PHASE BOUNDARIES MOTION PRESERVING THE VOLUME OF EACH CONNECTED COMPONENT

Georgia Karali
Department of Mathematics and Statistics
University of Massachusetts, Amherst
MA 01003-4515, USA.

Abstract. We consider a finite number of particles initially close to spherical but of varying size exhibiting the property of preserving the measure of each connected component. We show that under the assumptions i) particle size/interparticle distance $\ll 1$ and ii) initial deviation from sphericity/particle size $\ll 1$ the particles retain their almost spherical shape and the dynamics of the system is determined by the motion of the centers. This is in sharp contrast with the Mullins-Sekerka free boundary problem where the particle centers remain almost fixed and the dynamics of the system is determined by the evolution of the radii.

1. Introduction

We study the effective dynamics of the following equation:

$$\begin{cases} \Delta u_i = 0, & in \quad \Omega_i \\ \Delta u_e = 0, & in \quad \Omega_e \\ \nabla u_i n^- + \nabla u_e n^+ = H, & on \quad \partial\Omega_i \\ u_e = 0, & on \quad \partial\Omega \end{cases} \tag{1.1}$$

$$V = \nabla u_i n^- . \tag{1.2}$$

Here Ω is a bounded smooth domain in $I\!R^3$, Ω_i is open, smooth under the assumption that $\partial\Omega_i \cap \partial\Omega = \emptyset$ and $\Omega_1, \cdots, \Omega_N$ are bounded connected components of Ω_i such that $\Omega_i := \cup_{h=1}^N \Omega_h$. u_e, u_i are the restrictions of u on Ω_e (the exterior) and Ω_i (the interior) in Ω and n^-, n^+ the unit exterior normal to Ω_e, Ω_i. H is the mean curvature of $\partial\Omega_i$ at x and V is the normal velocity. The sign convention for H and V is that H and V are positive for Ω_i a shrinking sphere.

We observe from (1.1) that u_i is harmonic in Ω_h and therefore the Green's identity imply that $\int_{\partial\Omega_h} \nabla u_i n^- = 0$. This and (1.2) conclude $\int_{\partial\Omega_h} V = 0$ and thus the conservation of the measure of Ω_h.

There are physical situations where a system composed of two immiscible phases A and B has the quite remarkable property, that during the evolution, not only the measure of each of the regions Ω_A, Ω_B occupied by the two phases is conserved, but also any connected subregion of either Ω_A or Ω_B evolves keeping its measure constant until eventually coalesces with another component or splits in two or more components through some kind of singularity. It is worth mentioning here, that despite the fact that (1.1), (1.2) is not a perimeter shortening law, hence there is no isoperimetric property, the spherical shape is preserved and is indeed stable.

Even though they do not directly model a specific physical phenomenon, equations (1.1), (1.2) exhibit the property of preserving the measure of each connected component and the dynamics defined by (1.1), (1.2) can be expected to be a paradigm for some of the typical nonlinear phenomena that are tightly related to that property. More specifically, those equations capture some qualitative aspects like the fact that the motion (the evolution) is dictated by the decreasing of an energy proportional to the surface or also the fact that in such situations one expects that the only stable equilibrium should correspond to the case of just one component.

Conservation of the measure of each connected component does not imply that the number N of the such components cannot change during the evolution but only requires that $N = N(t)$ be locally constant. Therefore, as we have observed above, N can only change through jump discontinuities at certain singular times. Besides local constancy of N and the existence of singularities, another qualitative feature of the dynamics that any mathematical model having the above conservation property should present is a *particles-like* behavior in the case of small volume fraction

$$\frac{|\Omega_A|}{|\Omega_B|} \ll 1$$

when a small amount of phase A is divided into many tiny particles dispersed in the matrix of phase B. The rationale behind this conjecture is that: (i) If the size of a connected region is small with respect to the distance from the other regions, then, under general isotropy assumptions, the spherical shape should be privileged and particles should therefore have a tendency to assume a spherical shape (*Stability of Spherical Shape*). (ii) If the particles have an almost fixed shape, and in particular a fixed volume, then the only parameters left for determining the whole geometry and therefore also the dynamics of the system are the centers $\xi_h, h = 1, .., N$ of the particles. Therefore it is natural to expect that, provided particle sizes are small and interparticle distances sufficiently large, some reduction of the P.D.E becomes possible resulting in an O.D.E for the evolution and the interaction of the centers of the particles. This should be contrasted with the Mullins-Sekerka dynamics

$$\begin{cases} -\Delta u = 0, & off \quad \Gamma(t), \quad in \quad \Omega \subset I\!R^m, \ m = 2, 3 \\ u = H, & on \quad \Gamma(t) \\ \dfrac{\partial u}{\partial \nu} = 0, & on \quad \partial\Omega \\ V = \left[\left[\dfrac{\partial u}{\partial n}\right]\right], & on \quad \Gamma(t) \end{cases}$$

which in a sense is the "dual" of (1.1). As it has been shown in [3], the almost spherical shape is stable. However, in that context the centers, to principal order stay fixed, and the radii evolve according to the set of ODE's ([1], [2], [4]):

$$
\begin{aligned}
\epsilon\frac{d\rho_i}{dt} &= \frac{1}{\epsilon\rho_i}\Big\{\left(\frac{1}{\epsilon\bar\rho}-\frac{1}{\epsilon\rho_i}\right)\\
&+\frac{1}{N\epsilon\bar\rho}\sum\nolimits'_{h,k}\frac{\epsilon\rho_h}{|\xi_h-\xi_k|}\left(\frac{\epsilon\rho_k}{\epsilon\bar\rho}-1\right)-\sum_j{}'\frac{1}{|\xi_j-\xi_i|}\left(\frac{\epsilon\rho_j}{\epsilon\bar\rho}-1\right)\\
&+\frac{1}{N}\sum_{i,h}4\pi\,\frac{\epsilon\rho_i}{\epsilon\bar\rho}\,\gamma(\xi_i,\xi_h)\left(\frac{\epsilon\rho_h}{\epsilon\bar\rho}-1\right)-\sum_h\gamma(\xi_i,\xi_h)4\pi\left(\frac{\epsilon\rho_h}{\epsilon\bar\rho}-1\right)+\frac{g_i}{\epsilon\rho_i}\Big\}+\cdots
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\xi_i}{dt} &= -3\sum\nolimits'_k\left(\frac{1}{\epsilon\bar\rho}-\frac{1}{\epsilon\rho_k}\right)\epsilon\rho_k\,\frac{\xi_k-\xi_i}{|\xi_k-\xi_i|^3}\\
&\quad -3\sum\nolimits_h\epsilon\rho_h\,\frac{\partial\gamma(\xi_i,\xi_h)}{\partial x}\left(\frac{1}{\epsilon\bar\rho}-\frac{1}{\epsilon\rho_h}\right)+\cdots
\end{aligned}
$$

where $\overline{\rho} = \frac{1}{N-i+1}\sum_{h=1}^{N}\rho_h$ and $\|r_i(t)\|_{C^{3+\alpha}(S^2)} < C_r$ as long as r_i is defined. The symbol $\sum'$ means summation avoiding equal indices. Here γ is the smooth part of the Green's function. Notice that ρ, and ξ form a closed system of equations if the highest order terms are ignored.

For the problem (1.1), (1.2) under similar assumptions, the situation is exactly the opposite. To principal order the radii of the particles stay fixed, and the centers evolve according to (1.4) below.

The scope of the present note is to give a rigorous proof of the above conjectures concerning particles dynamics in the context of the P.D.E (1.1), (1.2). In particular we show that under the assumption that

$$\frac{\text{particle size}}{\text{interparticle distance}} \ll 1,$$

$$\frac{\text{initial deviation from sphericity}}{\text{particle size}} \ll 1, \tag{1.3}$$

the particles retain their almost spherical shape and their centers, in the limit $\epsilon \to 0^+$, evolve according to the O.D.E

$$\frac{d\xi_h}{dt} = \sum_{\substack{k=1\\h\neq k}}^{N}\rho_h\rho_k\frac{\xi_h-\xi_k}{|\xi_h-\xi_k|^3} - \sum_{h=1}^{N}4\pi\rho_h\rho_k\frac{\partial\gamma(\xi_h,\xi_k)}{\partial x}, \tag{1.4}$$

where $\rho_h, h = 1,..,N$ are the radii of the particles. Equation (1.4) is the gradient system in the Euclidean metric of the potential energy

$$\mathcal{P}(\xi) = \sum_{\substack{h,k=1\\h\neq k}}^{N}\rho_h\rho_k G(\xi_h,\xi_k) - \sum_{h=1}^{N}\rho_h^2\gamma(\xi_h,\xi_h),$$

where $G(x,y) = \frac{1}{4\pi|x-y|} + \gamma(x,y)$ is the Green function for the Dirichlet problem

$$\begin{cases} -\Delta_x G(x,y) = \delta_y(x), & x \in \Omega, \quad y \in \Omega \\ G(x,y) = 0 & y \in \Omega, \quad x \in \partial\Omega. \end{cases}$$

Equation (1.4) can be formally derived if one accepts that, under assumptions (1.3), the solution of (1.1) is well represented by the *Monopole Approximation*

$$u(x) = \sum_{k=1}^{N} c_k G(x, \xi_k), \quad x \in \Omega_e, \tag{1.5}$$

$$u(x) = c_h(\frac{1}{4\pi\rho_h} + \gamma(x,\xi_h)) + \sum_{\substack{k=1 \\ k\neq h}}^{N} c_k G(x,\xi_k), \quad x \in \Omega_{i,h}, \ h = 1, ..N.$$

Then, imposing the interior boundary condition $(1.1)_3$ yields

$$(\nabla u(x), n^-) + (\nabla u(x), n^+) \simeq (\nabla u(x), n^+) \simeq c_h \frac{1}{4\pi\rho_h^2} \simeq \frac{1}{\rho_h}, \quad x \in \partial\Omega_{i,h}$$

and therefore

$$c_h = 4\pi\rho_h. \tag{1.6}$$

For a ball that moves rigidly with speed $\frac{d\xi_h}{dt}$ the normal velocity V at a point of the surface where the exterior normal is n^- is given by $V = -(\frac{d\xi_h}{dt}, n^-)$ on the other hand from (1.5) and (1.2), we get $V = \sum_{k,k\neq h} c_k \rho_h (\nabla_x G(x,\xi_k)|_{x=\xi_h}, n^-)$ where we have used $\nabla_x G(x,\xi_k)|_{x=\xi_h+\rho_h n^-} \simeq \nabla_x G(x,\xi_k)|_{x=\xi_h}$. Comparing the two expressions of V above, using the fact that n^- can be chosen arbitrarily and using also (1.6) yield (1.4).

