{\varepsilon^2}t - 1\right)\right).$$

Then the function in (4.8) and (4.9) are written as

$$u^\pm(x,t) = p\left(\frac{h(x)}{\varepsilon^2}t, \tilde{\xi}_\varepsilon^\pm(x), \tilde{\eta}_\varepsilon^\pm(x)\right), \quad v^\pm(x,t) = q\left(\frac{h(x)}{\varepsilon^2}t, \tilde{\xi}_\varepsilon^\pm(x), \tilde{\eta}_\varepsilon^\pm(x)\right).$$

By the assumption (4.11), we have

$$\begin{pmatrix}\xi_\varepsilon^-(x)\\ \eta_\varepsilon^-(x)\end{pmatrix} \preceq \begin{pmatrix}\tilde{\xi}_\varepsilon^-(x,t)\\ \tilde{\eta}_\varepsilon^-(x,t)\end{pmatrix} \preceq \begin{pmatrix}\tilde{\xi}_\varepsilon^+(x,t)\\ \tilde{\eta}_\varepsilon^+(x,t)\end{pmatrix} \preceq \begin{pmatrix}\xi_\varepsilon^+(x)\\ \eta_\varepsilon^+(x)\end{pmatrix} \quad \text{for } x \in \Omega,\ t \in [0, t^*(x) + B\varepsilon^2].$$

The first inequality and Lemma 3.1 yield

$$p(\tau; \xi_\varepsilon^-(x), \eta_\varepsilon^-(x)) \le p(\tau; \tilde{\xi}_\varepsilon^-(x,t), \tilde{\eta}_\varepsilon^-(x,t)),$$
$$q(\tau; \xi_\varepsilon^-(x), \eta_\varepsilon^-(x)) \ge q(\tau; \tilde{\xi}_\varepsilon^-(x,t), \tilde{\eta}_\varepsilon^-(x,t)).$$

Setting $\tau = \frac{h(x)}{\varepsilon^2}t$, (here t is considered a fixed parameter and τ is an independent variable) we obtain

$$\begin{pmatrix}P^-(x,t)\\ Q^-(x,t)\end{pmatrix} \preceq \begin{pmatrix}u^-(x,t)\\ v^-(x,t)\end{pmatrix}.$$

By the same argument, we have

$$\begin{pmatrix}P^-(x,t)\\ Q^-(x,t)\end{pmatrix} \preceq \begin{pmatrix}u^-(x,t)\\ v^-(x,t)\end{pmatrix} \preceq \begin{pmatrix}u^+(x,t)\\ v^+(x,t)\end{pmatrix} \preceq \begin{pmatrix}P^+(x,t)\\ Q^+(x,t)\end{pmatrix} \quad \text{for } t \le t^*(x) + B\varepsilon^2. \tag{4.12}$$

Let us consider $x \in \Omega$ satisfying

$$\operatorname{dist}(x, \Gamma_0) \ge \frac{4\widetilde{C}_1(c_1+c_2)\varepsilon}{A_0}$$

where A_0 is the positive constant in Assumption 5 in Section 1. This implies that

$$\operatorname{dist}((u_0(x), v_0(x)), S) \ge 4\widetilde{C}_1(c_1+c_2)\varepsilon.$$

It holds that

$$\operatorname{dist}((u_0(x) + 2c_1\widetilde{C}_1\varepsilon, v_0(x) - 2c_2\widetilde{C}_1\varepsilon), S) \ge 2\widetilde{C}_1(c_1+c_2)\varepsilon, \tag{4.13}$$
$$\operatorname{dist}((u_0(x) - 2c_1\widetilde{C}_1\varepsilon, v_0(x) + 2c_2\widetilde{C}_1\varepsilon), S) > 2\widetilde{C}_1(c_1+c_2)\varepsilon. \tag{4.14}$$

Choose $c_1, c_2 > 0$ sufficiently large such that

$$2\widetilde{C}_1(c_1 + c_2) > \ell, \tag{4.15}$$

where ℓ is as in Lemma 3.8. We apply Lemma 3.8 to (P^+, Q^+), (P^-, Q^-) with (4.13), (4.14) and $\tau = \frac{h(x)t}{\varepsilon^2}$. Then we have either

$$\{(P^+(x,t), Q^+(x,t)), (P^-(x,t), Q^-(x,t))\} \in \mathscr{B}_{\sigma_1}(R_1, 0),$$

or

$$\{(P^+(x,t), Q^+(x,t)), (P^-(x,t), Q^-(x,t))\} \in \mathscr{B}_{\sigma_1}(0, R_2),$$

for $t \geq t^*(x) + B_0\varepsilon^2$. This fact and (4.12) with $\widetilde{C} = \frac{4\widetilde{C}_1(c_1+c_2)}{A_0}$ proves the lemma.

Proof of Lemma 4.4. We will show that (u^+, v^+), (u^-, v^-) satisfy inequalities (4.3) and (4.2) respectively. We set

$$\begin{aligned} \mathscr{L}_1(u^\pm, v^\pm) &= u_t^\pm - \nabla \cdot (k(x)\nabla u^\pm) - \frac{1}{\varepsilon^2} h(x)(R_1 - u^\pm - bv^\pm)u^\pm, \\ \mathscr{L}_2(u^\pm, v^\pm) &= v_t^\pm - D\nabla \cdot (k(x)\nabla v^\pm) - \frac{1}{\varepsilon^2} h(x)(R_2 - au^\pm - v^\pm)v^\pm. \end{aligned} \tag{4.16}$$

Our goal is to show that

$$\mathscr{L}_1(u^+, v^+) \geq 0, \quad \mathscr{L}_1(u^-, v^-) \leq 0, \quad \mathscr{L}_2(u^+, v^+) \leq 0, \quad \mathscr{L}_2(u^-, v^-) \geq 0.$$

We will only prove $\mathscr{L}_1(u^+, v^+) \geq 0$, since the other inequalities can be proved similarly. Substituting (4.8) into (4.16), we obtain

$$\begin{aligned} \mathscr{L}_1(u^+, v^+) &= \alpha^* h(x) \exp(\alpha^*\tau)(c_1 p_\xi - c_2 p_\eta) \\ &\quad - p_\xi A_1 - p_\eta A_2 - q_\xi A_3 - q_\eta A_4 + A_5 + A_6, \end{aligned} \tag{4.17}$$

where

$$\begin{aligned} A_1 &= \nabla k(x)\left(\nabla u_0 + c_1\alpha^*\tau \frac{\varepsilon^2}{h(x)} \exp(\alpha^*\tau)\nabla h\right) \\ &\quad + k(x)\left(\Delta u_0 + c_1\alpha^* \frac{\varepsilon^2\tau}{h(x)} \exp(\alpha^*\tau)\Delta h + c_1\alpha^{*2} \frac{\varepsilon^2}{h(x)^2}\tau^2 \exp(\alpha^*\tau)|\nabla h|^2\right) \\ &\quad + 2f_p(p,q)k(x)\left(\nabla u_0 + c_1\alpha^* \frac{\varepsilon^2\tau}{h(x)} \exp(\alpha^*\tau)\nabla h\right)\left(\frac{\alpha^*\tau\nabla h}{h(x)}\right), \\ A_2 &= \nabla k(x)\left(\nabla v_0 - c_2\alpha^* \frac{\varepsilon^2\tau}{h(x)} \exp(\alpha^*\tau)\nabla h\right) \\ &\quad + k(x)\left(\Delta v_0 - c_2\alpha^* \frac{\varepsilon^2\tau}{h(x)} \exp(\alpha^*\tau)\Delta h - c_2\alpha^{*2} \frac{\varepsilon^2}{h(x)}\tau^2(\alpha^*\tau)|\nabla h|^2\right) \\ &\quad + 2f_p(p,q)k(x)\left(\nabla v_0 - c_2\alpha^* \frac{\varepsilon^2\tau}{h(x)} \exp(\alpha^*\tau)\nabla h\right)\left(\frac{\alpha^*\tau\nabla h}{h(x)}\right), \end{aligned}$$

$$A_3 = 2f_q(p,q)k(x)\left(\nabla u_0 + c_1\alpha^*\frac{\varepsilon^2\tau}{h(x)}\exp(\alpha^*\tau)\nabla h\right)\left(\frac{\alpha^*\tau\nabla h}{h(x)}\right),$$

$$A_4 = 2f_q(p,q)k(x)\left(\nabla v_0 - c_2\alpha^*\frac{\varepsilon^2\tau}{h(x)}\exp(\alpha^*\tau)\nabla h\right)\left(\frac{\alpha^*\tau\nabla h}{h(x)}\right),$$

$$\begin{aligned} A_5 = &-p_{\xi\xi}k(x)\left|\nabla u_0 + c_1\alpha^*\frac{\varepsilon^2\tau}{h(x)}\exp(\alpha^*\tau)\nabla h\right|^2 \\ &- p_{\eta\eta}k(x)\left|\nabla v_0 - c_2\alpha^*\frac{\varepsilon^2\tau}{h(x)}\exp(\alpha^*\tau)\nabla h\right|^2 \\ &- p_{\xi\eta}2k(x)\left(\nabla u_0 + c_1\alpha^*\frac{\varepsilon^2\tau}{h(x)}\exp(\alpha^*\tau)\nabla h\right)\left(\nabla v_0 - c_2\alpha^*\frac{\varepsilon^2\tau}{h(x)}\exp(\alpha^*\tau)\nabla h\right), \end{aligned}$$

$$A_6 = -(f_pf(p,q) + f_qg(p,q))\frac{\tau^2}{h(x)^2}|\nabla h|^2k(x) - f(p,q)\frac{\tau}{h(x)}\{k(x)\Delta h + \nabla k(x)\cdot\nabla h\},$$

where $\tau = \frac{h(x)t}{\varepsilon^2}$. Here we have used the fact that

$$p_{\xi\tau} = f_pp_\xi + f_qq_\xi,\quad p_{\eta\tau} = f_pp_\eta + f_qq_\eta$$
$$p_{\tau\tau} = f_pp_\tau + f_qq_\tau = f_pf + f_qg.$$

Since $\tau \le \frac{1}{\alpha^*}\log\frac{1}{\varepsilon} + Bh(x)$, there exists $B_1' > 0$ such that

$$|A_1| + |A_2| + |A_3| + |A_4| \le B_1'.$$

Applying Lemma 3.9i), ii) with $C > B\max_{x\in\Omega}h(x)$, we have

$$|-p_\xi A_1 - p_\eta A_2 - q_\xi A_3 - q_\eta A_4| \le B_1'(B_2 + B_3 + B_4 + B_6)\exp(\alpha^*\tau). \tag{4.18}$$

In the same way, there exists $B_2' > 0$ such that

$$|A_5| \le B_2'(|p_{\xi\xi}| + |p_{\xi\eta}| + |p_{\eta\eta}|).$$

By this fact and Lemma 3.9iii), we have

$$|A_5| \le B_2'B_7\exp(2\alpha^*\tau). \tag{4.19}$$

Comparing τ, τ^2 with $\exp(\alpha^*\tau)$, we find $B_3' > 0$ such that

$$|A_6| \le B_3'\exp(\alpha^*\tau). \tag{4.20}$$

On the other hand, it follows from Lemma 3.9i), ii) that

$$c_1p_\xi - c_2p_\eta \ge (c_1B_1 + c_2B_4)\exp(\alpha^*\tau). \tag{4.21}$$

Substituting (4.18)–(4.21) into (4.17), we conclude that $\mathscr{L}_1(u^+, v^+) \ge 0$ for sufficiently large $c_1, c_2 > 0$. Since $u_0(x), v_0(x)$ satisfy the Neumann zero boundary condition, we can easily see that u^+, v^+, u^-, v^- satisfy the Neumann zero boundary condition. The proof of the lemma is complete.

Remark 4.7. It remains to consider the case where $u_0(x)$ and $v_0(x)$ do not satisfy the Neumann boundary conditions. In this case the function $(u^\pm, v^\pm)$ in (4.8) and (4.9) are no longer upper and lower solutions since they do not satisfy the Neumann boundary conditions on $\partial\Omega$. We therefore modify $(u^\pm, v^\pm)$ near $\partial\Omega$ as follows. Choose a small constant $d_N > 0$ such that $d(x, \partial\Omega)$ is a smooth function in the region

$$\mathcal{N}(\partial\Omega) = \{x \in \Omega \mid \mathrm{dist}(x, \partial\Omega) \leq d_N\}. \tag{4.22}$$

We then define a smooth function $\theta(x)$ on $\partial\Omega$ satisfying $0 \leq \theta \leq 1$ and

$$\theta(x) = \begin{cases} 1 & \text{if } \mathrm{dist}(x, \partial\Omega) \geq 2d_N, \\ 0 & \text{if } \mathrm{dist}(x, \partial\Omega) \leq d_N. \end{cases} \tag{4.23}$$

For each $x \in \mathcal{N}(\partial\Omega)$, we define $z(x) \in \partial\Omega$ by

$$|z(x) - x| = dist(x, \partial\Omega).$$

Note that $z(x)$ is uniquely determined since d_N is sufficiently small. Now choose a constant $\eta_1 > 0$ such that

$$\eta_1 > \max\left\{ \max_{x\in\mathcal{N}(\partial\Omega)} u_0(x) - \min_{x\in\mathcal{N}(\partial\Omega)} u_0(x),\ \max_{x\in\mathcal{N}(\partial\Omega)} v_0(x) - \min_{x\in\mathcal{N}(\partial\Omega)} v_0(x) \right\},$$

and define

$$\begin{aligned} u_0^+(x) &= \theta(x)u_0(x) + (1-\theta(x))(u_0(z(x)) + \eta_1), \\ v_0^+(x) &= \theta(x)v_0(x) + (1-\theta(x))(v_0(z(x)) - \eta_1), \\ u_0^-(x) &= \theta(x)u_0(x) + (1-\theta(x))(u_0(z(x)) - \eta_1), \\ v_0^-(x) &= \theta(x)v_0(x) + (1-\theta(x))(v_0(z(x)) + \eta_1). \end{aligned}$$

Note that we have

$$\frac{\partial u_0^+}{\partial \nu} = \frac{\partial v_0^+}{\partial \nu} = \frac{\partial u_0^-}{\partial \nu} = \frac{\partial v_0^-}{\partial \nu} = 0 \quad \text{on } \partial\Omega.$$

To see this, we remark that

$$\theta = 0, \quad \frac{\partial \theta}{\partial \nu} = 0, \quad \frac{\partial u_0(z(x))}{\partial \nu} = 0 \quad \text{on } \partial\Omega.$$

Then it follows that

$$\frac{\partial}{\partial \nu}(u_0^+(x)) = \frac{\partial \theta}{\partial \nu} u_0(x) + \theta(x)\frac{\partial u_0(x)}{\partial \nu} - \frac{\partial \theta}{\partial \nu}(u_0(z(x)) + \eta_1) = 0 \quad \text{on } \partial\Omega.$$

The same argument applies to v_0^+, u_0^-, v_0^-. Now we replace u_0 and v_0 by u_0^+ and v_0^+ respectively, in the definition of u^+, v^+ in (4.8). Similarly, we replace u_0 and v_0 by u_0^- and v_0^- respectively, in the definition of (u^-, v^-) in (4.9). Then it is not difficult to see that the new function (u^+, v^+) is an upper solution, while the new (u^-, v^-) is a lower solution. The proof is basically the same as that of Lemmas 4.4 and 4.5, so we omit the details.

5. Construction of Upper and Lower Solutions

In this section we construct upper and lower solutions $(U^\pm, V^\pm)$ of (1.2) that have steep internal layers near $\Gamma(t)$, the solution of the interface equation (1.8). These upper and lower solutions are used to estimate the distance between $\Gamma^\varepsilon(t)$ and $\Gamma(t)$ after the formation of internal layers.

In Section 5.1, we first construct $(U^\pm, V^\pm)$ in a tubular neighborhood of $\Gamma(t)$ by modifying the first two terms of the asymptotic expansion (2.5). We will then extend those upper and lower solutions to the entire domain Ω.

Though we borrow some symbols from Section 2 in order to clarify the meaning of certain terms that appear in the expression of upper and lower solutions, this does not mean that our argument relies on the formal (unproven) results in Section 2. All the results we present here and later sections are rigorous.

