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Ostwald ripening in two dimensions—the rigorous derivation of the equations from the Mullins–Sekerka dynamics

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Abstract

We consider a dilute mixture of a finite number of particles and we are interested in the coarsening of the spatial distribution in two space dimensions under Mullins–Sekerka dynamics. Under the appropriate scaling hypotheses we associate radii and centers to each particle and derive equations for the whole evolution.

Keywords: Ostwald ripening; Mullins–Sekerka; Asymptotics; Stability

1. Introduction

Ostwald ripening or coarsening is a diffusion process occurring in the last stage of a first-order phase transformation. Usually, any first-order phase transformation process results in a two-phase mixture with a dispersed second phase in a matrix. Initially, the average size of the dispersed particles is very small and therefore the interfacial energy of the system is large and the mixture is not in thermodynamical equilibrium. The force that drives the system towards equilibrium is the gradient of the chemical potential that, according to the Gibbs–Thomson condition, on the

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interface, is proportional to its mean curvature. As a result matter diffuses from regions of higher curvature to regions of lower curvature and large particles grow at the expense of smaller particles that eventually shrink to nothing. The outcome of this process, known as Ostwald ripening is an increasing of the average size of particles and a reduction of the number of them that makes the mixture coarser. A quantitative description of Ostwald ripening [34] has been developed by Lifschitz and Slyozov and independently by Wagner under the assumption that the relative fraction of the dispersed phase is very small. Their theory (LSW theory in the following) is in three dimensions and assumes that there are many particles in the system with the size of the particles small compared to the distance between them. LSW is a mean field theory and is derived as follows: The point of departure is the quasi-static Stefan problem with surface tension (otherwise known as Mullins–Sekerka free boundary problem, see (1.7)). From this problem a differential equation (1.2) describing the evolution of the size of a typical particle is formally derived. The derivation is based on the assumption that particles are exact spheres and that their centers stay fixed in time. The evolution is characterized by the particle distribution $n(R,t)\,dR$ which is defined as the number of particles which at time t have radius in $[R, R+dR]$. More specifically the LSW theory provides the equation

$$\frac{\partial n(R,t)}{\partial t} + \frac{\partial}{\partial R}\left(\frac{dR}{dt} n(R,t)\right) = 0 \tag{1.1}$$

with

$$\frac{dR}{dt} = \frac{1}{R(t)}\left(\frac{1}{\bar{R}(t)} - \frac{1}{R(t)}\right), \tag{1.2}$$

where $\bar{R}(t)$ is the average radius size

$$\bar{R}(t) = \frac{\int R n(R,t)\,dR}{\int n(R,t)\,dR}. \tag{1.3}$$

System (1.1)–(1.3) is analyzed in [27,41] and it is shown that there exist infinitely many self-similar solutions, but only one is believed to describe the typical behavior of the system for large times

$$n(R,t) \cong \frac{1}{t^{\frac{4}{3}}} g\left(\frac{R(t)}{\bar{R}(t)}\right). \tag{1.4}$$

The theory predicts also the following temporal laws for the average radius and the total number of particles:

$$\bar{R}(t) \cong \left(\bar{R}^3(0) + \tfrac{4}{9}t\right)^{\frac{1}{3}}, \tag{1.5}$$

$$N(t) = \left(\bar{R}^3(0) + \tfrac{4}{9}t\right)^{-1}. \tag{1.6}$$

Niethammer [32] has rigorously derived Eqs. (1.1) and (1.2) through a homogenization procedure starting by a suitable modification of the Mullins–Sekerka problem. Alikakos and Fusco [3–5] obtained precise expressions for the equations of the centers and the radii by taking also into account the geometry of the distribution thus removing these restrictive hypotheses. These results will make it possible to pass rigorously to the limit [7,33].