It is interesting to consider some particular cases of (1.4). If we take $N = 2$, $\rho_1 = \rho_2 = \rho$; $\xi_2 = -\xi_1$; $\xi_1 = (s,0,0)$, and $\Omega = I\!R^3$, then from (1.4), taking also into account that now $\gamma \equiv 0$, we obtain

$$\frac{ds}{dt} = \frac{\rho^2}{4s^2} \Rightarrow s(t) = (s_0^3 + \frac{3}{4}\rho^2 t)^{\frac{1}{3}}$$

which shows that the interaction between two particles has a *repelling* character. Similarly we can show that the boundary *attracts* the particles. To see this we take $N = 1$; $\xi = (s,0,0)$; $\Omega = \{x = (x_1,x_2,x_3) : x_1 > 0\}$. Then we get $G(x,y) = \frac{1}{4\pi|x-y|} - \frac{1}{4\pi|x+y|}$ and (1.4) yields

$$\frac{ds}{dt} = -\frac{\rho^2}{4s^2} \Rightarrow s(t) = (s_0^3 - \frac{3}{4}\rho t)^{\frac{1}{3}}.$$

Another interesting aspect of the model in discussion is the lack of knowledge of the shape of the equilibria, since it is not straightforward what the equilibrium states are. For the Mullins-Sekerka model if $H = const.$ on $\Gamma(t)$ then $u(x) = H = const.$, $\forall x \in \Omega$ and $V \equiv 0$ solves the problem. Therefore Ω_i is the union of equal balls with the same radius which are equilibria, something which is not the case here.

However, a main feature of the dynamics of (1.1), (1.2) is the generation of singularities. After a singularity occurs a new phase of smooth motion begins. As the evolution continues and after a finite number of singularities just one particle persists and moves according to the O.D.E. We can conjecture that this unique particle approaches asymptotically an equilibrium position which is determined by the geometry of Ω.

Our methodology and analysis follows [3]. In section 2 we present some general properties. We discuss the operator L which will be the main part in the linear part of the evolutionary equation for r. Moreover we introduce a suitable representation for $\Gamma = \Gamma(t)$ and show in Proposition 2.1 that there are uniquely determined ξ_i, ρ_i, r_i such that $\Gamma = \{x/x = \xi + \epsilon\rho(1+\epsilon r(u))u, u \in S^2\}$. In Proposition 2.2 we give the expression of H in terms of ρ and r. In section 3 (cfr. Proposition 3.1), we derive an expression for V_i, as a function of the time derivatives $\frac{d\xi_i}{dt}$, $\frac{d\rho_i}{dt}$, $\frac{dr_i}{dt}$ of the unknowns ξ_i, ρ_i, r_i. Moreover, we solve the linear equation $V = \frac{1}{2}H(x) - \int_\Gamma \frac{\partial G(x,y)}{\partial n_x} H(y)dy$ which is the central dynamical problem. The main result is in Proposition 3.2 where we solve this problem by assuming that H is known and we obtain a system of ξ, ρ and r equations with estimates for the higher order terms. At the end of section 3, we obtain an explicit formula for V_i as a function of ξ, ρ, r (cfr. Proposition 3.3). This function is inserted in the place of Z_i in Proposition 3.1 and so we obtain the sought evolutionary system (cfr. Proposition 3.4). Finally in section 4, we show that (1.1) is well posed globally in time, and we derive a bound on r which implies the robustness of the spherical shape. The main step is to prove that r is bounded (cfr. Proposition 4.2) and in order to do so we use a suitable functional-analytic setting by making use of the optimal regularity theory of Da Prato and Grisvard.

2. General properties

The operator L

Consider $\Omega \subset I\!R^3$ a bounded, smooth, connected set in $I\!R^3$, and Γ a $C^{1+\alpha}$ closed, orientable surface in Ω. We consider the Jacobi operator defined as $Lr = \Delta^s r + 2r$ on the two dimensional unit sphere. So, in spherical coordinates (coordinatizing with ϕ: longitude, θ: colatitude), we have the representation

$$L\chi = \frac{\chi_{\phi\phi}}{\sin^2\theta} + \frac{(\sin\theta\chi_\theta)_\theta}{\sin\theta} + 2\chi \tag{2.1}$$

$$\langle \chi, \psi \rangle_{L^2} = \int_0^\pi \int_0^{2\pi} \chi\psi \sin\theta \, d\phi \, d\theta \tag{2.2}$$

L is a self adjoint operator. Indeed, by integration by parts it follows

$$
\begin{aligned}
&-\int_0^{\pi}\int_0^{2\pi}(L\chi)\psi\sin\theta\,d\phi\,d\theta \\
&= \int_0^{\pi}\int_0^{2\pi}\left(\frac{\chi_\phi\psi_\phi}{\sin^2\theta}+\chi_\theta\psi_\theta-2\chi\psi\right)\sin\theta\,d\phi\,d\theta =: \mathcal{B}(\chi,\psi)
\end{aligned}
\tag{2.3}
$$

By direct calculation we obtain some information on the spectrum of $-L$:

$$\mu_0=-2, \qquad \mu_1=\mu_2=\mu_3=0, \qquad \mu_4>0 \tag{2.4}$$

with the first few eigenfunctions given by

$$
\begin{aligned}
w_0 &= \tfrac{1}{2\sqrt{\pi}}, \\
w_1 &= \tfrac{1}{2}\sqrt{\tfrac{3}{\pi}}\langle u,e_1\rangle = \tfrac{1}{2}\sqrt{\tfrac{3}{\pi}}\cos\phi\sin\theta, \\
w_2 &= \tfrac{1}{2}\sqrt{\tfrac{3}{\pi}}\langle u,e_2\rangle = \tfrac{1}{2}\sqrt{\tfrac{3}{\pi}}\sin\phi\sin\theta, \\
w_3 &= \tfrac{1}{2}\sqrt{\tfrac{3}{\pi}}\langle u,e_3\rangle = \tfrac{1}{2}\sqrt{\tfrac{3}{\pi}}\cos\theta.
\end{aligned}
\tag{2.5}
$$

The Coordinate System

The introduction of a suitable representation for the interface $\Gamma(t)$ is very important. Given an interface close to spherical, we would like to associate to it a unique sphere and view the interface as a small perturbation of this sphere. So, an appropriate choice of the coordinates used to represent $\Gamma_i(t)$ is essential for the analysis. In particular, to each interface we associate a center ξ, a radius ρ_i and a function r_i that describes the "distortion from sphericity" and we show that ξ_i, ρ_i, r_i are uniquely determined.

Proposition 2.1 ([3])**.** *Given an interface Γ in a small C^1 neighborhood of a sphere $S_{\bar\xi,\bar\rho}$, there are unique $\xi\in\mathbb{R}^3, \rho>0, r\in C^1(S^2)$ such that*

$$\Gamma=\{x/x=\xi+\epsilon\rho(1+\epsilon r(u))u, u\in S^2\} \tag{2.6}$$

satisfying the orthogonality conditions

$$\int_{S^2} r(u)\,du=0, \qquad \int_{S^2} r(u)\langle u,e_i\rangle\,du=0, \qquad i=1,2,3 \tag{2.7}$$

where $S^2\subset\mathbb{R}^3$ is the unit sphere and $\{e_1,e_2,e_3\}$ the standard basis in $\mathbb{R}^3$.

□

The Mean Curvature in Special Coordinates

Proposition 2.2 ([3])**.** *Assume* $\Gamma = \{x | x = X(u) := \xi + \epsilon\rho(1 + \epsilon r(u))u, u \in S^2\}$ *with* $r \in C^{2+\alpha}(S^2)$*. Then the mean curvature* $H(X(u))$ *of* Γ *at the point* $X(u)$ *is given by*

$$H(X(u)) = \frac{1}{\epsilon\rho}(1 - \epsilon Lr + B), \tag{2.8}$$

where L *is the Jacobi operator on* S^2*, that is*

$$Lr = \Delta^S r + 2r, \tag{2.9}$$

Δ^S *being the Laplace-Beltrami operator on* S^2*, and* B *of the form* $B = b(\epsilon r, \epsilon Jr, \epsilon J^2 r)$ *with* $b(z, p, P)$ *a smooth function which is linear in* P *and, under the assumption* $|z| < \delta$*, satisfies the estimate*

$$|b(z, p, P)| \leq C(|z|^2 + |p|^2 + (|z| + |p|)|P|). \tag{2.10}$$

□

The Green's function in three space dimensions is of the form

$$G(x, y) = \frac{1}{4\pi|x - y|} + \gamma(x, y)$$

which is associated to the problem

$$\begin{cases} -\Delta_x G(x, y) = \delta_y(x), & x \in \Omega, \quad y \in \Omega \\ G(x, y) = 0 & x \in \partial\Omega, \quad y \in \Omega \end{cases} \tag{2.11}$$

and γ is the smooth part of the Green's function that captures the effect of the boundary and satisfies

$$\begin{cases} -\Delta_x \gamma(x, y) = 0, & x \in \Omega, \quad y \in \Omega \\ \gamma(x, y) = -\frac{1}{4\pi}|x - y|, & x \in \partial\Omega, \quad y \in \Omega \end{cases} \tag{2.12}$$

where Ω is the container of the mixture and $\delta_x(y)$ is the Dirac δ supported at $y \in \Omega$.

Lemma 2.3. *The following estimates hold true*

$$|\gamma(x, y)| \leq \frac{C}{dist(x, \partial\Omega)}, \quad \left|\frac{\partial\gamma(x, y)}{\partial y}\right| \leq \frac{C}{dist^2(x, \partial\Omega)} \tag{2.13}$$

Proof. The above estimates can be proved by making use of classical elliptic theory [10].