5.1. *Construction of Upper and Lower Solutions Near the Interface*

As mentioned above, we will first construct $(U^\pm, V^\pm)$ in a small neighborhood of $\Gamma(t)$. Recall the distance function defined in (2.2). Since $\Gamma(t)$ is a smooth hypersurface that depends smoothly on t, $d(x,t)$ is a smooth function of (x,t) near $\Gamma(t)$. Let

$$\mathcal{N}_d(\Gamma(t)) := \{x \in \Omega \mid \operatorname{dist}(x, \Gamma(t)) \le d\}. \tag{5.1}$$

In what follows we fix a constant $d^* > 0$ such that $d(x,t)$ is smooth in the $(N+1)$-dimensional tubular neighborhood $\mathcal{N}_{3d^*}$. Because of the smoothness of $\Gamma(t)$ such a neighborhood exists. Note that $|\nabla d| = 1$ in this neighborhood.

Now we will find $(U^\pm, V^\pm)$ in the following form:

$$\begin{aligned}
U^+(x,t) &= U_0\left(\frac{d^+(x,t)}{\varepsilon}, x\right) + \varepsilon U_1\left(\frac{d^+(x,t)}{\varepsilon}, x, t\right) + Q\left(\frac{d^+(x,t)}{\varepsilon}, x, t\right),\\
V^+(x,t) &= V_0\left(\frac{d^+(x,t)}{\varepsilon}, x\right) + \varepsilon V_1\left(\frac{d^+(x,t)}{\varepsilon}, x, t\right) + \widehat{Q}\left(\frac{d^+(x,t)}{\varepsilon}, x, t\right),\\
U^-(x,t) &= U_0\left(\frac{d^-(x,t)}{\varepsilon}, x\right) + \varepsilon U_1\left(\frac{d^-(x,t)}{\varepsilon}, x, t\right) + Q\left(\frac{d^-(x,t)}{\varepsilon}, x, t\right),\\
V^-(x,t) &= V_0\left(\frac{d^-(x,t)}{\varepsilon}, x\right) + \varepsilon V_1\left(\frac{d^-(x,t)}{\varepsilon}, x, t\right) + \widehat{Q}\left(\frac{d^-(x,t)}{\varepsilon}, x, t\right).
\end{aligned} \tag{5.2}$$

Here (U_0, V_0) and (U_1, V_1) are functions defined in (2.14) and (2.23), namely

$$U_0(\zeta, x) = \phi_0(K(x)\zeta), \quad V_0(\zeta, x) = \psi_0(K(x)\zeta), \tag{5.3}$$

$$U_1(\zeta, x, t) = \frac{\nabla K \cdot \nabla d}{K^2}\phi_1(K(x)\zeta, x, t), \quad V_1(\zeta, x, t) = \frac{\nabla K \cdot \nabla d}{K^2}\psi_1(K(x)\zeta, x, t), \tag{5.4}$$

where (ϕ_0, ψ_0) and (ϕ_1, ψ_1) are solutions of (2.15) and (2.24), respectively, and

$$d^{\pm}(x,t) = d(x,t) \pm \varepsilon P(x,t).$$

From the above definition it is clear that both (U^+, V^+) and (U^-, V^-) have a steep internal layer within a 'small' neighborhood—of order εp—of $\Gamma(t)$. Our goal is to find smooth functions $P(x,t) > 0$, $Q(\zeta, x, t) > 0$, and $\widehat{Q}(\zeta, x, t) > 0$ such that for sufficiently small $\varepsilon > 0$, the following hold:

- $(\widetilde{U}^+, \widetilde{V}^+)$ and $(\widetilde{U}^-, \widetilde{V}^-)$ are an upper and a lower solution of (1.2), respectively in a neighborhood of $\Gamma(t)$ for $t \in [t^*(x), T]$, where $t^*(x)$ is defined in (4.10).
- $\widetilde{U}^+(x, t^*(x)) \geq \widetilde{U}^-(x, t^*(x))$, $\widetilde{V}^+(x, t^*(x)) \leq \widetilde{V}^-(x, t^*(x))$, moreover the graphs of $(\widetilde{U}^+, \widetilde{V}^+)$ and $(\widetilde{U}^-, \widetilde{V}^-)$ are well-separated in a certain sense to be specified later.

Here, roughly speaking, $t^*(x)$ represents the timing for transition from the initial stage of interface formation to the later stage of interface motion, which we have explained near the end of Section 1. $(u^{\pm}, v^{\pm})$ which we construct in Section 4, is defined for the initial stage $0 \leq t \leq t^*(x)$, while the other pair of upper and lower solutions $(U^{\pm}, V^{\pm})$ is defined for the later stage $t^*(x) \leq t \leq T$.

Define the operator $\mathscr{L}$ by

$$\mathscr{L}\begin{pmatrix} U \\ V \end{pmatrix} = \varepsilon \begin{pmatrix} U_t \\ V_t \end{pmatrix} - \varepsilon \begin{pmatrix} \nabla \cdot (k(x)\nabla U) \\ D\nabla \cdot (k(x)\nabla V) \end{pmatrix} - \frac{h(x)}{\varepsilon} \begin{pmatrix} f(U,V) \\ g(U,V) \end{pmatrix}. \tag{5.5}$$

Then we want (U^+, V^+) to satisfy $\mathscr{L}\left(\begin{smallmatrix} U^+ \\ V^+ \end{smallmatrix}\right) \geq 0$. Straightforward calculation shows

$$\mathscr{L}\begin{pmatrix} U^+ \\ V^+ \end{pmatrix} = E_1 + E_2 + E_3 + E_4 + E_5 + E_6 + E_7 + E_8 + E_9,$$

where $E_i = E_i(x,t)$ $(i = 1, 2, \ldots, 7)$ are defined by

$$E_1 = -\frac{k(x)|\nabla d|^2}{\varepsilon} \begin{pmatrix} U_{0\zeta\zeta} \\ DV_{0\zeta\zeta} \end{pmatrix} - \frac{h(x)}{\varepsilon} \begin{pmatrix} f(U_0, V_0) \\ g(U_0, V_0) \end{pmatrix},$$

$$E_2 = \varepsilon P_t \begin{pmatrix} U_{0\zeta} \\ V_{0\zeta} \end{pmatrix} + \varepsilon \begin{pmatrix} Q_t \\ \widehat{Q}_t \end{pmatrix} - \frac{h(x)}{\varepsilon} \begin{pmatrix} f_u(U_0, V_0) & f_v(U_0, V_0) \\ g_u(U_0, V_0) & g_v(U_0, V_0) \end{pmatrix} \begin{pmatrix} Q \\ \widehat{Q} \end{pmatrix}$$
$$+ \frac{2h(x)}{\varepsilon} \begin{pmatrix} Q^2 + bQ\widehat{Q} \\ \widehat{Q}^2 + aQ\widehat{Q}, \end{pmatrix}$$

$$E_3 = \varepsilon P_t \begin{pmatrix} Q_\zeta + \varepsilon U_{1\zeta} \\ \widehat{Q}_\zeta + \varepsilon V_{1\zeta} \end{pmatrix} - \frac{k(x)|\nabla d|^2}{\varepsilon} \begin{pmatrix} Q_{\zeta\zeta} \\ D\widehat{Q}_{\zeta\zeta} \end{pmatrix},$$

$$E_4 = -k(x)|\nabla d|^2 \begin{pmatrix} U_{1\zeta\zeta} \\ DV_{1\zeta\zeta} \end{pmatrix} - h(x) \begin{pmatrix} f_u(U_0, V_0) \; f_v(U_0, V_0) \\ g_u(U_0, V_0) \; g_v(U_0, V_0) \end{pmatrix} \begin{pmatrix} U_1 \\ V_1 \end{pmatrix}$$
$$+ (d_t - \nabla \cdot (k(x)\nabla d)) \begin{pmatrix} U_{0\zeta} \\ DV_{0\zeta} \end{pmatrix} - 2k(x) \begin{pmatrix} \nabla_x U_{0\zeta} \cdot \nabla d \\ D\nabla_x V_{0\zeta} \cdot \nabla d \end{pmatrix},$$

$$E_5 = 2h(x) \begin{pmatrix} 2U_1 Q + bU_1 \widehat{Q} + bV_1 Q \\ 2V_1 \widehat{Q} + 2aV_1 Q + 2aU_1 \widehat{Q} \end{pmatrix} + (d_t - \nabla \cdot (k(x)\nabla d)) \begin{pmatrix} Q_\zeta \\ D\widehat{Q}_\zeta \end{pmatrix},$$

$$E_6 = -2k(x)\nabla d \cdot \nabla P \begin{pmatrix} U_{0\zeta\zeta} + Q_{\zeta\zeta} \\ D(V_{0\zeta\zeta} + \widehat{Q}_{\zeta\zeta}) \end{pmatrix} - 2k(x) \begin{pmatrix} \nabla_x Q_\zeta \cdot \nabla d \\ D\nabla_x \widehat{Q}_\zeta \cdot \nabla d \end{pmatrix},$$

$$E_7 = -\varepsilon k(x)|\nabla P|^2 \begin{pmatrix} U_{0\zeta\zeta} + \varepsilon U_{1\zeta} + Q_{\zeta\zeta} \\ D(V_{0\zeta\zeta} + \varepsilon V_{1\zeta} + \widehat{Q}_{\zeta\zeta}) \end{pmatrix} - \varepsilon \nabla k \cdot \nabla P \begin{pmatrix} U_{0\zeta} + U_{1\zeta} + Q_\zeta \\ D(V_{0\zeta} + V_{1\zeta} + \widehat{Q}_\zeta) \end{pmatrix}$$
$$- 2\varepsilon k(x) \begin{pmatrix} (\nabla_x U_{0\zeta} + \varepsilon \nabla_x U_{1\zeta} + \nabla_x Q_\zeta) \cdot \nabla P \\ D(\nabla_x V_{0\zeta} + \varepsilon \nabla_x V_{1\zeta} + \nabla_x \widehat{Q}(\zeta)) \cdot \nabla P \end{pmatrix}$$
$$- 2\varepsilon \nabla d \cdot \nabla P \begin{pmatrix} U_{1\zeta\zeta} \\ DV_{1\zeta\zeta} \end{pmatrix} - \varepsilon \begin{pmatrix} \nabla k \cdot \nabla_x Q \\ D\nabla k \cdot \nabla_x \widehat{Q} \end{pmatrix},$$

$$E_8 = -\varepsilon k \Delta P \begin{pmatrix} U_{0\zeta} + \varepsilon U_{1\zeta} + Q_\zeta \\ D(V_{0\zeta} + \varepsilon V_{1\zeta} + \widehat{Q}_\zeta) \end{pmatrix} - \varepsilon \begin{pmatrix} k\Delta_x Q \\ Dk\Delta_x \widehat{Q} \end{pmatrix},$$

$$E_9 = \varepsilon(d_t - \nabla \cdot (k(x)\nabla d)) \begin{pmatrix} U_{1\zeta} \\ DV_{1\zeta} \end{pmatrix} - \varepsilon \begin{pmatrix} \nabla \cdot (k[\nabla_x U_0 + \varepsilon \nabla_x U_1]) \\ D\nabla \cdot (k[\nabla_x V_0 + \varepsilon \nabla_x V_1]) \end{pmatrix}$$
$$- \varepsilon \begin{pmatrix} k \cdot (\Delta_x U_0 + \varepsilon \Delta_x U_1) \\ Dk \cdot (\Delta_x V_0 + \varepsilon \Delta_x V_1) \end{pmatrix} - 2\varepsilon k(x) \begin{pmatrix} \nabla_x U_{1\zeta} \cdot \nabla d \\ D\nabla_x V_{1\zeta} \cdot \nabla d \end{pmatrix} + 2\varepsilon h(x) \begin{pmatrix} U_1^2 + bU_1V_1 \\ V_1^2 + aU_1V_1 \end{pmatrix}.$$

with ζ being substituted by

$$\zeta = \frac{d^+(x,t)}{\varepsilon}. \tag{5.6}$$

Here $\nabla_x U_0$ denotes the derivative with respect to x when we regard $U_0(\zeta, x)$ as a function of two variables ζ and x. The symbol Δ_x is defined similarly and this convention applies to U_0, U_ζ, V_1, Q, $\widehat{Q}$ as well.

Our first goal in this section is to show that E_2 dominates all the other terms $E_1, E_3, \ldots, E_9$ in a neighborhood of the interface $\Gamma(t)$, which implies that (U^+, V^+) in (5.2) is an upper solution near $\Gamma(t)$. In the second part of this section, we will modify (U^+, V^+) away from $\Gamma(t)$, to obtain an upper solution in the entire region Ω. In what follows, we will regard E_1 to E_9 as functions of (x, t, ζ) without the substitution (5.6) and derive estimates that hold for all $(x,t) \in \overline{\Omega} \times [0, T]$ and $\zeta \in \mathbf{R}$. This will reduce the notational complexity considerably. Naturally, all these estimates remain valid after substituting (5.6). For simplicity, we introduce the notation

$$R(\zeta, x) := \begin{pmatrix} f_u(U_0(\zeta,x), V_0(\zeta,x)) & f_v(U_0(\zeta,x), V_0(\zeta,x)) \\ g_u(U_0(\zeta,x), V_0(\zeta,x)) & g_v(U_0(\zeta,x), V_0(\zeta,x)) \end{pmatrix}.$$

Note that by (2.11) and (2.14), we have

$$\lim_{\zeta \to -\infty} R(\zeta, x) = -\begin{pmatrix} R_1 & R_1 b \\ 0 & R_1 a - R_2 \end{pmatrix}, \quad \lim_{\zeta \to +\infty} R(\zeta, x) = -\begin{pmatrix} R_2 b - R_1 & 0 \\ R_2 a & R_2 \end{pmatrix}, \tag{5.7}$$

uniformly in $x \in \overline{\Omega}$.

By the estimates in Lemmas 2.1 and 2.3, we can find constants $\mu > 0$, $M > 0$ such that

$$U_{0\zeta} + \varepsilon U_{1\zeta} \le -\mu, \quad V_{0\zeta} + \varepsilon V_{1\zeta} \ge \mu \quad \text{for } -M < \zeta < M. \tag{5.8}$$

Hereafter we fix such $M > 0$ and $\mu > 0$. Now we specify $P, Q, \widehat{Q}$ in (5.2) as follows:

$$P(x,t) = \delta\left(\exp\left(-\frac{\beta h(x)t}{\varepsilon^2} + \beta\tau^*(x)\right) - 1\right) - e^{\gamma t} - \gamma_0, \tag{5.9}$$

$$Q(\zeta,x,t) = \sigma J(\zeta)\delta\exp\left(-\frac{\beta h(x)t}{\varepsilon^2} + \beta\tau^*(x)\right) + \varepsilon^2\rho J(\zeta)e^{\gamma t}, \tag{5.10}$$

$$\widehat{Q}(\zeta,x,t) = \sigma \widehat{J}(\zeta)\delta\exp\left(-\frac{\beta h(x)t}{\varepsilon^2} + \beta\tau^*(x)\right) + \varepsilon^2\rho \widehat{J}(\zeta)e^{\gamma t}, \tag{5.11}$$

where $\tau^*(x)$ is as defined in (4.10), β is a positive constant satisfying

$$\beta < \frac{1}{2}\min\{R_1, R_2, aR_1 - R_2, bR_2 - R_1\} \tag{5.12}$$

and the constants $\delta, \sigma, \rho, \gamma, \gamma_0$ will be specified later. The coefficients $J(\zeta)$, $\hat{J}(\zeta)$ are smooth functions satisfying

$$\begin{pmatrix} J(\zeta) \\ \widehat{J}(\zeta) \end{pmatrix} = \begin{pmatrix} 2bR_1 \\ -(R_1 - \beta) \end{pmatrix} \quad \text{for } \zeta \le -M, \tag{5.13}$$

$$\begin{pmatrix} J(\zeta) \\ \widehat{J}(\zeta) \end{pmatrix} = \begin{pmatrix} R_2 - \beta \\ -2aR_2 \end{pmatrix} \quad \text{for } \zeta \ge M. \tag{5.14}$$

By (5.7), we have

$$\lim_{\zeta\to-\infty} R(\zeta,x)\begin{pmatrix} J(\zeta) \\ \widehat{J}(\zeta) \end{pmatrix} = \begin{pmatrix} R_1 b\beta + bR_1^2 \\ -(R_1 a - R_2)(R_1 - \beta) \end{pmatrix},$$

$$\lim_{\zeta\to+\infty} R(\zeta,x)\begin{pmatrix} J(\zeta) \\ \widehat{J}(\zeta) \end{pmatrix} = \begin{pmatrix} (R_2 b - R_1)(R_2 - \beta) \\ -R_2 a\beta - aR_2^2 \end{pmatrix}$$

uniformly in $x \in \overline{\Omega}$. In view of this and (5.12), and by replacing M by a larger constant if necessary, we see that there exists a constant $C > 0$ such that

$$-R(\zeta,x)\begin{pmatrix} J(\zeta) \\ \hat{J}(\zeta) \end{pmatrix} \ge C\begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{for } |\zeta| \ge M, \ \ x \in \overline{\Omega}. \tag{5.15}$$

The terms E_2 and E_3.