The quasistatic Stefan problem or Mullins–Sekerka model [31] in dimensionless variables takes the form

$$\begin{cases} -\Delta u = 0 & \text{off } \Gamma(t), \quad \text{in } \Omega \subset \mathbb{R}^m, \ m = 2, 3, \\ u = H & \text{on } \Gamma(t), \\ \dfrac{\partial u}{\partial v} = 0 & \text{on } \partial\Omega, \\ V = \left[\left[\dfrac{\partial u}{\partial n}\right]\right] & \text{on } \Gamma(t), \end{cases} \tag{1.7}$$

where u is the chemical potential, H is the mean curvature of Γ; the sign convention for H is that H is positive for a shrinking sphere; $\vec{v}$ is the outward normal to $\partial\Omega$; V is the normal velocity positive for a shrinking sphere; $[[\frac{\partial u}{\partial n}]] = \frac{\partial u^+}{\partial n^+} + \frac{\partial u^-}{\partial n^-}$ is the jump of the derivative of u in the normal direction to $\Gamma(t)$ where u^+, u^- are the restrictions of u on the exterior $\Omega^+(t)$ and the interior $\Omega^-(t)$ of $\Gamma(t)$ in Ω and n^-, n^+ the unit exterior normal to $\Omega^+(t)$, $\Omega^-(t)$. Here $\Gamma(t) = \bigcup_{i=1}^N \Gamma_i(t)$ is the union of the boundaries of the N particles and Ω is a bounded, smooth domain (the container of the mixture). The Mullins–Sekerka model is a nonlocal evolution law in which the normal velocity of a propagating interface depends on the jump across the interface of the normal derivative of a function which is harmonic on either side and which equals the mean curvature on the propagating interface. If $H = const.$ on $\Gamma(t)$ then $u(x) = H = \text{const.}, \forall x \in \Omega$ and $V \equiv 0$ solve (1.7). Therefore Ω^- is the union of $N \geqslant 1$ equal balls with the same radius which are equilibria for (1.7).

The Mullins–Sekerka model arises as a singular limit for the Cahn–Hilliard equation, a fourth-order parabolic equation which is used as a model for phase separation and coarsening phenomena in a melted binary alloy [1,12,13,35]. In this paper we are interested in the evolution in two space dimensions. This problem has also some physical interest, for example, in the theory of thin films. We refer to [30,36,38]. We denote by $Per(\Gamma(t))$, $Vol(\Omega^-(t))$, the surface area and the enclosed volume (perimeter, enclosed area for $n = 2$), respectively, then (1.7) is a volume preserving, perimeter shortening law. A standard computation (e.g. [14]) shows that

$$\frac{d}{dt} Per(\Gamma(t)) = -\int_\Gamma HV = -\int_\Gamma u\left[\frac{\partial u}{\partial n}\right] = -\int_\Omega |\nabla u|^2 \leqslant 0, \tag{1.8}$$

$$\frac{d}{dt} Vol(\Omega^-(t)) = -\int_\Gamma V = \int_\Gamma \left[\frac{\partial u}{\partial n}\right] = \int_{\Omega\setminus\Gamma} \Delta u = 0. \tag{1.9}$$

In this paper, we consider the case where Ω^- is the union of $N \geqslant 1$ small "particles" which are initially very close to balls B_{ξ_i,R_i}, $i = 1, \dots, N$ with center ξ_i and radii R_i. We assume that the radii are small with respect to interparticle distances. These assumptions imply in particular that we work under the premise that the "volume fraction", that is the ratio φ between the volume of the dispersed phase $|\Omega^-| \cong \sum_{i=1}^N \pi R_i^2$ and the measure of the set Ω is a small quantity

$$\varphi = \frac{|\Omega^-|}{|\Omega|} \cong \frac{\sum_{i=1}^N \pi R_i^2}{|\Omega|} \ll 1. \tag{1.10}$$