□

3. Solving the equation $V = \frac{1}{2}H(x) - \int_{\partial\Omega_i} \frac{\partial G(x,y)}{\partial n_x} H(y)dy$ for given H

In this section, we first give a decomposition result for a general V in terms of $\frac{d\rho}{dt}$, $\frac{d\xi}{dt}$, $\rho\frac{dr}{dt}$ and then we solve the linear equation

$$V = \frac{1}{2}H(x) - \int_\Gamma \frac{\partial G(x,y)}{\partial n_x} H(y)dy \quad \text{for given} \quad H.$$

So, firstly we are interested in obtaining a decomposition for V in terms of $\frac{d\rho}{dt}$, $\frac{d\xi}{dt}$, $\rho\frac{dr}{dt}$ for interfaces with the representation (2.6). We let $V = V(u,t)$ to be the speed of $\Gamma(t)$ in the orthogonal direction to $\Gamma(t)$ at the point $x \in \Gamma(t)$ and we study the relationship between V and $\frac{d\rho}{dt}$, $\frac{d\xi}{dt}$, $\rho\frac{dr}{dt}$.

Proposition 3.1 ([3])**.** *Assume that $\epsilon\|r\|_{C^{1+\alpha}(S^1)} < \delta$ for $\delta > 0$ a small fixed number, so that Proposition 2.2 holds. Then V is a linear combination of $\epsilon\frac{d\rho}{dt}$, $\epsilon^2\rho\frac{dr}{dt}$, $\frac{d\xi}{dt}$ and the equation $V = Z$ with $Z \in C^\alpha(S^2)$ a given function, determines uniquely $\epsilon\frac{d\rho}{dt}$, $\epsilon^2\rho\frac{dr}{dt}$, $\frac{d\xi}{dt}$. Moreover, the following estimates hold true:*

$$\begin{cases} |2\sqrt{\pi}\epsilon\frac{d\rho}{dt} + \langle Z, w_0\rangle_{L^2(S^2)}| \le C\epsilon\left(\epsilon\|r\|^2_{C^{1+\alpha}(S^2)}\|Z\|_{C^\alpha(S^2)} + \|r\|_{C^{1+\alpha}(S^2)}\sum_{h=1}^3 \left|\langle Z, w_h\rangle_{L^2(S^2)}\right|\right) \\ |2\sqrt{\frac{\pi}{3}}\frac{d\xi_j}{dt} + \langle Z, w_j\rangle_{L^2(S^2)}| \le C\epsilon\left(\epsilon\|r\|^2_{C^{1+\alpha}(S^2)}\|Z\|_{C^\alpha(S^2)} + \|r\|_{C^{1+\alpha}(S^2)}\sum_{h=1}^3 \left|\langle Z, w_h\rangle_{L^2(S^2)}\right|\right) \\ \|\epsilon^2\rho\frac{dr}{dt} + Z - \sum_{j=0}^3 \langle Z, w_j\rangle_{L^2(S^2)}w_j - \epsilon\langle Z, w_0\rangle_{L^2(S^2)}w_0 r\|_{C^\alpha(S^2)} \\ \quad \le C\epsilon\left(\epsilon\|r\|^2_{C^{1+\alpha}(S^2)}\|Z\|_{C^\alpha(S^2)} + \|r\|^2_{C^\alpha(S^2)}\sum_{h=1}^3 \left|\langle Z, w_h\rangle_{L^2(S^2)}\right|\right) \end{cases} \tag{3.1}$$

for some constant $C > 0$ and w_j, $j = 0, 1, 2$ defined in (2.5).

□

Problem (1.1), (1.2) can be formulated via potential theory ([9], [18]) as an integral equation

$$V = \frac{1}{2}H(x) - \int_\Gamma \frac{\partial G(x,y)}{\partial n_x} H(y)dy \tag{3.2}$$

where $\frac{\partial G(x,y)}{\partial n_x} = \frac{d}{ds}G(x+sn^-(x),y)\Big|_{s=0}$, V is the normal velocity and H is the mean curvature. Here t is suppressed and we write Γ instead of $\Gamma(t)$ with $\Gamma_i = \{x/x = X^i(u) := \xi_i + \epsilon\rho_i(1+\epsilon r_i(u))u, u \in S^2\}$ and $\Gamma = \bigcup_{i=1}^N \Gamma_i$. If $\epsilon > 0$ is small, the map $X^i : S^2 \to \Gamma_i$ is a diffeomorphism with the same regularity as r_i. We let $u^i : \Gamma_i \to S^2$ be the inverse of X^i. The above expression can be written in the form

$$V = \frac{1}{2}H(x) - \sum_{h=1}^{N} \int_{\Gamma_h} \frac{\partial G(x,y)}{\partial n_x} H_h(u^h(y))dy, \quad x \in \Gamma_i,\ i = 1, \dots N \tag{3.3}$$

where $H_h(u^h(y))$ is the restriction of H to Γ_h and

$$\frac{\partial G(x,y)}{\partial n_x} = \frac{-(x-y)n^-(x)}{4\pi|x-y|^3} + \frac{\partial \gamma(x,y)}{\partial n_x}.$$

Proposition 3.2. *Let $\xi_i \in \Omega$, $\rho_i > 0$, $r_i \in C^{1+\alpha}(S^2)$, $W_i \in C^{1+\alpha}(S^2)$, $i = 1, \dots, N$ be given and assume $\xi_i \neq \xi_j$, for $i \neq j$. Then, for small $\epsilon > 0$, the system*

$$V = \frac{1}{2}H(x) - \sum_{h=1}^{N} \int_{\Gamma_h} \frac{\partial G(x,y)}{\partial n_x} H_h(u^h(y))dy, \quad x \in \Gamma_i, \quad i = 1, \dots, N \tag{3.4}$$

has a unique solution $V_i \in C^\alpha(S^2)$. Moreover,

$$\|V_i - \frac{1}{2}H_i + K\|_{C^\alpha(S^2)} \leq \|F\|_{C^\alpha(S^2)} \tag{3.5}$$

where

$$K = \frac{1}{2}\epsilon\rho_h \int_{S^2} \frac{3r_i(v) - r_i(\cdot)}{4\pi|\cdot - v|} H_i(v)dv - \sum_{h\neq i} \epsilon^2 {\rho_h}^2 \frac{\langle \xi_i - \xi_h, \epsilon\rho_i u\rangle}{4\pi|\xi_i - \xi_h|^3} \int_{S^2} H_h(v)dv$$

$$-\sum_{h\neq i} \int_{S^2} \frac{2\epsilon^3 {\rho_h}^2 r_h(v)\langle \xi_i - \xi_h, \epsilon\rho_i u\rangle}{4\pi|\xi_i - \xi_h|^3} H_h(v)dv$$

$$+\sum_{h\neq i} \epsilon^2 {\rho_h}^2 \frac{1}{4\pi|\xi_i - \xi_h|^3} 3\left\langle \epsilon\rho_i u, \frac{\xi_i - \xi_h}{|\xi_i - \xi_h|}\right\rangle^2 \int_{S^2} H_h(v)dv$$

$$-\sum_{h\neq i} \epsilon^2 {\rho_h}^2 \frac{\langle \xi_i - \xi_h, \epsilon\rho_i u\rangle}{4\pi|\xi_i - \xi_h|^4} \int_{S^2} 3\left\langle \epsilon\rho_h v, \frac{\xi_i - \xi_h}{|\xi_i - \xi_h|}\right\rangle H_h(v)dv + \sum_{h=1}^{N} \epsilon^2 {\rho_h}^2 \int_{S^2} \langle \frac{\partial \gamma(\xi_i,\xi_h)}{\partial x}, \epsilon\rho_i u\rangle H_h(v)dv$$

$$+\sum_{h=1}^{N} \epsilon^2 \rho_h^2 \int_{S^2} \left\langle \frac{\partial \gamma(\xi_i,\xi_h)}{\partial x}, \epsilon\rho_i u\right\rangle \left\langle \frac{\partial \gamma(\xi_i,\xi_h)}{\partial y}, \epsilon\rho_h v\right\rangle H_h(v)dv$$

$$+\sum_{h=1}^{N}\epsilon^2\rho_h^2\int_{S^2}\left\langle\frac{\partial\gamma(\xi_i,\xi_h)}{\partial x},\epsilon\rho_i u\right\rangle^2 H_h(v)dv$$

and $\|F\|_{C^\alpha(S^2)}$ *includes precise estimates for higher order terms*

$$\|F\|_{C^\alpha(S^2)}=\epsilon^2{\rho_h}^2 O_{C^{1+\alpha}(S^2)}\Bigg(\frac{\epsilon^2\|r_h\|^2_{C^{1+\alpha}(S^2)}}{|\xi_i-\xi_h|^3}+\frac{\epsilon^2\|r_h\|^2_{C^{1+\alpha}(S^2)}}{|\xi_i-\xi_h|^4}$$

$$+\frac{\epsilon^2\|r_h\|_{C^{1+\alpha}(S^2)}}{|\xi_i-\xi_h|^4}(\rho_i+\rho_h)+\epsilon^2\left(\frac{\rho_i^2+\rho_h^2}{|\xi_i-\xi_h|^5}+\frac{\rho_i\|r_i\|_{C^{1+\alpha}(S^2)}+\rho_h\|r_h\|_{C^{1+\alpha}(S^2)}}{|\xi_i-\xi_h|^4}\right)$$

$$+\epsilon^2{\rho_h}^2 O_{C^{1+\alpha}(S^2)}(\epsilon^3{\rho_i}^3\frac{\partial}{\partial x}\frac{\partial\gamma(\xi_i,\xi_h)}{\partial x}+\epsilon\|r_h\|_{C^{1+\alpha}(S^2)}\frac{\partial}{\partial x}\frac{\partial\gamma(\xi_i,\xi_h)}{\partial x})\Bigg)\|H_h\|_{C^\alpha(S^2)}.$$