Lemma 5.1. *Let* (5.27), (5.32), *and* (5.49) *hold, and let M be the constant appearing in* (5.8), (5.13), (5.14). *Then*

$$E_2 \succeq \widetilde{C}_1\left(\frac{\sigma\delta}{\varepsilon}\exp\left(-\frac{\beta h(x)t}{\varepsilon^2} + \beta\tau^*(x)\right) + \varepsilon\rho e^{\gamma t}\right)\begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \textit{if } |\zeta| \ge M, \tag{5.16}$$

$$E_2 \succeq \widetilde{C}_3\left(\frac{\delta}{\varepsilon}\exp\left(-\frac{\beta h(x)t}{\varepsilon^2} + \beta\tau^*(x)\right) + \varepsilon\gamma e^{\gamma t}\right)\begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \textit{if } |\zeta| \le M, \tag{5.17}$$

$$|E_3| \le \varepsilon \widetilde{C}_2\left(\frac{\delta}{\varepsilon}\exp\left(-\frac{\beta h(x)t}{\varepsilon^2}+\beta\tau^*(x)\right)+\varepsilon\gamma e^{\gamma t}\right) \quad \text{if } |\zeta| \ge M, \tag{5.18}$$

$$|E_3| \le \widetilde{C}_4\left(\frac{\sigma\delta}{\varepsilon}\exp\left(-\frac{\beta h(x)t}{\varepsilon^2}+\beta\tau^*(x)\right)+\varepsilon\rho e^{\gamma t}\right) \quad \text{if } |\zeta| \le M, \tag{5.19}$$

where $\widetilde{C}_i$, $i = 1, 2, 3, 4$ *are positive constants independent of* $\delta, \sigma, \rho, \gamma$.

Proof. Since $P_t < 0$, $U_{0\zeta} < 0$, $V_{0\zeta} > 0$, we have

$$E_2 \succeq \varepsilon\begin{pmatrix} Q_t \\ \widehat{Q}_t\end{pmatrix} - \frac{h(x)}{\varepsilon}R(\zeta,x)\begin{pmatrix} Q \\ \widehat{Q}\end{pmatrix} + \frac{2h(x)}{\varepsilon}\begin{pmatrix} Q^2 + bQ\widehat{Q} \\ \widehat{Q}^2 + aQ\widehat{Q}\end{pmatrix} := I(\zeta, Q, \widehat{Q}). \tag{5.20}$$

From the definition of Q, $\widehat{Q}$, there exists $C_0' > 0$ such that

$$\max\{a, b, 1\}\big(|Q| + |\widehat{Q}|\big) \le C_0'\big(\sigma\delta + \varepsilon^2\rho e^{\gamma T}\big). \tag{5.21}$$

Therefore

$$I(\zeta, Q, \widehat{Q}) \succeq \varepsilon\begin{pmatrix} Q_t \\ \widehat{Q}_t\end{pmatrix} - \frac{h(x)}{\varepsilon}\left(R(\zeta,x) - C_0'\big(\sigma\delta+\varepsilon^2\rho e^{\gamma T}\big)\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}\right)\begin{pmatrix} Q \\ \widehat{Q}\end{pmatrix}. \tag{5.22}$$

Note that

$$Q_t(\zeta,x,t) = -\frac{\sigma J(\zeta)\delta\beta h(x)}{\varepsilon^2}\exp\left(-\frac{\beta h(x)t}{\varepsilon^2}+\beta\tau^*(x)\right) + \varepsilon^2\gamma\rho J(\zeta)e^{\gamma t}, \tag{5.23}$$

$$\widehat{Q}_t(\zeta,x,t) = -\frac{\sigma \hat{J}(\zeta)\delta\beta h(x)}{\varepsilon^2}\exp\left(-\frac{\beta h(x)t}{\varepsilon^2}+\beta\tau^*(x)\right) + \varepsilon^2\gamma\rho \hat{J}(\zeta)e^{\gamma t}. \tag{5.24}$$

Consequently,

$$\begin{aligned} I(\zeta, Q, \widehat{Q}) \succeq\ & \frac{h(x)\sigma\delta}{\varepsilon}\exp\left(-\frac{\beta h(x)t}{\varepsilon^2}+\beta\tau^*(x)\right) \\ & \times\left(-\big(\beta + C_0'(\sigma\delta+\varepsilon^2\rho e^{\gamma T})\big)\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix} - R(\zeta,x)\right)\begin{pmatrix} J(\zeta) \\ \hat{J}(\zeta)\end{pmatrix} \\ & - \varepsilon\rho h(x)e^{\gamma t}\left(C_0'\big(\sigma\delta+\varepsilon^2\rho e^{\gamma T}\big)\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix} + R(\zeta,x)\right)\begin{pmatrix} J(\zeta) \\ \hat{J}(\zeta)\end{pmatrix} \\ & + \varepsilon^3\rho\gamma e^{\gamma t}\begin{pmatrix} J(\zeta) \\ \hat{J}(\zeta)\end{pmatrix}. \end{aligned} \tag{5.25}$$

Observe that (5.15) implies

$$h(x)\left(-(\beta + C_0'(\sigma\delta+\varepsilon^2\rho e^{\gamma T}))\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix} - R(\zeta,x)\right)\begin{pmatrix} J(\zeta) \\ \hat{J}(\zeta)\end{pmatrix} \succeq C_1'\begin{pmatrix} 1 \\ -1\end{pmatrix} \tag{5.26}$$

and

$$-h(x)e^{\gamma t}\Big(C_0'\big(\sigma\delta+\varepsilon^2\rho e^{\gamma T}\big)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+R(\zeta,x)\Big)\begin{pmatrix}J(\zeta)\\ \hat{J}(\zeta)\end{pmatrix}\succeq C_2'e^{\gamma t}\begin{pmatrix}1\\ -1\end{pmatrix} \tag{5.27}$$

for some constants C_1', C_2' that are independent of $\sigma, \delta, \gamma, \rho$. If the constants $\sigma, \delta, \gamma, \rho, \varepsilon$ are chosen small enough to satisfy

$$\sigma\delta+\varepsilon^2\rho e^{\gamma T}\ll 1, \tag{5.28}$$

then (5.22)–(5.27) imply

$$I(\zeta,Q,\widehat{Q})\succeq\Big(C_1'\frac{\sigma\delta}{\varepsilon}\exp\Big(-\frac{\beta h(x)t}{\varepsilon^2}+\beta\tau^*(x)\Big)+\varepsilon C_2'\rho e^{\gamma t}\Big)\begin{pmatrix}1\\ -1\end{pmatrix} \tag{5.29}$$

for $|\zeta|\geq M$. This and (5.20) imply (5.16). Next we recall that J and $\widehat{J}$ are constant in the region $|\zeta|\geq M$. Consequently,

$$|E_3|=\varepsilon^2\left|P_t\begin{pmatrix}U_{1\zeta}\\ V_{1\zeta}\end{pmatrix}\right|\leq\varepsilon\widetilde{C}_2\Big(\frac{\delta}{\varepsilon}\exp\Big(-\frac{\beta h(x)t}{\varepsilon^2}+\beta\tau^*(x)\Big)+\varepsilon\gamma e^{\gamma t}\Big)\quad\text{for } |\zeta|\geq M,$$

where $\widetilde{C}_2$ is a constant independent of $\sigma, \delta, \gamma, \rho$. This proves (5.18).

We next consider the case $-M<\zeta<M$. In view of (5.8) and the fact that

$$P_t=-\frac{\delta\beta h(x)}{\varepsilon^2}\exp\Big(-\frac{\beta h(x)t}{\varepsilon^2}+\beta\tau^*(x)\Big)-\gamma e^{\gamma t}<0,$$

we obtain

$$P_t\begin{pmatrix}U_{0\zeta}+\varepsilon U_{1\zeta}\\ V_{0\zeta}+\varepsilon V_{1\zeta}\end{pmatrix}\succeq\mu\Big(\frac{\beta\delta h(x)}{\varepsilon^2}\exp\Big(-\frac{\beta h(x)t}{\varepsilon^2}+\beta\tau^*(x))+\gamma e^{\gamma t}\Big)\begin{pmatrix}1\\ -1\end{pmatrix}. \tag{5.30}$$

Since $|J'|$, $|\widehat{J}'|$, $|J''|$, $|\widehat{J}''|$ are continuous, they are bounded for $-M<\zeta<M$. Consequently there exists $C_4'>0$, $C_5'>0$ such that

$$\begin{aligned}&\varepsilon\begin{pmatrix}Q_t\\ \widehat{Q}_t\end{pmatrix}-\frac{h(x)}{\varepsilon}\begin{pmatrix}f_u(U_0,V_0) & f_v(U_0,V_0)\\ g_u(U_0,V_0) & g_v(U_0,V_0)\end{pmatrix}\begin{pmatrix}Q\\ \widehat{Q}\end{pmatrix}+\frac{2h(x)}{\varepsilon}\begin{pmatrix}Q^2+bQ\widehat{Q}\\ \widehat{Q}^2+aQ\widehat{Q}\end{pmatrix}\\ &\succeq-\frac{C_4'}{\varepsilon}\begin{pmatrix}Q\\ \widehat{Q}\end{pmatrix}=-C_5'\Big(\frac{\sigma\delta}{\varepsilon}\exp\Big(-\frac{\beta h(x)t}{\varepsilon^2}+\beta\tau^*(x)\Big)+\varepsilon\rho e^{\gamma t}\Big)\begin{pmatrix}1\\ -1\end{pmatrix},\end{aligned} \tag{5.31}$$

provided that (5.28) holds. Here we have used (5.23) and (5.24). Using (5.30) and (5.31) and choosing positive constants γ, δ, ρ, and σ satisfying

$$\sigma\ll 1,\quad \rho\ll\gamma,\quad \delta\ll 1, \tag{5.32}$$

we obtain (5.17) for $-M<\zeta<M$.

On the other hand, a straightforward calculation shows that there exists $\widetilde{C}_4 > 0$ such that (5.19) holds. The lemma is proved.

The terms E_6 and E_7. By straightforward calculation, we have

$$\nabla P = \left(-\frac{\beta \nabla h t}{\varepsilon^2} - \frac{\beta \nabla \delta^*}{\alpha^* \delta^*}\right)\delta \exp\left(-\frac{\beta h(x)t}{\varepsilon^2} + \beta\tau^*(x)\right), \tag{5.33}$$

where $\delta^* = \delta^*(u_0(x), v_0(x))$. Note that

$$|\nabla \delta^*| = \left|\frac{\partial \delta^*}{\partial u}\nabla u_0 + \frac{\partial \delta^*}{\partial v}\nabla v_0\right| \le |\widetilde{\nabla}\delta^*|\sqrt{|\nabla u_0|^2 + |\nabla v_0|^2}, \tag{5.34}$$

where $|\widetilde{\nabla}\delta^*| = \sqrt{\left(\frac{\partial \delta^*}{\partial u}\right)^2 + \left(\frac{\partial \delta^*}{\partial v}\right)^2}$. Since $|\widetilde{\nabla}\delta^*|$, $|\nabla u_0|$, $|\nabla v_0|$ are independent of $\varepsilon > 0$, the right-hand side of the above inequality is independent of $\varepsilon > 0$ as x varies in $\overline{\Omega}$. By the definition of δ^* in (4.9), we have $\delta^*(u_0(x), v_0(x)) \ge \varepsilon\ell$. Combining these, we get

$$|\nabla P| \le C_6'\left(\frac{t}{\varepsilon^2} + \frac{1}{\varepsilon\ell}\right)\delta \exp\left(-\frac{\beta h(x)t}{\varepsilon^2} + \beta\tau^*(x)\right). \tag{5.35}$$

Since $\min_{x\in\bar{\Omega}} h(x) > 0$, for any small constant $\eta_1 > 0$, we can choose $\varepsilon > 0$ sufficiently small such that

$$\begin{aligned}
&\frac{t}{\varepsilon^2}\exp\left(-\frac{\beta h(x)t}{\varepsilon^2} + \beta\tau^*(x)\right) < \eta_1\varepsilon && \text{for } t \ge \frac{1}{2}\eta_1\varepsilon,\\
&\frac{t}{\varepsilon^2}\exp\left(-\frac{\beta h(x)t}{\varepsilon^2} + \beta\tau^*(x)\right) < \frac{\eta_1}{\varepsilon}\exp\left(-\frac{\beta h(x)t}{\varepsilon^2} + \beta\tau^*(x)\right) && \text{for } t < \frac{1}{2}\eta_1\varepsilon.
\end{aligned} \tag{5.36}$$

Inequalities (5.35) and (5.36) yield

$$|\nabla P| \le \frac{C_7'}{\varepsilon}\left(\eta_1 + \frac{1}{\ell}\right)\delta \exp\left(-\frac{\beta h(x)t}{\varepsilon^2} + \beta\tau^*(x)\right) + C_8'\eta_1\varepsilon \tag{5.37}$$

for sufficiently small ε and $t \ge \frac{\varepsilon^2}{\beta h(x)}\tau^*(x)$ with $C_7', C_8' > 0$ independent of $\varepsilon > 0$.

On the other hand, it is not difficult to see that

$$\begin{aligned}
\nabla_x Q_\zeta &= \sigma J'(\zeta)\nabla P, & \nabla_x \widehat{Q}_\zeta &= \sigma \hat{J}'(\zeta)\nabla P,\\
\nabla_x Q_{\zeta\zeta} &= \sigma J''(\zeta)\nabla P, & \nabla_x \widehat{Q}_{\zeta\zeta} &= \sigma \hat{J}''(\zeta)\nabla P.
\end{aligned} \tag{5.38}$$

Note that U_0, V_0, U_1, V_1 and all their derivatives with respect to ζ are bounded from above by some constant independent of $\varepsilon > 0$. Therefore, (5.37) and (5.38) show that there exists a constant C_{11}', C_{12}' independent of $\varepsilon > 0$ such that

$$|E_6| + |E_7| \le \frac{C_{11}'}{\varepsilon}\left(\eta_1 + \frac{1}{\ell}\right)(1+\sigma)\delta \exp\left(-\frac{\beta h(x)t}{\varepsilon^2} + \beta\tau^*(x)\right) + C_{12}'(1+\sigma)\eta_1\varepsilon.$$

The term E_8. Differentiating ∇P with respect to x, we have

$$\Delta P = \left(-\frac{\beta\Delta ht}{\varepsilon^2} - \frac{\beta\Delta\delta^*}{\alpha^*\delta^*} + \frac{\beta|\nabla\delta^*|^2}{\alpha^*\delta^{*2}} + \left(\frac{\beta\nabla ht}{\varepsilon^2} + \frac{\beta\nabla\delta^*}{\alpha^*\delta^*}\right)^2\right)\delta\exp\left(-\frac{\beta h(x)t}{\varepsilon^2} + \beta\tau^*(x)\right), \tag{5.39}$$

where $\delta^* = \delta^*(u_0(x), v_0(x))$. This and the fact that

$$\Delta\delta^* = \frac{\partial^2\delta^*}{\partial u^2}|\nabla u_0|^2 + \frac{\partial\delta^*}{\partial u}\Delta u_0 + \frac{\partial\delta^*}{\partial v^2}|\nabla v_0|^2 + \frac{\partial\delta^*}{\partial v}\Delta v + 2\frac{\partial^2\delta^*}{\partial u\partial v}\nabla u_0\nabla v_0, \tag{5.40}$$

yield

$$|\Delta P| \leq \frac{\widehat{C}_1}{\varepsilon^2}\left(t + \frac{1}{\ell^2}\right)\delta\exp\left(-\frac{\beta h(x)t}{\varepsilon^2} + \beta\tau^*(x)\right) \tag{5.41}$$

for sufficiently small $\varepsilon > 0$. Hereafter, $\widehat{C}_i$ will denote a positive constant independent of ε. By (5.36),

$$|\Delta P| \leq \widehat{C}_1\left(\frac{\eta_1}{\varepsilon} + \frac{1}{\varepsilon^2\ell^2}\right)\delta\exp\left(-\frac{\beta h(x)t}{\varepsilon^2} + \beta\tau^*(x)\right) + \widehat{C}_4\eta_1\varepsilon. \tag{5.42}$$

By the fact that

$$\Delta_x Q = \sigma J(\zeta)\Delta P, \quad \Delta_x\widehat{Q} = \sigma\hat{J}(\zeta)\Delta P \tag{5.43}$$

and (5.41), it follows that

$$|E_8| \leq \widehat{C}_3\left(\eta_1 + \frac{1}{\varepsilon\ell^2}\right)\delta\exp\left(-\frac{\beta h(x)t}{\varepsilon^2} + \beta\tau^*(x)\right) + \widehat{C}_4\eta_1\varepsilon^2. \tag{5.44}$$

The term E_5. By the definition of Q and $\widehat{Q}$, we have

$$|E_5| \leq \widehat{C}_4\left(\sigma\delta\exp\left(-\frac{\beta h(x)t}{\varepsilon^2} + \beta\tau^*(x)\right) + \varepsilon^2\rho e^{\gamma t}\right).$$

The term E_9. Since U_0, V_0, U_1, V_1 and all their derivatives are bounded, we have

$$|E_9| \leq \widehat{C}_5\varepsilon$$

for some positive constant $\widehat{C}_5$.