We show that under these restrictions for the initial condition the particles retain their almost circular shape until one singularity occurs which always results in the fact that one or more particles disappear shrinking to nothing. We derive corrected equations for the radii which take into account the distance and the size of the neighboring particles and also equations for the motion of the centers of the particles. Moreover, the robustness of the circular shape is established. In the case that the boundary $\partial\Omega$ is removed, the equations of the radii and the centers take the form:

$$\begin{aligned}\dot{R}_i = &\frac{2}{|\log \varphi|}\frac{1}{R_i}\Bigg\{\left(\frac{1}{\bar{R}} - \frac{1}{R_i}\right) \\ &+ \frac{2}{|\log \varphi|}\left[\log \frac{R_i}{\varphi^{\frac{1}{2}}}\left(\frac{1}{\bar{R}} - \frac{1}{R_i}\right) + \sum_{h \neq i} \log|\xi_i - \xi_h|\left(\frac{1}{\bar{R}} - \frac{1}{R_h}\right) - E\right]\Bigg\} + \cdots,\end{aligned} \tag{1.11}$$

$$\dot{\xi}_i = -\frac{4}{|\log \varphi|}\sum_{h \neq i}\left(\frac{1}{\bar{R}} - \frac{1}{R_h}\right)\frac{\xi_h - \xi_i}{|\xi_h - \xi_i|^2} + \cdots, \tag{1.12}$$

where "dots" denote higher order terms, $\bar{R}$ is the harmonic mean of R_i defined by

$$\frac{1}{\bar{R}} = \frac{1}{N}\sum_{j=1}^N \frac{1}{R_j} \tag{1.13}$$

and E is determined by the conservation of volume and is given by

$$E = \frac{1}{N}\sum_{k=1}^N \log\frac{R_k}{\varphi^{\frac{1}{2}}}\left(\frac{1}{\bar{R}} - \frac{1}{R_k}\right) + \frac{1}{N}\sum_{k=1}^N \sum_{h \neq k} \log|\xi_h - \xi_k|\left(\frac{1}{\bar{R}} - \frac{1}{R_h}\right). \tag{1.14}$$

We remark that the above products and sums are taken only over the particles which have positive radius. This applies to all formulas throughout the paper. As it is well known the Mullins–Sekerka problem has an invariance property. Indeed, if $t \to \Omega^-(t)$ is a solution and $\mu > 0$ a positive constant then $t \to \tilde{\Omega}^-(t) =: \mu\Omega^-(\frac{t}{\mu^3})$ is again a solution. The equations

$$\dot{R}_i = \frac{2}{|\log \varphi|} \frac{1}{R_i} \left(\frac{1}{\bar{R}} - \frac{1}{R_i} \right), \tag{1.15}$$

$$\dot{\xi}_i = -\frac{4}{|\log \varphi|} \sum_{h \neq i} \left(\frac{1}{\bar{R}} - \frac{1}{R_h} \right) \frac{\xi_h - \xi_i}{|\xi_h - \xi_i|^2} \tag{1.16}$$

which are obtained by (1.11) and (1.12) by retaining the principal terms enjoy the same invariance property: if $t \to (R(t), \xi(t))$ is a solution then $t \to (\tilde{R}(t), \tilde{\xi}(t)) =: (\mu R(\frac{t}{\mu^3}), \mu \xi(\frac{t}{\mu^3}))$ is again a solution. The complete equations (1.11) do exhibit exactly the invariance property shared by Eqs. (1.7) and (1.15).

Eqs. (1.13), (1.15) state that given t, the radius $R_i(t)$ of the ith particle, decreases or increases according to whether at that time it is below or above the average value $\bar{R}$. Moreover, Eqs. (1.15) preserve the total enclosed area and reduce the total perimeter,

$$\frac{d}{dt} \sum_{i=1}^{N} R_i^2 = 0, \tag{1.17}$$

$$\frac{d}{dt} \sum_{i=1}^{N} R_i \leqslant 0 \tag{1.18}$$

reflecting the enclosed area preserving and perimeter shortening properties of Mullins–Sekerka problem. We assume

$$0 < R_1(0) < R_2(0) < \cdots < R_N(0),$$

and show (see Proposition 3.1) that the solution of (1.15) preserves the order: $R_1(t) < \cdots < R_N(t)$ in its maximal interval of existence $[0, T_1)$ and that T_1 is bounded and characterized by the fact that