Proof. We are interested in solving equation (3.4). We have,

$$\int_{\Gamma_h}\frac{\partial G(x,y)}{\partial n_x}H_h(u^h(y))dy=\int_{\Gamma_h}\frac{-(x-y)n^-(x)}{4\pi|x-y|^3}H_h(u^h(y))dy+\int_{\Gamma_h}\frac{\partial\gamma(x,y)}{\partial n_x}H_h(u^h(y))dy. \tag{3.6}$$

Step A

We consider the case $h=i$, $x\in\Gamma_i$ and we are interested in the analysis of the two integrals on the right hand side of Eq. (3.6).
a) We start our analysis with the study of the first of the two integrals.
Let $\Omega_h=\{z/z=\lambda u,\ 0\le\lambda<1+\epsilon r_h(u),u\in S^2\}$ and we consider the function $U^i:\Omega_i\to I\!R$ defined by

$$U^i(z):=\int_{\Gamma_i}-\frac{1}{4\pi}\frac{(\xi_i+\epsilon\rho_i z-y)z(x)}{|\xi_i+\epsilon\rho_i z-y|^3}H_i(u^i(y))dy$$

$$=\epsilon^2{\rho_i}^2\int_{\partial\Omega_i}-\frac{1}{4\pi}\frac{\epsilon\rho_i(z-z')z(x)}{\epsilon^3{\rho_i}^3|z-z'|^3}H_i(u^i(\xi_i+\epsilon\rho_i z'))dz'$$

$$=\int_{\partial\Omega_i}-\frac{1}{4\pi}\frac{1}{|z-z'|}H_i(u^i(\xi_i+\epsilon\rho_i z'))dz'. \tag{3.7}$$

Therefore by theorem 2.I pg. 307 in [15] applied to the derivatives of U^i, since $r_i\in C^{1+\alpha}(S^2)$, $\partial\Omega_i$ is a surface of class $C^{1+\alpha}(S^2)$, U^i can be extended as a $C^{1+\alpha}$ function to the closure $\overline{\Omega}_i$ of Ω_i, with the following estimate

$$\|U^i(\cdot)\|_{C^{1+\alpha}(\bar{\Omega}_i)}\le\epsilon\rho_i C\|H_i\left(u^i\left(\xi_i+\epsilon\rho_i\cdot\right)\right)\|_{C^\alpha(\partial\Omega_i)} \tag{3.8}$$

where C is $O\left(1+\epsilon\|r_i\|_{C^{1+\alpha}(S^2)}\right)$ and can be considered as a constant independent of r under the standing assumption $\|r_i\|_{C^{1+\alpha}(S^2)} < \frac{\delta}{\epsilon}$. The map $\partial\Omega_i \ni z \to u^i(\xi_i + \epsilon\rho_i z) \in S^2$ is a $C^{1+\alpha}$ diffeomorphism and

$$\|u^i\left(\xi_i + \epsilon\rho_i\,\cdot\right)\|_{C^{1+\alpha}(\partial\Omega_i)} < \text{ Const } \left(1+\epsilon\|r_i\|_{C^{1+\alpha}(S^2)}\right) < C$$

and a similar statement holds true for the inverse map $u \to z$. It follows that

$$\|H_i\left(u^i\left(\xi_i+\epsilon\rho_i\cdot\right)\right)\|_{C^\alpha(\partial\Omega_i)} \leq C\|H_i\|_{C^\alpha(S^2)}. \tag{3.9}$$

From (3.7) and the discussion after it and in particular from (3.8) and (3.9) we have a map $H_i \in C^\alpha(S^2) \to U^i|_{\partial\Omega_i} \in C^{1+\alpha}(S^2)$. From this and the properties of the diffeomorphism $u \to z(u) := \dfrac{X^i(u)-\xi_i}{\epsilon\rho_i}$ discussed above we can define a map $I_1^i : C^\alpha(S^2) \to C^{1+\alpha}(S^2)$ by setting

$$\left(I_1^i H_i\right)(u) = U^i\left(\frac{X^i(u)-\xi_i}{\epsilon\rho_i}\right), \tag{3.10}$$

and that

$$\|I_1^i H_i\|_{C^{1+\alpha}(S^2)} \leq C\|H_i\|_{C^\alpha(S^2)}. \tag{3.11}$$

Besides this estimate we also need to compute the main term in $I_1^i H_i$. From (3.7) and

$$dz' = \left(1 + 2\epsilon r_i + \; O_{C^\alpha(S^2)}\left(\epsilon^2\,\|r_i\|^2_{C^{1+\alpha}(S^2)}\right)\right)\, du \tag{3.12}$$

it follows that

$$\begin{aligned}
\left(I_1^i H_i\right)(u) \;&=\; -\int_{S^2} \frac{1+2\epsilon r_i(v) + \; O_{C^\alpha(S^2)}\left(\epsilon^2\,\|r_i\|^2_{C^{1+\alpha}(S^2)}\right)(v)}{4\pi\left|\frac{X^i(u)-X^i(v)}{\epsilon\rho_i}\right|} H_i(v)\,dv \qquad (3.13)\\
&=\; -\int_{S^2}\frac{1}{4\pi|u-v|}H_i(v)\,dv - \epsilon\int_{S^2}\frac{2r_i(v)H_i(v)}{4\pi|u-v|}\,dv\\
&\quad -\epsilon^2\int_{S^2}\frac{O_{C^\alpha(S^2)}\left(\|r_i\|^2_{C^{1+\alpha}(S^2)}\right)(v)H_i(v)}{4\pi|u-v|}\,dv\\
&\quad -\epsilon\int_{S^2}\frac{1}{4\pi|u-v|}\left(\frac{|u-v| - \left|\frac{X^i(u)-X^i(v)}{\epsilon\,\rho_i}\right|}{\epsilon\left|\frac{X^i(u)-X^i(v)}{\epsilon\rho_i}\right|}\right)\\
&\quad \left(1+2\epsilon r_i(v) + O_{C^\alpha(S^2)}\left(\epsilon^2\,\|r_i\|^2_{C^{1+\alpha}(S^2)}\right)(v)\right)H_i(v)\,dv.
\end{aligned}$$

By the result in [15] quoted above, $I_1^i H_i$ as well as the first 3 integrals on the right hand side of (3.13) are $C^{1+\alpha}(S^2)$ functions. Therefore also the last integral on the right hand side of (3.13) belongs to $C^{1+\alpha}(S^2)$. Let $\epsilon\ (\Im H_i)(u)$ be this last integral. We have

$$\begin{aligned}
&(\Im H_i)(u) \\
= &\int_{S^2} \frac{1}{4\pi|u-v|}\left(\frac{|u-v|-|u-v+\epsilon\ (r_i(u)u-r_i(v)v)|}{\epsilon\ |u-v+\epsilon\ (r_i(u)u-r_i(v)v)|}\right) H_i(v)\,dv \\
+ &\int_{S^2} \frac{1}{4\pi|u-v|}\left(\frac{|u-v|-|u-v+\epsilon\ (r_i(u)u-r_i(v)v)|}{\epsilon\ |u-v+\epsilon\ (r_i(u)u-r_i(v)v)|}\right) \\
&\quad O_{C^\alpha(S^2)}\left(\epsilon\, \|r_i\|_{C^{1+\alpha}(S^2)}\right)(v) H_i(v)\,dv \\
=: &\ (\Im_1 H_i)(u) + (\Im_2 H_i)(u).
\end{aligned} \tag{3.14}$$

From (3.14) it follows that

$$\|\,(\Im_2 H_i)\,\|_{C^{1+\alpha}(S^2)} \leq C\epsilon\, \|r_i\|^2_{C^\alpha(S^2)} \|H_i\|_{C^\alpha(S^2)} \tag{3.15}$$

where we have used the fact that

$$\begin{aligned}
&|u-v|-|u-v+\epsilon\ (r_i(u)u-r_i(v)v)| \\
&\quad = -\epsilon\frac{\langle r_i(u)u-r_i(v)v, u-v\rangle}{|u-v|} + \epsilon^2\, O\left(|r_i(u)u-r_i(v)v|^2\right).
\end{aligned} \tag{3.16}$$

For $|u|=|v|=1$ we have

$$\langle r_i(u)u-r_i(v)v, u-v\rangle = \frac{1}{2}\left(r_i(u)+r_i(v)\right)|u-v|^2, \tag{3.17}$$

and so (3.17) implies

$$\begin{aligned}
\Im_1 H_i &= -\frac{1}{2}\int_{S^2} \frac{r_i(\cdot)+r_i(v)}{4\pi|\cdot - v|} H_i(v)\,dv \\
&\qquad + \epsilon\, O_{C^{1+\alpha}(S^2)}\left(\|r_i\|^2_{C^{1+\alpha}(S^2)} \|H_i\|_{C^\alpha(S^2)}\right).
\end{aligned} \tag{3.18}$$

b) We now turn to the analysis of the second integral on the right hand side of (3.6) for the case $x\in\Gamma_i, h=i$. We have,

$$\begin{aligned}
&\int_{\Gamma_i} \frac{\partial\gamma(X^i(u),y)}{\partial n_x} H_i(u^i(y))dy = \epsilon^2{\rho_i}^2 \int_{\partial\Omega_i} \frac{\partial\gamma(X^i(u),y)}{\partial x} n^-(X^i(u)) H_i(u^i(y))dy \\
&= \epsilon^2{\rho_i}^2 \int_{\partial\Omega_i} \frac{\partial\gamma(X^i(u),\xi_i+\epsilon\rho_i z')}{\partial x} n^-(\xi_i+\epsilon\rho_i z) H_i(u^i(\xi_i+\epsilon\rho_i z'))dz' =: \epsilon^2{\rho_i}^2 (I_2^i H_i)(u).
\end{aligned} \tag{3.19}$$