The terms E_1 and E_4.

Lemma 5.2. *Let $\mathcal{N}_{d^*}(\Gamma(t))$ be as in* (5.1). *Then*

$$E_1 = \begin{pmatrix}0\\0\end{pmatrix} \quad \text{in } \mathcal{N}_{d^*}(\Gamma(t))$$

and there exists a constant $M > 0$ *such that*

$$|E_4| \le \varepsilon M \quad in\ \mathcal{N}_{d^*}(\Gamma(t)). \tag{5.45}$$

Proof. We first consider E_1. Note that $|\nabla d| = 1$ in the range where $|d(x,t)| \le d^*$. Therefore (2.15) implies $E_1 = \binom{0}{0}$.

We next consider E_4. Recall that d satisfies (2.21) on $\Gamma(t)$:

$$d_t - \nabla \cdot (k(x)\nabla d) = 2(C+1)\frac{k\nabla d \cdot \nabla K}{K}.$$

Here C is the constant in (2.22). By the smoothness of Γ, the functions d_t, ∇d, Δd, and ∇k are Lipschitz continuous in x. Consequently we have

$$\left| d_t - \nabla \cdot (k(x)\nabla d) - \frac{2(C+1)k\nabla d \cdot \nabla K}{K} \right| \le M_0|d| \tag{5.46}$$

for some constant $M_0 > 0$.

By the definition (5.4) and (2.24) with z replaced by $\zeta K(x)$, we have

$$\begin{aligned} &-k(x)|\nabla d|^2 \begin{pmatrix} U_{1\zeta\zeta} \\ DV_{1\zeta\zeta} \end{pmatrix} - h(x)\begin{pmatrix} f_u(U_0,V_0) & f_v(U_0,V_0) \\ g_u(U_0,V_0) & g_v(U_0,V_0) \end{pmatrix}\begin{pmatrix} U_1 \\ V_1 \end{pmatrix} \\ &\quad = -k(x)\nabla K \cdot \nabla d\left(-2\zeta K(x)\begin{pmatrix} \phi_{0zz} \\ D\psi_{0zz} \end{pmatrix} + 2C\begin{pmatrix} \phi_{0z} \\ D\psi_{0z} \end{pmatrix}\right). \end{aligned}$$

Using the fact that $\nabla U_{0\zeta} = \phi_{0zz}\nabla K + \phi_{0z}\zeta K(x)\nabla K$, we obtain

$$E_4 = \left\{ d_t - \nabla \cdot (k(x)\nabla d) - \frac{2k\nabla d \cdot \nabla K}{K}(C+1)\right\}\begin{pmatrix} U_{0\zeta} \\ DV_{0\zeta} \end{pmatrix}.$$

This and (5.46) imply

$$|E_4| \le M_0|d|\left|\left(U_{0\zeta}\left(\frac{d^+(x,t)}{\varepsilon}, x\right), DV_{0\zeta}\left(\frac{d^+(x,t)}{\varepsilon}, x\right)\right)\right|.$$

Note that $U_{0\zeta} = K(x)\phi_0'$, $V_{0\zeta} = K(x)\psi_0'$ and that (ϕ_0', ψ_0') is a solution of (2.19) with $A_1 \equiv A_2 \equiv 0$. By Lemma 2.3, there exists $\hat{\delta} > 0$ such that

$$\phi_0'(z) = o\big(\exp(-\hat{\delta}|z|)\big), \quad \psi_0'(z) = o\big(\exp(-\hat{\delta}|z|)\big), \tag{5.47}$$

which yields

$$|E_4| \le \widetilde{M}_0|d(x,t)|\exp\left(-\frac{\hat{\delta}k(x)(d(x,t) + \varepsilon P(x,t))}{\varepsilon}\right).$$

We can find some constants $\widetilde{M}_1, \widetilde{M}_2 > 0$ such that

$$\begin{aligned}|E_4| &\le \widetilde{M}_1 |d(x,t)| \exp(|P(x,t)|) \exp\left(-\frac{\hat{\delta}|d(x,t)|}{\varepsilon}\right)\\ &\le \varepsilon \widetilde{M}_1 \exp(|P(x,t)| \max_{\zeta\ge 0}\{\zeta \exp(-\hat{\delta}\zeta)\}\\ &\le \varepsilon \widetilde{M}_2 \exp(|P(x,t)|).\end{aligned}$$

The lemma is proved.

Now let us consider the sum of E_i, $i = 1, 2, \dots, 9$. By Lemma 5.1, E_3 is much smaller than E_2 as long as we have (5.28), (5.32), and

$$\varepsilon \ll \sigma, \quad \varepsilon\gamma \ll \rho. \tag{5.48}$$

Also, by what we have shown above, $E_1 = \binom{0}{0}$, and the terms E_4 to E_9 are much smaller than E_2 if

$$\sigma \ll 1, \quad \eta_1 + \frac{1}{\ell} \ll \sigma \tag{5.49}$$

and if $\varepsilon > 0$ sufficiently small. This yields $\mathscr{L}(U^+, V^+) \ge (0, 0)$ for $\sigma, \rho, \gamma, \eta, \ell$ satisfying (5.32), (5.49) and $\varepsilon > 0$ satisfying (5.28) and (5.48). Similarly $\mathscr{L}(U^-, V^-) \le (0, 0)$.

Summarizing, we obtain the following main lemma of this subsection.

Lemma 5.3. *Assume* (5.32) *and* (5.49). *For $\varepsilon > 0$ sufficiently small such that* (5.28) *and* (5.48) *hold,* (U^+, V^+) *and* (U^-, V^-) *are an upper and a lower solution respectively in a neighborhood of* $\Gamma(t)$. *More precisely, they satisfy* (4.3), (4.4), *respectively, for* (x, t) *satisfying*

$$t > t^*(x) \quad \text{and} \quad |d(x,t)| \le d^*,$$

where $t^*(x)$ is as defined in (4.10).

5.2. *Cut-Off Function*

So far, we have constructed the upper and lower solutions $(U^\pm, V^\pm)$ in a neighborhood of $\Gamma(t)$ where the distance function $d(x, t)$ is smooth. In order to extend the domain of definition of $(U^\pm, V^\pm)$ over the entire region $\overline{\Omega}$, we introduce a cut-off function. Let ξ be a smooth function such that

$$\begin{cases} \xi(d) = d & \text{if } |d| \le d^*,\\ d^* \le |\xi(d)| \le 2d^* & \text{if } d^* \le |d| \le 2d^*,\\ |\xi(d)| = 2d^* & \text{if } |d| \ge 2d^*,\end{cases} \tag{5.50}$$

and that $\xi'(d) \ge 0$. Set

$$\tilde{d}^\pm(x,t) = \xi(d(x,t) \pm \varepsilon P(x,t)).$$

We will first consider the case where $h(x)$ satisfies the zero Neumann boundary condition. The newly constructed upper and lower solutions given below will then satisfy the Neumann boundary condition on $\partial\Omega$ automatically. To be more precise, we set

$$\begin{aligned}\widetilde{U}^{+}(x,t) &= U_0\left(\frac{\tilde{d}^{+}(x,t)}{\varepsilon},x\right)+\varepsilon U_1\left(\frac{\tilde{d}^{+}(x,t)}{\varepsilon},x,t\right)+Q\left(\frac{\tilde{d}^{+}(x,t)}{\varepsilon},x,t\right),\\ \widetilde{V}^{+}(x,t) &= V_0\left(\frac{\tilde{d}^{+}(x,t)}{\varepsilon},x\right)+\varepsilon V_1\left(\frac{\tilde{d}^{+}(x,t)}{\varepsilon},x,t\right)+\widehat{Q}\left(\frac{\tilde{d}^{+}(x,t)}{\varepsilon},x,t\right),\end{aligned} \tag{5.51}$$

$$\begin{aligned}\widetilde{U}^{-}(x,t) &= U_0\left(\frac{\tilde{d}^{-}(x,t)}{\varepsilon},x\right)+\varepsilon U_1\left(\frac{\tilde{d}^{-}(x,t)}{\varepsilon},x,t\right)+Q\left(\frac{\tilde{d}^{-}(x,t)}{\varepsilon},x,t\right),\\ \widetilde{V}^{-}(x,t) &= V_0\left(\frac{\tilde{d}^{-}(x,t)}{\varepsilon},x\right)+\varepsilon V_1\left(\frac{\tilde{d}^{-}(x,t)}{\varepsilon},x,t\right)+\widehat{Q}\left(\frac{\tilde{d}^{-}(x,t)}{\varepsilon},x,t\right).\end{aligned} \tag{5.52}$$

The following lemma is the main result of this section.

Lemma 5.4 (Main Lemma). *Assume* (5.32) *and* (5.49). *For sufficiently small* ε *such that* (5.28) *and* (5.48) *hold,* $(\widetilde{U}^{+},\widetilde{V}^{+})$ *and* $(\widetilde{U}^{-},\widetilde{V}^{-})$ *are an upper and a lower solution, respectively, in the region* $\{(x,t)\in\overline{\Omega}\times[0,T]\,|\,t>t^{*}(x)\}$.

Proof of Main Lemma. We only prove that $(\widetilde{U}^{+},\widetilde{V}^{+})$ is an upper solution. To do so, it suffices to show that there exist $\widetilde{C}_1,\widetilde{C}_2>0$ such that for sufficiently small $\varepsilon>0$ the following holds:

$$\mathscr{L}\begin{pmatrix}\widetilde{U}^{+}\\ \widetilde{V}^{+}\end{pmatrix}\succeq\left(\frac{\widetilde{C}_1}{\varepsilon}\exp\left(\frac{-\beta h(x)t}{\varepsilon^{2}}+\beta\tau^{*}(x)\right)+\varepsilon\widetilde{C}_2e^{\gamma t}\right)\begin{pmatrix}1\\-1\end{pmatrix}. \tag{5.53}$$

In the case where $|\tilde{d}^{+}|\le d^{*}$, we have $(\widetilde{U}^{+},\widetilde{V}^{+})=(U^{+},V^{+})$. Therefore (5.53) follows directly from Lemma 5.3.

In the case where $|\tilde{d}^{+}|\ge d^{*}$, $(\widetilde{U}^{+},\widetilde{V}^{+})$ is close to $(R_1,0)$ if $\tilde{d}^{+}\le -d^{*}$, while it is close to $(0,R_2)$ if $\tilde{d}^{+}\ge d^{*}$. We only consider the case where $\tilde{d}^{+}\le -d^{*}$ since the other case is treated in the same way. To complete the proof, we need to prove the following two inequalities.

In Step 1, we will show that

$$\left|\mathscr{L}\begin{pmatrix}\widetilde{U}^{+}\\ \widetilde{V}^{+}\end{pmatrix}-\mathscr{L}\begin{pmatrix}R_1+Q\\ \widehat{Q}\end{pmatrix}\right|\le\varepsilon\widetilde{C}_3\exp\left(-\frac{M_0d^{*}}{\varepsilon}\right) \tag{5.54}$$

under the condition $\tilde{d}^{+}(x,t)\le -d^{*}$ where

$$Q=Q\left(\frac{\tilde{d}^{+}(x,t)}{\varepsilon},x,t\right),\quad \widehat{Q}=\widehat{Q}\left(\frac{\tilde{d}^{+}(x,t)}{\varepsilon},x,t\right).$$

In Step 2, we will prove

$$\mathscr{L}\begin{pmatrix} R_1 + Q \\ \widehat{Q} \end{pmatrix} \succeq \left(\frac{\widetilde{C}_4 \sigma \delta}{\varepsilon} \exp\left(-\frac{\beta h(x) t}{\varepsilon^2} + \beta \tau^*(x) \right) + \varepsilon \widetilde{C}_5 \rho e^{\gamma t} \right) \begin{pmatrix} 1 \\ -1 \end{pmatrix} \succeq 0. \tag{5.55}$$

Hereafter $\widetilde{C}_i$, M_i ($i \in \mathbf{N}$) are constants independent of $\varepsilon > 0$.

Step 1. By Lemma 2.1 and (5.47), ϕ_0' and ψ_0' decay exponentially as $|z| \to \infty$. If we recall that ϕ_0, ψ_0 satisfy (2.15), we obtain that ϕ_0'', ψ_0'' also have the same decay rate as $R_1 - \phi_0$ and ψ_0. From this fact, there exist $\widetilde{C}_6$, $M_1 > 0$ such that

$$|U_0 - R_1| + |V_0| + |U_{0\zeta}| + |V_{0\zeta}| + |U_{0\zeta\zeta}| + |V_{0\zeta\zeta}| \leq \widetilde{C}_6 \exp\left(\frac{-M_1 |\tilde{d}^+|}{\varepsilon} \right). \tag{5.56}$$

Next we estimate U_1, V_1 and their derivatives. Combining (2.17) and Lemma 2.3, we see that ϕ_1, ψ_1 decay exponentially. Differentiating (2.17) and applying Lemma 2.3, we see that ϕ_1', ψ_1' also decay exponentially. The same argument holds for ϕ_1'' and ψ_1''. Consequently there exist $M_2 > 0$ and $\widetilde{C}_7 > 0$ such that

$$|U_1| + |V_1| + |U_{1\zeta}| + |V_{1\zeta}| + |U_{1\zeta\zeta}| + |V_{1\zeta\zeta}| \leq \widetilde{C}_7 \exp\left(\frac{-M_2 |\tilde{d}^+|}{\varepsilon} \right). \tag{5.57}$$

By Taylor's theorem we have

$$\begin{aligned} \mathscr{L}\begin{pmatrix} \widetilde{U}^+ \\ \widetilde{V}^+ \end{pmatrix} &= \mathscr{L}\begin{pmatrix} R_1 + Q \\ \widehat{Q} \end{pmatrix} + \varepsilon \begin{pmatrix} U_0 - R_1 + \varepsilon U_1 \\ V_0 - R_2 + \varepsilon U_2 \end{pmatrix}_t - \varepsilon \nabla \cdot \left(k(x) \nabla \begin{pmatrix} U_0 - R_1 + \varepsilon U_1 \\ V_0 + \varepsilon V_1 \end{pmatrix} \right) \\ &\quad - \frac{h(x)}{\varepsilon} \begin{pmatrix} f_u(v_1, \mu_1) & f_v(v_2, \mu_2) \\ g_u(v_3, \mu_3) & g_v(v_4, \mu_4) \end{pmatrix} \begin{pmatrix} U_0 + \varepsilon U_1 - R_1 \\ V_0 + \varepsilon V_1 \end{pmatrix}, \end{aligned} \tag{5.58}$$

where v_i, $i = 1, 2, 3, 4$ are some values between $U_0 + \varepsilon U_1 + Q$ and $R_1 + Q$, and μ_i, $i = 1, 2, 3, 4$, are some values between $V_0 + \varepsilon V_1 + \widehat{Q}$ and $\widehat{Q}$. From (5.56)–(5.58) we obtain (5.54).