$$\lim_{t \to T_1^-} R_1(t) = 0.$$

After $t = T_1$, the solution of (1.15) can be continued to an interval $[T_1, T_2)$ by removing the first equation; changing N to $N-1$, and taking $R_j(T_1), j = 2, \ldots, N$ as initial condition. Proceeding in this way one defines a sequence of times $T_1 < T_2 < \cdots < T_{N-1}$ characterized by the fact that $\lim_{t \to T_j^-} R_j(t) = 0$. We show that under the above assumption that $\Omega^-(0) \cong \bigcup_{i=1}^N B_{\xi_i(0), R_i(0)}$, something similar holds true also for the full Mullins–Sekerka free boundary problem. Indeed we prove (see Theorem 2.1) that there are times $\hat{T}_1 < \hat{T}_2 <, \ldots, < \hat{T}_{N-1}$ near to $T_1, \ldots, T_{N-1}$ such that at time $\hat{T}_j$ a singularity occurs and the jth particle shrinks to nothing. Eq. (1.15) for the case of 2 particles where first derived in [43] by the method of images Eq. (1.15) were also obtained in [33] starting from a variation of the Mullins–Sekerka model which assumes the Gibbs–Thomson relation in an averaged integral form:

Comparison of principal terms

3 Dimensions	2 Dimensions
$\dot{R}_i = \frac{1}{R_i}\left(\frac{1}{\bar{R}} - \frac{1}{R_i}\right)$	$\dot{R}_i = \frac{1}{R_i}\frac{2}{\lvert\log\varphi\rvert}\left(\frac{1}{\bar{R}} - \frac{1}{R_i}\right)$
$\bar{R} = \frac{1}{N}\sum_{j=1}^{N} R_j$	$\bar{R} = \left(\frac{1}{N}\sum_{j=1}^{N}\frac{1}{R_j}\right)^{-1}$
arithmetic mean	harmonic mean
independent of distance	independent of distance
scale invariance compatible with $t^{\frac{1}{3}}$ law	scale invariance compatible with $t^{\frac{1}{3}}$ law
$x \to \mu^{-\frac{1}{3}}x,\ t \to \mu t$	$x \to \mu^{-\frac{1}{3}}x,\ t \to \mu t$
No singularity for $\frac{1}{\bar{R}}$	$\frac{1}{\bar{R}}$ becomes singular at the extinction times
Singularity like that of $\dot{R}_i = -\frac{1}{R_i^2}$	Singularity like that of $\dot{R}_i = -\frac{1}{R_i^2}\frac{2}{\lvert\log\varphi\rvert}$ milder than in 3D
No crossing of time lines	No crossing of time lines
No effect of neighbors	No effect of neighbors

We now present a formal derivation of the equations of the radii and the centers in two dimensions. In the following, we will show that in the limit of small size the distortion from sphericity measured by the function r_i introduced in Section 5 does not affect to principal order the evolution of the centers and the radii of the particles. Therefore in this paragraph in doing the formal derivation we will assume that Ω^- is the union of $N \geqslant 1$ perfect balls of center ξ_i and radius $R_i > 0$, $i = 1, \dots, N$ that is $\Omega^- = \bigcup_{i=1}^{N} B_{\xi_i,R_i}$. We represent the boundary $\partial B_{\xi_i,R_i}$ of B_{ξ_i,R_i} through the map $X^i : S^1 \to \mathbb{R}^2$ defined by

$$S^1 \ni u \to x = X^i(u) := \xi_i + R_i u. \tag{1.19}$$

We work under the assumption that the radii R_i are small, that is we assume $R_i \ll 1$, $i = 1, \dots, N$. For this reason in the analysis that follows we will work with the rescaled radii ρ_i defined by

$$R_i = \varepsilon\rho_i,$$

where $\varepsilon > 0$ is a small parameter $\varepsilon \ll 1$. The parameter ε can be identified with the square root of the volume fraction φ

$$\varphi = \frac{|\Omega^-|}{|\Omega|} \cong \frac{\varepsilon^2 \sum_{i=1}^N \pi\rho_i^2}{|\Omega|}.$$