By taking into account (3.12) and the fact that $z(u) := (X^i(u) - \xi_i)/\epsilon\rho_i$, $(I_2^i H_i)(u)$ takes the form

$$(I_2^i H_i)(u) = \int_{S^2} \frac{\partial\gamma(X^i(u), X^i(v))}{\partial x} n^-(X^i(u))(1+2\epsilon r_i(v)+ O_{C^\alpha(S^2)}\left(\epsilon^2 \|r_i\|^2_{C^{1+\alpha}(S^2)}\right)(v))H_i(v)dv$$

$$= \int_{S^2} \langle \frac{\partial\gamma(\xi_i,\xi_i)}{\partial x}, \epsilon\rho_i u\rangle H_i(v)dv + \int_{S^2} \langle \frac{\partial\gamma(\xi_i,\xi_i)}{\partial x}, \epsilon\rho_i u\rangle \langle \frac{\partial\gamma(\xi_i,\xi_i)}{\partial y}, \epsilon\rho_i v\rangle H_i(v)dv$$

$$+ \int_{S^2} \langle \frac{\partial\gamma(\xi_i,\xi_i)}{\partial x}, \epsilon\rho_i u\rangle^2 H_i(v)dv + \frac{1}{2}\int_{S^2} \langle \frac{\partial\gamma(\xi_i,\xi_i)}{\partial x}, \epsilon\rho_i u\rangle^3 H_i(v)dv$$

$$+ \int_{S^2} \langle \frac{\partial\gamma(\xi_i,\xi_i)}{\partial x}, \epsilon\rho_i u\rangle^2 \langle \frac{\partial\gamma(\xi_i,\xi_i)}{\partial y}, \epsilon\rho_i v\rangle H_i(v)dv + \frac{1}{2}\int_{S^2} \langle \frac{\partial\gamma(\xi_i,\xi_i)}{\partial y}, \epsilon\rho_i v\rangle^2 \langle \frac{\partial\gamma(\xi_i,\xi_i)}{\partial x}, \epsilon\rho_i u\rangle H_i(v)dv$$

$$+O_{C^{1+\alpha}(S^2)}(\epsilon^3 {\rho_i}^3 \frac{\partial\gamma(\xi_i,\xi_i)}{\partial x} + \epsilon\|r_i\|_{C^{1+\alpha}(S^2)} \frac{\partial\gamma(\xi_i,\xi_i)}{\partial x})\|H_i\|_{C^\alpha(S^2)}. \tag{3.20}$$

From the above analysis and in particular from Eqs. (3.13), (3.15), (3.18) and (3.20)

$$\int_{\Gamma_i} \frac{\partial G(x,y)}{\partial n_x} H_i(u^i(y))dy = \int_{S^2} \frac{1}{4\pi|u-v|} H_i(v)dv + \epsilon\rho_i (I^{ii} H_i)(u) \tag{3.21}$$

where I^{ii} is a linear operator that satisfies

$$\|I^{ii} H_i\|_{C^{1+\alpha}(S^2)} \leq C\|Hi\|_{C^\alpha(S^2)}. \tag{3.22}$$

Step B

We now consider the case $x \in \Gamma_i, h \neq i$ in Eq. (3.6). For $h \neq i$ and $x = X^i(u)$ both integrals on the right hand side of (3.6) have, as functions of $u \in S^2$, the same smoothness as X^i. We will analyze how the $C^{1+\alpha}(S^2)$ norm of these functions depends on ϵ, ρ, r. We can write

$$\int_{\Gamma_h} \frac{\partial}{\partial n_x}\left(\frac{1}{4\pi|X^i(u)-y|}\right) H_h(u^h(y))dy = -\int_{\Gamma_h} \frac{\langle X^i(u)-y, n^-(x)\rangle}{4\pi|X^i(u)-y|^4}|X^i(u)-y|H_h(u^h(y))dy$$

$$= -\epsilon^2 {\rho_h}^2 \int_{S^2} \frac{\langle X^i(u)-X^h(v), n^-(X^i(u))\rangle}{4\pi|X^i(u)-X^h(v)|^4}|X^i(u)-X^h(v)|$$

$$\cdot(1+2\epsilon r_h(v) + O_{C^\alpha(S^2)}\left(\epsilon^2 \|r_h\|^2_{C^{1+\alpha}(S^2)}\right)(v))H_h(v)dv$$

$$= -\epsilon^2 {\rho_h}^2 \int_{S^2} \left[\frac{\langle \xi_i-\xi_h, \epsilon\rho_i u\rangle}{4\pi|\xi_i-\xi_h|^3} + \frac{\langle \epsilon\rho_i u - \epsilon\rho_h v, \epsilon\rho_i u\rangle}{4\pi|\xi_i-\xi_h|^3}\right]\left(1 - 3\left\langle \frac{\xi_i-\xi_h}{|\xi_i-\xi_h|^2}, \epsilon\rho_i u - \epsilon\rho_h v\right\rangle\right)$$

$$\cdot(1+2\epsilon r_h(v) + O_{C^\alpha(S^2)}\left(\epsilon^2 \|r_h\|^2_{C^{1+\alpha}(S^2)}\right)(v))H_h(v)dv$$

$$= -\epsilon^2{\rho_h}^2\frac{\langle \xi_i-\xi_h,\epsilon\rho_i u\rangle}{4\pi|\xi_i-\xi_h|^3}\int_{S^2}(1+2\epsilon r_h(v)+\ O_{C^\alpha(S^2)}\left(\epsilon^2\left\|r_h\right\|^2_{C^{1+\alpha}(S^2)}\right)(v))H_h(v)dv$$

$$+\epsilon^2{\rho_h}^2\frac{\langle \xi_i-\xi_h,\epsilon\rho_i u\rangle}{4\pi|\xi_i-\xi_h|^3}\int_{S^2}3\left\langle\frac{\xi_i-\xi_h}{|\xi_i-\xi_h|^2},\epsilon\rho_i u-\epsilon\rho_h v\right\rangle$$

$$\cdot(1+2\epsilon r_h(v)+\ O_{C^\alpha(S^2)}\left(\epsilon^2\left\|r_h\right\|^2_{C^{1+\alpha}(S^2)}\right)(v))H_h(v)dv$$

$$-\epsilon^2{\rho_h}^2\int_{S^2}\frac{\langle\epsilon\rho_i u-\epsilon\rho_h v,\epsilon\rho_i u\rangle}{4\pi|\xi_i-\xi_h|^3}(1+2\epsilon r_h(v)+\ O_{C^\alpha(S^2)}\left(\epsilon^2\left\|r_h\right\|^2_{C^{1+\alpha}(S^2)}\right)(v))H_h(v)dv$$

$$+\epsilon^2{\rho_h}^2\int_{S^2}\frac{\langle\epsilon\rho_i u-\epsilon\rho_h v,\epsilon\rho_i u\rangle}{4\pi|\xi_i-\xi_h|^3}\cdot 3\left\langle\frac{\xi_i-\xi_h}{|\xi_i-\xi_h|^2},\epsilon\rho_i u-\epsilon\rho_h v\right\rangle$$

$$\cdot(1+2\epsilon r_h(v)+\ O_{C^\alpha(S^2)}\left(\epsilon^2\left\|r_h\right\|^2_{C^{1+\alpha}(S^2)}\right)(v))H_h(v)dv$$

$$= -\epsilon^2{\rho_h}^2\frac{\langle \xi_i-\xi_h,\epsilon\rho_i u\rangle}{4\pi|\xi_i-\xi_h|^3}\int_{S^2}H_h(v)dv-\int_{S^2}\frac{2\epsilon^3{\rho_h}^2 r_h(v)\langle\xi_i-\xi_h,\epsilon\rho_i u\rangle}{4\pi|\xi_i-\xi_h|^3}H_h(v)dv$$

$$+\frac{\epsilon^2{\rho_h}^2}{4\pi|\xi_i-\xi_h|^3}3\left\langle\epsilon\rho_i u,\frac{\xi_i-\xi_h}{|\xi_i-\xi_h|}\right\rangle^2\int_{S^2}H_h(v)dv-\frac{\epsilon^2{\rho_h}^2\langle\xi_i-\xi_h,\epsilon\rho_i u\rangle}{4\pi|\xi_i-\xi_h|^4}\int_{S^2}3\left\langle\epsilon\rho_h v,\frac{\xi_i-\xi_h}{|\xi_i-\xi_h|}\right\rangle H_h(v)dv$$

$$+\frac{2\epsilon^2{\rho_h}^2\langle\xi_i-\xi_h,\epsilon\rho_i u\rangle}{4\pi|\xi_i-\xi_h|^3}\int_{S^2}3\left\langle\epsilon\rho_i u-\epsilon\rho_h v,\frac{\xi_i-\xi_h}{|\xi_i-\xi_h|}\right\rangle\epsilon r_h H_h(v)dv$$

$$-\frac{\epsilon^2{\rho_h}^2}{4\pi|\xi_i-\xi_h|^3}\int_{S^2}\langle\epsilon\rho_i u-\epsilon\rho_h v,\epsilon\rho_i u\rangle H_h(v)dv$$

$$-\frac{2\epsilon^2\rho_h^2}{4\pi|\xi_i-\xi_h|^3}\int_{S^2}\langle\epsilon\rho_i u-\epsilon\rho_h v,\epsilon\rho_i u\rangle\cdot\epsilon r_h H_h(v)dv+\epsilon^2{\rho_h}^2 O_{C^{1+\alpha}(S^2)}\left(\frac{\epsilon^2\|r_h\|^2_{C^{1+\alpha}(S^2)}}{|\xi_i-\xi_h|^3}+\frac{\epsilon^2\|r_h\|^2_{C^{1+\alpha}(S^2)}}{|\xi_i-\xi_h|^4}\right.$$

$$\left.+\frac{\epsilon^2\|r_h\|_{C^{1+\alpha}(S^2)}}{|\xi_i-\xi_h|^4}(\rho_i+\rho_h)+\epsilon^2\left(\frac{\rho_i^2+\rho_h^2}{|\xi_i-\xi_h|^5}+\frac{\rho_i\|r_i\|_{C^{1+\alpha}(S^2)}+\rho_h\|r_h\|_{C^{1+\alpha}(S^2)}}{|\xi_i-\xi_h|^4}\right)\right)\|H_h\|_{C^\alpha(S^2)}. \tag{3.23}$$