Step 2. Let $\xi := \tilde{d}^+/\varepsilon$. Since we are assuming $|\tilde{d}^+| \geq d^*$, we have $|\xi| \geq M$ for ε sufficiently small, where M is as in (5.13), (5.14). Consequently the terms Q, $\widehat{Q}$ do not depend on $\zeta := \tilde{d}^+/\varepsilon$ (see (5.10), (5.11), (5.13), and (5.14)). Therefore, we have

$$\begin{aligned} \mathscr{L}\begin{pmatrix} R_1 + Q \\ \widehat{Q} \end{pmatrix} &= \varepsilon \begin{pmatrix} Q_t \\ \widehat{Q}_t \end{pmatrix} - \frac{h(x)}{\varepsilon} \begin{pmatrix} f_u(R_1, 0) & f_v(R_1, 0) \\ g_u(R_1, 0) & g_v(R_1, 0) \end{pmatrix} \begin{pmatrix} Q \\ \widehat{Q} \end{pmatrix} + \frac{2h(x)}{\varepsilon} \begin{pmatrix} Q^2 + bQ\widehat{Q} \\ \widehat{Q}^2 + aQ\widehat{Q} \end{pmatrix}, \\ &\quad - \varepsilon \begin{pmatrix} \nabla k \cdot \nabla_x Q \\ D \nabla k \cdot \nabla_x \widehat{Q} \end{pmatrix} - \varepsilon \begin{pmatrix} k \Delta_x Q \\ D k \Delta_x \widehat{Q} \end{pmatrix}. \end{aligned} \tag{5.59}$$

Note that the sum of the first three terms on the right-hand side is equal to $I(\zeta, Q, \widehat{Q})$ with $\zeta = -\infty$ (see (5.20)). Clearly (5.29) holds for $\zeta = -\infty$. Using the same argument as we have used to estimate $|E_6| + |E_7|$ and $|E_8|$, we obtain

$$\left|\varepsilon\begin{pmatrix}\nabla k\cdot\nabla_x Q\\ D\nabla k\cdot\nabla_x\widehat{Q}\end{pmatrix}\right| \le \frac{C'_{11}}{\varepsilon}\left(\eta_1+\frac{1}{\ell}\right)(1+\sigma)\delta\exp\left(-\frac{\beta h(x)t}{\varepsilon^2}+\beta\tau^*(x)\right) + C'_{12}(1+\sigma)\eta_1\varepsilon, \tag{5.60}$$

$$\left|\varepsilon\begin{pmatrix}k\Delta_x Q\\ Dk\Delta_x\widehat{Q}\end{pmatrix}\right| \le \widehat{C}_3\left(\eta_1+\frac{1}{\varepsilon\ell^2}\right)\delta\exp\left(-\frac{\beta h(x)t}{\varepsilon^2}+\beta\tau^*(x)\right)+\widehat{C}_4\eta_1\varepsilon^2. \tag{5.61}$$

Combining (5.29) and (5.59)–(5.61), we obtain (5.55). This together with (5.54) yields (5.53). The lemma is proved.

6. Transition from Stage 1 to Stage 2

In Sections 4 and 5, we have constructed two pairs of upper and lower solutions. More precisely:

- **Stage 1** (Generation of interface phase): (u^+, v^+) and (u^-, v^-),
- **Stage 2** (Motion of interface phase): (U^+, V^+) and (U^-, V^-).

In this section we will 'glue' these pairs together, to form a single pair of upper and lower solutions $(\hat{u}^+, \hat{v}^+)$ and $(\hat{u}^-, \hat{v}^-)$ on the entire period $0 \le t \le T$. The timing for transition from Stage 1 to Stage 2, denoted by $t^*(x)$, varies from place to place within Ω, due to the spatial-inhomogeneity of the equation as well as that of the initial data. To deal with this non-uniform transition, we define $(\hat{u}^\pm, \hat{v}^\pm)$ as follows:

Definition 6.1. For $x \in \Omega$ and $t \ge 0$ we set

$$\hat{u}^+(x,t) = \left\{1-\lambda\left(\frac{h(x)t}{\varepsilon^2}-\tau^*(x)\right)\right\}u^+(x,t)+\lambda\left(\frac{h(x)t}{\varepsilon^2}-\tau^*(x)\right)\widetilde{U}^+(x,t), \tag{6.1}$$

$$\hat{v}^+(x,t) = \left\{1-\lambda\left(\frac{h(x)t}{\varepsilon^2}-\tau^*(x)\right)\right\}v^+(x,t)+\lambda\left(\frac{h(x)t}{\varepsilon^2}-\tau^*(x)\right)\widetilde{V}^+(x,t), \tag{6.2}$$

where $\tau^*(x)$ is as defined in (4.10) and $\lambda(s) = g(\alpha s)$. Here α is a positive constant to be specified later, and g is defined as follows:

$$g(s) = 0 \text{ if } s<0, \quad g(s) = 1 \text{ if } s>1,$$
$$g(s) = \frac{\int_0^s r^m(1-r)^m\,dr}{\int_0^1 r^m(1-r)^m\,dr} \quad \text{if } 0\le s\le 1,$$

with m being an arbitrary constant which is larger than 1.

We recall

$$\begin{aligned}\mathscr{L}_1(\hat{u}^\pm,\hat{v}^\pm) &= \varepsilon\hat{u}^\pm_t - \varepsilon\nabla\cdot(k(x)\nabla\hat{u}^\pm) - \frac{h(x)}{\varepsilon}f(\hat{u}^\pm,\hat{v}^\pm),\\ \mathscr{L}_2(\hat{u}^\pm,\hat{v}^\pm) &= \varepsilon\hat{v}^\pm_t - \varepsilon D\nabla\cdot(k(x)\nabla\hat{v}^\pm) - \frac{h(x)}{\varepsilon}f(\hat{u}^\pm,\hat{v}^\pm).\end{aligned} \tag{6.3}$$

Now we specify the constant γ_0 in (5.8) as

$$\gamma_0 = 4\widetilde{C} \tag{6.4}$$

and choose other parameters to satisfy

$$\gamma_0 \gg \widetilde{C}, \quad \sigma_1 + \exp\left(\frac{-M_1\gamma_0}{2}\right) + \frac{1}{\ell} \ll \min\{\sigma, \hat{\sigma}\}\delta, \quad B > \frac{1}{\min_{x\in\Omega} h(x)\alpha}, \tag{6.5}$$

where $\widetilde{C}, B, \sigma_1$ are as in Lemma 4.5 and M_1 is as in (5.57). The constants σ and δ satisfy (5.32) and (5.49).

Lemma 6.2. *Under the conditions* (6.4), (6.5), *the following holds for sufficiently small* $\varepsilon > 0$:

$$\mathscr{L}_1(\hat{u}^+, \hat{v}^+) \geq 0, \quad \mathscr{L}_1(\hat{u}^-, \hat{v}^-) \leq 0, \quad \mathscr{L}_2(\hat{u}^+, \hat{v}^+) \leq 0, \quad \mathscr{L}_2(\hat{u}^-, \hat{v}^-) \geq 0 \tag{6.6}$$

for $x \in \overline{\Omega}$ *and* $t \in [0, T]$. *Consequently*, $(\hat{u}^+, \hat{v}^+)$ (*resp.* $(\hat{u}^-, \hat{v}^-)$) *is an upper solution* (*resp. lower solution*) *of* (1.2) *in* $\overline{\Omega} \times [0, T]$.

To prove the above lemma, we present some properties of λ.

Lemma 6.3. *λ is a C^2-class function satisfying*

i) $\lambda(s) = 0$ *for* $s < 0$, $\lambda(s) = 1$ *for* $s > \frac{1}{\alpha}$, $0 \leq \lambda(s) \leq 1$ *for* $0 \leq s \leq \frac{1}{\alpha}$,
ii) $\lambda'(0) = 0$, $\lambda'(s) \geq 0$ *for* $s \in \left(0, \frac{1}{\alpha}\right]$,
iii) $\lambda(s)(1-\lambda(s)) \leq \frac{1}{2\alpha}\lambda'(s)$ *for all* $s \in \left[0, \frac{1}{\alpha}\right]$,
iv) *There exists positive constant* $C > 0$ *such that* $\lambda''(s) \leq C\alpha^{m+1}s^{m-1}$.

The proof of this lemma will be given below.

Proof of Lemma 6.3. i) It follows from the definition of λ.

ii) Differentiating λ, we obtain

$$\lambda'(s) = \alpha g'(\alpha s) = \frac{\alpha^{m+1}s^m(1-\alpha s)^m}{\int_0^1 r^m(1-r)^m dr}. \tag{6.7}$$

This yields ii).

iii) Suppose that $0 \leq s \leq \frac{1}{2\alpha}$. It holds that

$$\lambda(s)(1-\lambda(s)) \leq \lambda(s) \leq s \sup_{r\in[0,s]} \lambda'(r) \leq \frac{1}{2\alpha}\lambda'(s).$$

Using ii), we see that λ' is monotone increasing in $[0, \frac{1}{2\alpha}]$. Therefore,

$$\lambda(s)(1-\lambda(s)) \leq \frac{1}{2\alpha}\lambda'(s) \quad \text{for } s \in \left[0, \frac{1}{2\alpha}\right].$$

Since $\lambda'(s)$ is symmetric with respect to $s = \frac{1}{2\alpha}$, the above estimate also holds for $s \in \left[\frac{1}{2\alpha}, \frac{1}{\alpha}\right]$. This proves iii).

iv) Differentiating (6.7), we have

$$\lambda''(s) = \alpha^2 g''(\alpha s) = \frac{m\alpha^{m+1}s^{m-1}(1-\alpha s)^{m-1}(1-2\alpha s)}{\int_0^1 r^m(1-r)^m dr}. \tag{6.8}$$

The desired result follows immediately from (6.8). Clearly from (6.7), (6.8) and the fact that λ is flat for $s < 0$, $s > \frac{1}{\alpha}$, we have $\lambda \in C^2(\mathbf{R})$. The lemma is proved.

Lemma 6.4. *Assume* (6.5). *There exists* $0 < \eta_0 < 1$ *such that for sufficiently small* $\varepsilon > 0$

$$\widetilde{U}^+(x,t) - u^+(x,t) \geq \eta_0\sigma\delta, \quad v^+(x,t) - \widetilde{V}^+(x,t) \geq \eta_0\hat{\sigma}\delta,$$
$$u^-(x,t) - \widetilde{U}^-(x,t) \geq \eta_0\sigma\delta, \quad \widetilde{V}^-(x,t) - v^-(x,t) \geq \eta_0\hat{\sigma}\delta,$$

for $t^*(x) \leq t \leq t^*(x) + \frac{\varepsilon^2}{h(x)\alpha}$, *where* $t^*(x)$ *is as defined in* (4.10).

Proof. Set

$$I(\alpha) := \left[t^*(x), t^*(x) + \frac{\varepsilon^2}{h(x)\alpha}\right] \tag{6.9}$$

and recall that d is smooth in $|d(x,t)| \leq d^*$. We will only show $\widetilde{U}^+(x,t) - u^+(x,t) \geq \eta_0\sigma\delta$, since the other inequalities can be proven in the same way. The proof consists of two steps.

Step 1. We first deal with (x,t) satisfying

$$d(x,t) < \frac{\gamma_0\varepsilon}{2} \tag{6.10}$$

where $\gamma_0 > 0$ is a positive constant in (5.9). Recall that $P(t) \leq -\gamma_0$ in $I(\alpha)$. The following holds for sufficiently small $\varepsilon > 0$:

$$\frac{\tilde{d}^+(x,t)}{\varepsilon} < \frac{1}{\varepsilon}\xi\big(d(x,t) + \varepsilon P(t)\big) < -\frac{\gamma_0}{2} \tag{6.11}$$

To estimate $\widetilde{U}^+(x,t)$ we will estimate U_0, U_1, Q respectively. Using (6.11) and (5.56), we get

$$U_0\left(\frac{\tilde{d}^+(x,t)}{\varepsilon}, x\right) \geq R_1 - C_1' \exp\left(-\frac{M|\tilde{d}^+(x,t)|}{2}\right) \geq R_1 - C_1' \exp\left(-\frac{M\gamma_0}{2}\right) \tag{6.12}$$

We next consider εU_1. Since U_1 is given by (5.4), there exists C_2' such that

$$\varepsilon U_1\left(\frac{\tilde{d}^+(x,t)}{\varepsilon}, x, t\right) \geq -\varepsilon C_2'. \tag{6.13}$$

We finally consider Q. There exists $C_3' > 0$ such that

$$Q(x, t) \geq C_3'\sigma\delta \quad \text{for } t \in I(\alpha) \tag{6.14}$$

by the definition of $Q(x, t)$. Combining (6.12)–(6.14), we obtain

$$\widetilde{U}^+(x, t) \geq R_1 - C_1' \exp\left(-\frac{M_1\gamma_0}{2}\right) - \varepsilon C_2' + C_3'\sigma\delta.$$

It follows from the assumption (6.5) that $\widetilde{U}^+(x, t) \geq R_1 + \frac{C_3'}{2}\sigma\delta$. On the other hand, $u^+(x, t) \leq R_1 + \sigma_1$ by Lemma 4.5

$$\widetilde{U}^+(x, t) - u^+(x, t) \geq \frac{C_3'}{2}\sigma\delta - \sigma_1.$$

Step 2. We next consider (x, t) satisfying

$$\frac{d(x, t)}{\varepsilon} \geq \frac{\gamma_0\varepsilon}{2}.$$

First we note that the following inequality holds near the interface:

$$|d(x, t) - d(x, 0)| \leq \left(\sup_{t\in I_0(\alpha)} |d_t|\right)t \leq C_4'\varepsilon^2|\log\varepsilon| \quad \text{for } x \in \mathcal{N}_{d^*}(\Gamma(t)), \ \ t \in I_0(\alpha), \tag{6.15}$$

where

$$I_0(\alpha) := \left[0, t^*(x) + \frac{\varepsilon^2}{h(x)\alpha}\right]$$

and C_4' is a positive constant independent of $\varepsilon > 0$. Consequently,

$$d(x, 0) \geq \frac{\gamma_0\varepsilon}{2} - C_4'\varepsilon^2|\log\varepsilon| \geq \frac{\gamma_0\varepsilon}{4} \tag{6.16}$$

for sufficiently small ε. Now we choose B satisfying $B > \left(\min_{x\in\Omega} h(x)\alpha\right)^{-1}$ as (6.5) so that Lemma 4.5 applies. Then we get

$$(u^+(x, t)), v^+(x, t)) \in \mathcal{B}_{\sigma_1}(R_1, 0) \cup \mathcal{B}_{\sigma_1}(0, R_2).$$

In view of this and the fact that $d(x, 0) \geq 0$, we obtain $(u^+(x, t), v^+(x, t)) \in \mathcal{B}_{\sigma_1}(R_1, 0)$. Especially it holds that

$$u^+ \leq \sigma_1 \quad \text{in } I(\alpha). \tag{6.17}$$

On the other hand, there exists $C_5' > 0$ such that

$$\widetilde{U}^+(x, t) \geq \varepsilon U_1\left(\frac{\tilde{d}^+(x, t)}{\varepsilon}, x, t\right) + Q(x, t) \geq C_5'(\sigma\delta - \varepsilon) \geq \frac{C_5'}{2}\sigma\delta \tag{6.18}$$

for $\varepsilon > 0$ sufficiently small. The assumption (6.5), (6.17), and (6.18) show that

$$\widetilde{U}^+(x,t) - u^+(x,t) \geq \frac{C_5'}{2}\sigma\delta - \sigma_1 \geq \frac{C_5'}{3}\sigma\delta.$$

Choosing $\eta_0 > 0$ in Step 1 such that $\eta_0 < \max\left\{\frac{C_3'}{2}, \frac{C_5'}{3}\right\}$ if necessary and combining Steps 1–3, we complete the proof of Lemma 6.4.