By regarding ξ_i and $R_i = \varepsilon\rho_i$ in (1.19) as functions of time t and by differentiating with respect to t we get $\dot{x} = \dot{\xi}_i + \varepsilon\dot{\rho}_i u$. By projecting this equation on the unit vector u which coincides with the exterior normal to B_{ξ_i,R_i} we get the following expression for the normal velocity V of $\partial B_{\xi_i,R_i}$ at $X^i(u)$:

$$V(u) = -\varepsilon\dot{\rho}_i - \sum_{j=1}^{2} \dot{\xi}_{ij}\langle u, e_j\rangle, \quad \dot{\xi}_{ij} = \langle \dot{\xi}_i, e_j\rangle, \tag{1.20}$$

where $\{e_1, e_2\}$ is the standard basis of $\mathbb{R}^2$, $\langle , \rangle$ the standard scalar product of $\mathbb{R}^2$ and the "–" sign is due to the sign convention that we assume V positive for a shrinking sphere. As we recall in Section 3, (1.7) can be reduced via potential theory (see also [43]) to a problem that lives entirely on the interface:

$$\int_{\Gamma(t)} g(x,y)V(y)\, dy = H(x) - E, \quad x \in \Gamma, \tag{1.21}$$

where $g(x,y) = -\frac{1}{2\pi}\log|x-y| + \gamma(x,y)$ is the Green function of the Neumann problem

$$\begin{cases} -\Delta u = f, \\ \dfrac{\partial u}{\partial v} = 0, \\ \int_\Omega u = \int_\Omega f = 0 \end{cases}$$

$\Gamma = \Gamma(t)$; $E = E(t)$, V and H also depend on time and $E(t)$ is to be chosen in order to ensure that

$$\int_\Gamma V = 0$$

which implies conservation of volume.

In the formal derivation that follows we do not consider the contribution of the smooth part of the Green's function. We will indicate at the end the extra terms that one gets when the contribution of γ is also accounted for.

If $\Gamma = \bigcup_{i=1}^N \partial B_{\xi_i,\varepsilon\rho_i}$ then, at $\partial B_{\xi_i,\varepsilon\rho_i} \ni x = \xi_i + \varepsilon\rho_i u$, $u \in S^1$, Eq. (1.21) can be written in the form

$$\varepsilon\rho_i \int_{S^1} -\frac{1}{2\pi}\log|\varepsilon\rho_i(u-v)|V_i(v)\, dv + \sum_{h\neq i} \varepsilon\rho_h \int_{S^1}\left[-\frac{1}{2\pi}\log|\xi_i - \xi_h + \varepsilon(\rho_i u - \rho_h v)|\right] \times V_h(v)\, dv = \frac{1}{\varepsilon\rho_i} - E, \quad \forall u \in S^1, \quad i = 1, \dots, N. \tag{1.22}$$

Substituting Eq. (1.20) into Eq. (1.22) for $x \in \Gamma_i(t)$, we obtain

$$\varepsilon\rho_i \int_{S^1} \left[-\frac{1}{2\pi}\log|\varepsilon\rho_i| - \frac{1}{2\pi}\log|u-v| \right] \left[-\varepsilon\dot\rho_i - \sum_{j=1}^{2} \dot\xi_{ij} \langle v, e_j \rangle \right] dv$$
$$+ \sum_{h \neq i} \varepsilon\rho_h \int_{S^1} \left[-\frac{1}{2\pi}\log|\xi_i - \xi_h| - \frac{\varepsilon}{2\pi}\rho_i \frac{\langle \xi_i - \xi_h, u \rangle}{|\xi_i - \xi_h|^2} + \frac{\varepsilon}{2\pi}\rho_h \frac{\langle \xi_i - \xi_h, v \rangle}{|\xi_i - \xi_h|^2} + O(\varepsilon^2) \right]$$
$$\times \left[-\varepsilon\dot\rho_h - \sum_{j=1}^{2} \dot\xi_{hj} \langle v, e_j \rangle \right] dv = \frac{1}{\varepsilon\rho_i} - E, \quad \forall u \in S^1, \quad i = 1, \dots, N. \tag{1.23}$$