For the other integral on the right hand side of Eq. (3.6) we have

$$\int_{\Gamma_h}\frac{\partial\gamma(X^i(u),y)}{\partial n_x}H_h(u^h(y))dy=\epsilon^2{\rho_h}^2\int_{\partial\Omega_h}\frac{\partial\gamma(X^i(u),y)}{\partial x}n^-(X^i(u))H_h(u^h(y))dy$$

$$=\epsilon^2{\rho_h}^2\int_{S^2}\frac{\partial\gamma(X^i(u),X^h(v))}{\partial x}n^-(X^i(u))(1+2\epsilon r_h(v)+O_{C^\alpha(S^2)}\left(\epsilon^2\left\|r_h\right\|^2_{C^{1+\alpha}(S^2)}\right)(v))H_h(v)dv$$

$$=\epsilon^2{\rho_h}^2\int_{S^2}\langle\frac{\partial\gamma(\xi_i,\xi_h)}{\partial x},\epsilon\rho_i u\rangle H_h(v)dv+\epsilon^2{\rho_h}^2\int_{S^2}\langle\frac{\partial\gamma(\xi_i,\xi_h)}{\partial x},\epsilon\rho_i u\rangle\langle\frac{\partial\gamma(\xi_i,\xi_h)}{\partial y},\epsilon\rho_h v\rangle H_h(v)dv$$

$$+\epsilon^2{\rho_h}^2\int_{S^2}\langle\frac{\partial\gamma(\xi_i,\xi_h)}{\partial x},\epsilon\rho_i u\rangle^2 H_h(v)dv+\frac{1}{2}\epsilon^2{\rho_h}^2\int_{S^2}\langle\frac{\partial\gamma(\xi_i,\xi_h)}{\partial y},\epsilon\rho_h v\rangle^2\langle\frac{\partial\gamma(\xi_i,\xi_h)}{\partial x},\epsilon\rho_i u\rangle H_h(v)dv$$

$$+\epsilon^2{\rho_h}^2\int_{S^2}\langle\frac{\partial\gamma(\xi_i,\xi_h)}{\partial x},\epsilon\rho_i u\rangle^2\langle\frac{\partial\gamma(\xi_i,\xi_h)}{\partial y},\epsilon\rho_h v\rangle H_h(v)dv+\frac{1}{2}\epsilon^2{\rho_h}^2\int_{S^2}\langle\frac{\partial\gamma(\xi_i,\xi_h)}{\partial x},\epsilon\rho_i u\rangle^3 H_h(v)dv$$

$$+\epsilon^2{\rho_h}^2O_{C^{1+\alpha}(S^2)}(\epsilon^3{\rho_i}^3\frac{\partial\gamma(\xi_i,\xi_h)}{\partial x}+\epsilon\|r_h\|_{C^{1+\alpha}(S^2)}\frac{\partial\gamma(\xi_i,\xi_h)}{\partial x})\|H_h\|_{C^\alpha(S^2)}. \tag{3.24}$$

From Eqs. (3.23), (3.24) it follows that

$$\int_{\Gamma_h}\frac{\partial G(x,y)}{\partial n_x}H_h(u^h(y))dy=\epsilon\rho_h(I^{ih}H_h)(u) \tag{3.25}$$

where I^{ih} is a linear operator that satisfies

$$\|I^{ih}H_h\|_{C^{1+\alpha}(S^2)}\le C\|H_h\|_{C^\alpha(S^2)}. \tag{3.26}$$

Under the assumption that $\epsilon\|r_h\|_{C^{1+\alpha}(S^2)}<\delta$ and $\epsilon\rho_h<\delta$, we have

$$V_i=\frac{1}{2}H_i-\sum_{h=1}^N\epsilon\rho_h I^{ih}H_h. \tag{3.27}$$

The above system has a unique solution which can be computed by iteration and moreover from Eqs. (3.13), (3.15), (3.18), (3.21) it follows that

$$\Big\|I^{ii}H_i-\frac{1}{2}\int_{S^2}\frac{3r_i(v)-r_i(\cdot)}{4\pi|\cdot-v|}H_i(v)dv$$

$$-\int_{S^2}\langle\frac{\partial\gamma(\xi_i,\xi_i)}{\partial x},\epsilon\rho_i u\rangle H_i(v)dv-\int_{S^2}\langle\frac{\partial\gamma(\xi_i,\xi_i)}{\partial x},\epsilon\rho_i u\rangle\langle\frac{\partial\gamma(\xi_i,\xi_i)}{\partial y},\epsilon\rho_i v\rangle H_i(v)dv$$

$$-\int_{S^2}\langle\frac{\partial\gamma(\xi_i,\xi_i)}{\partial x},\epsilon\rho_i u\rangle^2 H_i(v)dv-\frac{1}{2}\int_{S^2}\langle\frac{\partial\gamma(\xi_i,\xi_i)}{\partial x},\epsilon\rho_i u\rangle^3 H_i(v)dv$$

$$-\int_{S^2}\langle\frac{\partial\gamma(\xi_i,\xi_i)}{\partial x},\epsilon\rho_i u\rangle^2\langle\frac{\partial\gamma(\xi_i,\xi_i)}{\partial y},\epsilon\rho_i v\rangle H_i(v)dv-\frac{1}{2}\int_{S^2}\langle\frac{\partial\gamma(\xi_i,\xi_i)}{\partial y},\epsilon\rho_i v\rangle^2\langle\frac{\partial\gamma(\xi_i,\xi_i)}{\partial x},\epsilon\rho_i u\rangle H_i(v)dv\Big\|$$

$$=O_{C^{1+\alpha}(S^2)}(\epsilon^3{\rho_i}^3\frac{\partial\gamma(\xi_i,\xi_i)}{\partial x}+\epsilon\|r_i\|_{C^{1+\alpha}(S^2)}\frac{\partial\gamma(\xi_i,\xi_i)}{\partial x})\|H_i\|_{C^\alpha(S^2)} \tag{3.28}$$

and Eqs. (3.23), (3.24), (3.25) imply that

$$\Big\|I^{ih}H_h+\frac{\epsilon\rho_h\langle\xi_i-\xi_h,\epsilon\rho_i u\rangle}{4\pi|\xi_i-\xi_h|^3}\int_{S^2}H_h(v)dv+\int_{S^2}\frac{2\epsilon^2\rho_h r_h(v)\langle\xi_i-\xi_h,\epsilon\rho_i u\rangle}{4\pi|\xi_i-\xi_h|^3}H_h(v)dv$$

$$-\frac{\epsilon\rho_h}{4\pi|\xi_i-\xi_h|^3}3\left\langle\epsilon\rho_i u,\frac{\xi_i-\xi_h}{|\xi_i-\xi_h|}\right\rangle^2\int_{S^2}H_h(v)dv+\frac{\epsilon\rho_h\langle\xi_i-\xi_h,\epsilon\rho_i u\rangle}{4\pi|\xi_i-\xi_h|^4}\int_{S^2}3\left\langle\epsilon\rho_h v,\frac{\xi_i-\xi_h}{|\xi_i-\xi_h|}\right\rangle H_h(v)dv$$

$$-\frac{2\epsilon\rho_h\langle\xi_i-\xi_h,\epsilon\rho_i u\rangle}{4\pi|\xi_i-\xi_h|^3}\int_{S^2}3\left\langle\epsilon\rho_i u-\epsilon\rho_h v,\frac{\xi_i-\xi_h}{|\xi_i-\xi_h|}\right\rangle\epsilon r_h H_h(v)dv$$

$$+\frac{\epsilon\rho_h}{4\pi|\xi_i-\xi_h|^3}\int_{S^2}\langle\epsilon\rho_i u-\epsilon\rho_h v,\epsilon\rho_i u\rangle H_h(v)dv+\frac{2\epsilon\rho_h}{4\pi|\xi_i-\xi_h|^3}\int_{S^2}\langle\epsilon\rho_i u-\epsilon\rho_h v,\epsilon\rho_i u\rangle\cdot\epsilon r_h H_h(v)dv$$

$$-\epsilon\rho_h\int_{S^2}\langle\frac{\partial\gamma(\xi_i,\xi_h)}{\partial x},\epsilon\rho_i u\rangle H_h(v)dv-\epsilon\rho_h\int_{S^2}\langle\frac{\partial\gamma(\xi_i,\xi_h)}{\partial x},\epsilon\rho_i u\rangle\langle\frac{\partial\gamma(\xi_i,\xi_h)}{\partial y},\epsilon\rho_h v\rangle H_h(v)dv$$

$$-\epsilon\rho_h\int_{S^2}\langle\frac{\partial\gamma(\xi_i,\xi_h)}{\partial x},\epsilon\rho_i u\rangle^2 H_h(v)dv-\frac{1}{2}\epsilon\rho_h\int_{S^2}\langle\frac{\partial\gamma(\xi_i,\xi_h)}{\partial y},\epsilon\rho_h v\rangle^2\langle\frac{\partial\gamma(\xi_i,\xi_h)}{\partial x},\epsilon\rho_i u\rangle H_h(v)dv$$

$$-\epsilon\rho_h\int_{S^2}\langle\frac{\partial\gamma(\xi_i,\xi_h)}{\partial x},\epsilon\rho_i u\rangle^2\langle\frac{\partial\gamma(\xi_i,\xi_h)}{\partial y},\epsilon\rho_h v\rangle H_h(v)dv-\frac{1}{2}\epsilon\rho_h\int_{S^2}\langle\frac{\partial\gamma(\xi_i,\xi_h)}{\partial x},\epsilon\rho_i u\rangle^3 H_h(v)dv\Bigg\|$$

$$=\epsilon\rho_h O_{C^{1+\alpha}(S^2)}\Bigg(\frac{\epsilon^2\|r_h\|^2_{C^{1+\alpha}(S^2)}}{|\xi_i-\xi_h|^3}+\frac{\epsilon^2\|r_h\|^2_{C^{1+\alpha}(S^2)}}{|\xi_i-\xi_h|^4}$$

$$+\frac{\epsilon^2\|r_h\|_{C^{1+\alpha}(S^2)}}{|\xi_i-\xi_h|^4}(\rho_i+\rho_h)+\epsilon^2\left(\frac{\rho_i^2+\rho_h^2}{|\xi_i-\xi_h|^5}+\frac{\rho_i\|r_i\|_{C^{1+\alpha}(S^2)}+\rho_h\|r_h\|_{C^{1+\alpha}(S^2)}}{|\xi_i-\xi_h|^4}\right)\Bigg)\|H_h\|_{C^\alpha(S^2)}$$

$$+\epsilon^2{\rho_h}^2 O_{C^{1+\alpha}(S^2)}(\epsilon^3{\rho_i}^3\frac{\partial\gamma(\xi_i,\xi_h)}{\partial x}+\epsilon\|r_h\|_{C^{1+\alpha}(S^2)}\frac{\partial\gamma(\xi_i,\xi_h)}{\partial x})\|H_h\|_{C^\alpha(S^2)}. \tag{3.29}$$