Proof of Lemma 6.2. We will only present the proof for $\mathscr{L}_1(\hat{u}^+, \hat{v}^+) \geq 0$. In the case where $t \geq t^*(x) + \frac{\varepsilon^2}{h(x)\alpha}$ or $t \leq t^*(x)$, the assertion is already shown in Sections 4 and 5, respectively. Therefore, we only consider the case where $t \in I(\alpha)$. Substituting (6.1) and (6.2) into (6.3), we obtain

$$\mathscr{L}_1(\hat{u}^+, \hat{v}^+) = \varepsilon(J_1 + J_2 + J_3 + J_4 + J_5), \tag{6.19}$$

where

$$\begin{aligned}
J_1 &= (1-\lambda)\left(u_t^+ - \nabla\cdot(k(x)\nabla u^+) - \frac{h(x)}{\varepsilon^2}f(u^+, v^+)\right),\\
J_2 &= \lambda\left((\widetilde{U}_t^+ - \nabla(k(x)\nabla\widetilde{U}^+) - \frac{h(x)}{\varepsilon^2}f(\widetilde{U}^+, \widetilde{V}^+)\right),\\
J_3 &= -\frac{h(x)}{\varepsilon^2}\{f\big((1-\lambda)u^+ + \lambda\widetilde{U}^+, (1-\lambda)v^+ + \lambda\widetilde{V}^+\big)\\
&\quad - (1-\lambda)f(u^+, v^+) - \lambda f(\widetilde{U}^+, \widetilde{V}^+)\} + \frac{h(x)}{\varepsilon^2}\lambda'(\widetilde{U}^+ - u^+),\\
J_4 &= \lambda'(\widetilde{U}^+ - u^+)\left(-\frac{t\nabla k\cdot\nabla h}{\varepsilon^2} + \frac{\nabla k\cdot\nabla\delta^*}{\alpha^*} - \frac{tk\Delta h}{\varepsilon^2} + \frac{k\Delta\delta^*}{\alpha^*\delta^*} - \frac{k|\nabla\delta^*|^2}{\alpha^*{\delta^*}^2}\right)\\
&\quad - 2k\lambda'(\nabla\widetilde{U}^+ - \nabla u^+)\cdot\left(\frac{t\nabla h}{\varepsilon^2} - \frac{\nabla\delta^*}{\alpha^*\delta^*}\right),\\
J_5 &= -k\lambda''\left(\frac{t\nabla h}{\varepsilon^2} - \frac{\nabla\delta^*}{\alpha^*\delta^*}\right)^2(\widetilde{U}^+ - u^+),
\end{aligned}$$

and

$$\lambda = \lambda\left(\frac{h(x)t}{\varepsilon^2} - \tau^*(x)\right).$$

To obtain (6.19) we recall the definitions of $\tau^*(x)$ and the following:

$$\nabla\tau^*(x) = \frac{\nabla\delta^*}{\alpha^*\delta^*}, \quad \Delta\tau^*(x) = \frac{1}{\alpha^*}\left(\frac{\Delta\delta^*}{\delta^*} - \frac{|\nabla\delta^*|^2}{{\delta^*}^2}\right),$$

where $\delta^* = \delta^*(u_0(x), v_0(x))$.

We will estimate separately each term: Set $s = \frac{h(x)t}{\varepsilon^2} - \tau^*(x)$. It is already shown in Section 5 that

$$J_1 \geq (1-\lambda(s))(c_1'p_\xi - c_2'p_\eta)\exp\left(\frac{\alpha^* h(x)t}{\varepsilon^2}\right),$$

for some large constants c_1', c_2' independent of $\varepsilon > 0$. If we use the estimates in Lemma 3.9, we have

$$J_1 \geq c_3'(1-\lambda(s))\exp\left(\frac{2\alpha^* h(x)t}{\varepsilon^2}\right) \geq \frac{c_3'(1-\lambda(s))}{\delta^{*2}} \tag{6.20}$$

for $t \in I(\alpha)$ and some constant c_3' independent of $\varepsilon > 0$. The inequality (5.53) yields

$$J_2 \geq \frac{c_4'}{\varepsilon^2}\lambda(s) \quad \text{for } t \in I(\alpha). \tag{6.21}$$

Now we will estimate J_3. Set

$$M = \|\widetilde{U}^+\|_{L^\infty} + \|u^+\|_{L^\infty} + b(\|\widetilde{V}^+\|_{L^\infty} + \|v^+\|_{L^\infty}), \tag{6.22}$$

where $\|v\|_{L^\infty} := \max_{x\in\overline{\Omega}} |v(x)|$. Straightforward calculation shows that

$$\begin{aligned}\frac{\varepsilon^2 J_3}{h(x)} &= \lambda'(s)(\widetilde{U}^+ - u^+) + (1-\lambda(s))u^+(R_1 - u^+ - bv^+) \\ &\quad + \lambda(s)[\widetilde{U}^+(R_1 - \widetilde{U}^+ - b\widetilde{V}^+) - u^+(R_1 - u^+ - bv^+)] \\ &\quad - (u^+ + \lambda(s)(\widetilde{U}^+ - u^+))\{R_1 - u^+ - \lambda(s)(\widetilde{U}^+ - u^+) - b(v^+ + \lambda(s)(\widetilde{V}^+ - v^+))\} \\ &= \lambda'(s)(\widetilde{U}^+ - u^+) - \lambda(s)(1-\lambda(s))\big(\widetilde{U}^+ - u^+ + b(\widetilde{V}^+ - v^+)\big)(\widetilde{U}^+ - u^+).\end{aligned}$$

Using Lemma 6.4, we have $\widetilde{U}^+ \geq u^+$. Therefore,

$$J_3 \geq \frac{h(x)}{\varepsilon^2}\big(\widetilde{U}^+ - u^+\big)\big(\lambda'(s) - \lambda(s)(1-\lambda(s))M\big),$$

Using Lemma 6.3iii), and Lemma 6.4

$$J_3 \geq \frac{h(x)}{\varepsilon^2}\lambda'(s)\left(1 - \frac{M}{2\alpha}\right)(\widetilde{U}^+ - u^+) \geq \frac{h(x)}{\varepsilon^2}\lambda'(s)\left(1 - \frac{M}{2\alpha}\right)\eta_0\sigma\delta \geq 0 \tag{6.23}$$

holds if

$$M < 2\alpha. \tag{6.24}$$

We next consider J_4. By the definition (4.9) of δ^*, we have $\delta^* \geq \varepsilon\ell$. In addition, by (5.34) and (5.40), we obtain that $|\Delta\delta^*|$ and $|\nabla\delta^*|$ are bounded from above by some constant independent of $\varepsilon > 0$. Therefore there exist $\widetilde{C}_1, \widetilde{C}_2 > 0$ such that for sufficiently small $\varepsilon > 0$, we have

$$|J_4| \leq \widetilde{C}_1\lambda'(s)\left((\widetilde{U}^+ - u^+)\left(\frac{t}{\varepsilon^2} + \frac{1}{\delta^{*2}}\right) + \widetilde{C}_2(\nabla\widetilde{U}^+ - \nabla u^+)\left(\frac{t}{\varepsilon^2} + \frac{1}{\delta^*}\right)\right).$$

By the definition of $\widetilde{U}^+$ and u^+, there exists $M' > 0$ such that

$$|\nabla\widetilde{U}^+| + |\nabla u^+| \leq \frac{M'}{\varepsilon}.$$

By the above two inequalities, there exist $C_3' > 0$ independent of $\varepsilon > 0$ such that

$$|J_4| \leq \widetilde{C}_3 \lambda'(s)\left(\frac{M}{\delta^{*2}} + \frac{M'}{\varepsilon\delta^*}\right). \tag{6.25}$$

For the term J_5, there exist $\widetilde{C}_4$ independent of $\varepsilon > 0$ such that

$$|J_5| \leq \widetilde{C}_4 |\lambda''(s)| \left(\tau^*(x) + \frac{1}{\delta^*}\right)^2 \leq \widetilde{C}_3 |\lambda''(s)| \left(\log \frac{1}{\delta^*} + \frac{1}{\delta^*}\right)^2, \tag{6.26}$$

where the definition of $\tau^*(x)$ has been used. We now consider $J_3 + J_4$. By (6.23), (6.25) and the fact that $\delta^* \geq \varepsilon\ell$, (see (4.9) we have

$$J_3 + J_4 \geq \lambda'(s)\left(\frac{h(x)}{\varepsilon^2}\left(1 - \frac{M}{2\alpha}\right)\eta_0\sigma\delta - \widetilde{C}_3\left(\frac{M}{\varepsilon^2\ell^2} + \frac{M'}{\varepsilon\ell}\right)\right). \tag{6.27}$$

By the assumption that $\sigma\delta \gg \frac{1}{\ell}$, the right-hand side of (6.27) is positive. Next we consider (6.20), (6.21), and (6.26) and obtain

$$J_1 + J_2 + J_5 \geq \frac{C_3'(1 - \lambda(s))}{\delta^{*2}} + \frac{C_4'}{\varepsilon^2}\lambda(s) - \widetilde{C}_4 |\lambda''(s)| \left(\log \frac{1}{\delta^*} + \frac{1}{\delta^*}\right)^2. \tag{6.28}$$

Recall that C_3' is chosen arbitrarily large (by choosing c_1, c_2 large enough). If $\lambda \leq \frac{1}{2}$ the first term of (6.28) is dominant, while for $\lambda > \frac{1}{2}$, the second term is dominant. Now, combining (6.27) and (6.28), we obtain

$$J_1 + J_2 + J_3 + J_4 + J_5 \geq 0.$$

This proves the inequalities (6.6). Since $(\hat{u}^+, \hat{v}^+)$ and $(\hat{u}^-, \hat{v}^-)$ satisfy the Neumann boundary conditions on $\partial\Omega$, they are, respectively, an upper and a lower solution of (1.2). The lemma is proved.

7. Proof of the Main Theorems

Let $(u^\varepsilon, v^\varepsilon)$ be a solution of (1.2). It follows from Lemma 6.2 and Proposition 4.1 that

$$\begin{pmatrix} \hat{u}^- \\ \hat{v}^- \end{pmatrix} \preceq \begin{pmatrix} u^\varepsilon \\ v^\varepsilon \end{pmatrix} \preceq \begin{pmatrix} \hat{u}^+ \\ \hat{v}^+ \end{pmatrix}, \quad 0 \leq t \leq T. \tag{7.1}$$

Set $\mathcal{N}_{2\gamma_0\varepsilon}(\Gamma(t)) = \{x \in \Omega \mid dist(x, \Gamma(t)) \leq 2\gamma_0\varepsilon\}$, where γ_0 is the same constant as in (5.9), and is defined by (6.16). By the definition of $\hat{u}^\pm$, $\hat{v}^\pm$, we see that

$$\begin{pmatrix} \hat{u}^+ \\ \hat{v}^+ \end{pmatrix} \preceq \begin{pmatrix} R_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2\sigma\delta \\ -2\sigma\delta \end{pmatrix}, \quad \begin{pmatrix} \hat{u}^- \\ \hat{v}^- \end{pmatrix} \succeq \begin{pmatrix} R_1 \\ 0 \end{pmatrix} - \begin{pmatrix} 2\sigma\delta \\ -2\sigma\delta \end{pmatrix}$$

for $x \in \Omega_{in}(t)\backslash\mathcal{N}_{2\gamma_0\varepsilon}(\Gamma(t))$ and for $t \geq t^*(x) + \frac{\varepsilon^2}{\alpha h(x)}$. Consequently

$$\begin{pmatrix} R_1 \\ 0 \end{pmatrix} - \begin{pmatrix} 2\sigma\delta \\ -2\sigma\delta \end{pmatrix} \preceq \begin{pmatrix} u^\varepsilon \\ v^\varepsilon \end{pmatrix} \preceq \begin{pmatrix} R_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2\sigma\delta \\ -2\sigma\delta \end{pmatrix} \quad \text{for } x \in \Omega_{in}(t)\backslash\mathcal{N}_{2\gamma_0\varepsilon}(\Gamma(t)). \tag{7.2}$$

Here Ω^ε_{in} and Ω^ε_{out} below are as in Section 1. In the same way we obtain

$$\begin{pmatrix} 0 \\ R_2 \end{pmatrix} - \begin{pmatrix} 2\sigma\delta \\ -2\sigma\delta \end{pmatrix} \preceq \begin{pmatrix} u^\epsilon \\ v^\varepsilon \end{pmatrix} \preceq \begin{pmatrix} 0 \\ R_2 \end{pmatrix} + \begin{pmatrix} 2\sigma\delta \\ -2\sigma\delta \end{pmatrix} \quad \text{for } x \in \Omega_{out}(t)\backslash\mathcal{N}_{2\gamma_0\varepsilon}(\Gamma(t)). \tag{7.3}$$

Setting

$$t^\varepsilon = \sup_{x\in\overline{\Omega}} \left(t^*(x) + \frac{\varepsilon^2}{\alpha h(x)} \right),$$

we obtain the conclusion of Theorem 1.

We next prove Theorem 2. By (7.2) and (7.3), we have

$$\Gamma^\varepsilon(t) := \left\{ x \in \Omega \;\middle|\; \begin{pmatrix} u^\varepsilon(x,t) \\ v^\varepsilon(x,t) \end{pmatrix} \in S \right\} \subset \mathcal{N}_{2\gamma_0\varepsilon}(\Gamma(t)) \tag{7.4}$$

for $t \in [t^\varepsilon, T]$. Let us show that (7.4) holds also for $t \in [0, t^\varepsilon]$.

In fact, by the definition of $t^*(x)$ in (4.10), we have

$$\varepsilon^2(\exp(\alpha^*\tau) - 1) = O(\varepsilon) \quad \text{for } 0 \leq t \leq t^*(x),$$

where α^* is the constant appearing in (4.7), (4.8), and $\tau = \frac{h(x)}{\varepsilon^2}t$. In view of this and the nondegeneracy condition (Assumption 5) on (u_0, v_0), we see that

$$\begin{pmatrix} u_0(x) \pm \varepsilon^2 C_1(\exp(\alpha^*\tau) - 1) \\ v_0(x) \pm \varepsilon^2 C_2(\exp(\alpha^*\tau) - 1) \end{pmatrix} \in \begin{cases} \Delta_1, & \text{if } x \in \Omega_{in}(0)\backslash\mathcal{N}_{\gamma_1\varepsilon}(\Gamma_0) \\ \Delta_2, & \text{if } x \in \Omega_{out}(0)\backslash\mathcal{N}_{\gamma_1\varepsilon}(\Gamma_0), \end{cases}$$

for $0 \leq t \leq t^*(x)$, where C_1, C_2 are the constants in (4.8) and (4.9), and γ_1 is a constant satisfying $\gamma_1 \leq \gamma_0$ and independent of $\varepsilon > 0$. Since both Δ_1 and Δ_2 are positively invariant regions for the system (1.4), we have

$$\begin{pmatrix} u^\pm(x,t) \\ v^\pm(x,t) \end{pmatrix} \in \begin{cases} \Delta_1, & \text{if } x \in \Omega_{in}(0)\backslash\mathcal{N}_{\gamma_1\varepsilon}(\Gamma_0) \\ \Delta_2, & \text{if } x \in \Omega_{out}(0)\backslash\mathcal{N}_{\gamma_1\varepsilon}(\Gamma_0), \end{cases} \tag{7.5}$$

for $0 \leq t \leq t^*(x)$. Now since $\Gamma(t)$ depends on t smoothly, we have

$$d_H(\Gamma_0, \Gamma(t)) = O(t^\varepsilon) = O(\varepsilon^2|\log\varepsilon|). \tag{7.6}$$