We now observe that

$$\int_{S^1} -\frac{1}{2\pi}\log|u-v|\, dv = 0, \quad \int_{S^1} -\frac{1}{2\pi}\log|u-v| \langle v, e_j \rangle\, dv = \frac{1}{2}\langle u, e_j \rangle,$$
$$\int_{S^1} du = 2\pi, \quad \int_{S^1} \langle u, e_j \rangle\, du = 0,$$
$$\int_{S^1} \langle u, e_j \rangle^2\, du = \pi, \quad \int_{S^1} \langle u, e_1 \rangle \langle u, e_2 \rangle\, du = 0.$$

Moreover, we can write

$$\langle \xi_i - \xi_h, u \rangle = \sum_{j=1}^{2} \langle \xi_i - \xi_h, e_j \rangle \langle u, e_j \rangle.$$

Utilizing the above expressions, we get from Eq. (1.23)

$$\varepsilon\rho_i \left[\varepsilon\dot\rho_i \log|\varepsilon\rho_i| - \frac{1}{2}\sum_{j=1}^{2} \dot\xi_{ij} \langle u, e_j \rangle \right] + \sum_{h\neq i} \varepsilon\rho_h \left[\varepsilon\dot\rho_h \log|\xi_i - \xi_h| \right.$$
$$+ \varepsilon^2 \rho_i \dot\rho_h \frac{1}{|\xi_i - \xi_h|^2} \sum_{j=1}^{2} \langle \xi_i - \xi_h, e_j \rangle \langle u, e_j \rangle + O(\varepsilon^3) - \frac{\varepsilon}{2}\rho_h$$
$$\left. \times \sum_{j=1}^{2} \dot\xi_{hj} \frac{\langle \xi_i - \xi_h, e_j \rangle}{|\xi_i - \xi_h|^2} \right] = \frac{1}{\varepsilon\rho_i} - E. \tag{1.24}$$

This equation implies

$$\begin{cases} \varepsilon\rho_i \varepsilon\dot\rho_i \log|\varepsilon\rho_i| + \sum_{h\neq i} \varepsilon\rho_h \left[\varepsilon\dot\rho_h \log|\xi_i - \xi_h| - \frac{\varepsilon}{2}\rho_h \sum_{j=1}^{2} \dot\xi_{hj} \frac{\langle \xi_i - \xi_h, e_j \rangle}{|\xi_i - \xi_h|^2} \right] = \frac{1}{\varepsilon\rho_i} - E, \\ \varepsilon\rho_i(-\frac{1}{2}\dot\xi_{ij}) + \sum_{h\neq i} \varepsilon\rho_h \varepsilon^2 \rho_i \dot\rho_h \frac{1}{|\xi_i - \xi_h|^2} \langle \xi_i - \xi_h, e_j \rangle = 0, \quad i = 1, \dots, N, \ j = 1, 2. \end{cases} \tag{1.25}$$

From Eq. $(1.25)_2$ we obtain

$$\dot{\xi}_{ij} = 2\varepsilon^2 \sum_{h\neq i} \rho_h \dot{\rho}_h \frac{\langle \xi_i - \xi_h, e_j \rangle}{|\xi_i - \xi_h|^2} \tag{1.26}$$

while Eq. $(1.25)_1$ is equivalent to

$$\begin{aligned}&\varepsilon\rho_i\varepsilon\dot{\rho}_i \log|\varepsilon\rho_i| + \sum_{h\neq i} \varepsilon\rho_h\varepsilon\dot{\rho}_h \log|\xi_i - \xi_h| \\ &\quad - \varepsilon\rho_h \sum_{j=1}^{2} \sum_{k\neq h} \varepsilon^2 \rho_k\dot{\rho}_k \frac{\langle \xi_h - \xi_k, e_j \rangle \langle \xi_i - \xi_h, e_j \rangle}{|\xi_h - \xi_k|^2 |\xi_i - \xi_k|^2} = \frac{1}{\varepsilon\rho_i} - E. \end{aligned}\tag{1.27}$$