Inserting the expressions of $I^{ii}H_i, I^{ih}H_h$ into Eq. (3.27) we conclude to Eq. (3.5). □

We are going to show next that the integral equation (3.2) is equivalent to a system of evolution equations in terms of $\rho=(\rho_1,\ldots,\rho_N)$, $r=(r_1,\ldots,r_N)$ and $\xi=(\xi_1,\ldots,\xi_N)$ with $N+N+3N$ unknowns. We first provide the following proposition

Proposition 3.3. *Equation (3.5) implies*

$$V_i=\frac{\epsilon}{\epsilon\rho_i}Lr_i+\sum_{h\neq i}\frac{\epsilon\rho_h\langle\xi_i-\xi_h,\epsilon\rho_i u\rangle}{4\pi|\xi_i-\xi_h|^3}-\sum_{h\neq i}\epsilon\rho_h\epsilon\rho_i\frac{1}{4\pi|\xi_i-\xi_h|^3}3\left\langle u,\frac{\xi_i-\xi_h}{|\xi_i-\xi_h|}\right\rangle^2$$

$$+\sum_{h\neq i}\epsilon^2{\rho_h}^2\frac{\langle\xi_i-\xi_h,\epsilon\rho_i u\rangle}{4\pi|\xi_i-\xi_h|^2}\int_{S^2}3\left\langle v,\frac{\xi_i-\xi_h}{|\xi_i-\xi_h|}\right\rangle dv-\epsilon\rho_h\int_{S^2}\langle\frac{\partial\gamma(\xi_i,\xi_h)}{\partial x},\epsilon\rho_i u\rangle dv$$

$$+\frac{1}{\epsilon\rho_i}O_{C^{1+\alpha}(S^2)}(\epsilon^2\|r_i\|^2_{C^{1+\alpha}(S^2)}+\epsilon\|r_i\|_{C^{1+\alpha}(S^2)}+\frac{\epsilon\rho_h}{|\xi_i-\xi_h|^4}\epsilon\|r_i\|_{C^{1+\alpha}(S^2)}+\epsilon^2\rho_h^2\frac{\partial}{\partial x}\frac{\partial\gamma(\xi_i,\xi_h)}{\partial x}). \tag{3.30}$$

Proof. By utilizing the expression for H given in Proposition 2.1 that is

$$H|_{\Gamma_i} = \frac{1}{\epsilon\rho_i}(1 - \epsilon L r_i + B)$$

and as long as L is a second order operator satisfying the estimates

$$\|\int_{S^2} \frac{\frac{3}{2}r_i(v) - \frac{1}{2}r_i(\cdot)}{4\pi|\cdot - v|}\epsilon L r_i(v)dv\|_{C^\alpha(S^2)} \leq C\|r_i\|^2_{C^{3+\alpha}(S^2)}$$

$$\|\epsilon L r_i(v)dv\|_{C^\alpha(S^2)} \leq C\|r_i\|^2_{C^{3+\alpha}(S^2)}.$$

Then equation (3.30) follows from (3.5). ☐

Proposition 3.4. *There is $\bar{\epsilon} > 0$ such that for $0 < \epsilon < \bar{\epsilon}$ the integral equation (3.2) is equivalent to the following system of evolution equations*

$$\begin{cases} \epsilon \dfrac{d\rho_i}{dt} = f_i^\rho(\rho, \xi, r) \\[2ex] \dfrac{dr_i}{dt} = \dfrac{1}{\epsilon^2 {\rho_i}^2} L r_i + f_i^r(\rho, \xi, r) \\[2ex] \dfrac{d\xi_i}{dt} = \displaystyle\sum_{h \neq i} \epsilon^2 \rho_h \rho_i \frac{\xi_i - \xi_h}{|\xi_i - \xi_h|^3} - \sum_{h=1}^N 4\pi\epsilon\rho_h\epsilon\rho_i \frac{\partial\gamma(\xi_i, \xi_h)}{\partial x} + f_i^\xi(\rho, \xi, r) \end{cases} \tag{3.31}$$

where $f_i^\rho(\rho, \xi, r)$, $f_i^r(\rho, \xi, r)$, $f_i^\xi(\rho, \xi, r)$ are smooth functions of $\rho = (\rho_1, \ldots, \rho_N)$, $\xi = (\xi_1, \ldots, \xi_N)$ and $r = (r_1, \ldots, r_N)$, $r \in C^{3+\alpha}(S^2)$ satisfying the following estimates:

$$f_i^\rho(\rho, \xi, r) = O_{C^{1+\alpha}(S^2)}\left(\frac{\epsilon\rho_h}{|\xi_i - \xi_h|} + \frac{\epsilon\rho_h}{|\xi_i - \xi_h|}\epsilon\|r_h\|_{C^{1+\alpha}(S^2)}\right)$$

$$+\frac{1}{\epsilon\rho_i} O_{C^{1+\alpha}(S^2)}(\epsilon^2\|r_i\|^2_{C^{1+\alpha}(S^2)})$$

$$f_i^r(\rho, \xi, r) = \frac{1}{\epsilon\rho_i} O_{C^{1+\alpha}(S^2)}\left(\epsilon^2\|r_i\|^2_{C^{1+\alpha}(S^2)} + \frac{\epsilon\rho_h}{|\xi_i - \xi_h|}\right)$$

$$f_i^\xi(\rho, \xi, r) = O_{C^{1+\alpha}(S^2)}\Bigg(\epsilon^2\|r_h\|_{C^{1+\alpha}(S^2)} + \frac{\epsilon^2\|r_h\|^2_{C^{1+\alpha}(S^2)}}{|\xi_i - \xi_h|^3} + \frac{\epsilon^2\rho_i\|r_i\|_{C^{1+\alpha}(S^2)} + \epsilon^2\rho_h\|r_h\|_{C^{1+\alpha}(S^2)}}{|\xi_i - \xi_h|^4}$$

$$+\epsilon^3\rho_h^3 \frac{\partial}{\partial x}\frac{\partial\gamma(\xi_i, \xi_h)}{\partial x}\Bigg).$$

Proof. Eqs. (3.31) are obtained from Proposition 3.1 when one identifies Z with the right hand side of Eqs. (3.30). Moreover, we note that under the standing assumption $\epsilon\|r_i\|_{C^{1+\alpha}(S^2)} < \delta$, the expression $\epsilon\|r_i\|_{C^{1+\alpha}(S^2)}\sum_{h=1}^{3}|\langle Z, w_h\rangle_{L^2(S^2)}|$ is estimated by the right hand side of Eqs. (3.30). We refer to [3] for further details. □

4. The ρ, r, ξ estimates

In this section, we prove that r is bounded by analyzing the following system

$$\begin{cases} \epsilon \dfrac{d\rho_i}{dt} = f_i^{\rho}(\rho,\xi,r) \\ \dfrac{dr_i}{dt} = \dfrac{1}{\epsilon^2 {\rho_i}^2} L r_i + f_i^r(\rho,\xi,r) \\ \dfrac{d\xi_i}{dt} = \displaystyle\sum_{h\neq i} \epsilon^2 \rho_h \rho_i \frac{\xi_i - \xi_h}{|\xi_i-\xi_h|^3} - \sum_{h=1}^{N} 4\pi\epsilon\rho_h\epsilon\rho_i \frac{\partial \gamma(\xi_i,\xi_h)}{\partial x} + f_i^{\xi}(\rho,\xi,r) \end{cases} \tag{4.1}$$

with initial data $\rho_i(0), r_i(0), \xi_i(0)$. To obtain this bound on r we use a suitable functional-analytic setting for the evolutionary equation for r in (3.31). We will use the optimal regularity theory of Da Prato and Grisvard [17], [8] which provides the appropriate semi-group setting and makes available the variation of constants formula.

Let $h^{k+\alpha}$ be the "little Hölder" space defined as the completion of the set of C^∞ functions with respect to $C^{k+\alpha}$ norm. It is known that the operator L has the optimal regularity property with respect to the pair $E_0 = h^\alpha(S^2)$, $E_1 = h^{2+\alpha}(S^2)$, [12] and it holds that the operator L is the generator of an analytic semigroup while the following estimate holds true:

$$\sup_{[0,\bar{t}]} \| \int_0^t e^{L(t-\vartheta)} g(\vartheta) d\vartheta \|_{E_1} \le c_{\bar{t}} \sup_{t\in[0,\bar{t}]} \|g(t)\|_{E_0} \tag{4.2}$$

with $g : [0,\bar{t}] \to E_0$ is continuous function.
In order to utilize the above estimate, we need estimates of the type

$$\|f_i^r(\rho,\xi,r)\|_{E_0} \le C\|r\|_{E_1}, \quad i = 1,\dots,N.$$

For obtaining such Hölder estimates, we use Th.2I in [15]. That theorem covers a class of operators

$$U(x) = \int_{\partial\Omega} E(y-x) f(y) dy$$

with E modelled after $\frac{\partial}{\partial x_i}(\frac{1}{|x-y|})$ and provides estimates of the type

$$\|U(\cdot)\|_{C^{1+\alpha}(\Omega)} \le \|f(\cdot)\|_{C^\alpha(\partial\Omega)}.$$

Lemma 4.1. *There exist constants $\mu > 0, M > 0, \beta \in (0,1)$ such that the semigroup generated by L in E_1 satisfies*

$$\|e^{Ls}\varphi\|_{C^{2+\alpha}} \leq Me^{-\mu s}\|\varphi\|_{C^{2+\alpha}(S^2)}, \quad \varphi \in E_1 \tag{4.3}$$

$$\|e^{Ls}\varphi\|_{C^{2+\alpha}(S^2)} \leq \frac{M}{s^\beta}e^{-\mu s}\|\varphi\|_{C^{2+\alpha}(S^2)},\ \varphi \in h^{1+\alpha}(S^2)\cap E_0, \beta = \frac{1}{3}. \tag{4.4}$$

Moreover if $\varphi : (0,\bar{s}] \to E_0$ is continuous

$$\sup_{0<s\leq\bar{s}} \left\| \int_0^s e^{L(s-\sigma)}\varphi(\sigma)\,ds \right\|_{C^{2+\alpha}(S^2)} \leq \bar{C} \sup_{0<s\leq\bar{s}} \|\varphi(s)\|_{C^\alpha(S^2)}, \tag{4.5}$$

where $\bar{C} > 0$ is a constant independent of $\bar{s}$.