Consequently,

$$\begin{pmatrix} u^\pm(x,t) \\ v^\pm(x,t) \end{pmatrix} \in \begin{cases} \Delta_1, & \text{if } x \in \Omega_{in}(t)\backslash\mathcal{N}_{2\gamma_1\varepsilon}(\Gamma(t)) \\ \Delta_2, & \text{if } x \in \Omega_{out}(t)\backslash\mathcal{N}_{2\gamma_1\varepsilon}(\Gamma(t)), \end{cases} \tag{7.7}$$

for $0 \le t \le t^*(x)$ provided that $\varepsilon > 0$ is chosen small enough so that $\gamma_1\varepsilon \gg \varepsilon^2|\log\varepsilon|$. During the transition period $t^*(x) \le t \le t^\varepsilon$, the upper and lower solutions $(\hat{u}^\pm, \hat{v}^\pm)$ take intermediate values between $(u^\pm, v^\pm)$ and $(U^\pm, V^\pm)$ with respect to the order relation $\succeq$. As is easily seen, (7.7) is valid for $t^*(x) \le t \le t^\varepsilon$. On the other hand, by the definition of $(U^\pm, V^\pm)$ it is clear that

$$\begin{pmatrix} \widetilde{U}^\pm(x,t) \\ \widetilde{V}^\pm(x,t) \end{pmatrix} \in \begin{cases} \Delta_1, & \text{if } x \in \Omega_{in}(t)\backslash \mathcal{N}_{2\gamma_1\varepsilon}(\Gamma(t)) \\ \Delta_2, & \text{if } x \in \Omega_{out}(t)\backslash \mathcal{N}_{2\gamma_1\varepsilon}(\Gamma(t)). \end{cases} \tag{7.8}$$

During this transition period $t^*(x) \le t \le t^\varepsilon$, the upper and lower solutions $(\hat{u}^\pm, \hat{v}^\pm)$ are given by

$$\hat{u}^\pm = (1-\lambda)u^\pm + \lambda\widetilde{U}^\pm$$
$$\hat{v}^\pm = (1-\lambda)v^\pm + \lambda\widetilde{V}^\pm.$$

Since by Lemma 6.4,

$$\begin{pmatrix} u^+ \\ v^+ \end{pmatrix} \preceq \begin{pmatrix} \widehat{U}^+ \\ \widehat{V}^+ \end{pmatrix}, \quad \begin{pmatrix} u^- \\ v^- \end{pmatrix} \succeq \begin{pmatrix} \widehat{U}^- \\ \widehat{V}^- \end{pmatrix},$$

we have

$$\begin{pmatrix} u^+ \\ v^+ \end{pmatrix} \preceq \begin{pmatrix} \hat{u}^+ \\ \hat{v}^+ \end{pmatrix} \preceq \begin{pmatrix} \widetilde{U}^+ \\ \widetilde{V}^+ \end{pmatrix}, \quad \begin{pmatrix} u^- \\ v^- \end{pmatrix} \succeq \begin{pmatrix} \hat{u}^- \\ \hat{v}^- \end{pmatrix} \succeq \begin{pmatrix} \widetilde{U}^- \\ \widetilde{V}^- \end{pmatrix}.$$

This and (7.8) along with Lemma 3.3 imply that

$$\begin{pmatrix} \hat{u}^\pm(x,t) \\ \hat{v}^\pm(x,t) \end{pmatrix} \in \begin{cases} \Delta_1, & \text{if } x \in \Omega_{in}(t)\backslash \mathcal{N}_{2\gamma_1\varepsilon}(\Gamma(t)) \\ \Delta_2, & \text{if } x \in \Omega_{out}(t)\backslash \mathcal{N}_{2\gamma_1\varepsilon}(\Gamma(t)). \end{cases}$$

Combining the above equation, (7.1) and using Lemma 3.3 again, we obtain

$$\begin{pmatrix} u^\varepsilon(x,t) \\ v^\varepsilon(x,t) \end{pmatrix} \in \begin{cases} \Delta_1, & \text{if } x \in \Omega_{in}(t)\backslash \mathcal{N}_{2\gamma_1\varepsilon}(\Gamma(t)) \\ \Delta_2, & \text{if } x \in \Omega_{out}(0)\backslash \mathcal{N}_{2\gamma_1\varepsilon}(\Gamma(t)), \end{cases}$$

for $0 \le t \le t^\varepsilon$. Hence

$$\Gamma^\varepsilon(t) \subset \mathcal{N}_{2\gamma_0\varepsilon}(\Gamma_0) \quad \text{for } 0 \le t \le t^\varepsilon. \tag{7.9}$$

Let x_0 be any point on $\Gamma(t)$. Then there exist points $x_1, x_2 \in \Omega$ such that

$$|x_0 - x_1| = |x_0 - x_2| = 2\gamma_0\varepsilon,$$
$$x_1 \in \partial\mathcal{N}_{2\gamma_0\varepsilon}(\Gamma(t)) \cap \Omega_{int}(t),$$
$$x_2 \in \partial\mathcal{N}_{2\gamma_0\varepsilon}(\Gamma(t)) \cap \Omega_{out}(t).$$

Then (7.2) and (7.3) imply that

$$\begin{pmatrix} u^\varepsilon(x_1,t) \\ v^\varepsilon(x_1,t) \end{pmatrix} \in \Delta_1, \quad \begin{pmatrix} u^\varepsilon(x_2,t) \\ v^\varepsilon(x_2,t) \end{pmatrix} \in \Delta_2.$$

Denote by $\overline{x_1x_0x_2}$ the piecewise linear curve consisting of the two line segments $\overline{x_1x_0}$ and $\overline{x_0x_2}$. As the point x moves from x_1 to x_2 along $\overline{x_1x_0x_2}$, the point $\binom{u^\varepsilon(x,t)}{v^\varepsilon(x,t)}$ moves from the region Δ_1 into Δ_2. Therefore there exists a point $x_3 \in \overline{x_1x_0x_2}$ with $x_3 \neq x_1, x_2$, such that $\binom{u^\varepsilon(x_3,t)}{v^\varepsilon(x_3,t)} \in S$. This means that $dist(x_0, \Gamma^\varepsilon(t)) < 2\gamma_0\varepsilon$, hence

$$\Gamma(t) \subset \mathcal{N}_{2\gamma_0\varepsilon}(\Gamma^\varepsilon(t)). \tag{7.10}$$

Combining (7.4) and (7.10), we obtain

$$d_H(\Gamma^\varepsilon(t), \Gamma(t)) \le 2\gamma_0\varepsilon.$$

This proves Theorem 1.

Next we prove Theorem 3. Choose $t > 0$ and $x_0 \in \Omega\backslash\Gamma(t)$ arbitrarily. Choose $\varepsilon_0 > 0$ small enough so that

$$t > t_\varepsilon(x_0) = O\left(\varepsilon^2 \log\frac{1}{\varepsilon}\right) \quad \text{for } 0 < \varepsilon < \varepsilon_0.$$

Then we have $\hat{u}^\pm(x_0,t) = \widetilde{U}^\pm(x_0,t)$, hence

$$\widetilde{U}^+(x_0,t) \ge u^\varepsilon(x_0,t) \ge \widetilde{U}^-(x_0,t), \tag{7.11}$$

where $\widetilde{U}^\pm$ are as defined in (5.51) and (5.52). It is easily seen that

$$\widetilde{U}^\pm(x_0,t) \to \begin{cases} (R_1, 0) & \text{if } x_0 \in \Omega_{in}(t) \\ (0, R_2) & \text{if } x_0 \in \Omega_{out}(t). \end{cases}$$

Therefore by the comparison principle and (7.11), we have

$$u^\varepsilon(x_0,t) \to \begin{cases} (R_1, 0) & \text{if } x_0 \in \Omega_{in}(t) \\ (0, R_2) & \text{if } x_0 \in \Omega_{out}(t). \end{cases}$$

The theorem is proved.

8. Appendix A. Proof of Lemma 3.9

In this appendix, we give the proof of Lemma 3.9. Set

$$\mathcal{A}_n = \{x \in \mathcal{B}(u^*, v^*) \mid \sigma_1^{2(n+1)} \le \text{dist}(x, S) \le \sigma_1^{2n}\}$$

and note that

$$\bigcup_{n=1}^{\infty} \mathcal{A}_n = \mathcal{B}(u^*, v^*).$$

The proof consists of three steps. In Steps 1 and 2, we prove the inequalities assuming some conditions on the initial data and the location of (p, q). Finally in Step 3 by combining Steps 1 and 2 we obtain the desired inequalities for arbitrarily chosen initial data.

Step 1. Let $(\xi, \eta) \in \mathscr{A}_n$ for $n = 1, 2, \ldots$. We will show that the solution (p, q) satisfies i) as long as $(p, q) \in \mathscr{A}_n$. By differentiating (1.4) with respect to ξ, we obtain

$$\frac{d}{d\tau}\begin{pmatrix} p_\xi \\ q_\xi \end{pmatrix} = \begin{pmatrix} f_p(p, q) & f_q(p, q) \\ g_p(p, q) & g_q(p, q) \end{pmatrix}\begin{pmatrix} p_\xi \\ q_\xi \end{pmatrix}, \quad \begin{pmatrix} p_\xi(0) \\ q_\xi(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{A.1}$$

Now, we consider (w_1, w_2) to be an upper solution of (A.1) satisfying

$$\frac{d}{d\tau}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \overline{T}_1\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \quad \begin{pmatrix} w_1(0) \\ w_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{A.2}$$

where

$$\overline{T}_1 = \begin{pmatrix} \max\limits_{(p,q)\in\mathscr{A}_n} f_p(p, q) & \min\limits_{(p,q)\in\mathscr{A}_n} f_q(p, q) \\ \min\limits_{(p,q)\in\mathscr{A}_n} g_p(p, q) & \max\limits_{(p,q)\in\mathscr{A}_n} g_q(p, q) \end{pmatrix}$$

and (w_3, w_4) to be a lower solution of (A.1) satisfying

$$\frac{d}{d\tau}\begin{pmatrix} w_3 \\ w_4 \end{pmatrix} = \underline{T_1}\begin{pmatrix} w_3 \\ w_4 \end{pmatrix}, \quad \begin{pmatrix} w_3(0) \\ w_4(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{A.3}$$

with

$$\underline{T_1} = \begin{pmatrix} \min\limits_{(p,q)\in\mathscr{A}_n} f_p(p, q) & \max\limits_{(p,q)\in\mathscr{A}_n} f_q(p, q) \\ \max\limits_{(p,q)\in\mathscr{A}_n} g_p(p, q) & \min\limits_{(p,q)\in\mathscr{A}_n} g_q(p, q) \end{pmatrix}.$$

Therefore it holds that

$$\begin{aligned} w_3(\tau) &\le p_\xi(\tau; \xi, \eta) \le w_1(\tau), \\ w_4(\tau) &\ge q_\xi(\tau; \xi, \eta) \ge w_2(\tau). \end{aligned} \tag{A.4}$$

It is clear that there exist $c_i > 0$, $i = 1, 2, 3, 4$ such that

$$\overline{T}_1 = \begin{pmatrix} -u^* + c_1{\sigma_1}^{2(n-1)} & -bu^* - c_2{\sigma_1}^{2(n-1)} \\ -av^* - c_3\sigma_1^{2(n-1)} & -v^* + c_4\sigma_1^{2(n-1)} \end{pmatrix}.$$

For sufficiently small $\sigma_1 > 0$, the matrix $\overline{T}_1$ has a positive and a negative eigenvalue denoted by $\bar{a}$ and $\bar{\beta}$ respectively. By straightforward computation there exists $c_5 > c_6 > 0$ such that

$$\alpha^* + c_6{\sigma_1}^{2(n-1)} < \bar{a} < \alpha^* + c_5{\sigma_1}^{2(n-1)}. \tag{A.5}$$

Now by solving (A.2) explicitly, we obtain

$$w_1(\tau) = \frac{\bar{a} + v^* + c_1{\sigma_1}^{2(n-1)}}{\bar{a} - \bar{\beta}} \exp(\bar{a}\tau) + \frac{\bar{a} + u^* - c_1{\sigma_1}^{2(n-1)}}{\bar{a} - \bar{\beta}} \exp(\bar{\beta}\tau),$$

$$w_2(\tau) = \frac{-(\bar{a} + u^* + c_1{\sigma_1}^{2(n-1)})(\bar{a} + v^* - c_1{\sigma_1}^{2(n-1)})}{(\bar{a} - \bar{\beta})(bu^* - c_2{\sigma_1}^{2(n-1)})} (\exp(\bar{a}\tau) - \exp(\bar{\beta}\tau)).$$

Since v is sufficiently small, there exist $B_1 > 0$ and $B_2 > 0$ such that

$$w_1(\tau) \leq B_1 \exp(\bar{a}\tau), \quad w_2(\tau) \geq -B_2 \exp(\bar{a}\tau). \tag{A.6}$$

From inequality (A.5) we get

$$w_1(\tau) \leq B_1 \exp(\bar{a}\tau) \leq B_1 \exp((\alpha^* + c_5{\sigma_1}^{2(n-1)})\tau). \tag{A.7}$$

In the same way we obtain

$$w_2(\tau) \geq -B_2 \exp((\alpha^* + c_5{\sigma_1}^{2(n-1)})\tau). \tag{A.8}$$

On the other hand, by solving (A.3) explicitly, we can easily obtain

$$w_3(\tau) \geq \tilde{c}_3 \exp(\alpha^*\tau), \quad w_4(\tau) \leq 0. \tag{A.9}$$

Equations (A.7)–(A.9) yield

$$\tilde{c}_3 \exp(\alpha^*\tau) \leq p_\xi(\tau; \xi, \eta) \leq B_1 \exp((\alpha^* + c_5{\sigma_1}^{2(n-1)})\tau), \tag{A.10}$$

$$-B_1 \exp((\alpha^* + c_5{\sigma_1}^{2(n-1)})\tau) \leq q_\xi(\tau; \xi, \eta) \leq 0. \tag{A.11}$$

Here we will show that the time when $(p(\tau; \xi, \eta), q(\tau; \xi, \eta))$ goes out of $\mathscr{A}_n$, say $\hat{\tau}_1$, satisfies

$$\hat{\tau}_1 < \frac{1}{\alpha^*}\left(\log \frac{1}{\sigma_1}\right) + C_5', \tag{A.12}$$

where $C_5' > 0$ is independent of n, σ_1. To prove this, we only need to apply Lemma 3.6 with $dist((\xi, \eta), S) \geq \sigma_1^{n+1}$ and $dist((p, q), S) \leq \sigma_1^n$.

By (A.12) and the fact that $\sigma_1 > 0$ is small there exists $\tilde{c}_1 > 0$ such that

$$\exp((\alpha^* + c_5{\sigma_1}^{2(n-1)})\tau) \leq \exp(\alpha^*\tau)\exp(c_5{\sigma_1}^{2(n-1)}\hat{\tau}_1) \leq \tilde{c}_0 \exp(\alpha^*\tau). \tag{A.13}$$

Combining (A.10), (A.11), and (A.13), we have

$$\tilde{c}_3 \exp(\alpha^*\tau) \leq p_\xi(\tau; \xi, \eta) \leq \tilde{c}_1 \exp(\alpha^*\tau), \tag{A.14}$$

$$-\tilde{c}_2 \exp(\alpha^*\tau) \leq q_\xi(\tau; \xi, \eta) \leq 0. \tag{A.15}$$

By following the same argument as above we can also obtain

$$0 \le p_\eta(\tau; \xi, \eta) \le \tilde{c}_4 \exp(\alpha^* \tau), \tag{A.16}$$

$$-\tilde{c}_5 \exp(\alpha^* \tau) \le q_\eta(\tau; \xi, \eta) \le -\tilde{c}_6 \exp(\alpha^* \tau), \tag{A.17}$$

which implies ii).