If in Eq. (1.27) we disregard the third term on the left-hand side we get

$$\varepsilon\rho_i\varepsilon\dot{\rho}_i \log\varepsilon + \varepsilon\rho_i\varepsilon\dot{\rho}_i \log\rho_i + \sum_{h\neq i} \varepsilon\rho_h\varepsilon\dot{\rho}_h \log|\xi_i - \xi_h| = \frac{1}{\varepsilon\rho_i} - E. \tag{1.28}$$

From this if we determine E by imposing $\sum_{k=1}^{N} \rho_k\dot{\rho}_k = 0$ we get

$$\begin{aligned}\varepsilon\dot{\rho}_i =& - \frac{1}{|\log\varepsilon|}\frac{1}{\varepsilon\rho_i}\left(\frac{1}{\varepsilon\rho_i} - \frac{1}{\varepsilon\bar{\rho}}\right) + \frac{1}{|\log\varepsilon|}\left[\varepsilon\dot{\rho}_i \log\rho_i + \frac{1}{\varepsilon\rho_i}\sum_{h\neq i} \varepsilon\rho_h\varepsilon\dot{\rho}_h \log|\xi_i - \xi_h|\right] \\ & - \frac{1}{\varepsilon\rho_i|\log\varepsilon|N}\left[\sum_{k=1}^{N} \varepsilon\rho_k\varepsilon\dot{\rho}_k \log\rho_k + \sum_{k,h\ h\neq k} \varepsilon\rho_h\varepsilon\dot{\rho}_h \log|\xi_k - \xi_h|\right]. \end{aligned}\tag{1.29}$$

Since we have $\sum_{k=1}^{N} \rho_k\dot{\rho}_k = 0$ if $t \to \rho_i(t)$ is a solution of (1.29) then $t \to \hat{\rho}_i(t) = \mu\rho_i(\frac{t}{\mu^3})$ is again a solution. Indeed if $L(\rho)$, $R(\rho)$ correspond to the left-hand side and right-hand side terms of (1.29) we have

$$L(\hat{\rho}) = \frac{1}{\mu^2} L(\rho),$$

$$R(\hat{\rho}) = \frac{1}{\mu^2} R(\rho) + \frac{\log\mu}{\varepsilon\rho_i|\log\varepsilon|}\sum_{h=1}^{N} \varepsilon\rho_h\varepsilon\dot{\rho}_h - \frac{\log\mu}{\varepsilon\rho_i|\log\varepsilon|N} N \sum_{h=1}^{N} \varepsilon\rho_h\varepsilon\dot{\rho}_h = \frac{1}{\mu^2} R(\rho).$$

From this computation it appears that the extra terms in the expression of $R(\hat{\rho})$ cancel out even if $\sum_{k=1}^{N} \rho_k\dot{\rho}_k \neq 0$. Let $Q_i(\rho)$ be any function which satisfies the following conditions:

$$c_1 - Q_i(\mu\rho) = \frac{1}{\mu^2} Q_i(\rho),$$

$$c_2 - \sum_{i=1}^{N} \rho_i Q_i(\rho) = 0.$$

From the above discussion it follows that if we replace in the right-hand side of (1.29) $\varepsilon\dot{\rho}_i$ with $Q_i(\varepsilon\rho)$ then the ODE that we obtain has the desired scale invariance. Clearly, the function

$$Q_i(\rho) = -\frac{1}{|\log\varepsilon|}\frac{1}{\rho_i}\left(\frac{1}{\rho_i} - \frac{1}{\bar{\rho}}\right)$$