Proof. Estimates (4.3)-(4.5) are well known consequences of the fact that L generates an analytic semigroup on E_1, and of basic interpolation properties of the "little" Hölder spaces. Estimate (4.4) states that L belongs to $M_1(E_0, E_1)$. We note that generally for an operator in $M_1(E_0, E_1)$ the inequality (4.5) holds with $\bar{C}$ replaced by a $C_{\bar{s}}$ which grows with $\bar{s}$. In our special case we can take a $\bar{C}$ independent of $\bar{s}$ because the spectrum of L is bounded above by a negative number. □

In the following we assume, as we can, that the constant M in Lemma (4.1) satisfies $M > 1$.

In terms of simplicity in what follows we define

$$s_i := \int_0^t \frac{dt'}{\epsilon^2\rho_i^2(t)}, \quad i = 1,\ldots,N.$$

So, after this transformation Eqs. (3.31) take the form

$$\begin{cases} \dfrac{d\rho_i}{ds_i} = g_i^\rho(\rho,\xi,r) \\ \\ \dfrac{dr_i}{ds_i} = Lr_i + \epsilon g_i^r(\rho,\xi,r) \\ \\ \dfrac{d\xi_i}{ds_i} = \epsilon^2\rho_i^2\left[\displaystyle\sum_{h\neq i}\epsilon^2\rho_h\rho_i\frac{\xi_i-\xi_h}{|\xi_i-\xi_h|^3} - \sum_{h=1}^N 4\pi\epsilon\rho_h\epsilon\rho_i\frac{\partial\gamma(\xi_i,\xi_h)}{\partial x}\right] + g_i^\xi(\rho,\xi,r) \end{cases} \tag{4.6}$$

where

$$g_i^\rho(\rho,\xi,r) = \epsilon\rho_i^2 f_i^\rho(\rho,\xi,r), \quad g_i^r(\rho,\xi,r) = \epsilon\rho_i^2 f_i^r(\rho,\xi,r), \quad g_i^\xi(\rho,\xi,r) = \epsilon^2\rho_i^2 f_i^\xi(\rho,\xi,r).$$

Proposition 4.2. *Assume $N \geq 2$. Then there exists $\bar{\epsilon} > 0$ and $\zeta > 0$, independent of ϵ such that, for $\epsilon \in (0,\bar{\epsilon})$ the following inequality holds true*

$$\|r_i(t)\|_{C^{2+\alpha}(S^2)} < \zeta.$$

Proof. We need to obtain estimates on r for $r_t = Lr + f(r(t))$.
It is known that if r is a solution of $r_t = Lr + f(r(t))$ then r satisfies the "variation of constants formula"

$$r(t) = e^{-Lt}r(0) + \int_0^t e^{-L(t-s)}f(r(s))ds.$$

Let $\zeta > 0$ any number that satisfies

$$\|r_i(0)\|_{C^{2+\alpha}(S^2)} < \zeta, \quad i = 1, \ldots, N.$$

Then from Eq. (4.4) it follows that

$$\sup_{0<s\leq s_i} \Big\| \int_0^s e^{L(s-\sigma)}\epsilon g_i^r ds\Big\|_{C^{2+\alpha}(S^2)} \leq \epsilon C_1 \zeta \sup_{0<s\leq s_i} \|r_i(t(s))\|_{C^{2+\alpha}(S^2)} \tag{4.7}$$

where C_1 a suitably chosen constant.
From the variation of constants formula applied to Eq. $(4.6)_2$, it follows via Eq. (4.7) and Eq. (4.3) that

$$\sup_{0<s\leq s_i} \|r_i(t(s))\|_{C^{2+\alpha}(S^2)} \leq M\|r_i(0)\|_{C^{2+\alpha}(S^2)} + \epsilon C_1(1+\zeta) + \epsilon C_1 \zeta \sup_{0<s\leq s_i} \|r_i(t(s))\|_{C^{2+\alpha}(S^2)}. \tag{4.8}$$

If we set

$$z_i = \sup_{0<s\leq s_i} \|r_i(t(s))\|_{C^{2+\alpha}(S^2)}, \quad z_i(0) = \|r_i(0)\|_{C^{2+\alpha}(S^2)},$$

the above equation takes the form

$$z_i \leq M z_i(0) + \epsilon C_1(1+\zeta) + \epsilon C_1 \zeta z_i. \tag{4.9}$$

We make a specific choice for ζ

$$\zeta = 8M(2M)^{\bar{k}} \max_i \|r_i(0)\|_{C^{2+\alpha}(S^2)} + 1 \tag{4.10}$$

and we assume $\bar{\epsilon} > 0$ so small such that for $\epsilon \in (0, \bar{\epsilon})$

$$2\epsilon C_1 \zeta < \frac{1}{4} \tag{4.11}$$

$$2\epsilon C_1 \sum_{k=0}^{\bar{k}-1} (2M)^k < \frac{1}{8M}. \tag{4.12}$$

From Eqs. (4.9), (4.11) it follows that

$$z_i < 2M z_i(0) + 2\epsilon C_1(1+\zeta)$$

equivalently

$$z_i < (2M)^2 z_i(0) + 2\epsilon C_1(1+\zeta)(2M+1)$$

and by iterating this procedure we get

$$z_i < (2M)^k z_i(0) + 2\epsilon C_1(1+\zeta)\sum_{h=0}^{k-1}(2M)^h$$

$$\leq (2M)^{\bar{k}} \max_i \|r_i(0)\|_{C^{2+\alpha}(S^2)} + 2\epsilon C_1 \sum_{h=0}^{\bar{k}-1}(2M)^h + 2\epsilon C_1 \sum_{h=0}^{\bar{k}-1}(2M)\zeta$$

$$\leq (2M)^{\bar{k}} \max_i \|r_i(0)\|_{C^{2+\alpha}(S^2)} + \frac{1}{8M} + \frac{\zeta}{8M} < \frac{\zeta}{4M}$$

where the definition of ζ has been utilized. So,

$$z_i(s_i) < \frac{\zeta}{4M} < \zeta, \quad i = 1, \ldots, N$$

which implies that $\|r_i(t)\|_{C^{2+\alpha}(S^2)} < \zeta$.

The proof of the proposition is now complete.

$\square$

References

[1] N.D. Alikakos, G. Fusco, The Equations of Ostwald Ripening for Dilute Systems, Journal of Statistical Physics, 95, 851-866, (1999).

[2] N.D. Alikakos, G. Fusco, The Effect of Distribution in Space in Ostwald Ripening. Centre de Recherches Mathematiques CRM Proceedings and Lecture Notes, Vol. 27, (2001).

[3] N.D. Alikakos, G. Fusco, Ostwald ripening for dilute systems under quasistationary dynamics, Comm. Math. Phys. 238, 429-479, (2003).

[4] N.D. Alikakos, G. Fusco, G. Karali, The effect of the geometry of the particle distribution in Ostwald ripening, Comm. Math. Phys. 238, 480-488, (2003).

[5] H. Amann, Linear and quasilinear parabolic. I. Abstract linear theory, Monogr. Mathem., 89, Birkhäuser Verlag, Basel, (1995).

[6] S.B. Angenent, Nonlinear analytic semigroups, Proc. Roy. Soc. Edinburgh A 115, 91-107, (1990).

[7] X. Chen, The Hele-Shaw problem and area-preserving curve shortening motion, Arch. Rational Mech. Anal. 123, 117-151, (1993).

[8] J. Escher and G. Simonett, Classical solutions for Hele Shaw models with surface tension, Adv. Differential Equations 2, 619-642, (1997).

[9] G. Folland, Introduction to Partial Differential Equations, Princeton University Press and University of Tokyo Press, Princeton N.J., 1976.

[10] D. Gilbarg, N. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer.

[11] D. Henry, Geometric Theory of Semilinear Parabolic Equations, Lecture Notes in Mathematics, Springer-Verlag, 1981.

[12] K. Sakamoto, Interface Equations with Nonlocal Effects, Nonlinear Diffusive Systems: Dynamics and Asymptotics, RIMS Kokyuroku 1178, Research Institute for Mathematical Sciences Kyoto University, Kyoto, Japan, December 2000.

[13] I.M. Lifschitz and V.V. Slyozov, The kinetics of precipitation from supersaturated solid solutions, J. Phys. Chem. Solids 19, 35-50, (1961).

[14] A. Lunardi, Analytic semigroups and optimal regularity for parabolic problems, Progr. Nonlinear Differential Equations Appl., 16, Birkhäuser Verlag, Basel, (1995).

[15] C. Miranda, Sulle proprieta di regolarita di certe trasformazioni integrali, Atti Accad. Naz. Lincei Mem. Cl. Sci. Fis. Mat. Natur. Sez. I (8) 7, 303-336, (1965).
[16] W. Ostwald, Z. Phys. Chem. 37, 385, (1901).
[17] G. da Prato and P. Grisvard, Equations d'evolution abstraites non lineaires de type parabolique, Ann. Mat. Pura Appl. (4) 120, 329-396, (1979).
[18] J. Zhu, X. Chen, and T.Y. Hou, An Efficient Boundary Integral Method for the Mullins-Sekerka Problem, Journal of Computational Physics **126** no2, 246-267 (1996).