We now prove estimate iii) for $(p, q) \in \mathscr{A}_n$ with initial data $(\xi, \eta) \in \mathscr{A}_n$. Differentiating (1.8) twice with respect to ξ, we obtain

$$\frac{d}{d\tau}\begin{pmatrix} p_{\xi\xi} \\ q_{\xi\xi} \end{pmatrix} = \begin{pmatrix} f_p(p,q) & f_q(p,q) \\ g_p(p,q) & g_q(p,q) \end{pmatrix}\begin{pmatrix} p_{\xi\xi} \\ q_{\xi\xi} \end{pmatrix} + H(p_\xi, q_\xi), \quad \begin{pmatrix} p_{\xi\xi}(0) \\ q_{\xi\xi}(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{A.18}$$

where

$$H(p_\xi, q_\xi) = \begin{pmatrix} (p_\xi \quad q_\xi)\begin{pmatrix} f_{pp}(p,q) & f_{pq}(p,q) \\ f_{pq}(p,q) & f_{qq}(p,q) \end{pmatrix}\begin{pmatrix} p_\xi \\ q_\xi \end{pmatrix} \\ (p_\xi \ q_\xi)\begin{pmatrix} g_{pp}(p,q) & g_{pq}(p,q) \\ g_{pq}(p,q) & g_{qq}(p,q) \end{pmatrix}\begin{pmatrix} p_\xi \\ q_\xi \end{pmatrix} \end{pmatrix}.$$

The solution of the system

$$\frac{d}{d\tau}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \overline{T}_1\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + H(p_\xi, q_\xi), \quad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{A.19}$$

is an upper solution of (A.18). A straightforward calculation yields

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \int_0^\tau \exp\big((\tau - s)\overline{T}_1\big)H(p_\xi, q_\xi)ds \le \int_0^\tau |\exp\big((\tau - s)\overline{T}_1\big)||H(p_\xi, q_\xi)|ds. \tag{A.20}$$

Utilizing (A.14), (A.15) and the fact that the second derivatives of f and g are bounded, we have

$$|H(p_\xi, q_\xi)| \le \widetilde{C}_1 \exp(2\alpha^* \tau), \tag{A.21}$$

for some positive constant $\widetilde{C}_1$ independent of σ_1, n. On the other hand it holds that

$$|\exp\big((t - s)\widetilde{T}_1\big)| \le \widetilde{C}_2 \exp(\bar{a}\tau) \tag{A.22}$$

for some constant $\widetilde{C}_2 > 0$ independent of σ_1, n. Inequalities (A.20)–(A.22) imply

$$\begin{aligned} \left|\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right| &\le \int_0^\tau \widetilde{C}_1 \widetilde{C}_2 \exp(\bar{a}(\tau - s)) \exp(2\alpha^* s) ds \\ &= \widetilde{C}_3\{\exp(2\alpha^* \tau) - \exp(\bar{a}(\tau))\} \le \widehat{C}_3 \exp(2\alpha^* \tau). \end{aligned} \tag{A.23}$$

Similarly, a solution of the system

$$\frac{d}{d\tau}\begin{pmatrix} y_3 \\ y_4 \end{pmatrix} = \underline{T_1}\begin{pmatrix} y_3 \\ y_4 \end{pmatrix} + H(p_\xi, q_\xi), \quad \begin{pmatrix} y_3(0) \\ y_4(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is a lower solution of (A.18). By following the same argument as before we obtain

$$\left| \begin{pmatrix} y_3 \\ y_4 \end{pmatrix} \right| \leq \widetilde{C}_4 \exp(2\alpha^* \tau). \tag{A.24}$$

Since (y_1, y_2) and (y_3, y_4) are upper and lower solution, respectively, we have

$$y_1 \geq p_{\xi\xi} \geq y_3, \quad y_2 \leq q_{\xi\xi} \leq y_4.$$

It follows from (A.23) and (A.24) that

$$|p_{\xi\xi}| \leq c_1 \exp(2\alpha^* \tau), \quad |q_{\xi\xi}| \leq c_1 \exp(2\alpha^* \tau).$$

Step 2. In this step we consider the solution of the ordinary differential equation with initial data $(\xi, \eta) \in \Theta$. We will show that the lemma holds for $\tau \leq \widehat{C}_2$ with any constant $\widehat{C}_2$ independent of the initial data. We first show i). In the same way as Step 1, we observe that the solution of the system

$$\frac{d}{d\tau}\begin{pmatrix} w_5 \\ w_6 \end{pmatrix} = \overline{T}_2\begin{pmatrix} w_5 \\ w_6 \end{pmatrix}, \quad \begin{pmatrix} w_5(0) \\ w_6(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{A.25}$$

where

$$\overline{T}_2 = \begin{pmatrix} \max\limits_{(p,q)\in\Theta} f_p(p,q) & \min\limits_{(p,q)\in\Theta} f_q(p,q) \\ \min\limits_{(p,q)\in\Theta} g_p(p,q) & \max\limits_{(p,q)\in\Theta} g_q(p,q) \end{pmatrix},$$

is an upper solution, while the solution of the system

$$\begin{gathered} \frac{d}{d\tau}\begin{pmatrix} w_7 \\ w_8 \end{pmatrix} = \underline{T_2}\begin{pmatrix} w_7 \\ w_8 \end{pmatrix}, \quad \begin{pmatrix} w_7(0) \\ w_8(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\ \underline{T}_2 = \begin{pmatrix} \min\limits_{(p,q)\in\Theta} f_p(p,q) & 0 \\ 0 & \min\limits_{(p,q)\in\Theta} g_q(p,q) \end{pmatrix} \end{gathered} \tag{A.26}$$

is a lower solution of (A.1). If we denote by M_1 the largest eigenvalue of the matrix $\overline{T}_2$, then (A.25) has a solution satisfying

$$w_5(\tau) \leq c_1 \exp(M_1 \tau), \quad w_6(\tau) \geq -c_2 \exp(M_1 \tau)$$

and the solution of (A.26) is given by

$$w_7(\tau) = c_3 \exp\Big(\tau \min_{(p,q)\in\Theta} f_p(p,q)\Big), \quad w_8(\tau) = c_4 \exp\Big(\tau \min_{(p,q)\in\Theta} g_q(p,q)\Big).$$

Since $w_8(0) = 0$, we have $c_4 = 0$. Therefore it holds that

$$c_3 \exp(-M_2\tau) = w_7(\tau) \le p_\xi(\tau;\xi,\eta) \le w_5(\tau) = c_1 \exp(M_1\tau),$$
$$-c_2 \exp(M_1\tau) \le q_\xi(\tau;\xi,\eta) \le 0,$$

where $-M_2 = \min_{(p,q)\in\Theta} f_p(p,q)$. On the other hand, since $\tau < \widehat{C}_2$, there exist positive constants c_5, c_6, c_7 such that

$$c_5 \le p_\xi(\tau;\xi,\eta) \le c_6, \quad -c_7 \le q_\xi(\tau;\xi,\eta) \le 0, \quad \text{for any } \tau \in [0,\hat{\tau}].$$

By choosing $c_8 > 0$ sufficiently small and c_9, $c_{10} > 0$ sufficiently large we conclude

$$c_8 \exp(\alpha^*\tau) \le p_\xi(\tau;\xi,\eta) \le c_9 \exp(\alpha^*\tau), \tag{A.27}$$
$$-c_{10} \exp(\alpha^*\tau) \le q_\xi(\tau;\xi,\eta) \le 0. \tag{A.28}$$

By following the same argument, we have

$$-c_{11} \exp(\alpha^*\tau) \le p_\xi(\tau;\xi,\eta) \le 0, \tag{A.29}$$
$$c_{12} \exp(\alpha^*\tau) \le q_\xi(\tau;\xi,\eta) \le c_{13} \exp(\alpha^*\tau) \tag{A.30}$$

for an initial data $(\xi,\eta) \in \Theta$.

We next show iii) for $(\xi,\eta) \in \Theta$. In the same way as in Step 1, we consider an upper and lower solution of (A.25) for $\tau \in [0, \widehat{C}_2)$. An upper solutions (y_5, y_6) is given by

$$\frac{d}{d\tau}\begin{pmatrix} y_5 \\ y_6 \end{pmatrix} = \overline{T}_2 \begin{pmatrix} y_5 \\ y_6 \end{pmatrix} + H(p_\xi, q_\xi), \quad \begin{pmatrix} y_5(0) \\ y_6(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which implies

$$\begin{pmatrix} y_5 \\ y_6 \end{pmatrix} = \int_0^\tau \exp\big((\tau - s)\overline{T}_2\big) H(p_\xi, q_\xi)\,ds. \tag{A.31}$$

On the other hand, a lower solution (y_7, y_8) is given by

$$\frac{d}{d\tau}\begin{pmatrix} y_7 \\ y_8 \end{pmatrix} = \underline{T_2} \begin{pmatrix} y_7 \\ y_8 \end{pmatrix} + H(p_\xi, q_\xi), \quad \begin{pmatrix} y_7(0) \\ y_8(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which implies

$$\begin{pmatrix} y_7 \\ y_8 \end{pmatrix} = \int_0^\tau \exp\big((\tau - s)\underline{T_2}\big) H(p_\xi, q_\xi)\,ds. \tag{A.32}$$

Note that $\tau < \widehat{C}_2$. We easily see from (A.31) and (A.32) that there exists $M_3 > 0$ such that

$$|y_i| < M_3 \quad \text{for } i = 5, 6, 7, 8.$$

This and the fact that (y_5, y_6) and (y_7, y_8) are a pair of upper and lower solutions imply that

$$|p_{\xi\xi}(\tau;\xi,\eta)| < M_3, \quad |q_{\xi\xi}(\tau;\xi,\eta)| < M_3.$$

This implies that iii) holds for $(\xi,\eta) \in \Theta$ if we choose $\widetilde{B}_7$ sufficiently large.

Step 3. We are interested in obtaining the lemma for any initial data $(\xi,\eta) \in [0,\widetilde{M}] \times [0,\widetilde{M}] \backslash \mathscr{B}_{\sigma_0}(0,0)$. Here we only consider the case where $(\xi,\eta) \in \Delta_1$ and *dist*$((\xi,\eta),S)$ is small so that (p,q) enters $\mathscr{B}_{\sigma_1}(u^*,v^*)$, since all the other cases are simpler. Note that (u^*,v^*) is a saddle point. The solution (p,q) starts from inside of Θ and goes through the following sets:

$$\Theta, \mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_{n-1}, \mathscr{A}_n, \mathscr{A}_{n-1}, \ldots, \mathscr{A}_2, \mathscr{A}_1, \Theta,$$

where n is some integer depending on *dist*$((\xi,\eta),S)$. Finally it arrives at $\mathscr{B}_{\sigma_1}(R_1,0)$. Let τ_1 be the time when (p,q) exits from Θ and enters $\mathscr{A}_1$ and τ_2 be the time when (p,q) exits $\mathscr{A}_1$ and enters $\mathscr{A}_2$. We have already shown in Step 2 that (p,q) with initial data $(\xi,\eta) \in \Theta$ satisfies (A.27)–(A.30) for $\tau \in (0,\tau_1]$. We will next show that these inequalities hold also true for $[\tau_1,\tau_2)$. Differentiating the equality

$$p(\tau;\xi,\eta) = p(\tau-\tau_1, p(\tau_1;\xi,\eta), q(\tau_1;\xi,\eta)), \tag{A.33}$$

with respect to ξ, we have

$$p_\xi(\tau;\xi,\eta) = p_\xi(\tau-\tau_1; p(\tau_1), q(\tau_1))p_\xi(\tau_1;\xi,\eta) + p_\eta(\tau-\tau_1; p(\tau_1), q(\tau_1))q_\xi(\tau_1;\xi,\eta), \tag{A.34}$$

where we write $p(\tau_1;\xi,\eta), q(\tau_1;\xi,\eta)$ as $(p(\tau_1), q(\tau_1))$. By replacing τ by $\tau-\tau_1$ and initial data (ξ,η) by $(p(\tau_1), q(\tau_1))$ in (A.14) and (A.17), we obtain

$$\tilde{c}_1 \exp(\alpha^*(\tau-\tau_1)) \le p_\xi(\tau-\tau_1; p(\tau_1), q(\tau_1)) \le \tilde{c}_2 \exp(\alpha^*(\tau-\tau_1)) \tag{A.35}$$

$$-\tilde{c}_7 \exp(\alpha^*(\tau-\tau_1)) \le q_\eta(\tau-\tau_1; p(\tau_1), q(\tau_1)) \le -\tilde{c}_8 \exp(\alpha^*(\tau-\tau_1)). \tag{A.36}$$

Substituting (A.27), (A.28) with $\tau = \tau_1$ and (A.35), (A.36) into (A.34), we obtain

$$B_1 \exp(\alpha^*\tau) \le p_\xi(\tau;\xi,\eta) \le B_2 \exp(\alpha^*\tau) \quad \text{for } \tau \in [\tau_1,\tau_2). \tag{A.37}$$

Similarly we can obtain estimates for q_ξ, p_η, q_η in $(\tau_1,\tau_2]$.

Next we define τ_3 to be the time when (p,q) exits $\mathscr{A}_2$ and enters $\mathscr{A}_3$. Then we will show that i) holds for $\tau \in (\tau_2,\tau_3]$ by the same argument as we get (A.37). Inductively by the same argument, we can show the first equation of i) for $\tau \in \left[0, \frac{1}{\alpha^*}\log\frac{1}{\mathrm{dist}((\xi,\eta),S)} + \widehat{C}_0\right]$, where $\widehat{C}_0$ is the same constant as in Lemma 3.9.

Obviously the second equation of i) and ii) can be shown in the same way.

We proceed with estimate iii). Since we have already shown in Step 2 that iii) holds for $\tau \in [0,\tau_1]$, we will only prove iii) for $\tau \in (\tau_1,\tau_2]$. Differentiating (A.33) by ξ, we have

$$\begin{aligned} p_{\xi\xi}(\tau;\xi,\eta) = {} & p_{\xi\xi}(\tau-\tau_1; p(\tau_1), q(\tau_1))(p_\xi(\tau_1;\xi,\eta))^2 \\ & + 2p_{\xi\eta}(\tau-\tau_1; p(\tau_1), q(\tau_1))p_\xi(\tau_1;\xi,\eta)q_\xi(\tau_1;\xi,\eta) \\ & + p_\xi(\tau-\tau_1; p(\tau_1), q(\tau_1))p_{\xi\xi}(\tau_1;\xi,\eta) \\ & + p_{\eta\eta}(\tau-\tau_1; p(\tau_1), q(\tau_1))(q_\xi(\tau_1;\xi,\eta))^2 \\ & + p_\eta(\tau-\tau_1; p(\tau_1), q(\tau_1))q_{\xi\xi}(\tau_1;\xi,\eta). \end{aligned} \tag{A.38}$$

On the other hand by utilizing the estimate $|p_{\xi\xi}| \le c_1 \exp(2\alpha^*\tau)$ with (ξ, η) replaced by $(p(\tau_1), q(\tau_1))$, we obtain

$$|p_{\xi\xi}(\tau-\tau_1; p(\tau_1), q(\tau_1))| \le \widetilde{B}_2 \exp(2\alpha^*(\tau-\tau_1)), \tag{A.39}$$

$$|p_{\xi\eta}(\tau-\tau_1; p(\tau_1), q(\tau_1))| \le \widetilde{B}_2 \exp(2\alpha^*(\tau-\tau_1)). \tag{A.40}$$

We substitute (A.39), (A.40) into (A.38) and get

$$|p_{\xi\xi}(\tau;\xi,\eta)| < B_7 \exp(2\alpha^*\tau) \tag{A.41}$$

for $\tau \in (\tau_1, \tau_2]$. Inductively by repeating the same argument, we obtain iii). The estimates for $p_{\xi\eta}$, $p_{\eta\eta}$, $q_{\xi\xi}$, $q_{\xi\eta}$, $q_{\xi\xi}$ can be obtained by following the same procedure.

The proof of the lemma is now completed.

References

Alexander, J., Gardner, R., Jones, C. (1990). A topological invariant arising in the stability analysis of travelling waves. *J. Reine Angew. Math.* 410:167–212.

Allen, S., Cahn, J. (1979). A microscopic theory for antiphase boundary motion and its application to antiphase domain coarsening. *Acta Metall.* 27:1084–1095.

Ei, S.-I., Yanagida, E. (1994). Dynamics of interfaces in competition diffusion systems. *SIAM J. Appl. Math.* 54:1355–1373.

Ei, S.-I., Iida, M., Yanagida, E. (1997). Dynamics of interfaces in a scalar parabolic equations with variable diffusion coefficients. *Japan J. Ind. Appl. Math.* 14(1):1–23.

Hilhorst, D., Matano, H., Schätzle, R. Motion of interfaces in a spatially inhomogeneous Allen–Cahn equation with equal well-depth potential. In preparation.

Hirsch, M. W., Smale, S. (1974). *Differential Equations, Dynamical Systems, and Linear Algebra*. Academic Press.

Kan-on, Y. (1995). Parameter dependence of propagation speed of traveling waves for competition–diffusion equations. *SIAM J. Math. Anal.* 26:340–363.

Nakamura, K., Matano, H., Hilhorst, D., Schätzle, R. (1999). Singular limit of a reaction–diffusion equation with a spatially inhomogeneous reaction term. *J. Stat. Phys.* 95:1165–1185.

Volpert, A. I., Volpert, V. A., Volpert, V. A. (1994). *Travelling Wave Solutions of Parabolic Systems*. Translations of Mathematical Monographs, 140. Providence, RI: AMS.