satisfies c_1 and c_2. This leads to the ODE

$$\begin{aligned}\varepsilon\dot{\rho}_i =& \frac{1}{\varepsilon\rho_i}\frac{1}{|\log\varepsilon|}\left\{\left(\frac{1}{\varepsilon\bar{\rho}} - \frac{1}{\varepsilon\rho_i}\right)\right.\\ &\left. + \frac{1}{|\log\varepsilon|}\left[\log\rho_i\left(\frac{1}{\varepsilon\bar{\rho}} - \frac{1}{\varepsilon\rho_i}\right) + \sum_{h\neq i}\log|\xi_i - \xi_h|\left(\frac{1}{\varepsilon\bar{\rho}} - \frac{1}{\varepsilon\rho_h}\right) - E\right]\right\},\end{aligned} \tag{1.30}$$

where

$$E = \frac{1}{N}\sum_{k=1}^{N}\log\rho_k\left(\frac{1}{\varepsilon\bar{\rho}} - \frac{1}{\varepsilon\rho_k}\right) + \frac{1}{N}\sum_{k=1}^{N}\sum_{h\neq k}\log|\xi_h - \xi_k|\left(\frac{1}{\varepsilon\bar{\rho}} - \frac{1}{\varepsilon\rho_h}\right)$$

and this equation satisfies the correct scaling law.

Finally, we obtain from (1.26)

$$\dot{\xi}_i = -2\sum_{h\neq i}\frac{1}{|\log\varepsilon|}\left(\frac{1}{\varepsilon\bar{\rho}} - \frac{1}{\varepsilon\rho_h}\right)\frac{\xi_h - \xi_i}{|\xi_h - \xi_i|^2}. \tag{1.31}$$

This concludes the derivation.

Remark. The above analysis can be extended to include also the effect of the boundary. With similar computations one finds that the boundary contributes with the following terms:

$$\begin{aligned}\varepsilon\dot{\rho}_i =& \frac{1}{|\log\varepsilon|}\frac{1}{\varepsilon\rho_i}\left\{\left(\frac{1}{\varepsilon\bar{\rho}} - \frac{1}{\varepsilon\rho_i}\right) + \frac{1}{|\log\varepsilon|}\left[(\log\rho_i + \gamma(\xi_i,\xi_i))\left(\frac{1}{\varepsilon\bar{\rho}} - \frac{1}{\varepsilon\rho_i}\right)\right.\right.\\ &\left.\left. + \sum_{h\neq i}(\log|\xi_i - \xi_h| + \gamma(\xi_i,\xi_h))\left(\frac{1}{\varepsilon\bar{\rho}} - \frac{1}{\varepsilon\rho_h}\right) - E\right]\right\},\end{aligned}$$

where

$$E = \frac{1}{N}\sum_{k=1}^{N} (\log \rho_k + \gamma(\xi_k, \xi_k))\left(\frac{1}{\varepsilon\bar{\rho}} - \frac{1}{\varepsilon\rho_k}\right) + \frac{1}{N}\sum_{k=1}^{N} \sum_{h \neq k} (\log|\xi_h - \xi_k| + \gamma(\xi_h, \xi_k))\left(\frac{1}{\varepsilon\bar{\rho}} - \frac{1}{\varepsilon\rho_h}\right)$$

and

$$\dot{\xi}_i = -\frac{2}{|\log \varepsilon|}\sum_{h \neq i} \left(\frac{1}{\varepsilon\bar{\rho}} - \frac{1}{\varepsilon\rho_h}\right)\left(\frac{\xi_h - \xi_i}{|\xi_h - \xi_i|^2} + \frac{\partial\gamma(\xi_i, \xi_h)}{\partial x}\right).$$

We now present some examples which illustrate the use of system (1.29), (1.31), (1.32).

The two-particle case (See Fig. 1).
We calculate

$$\begin{cases} \dot{R}_1 = \dfrac{2}{|\log \varphi|}\dfrac{1}{R_1}\left(\dfrac{1}{\bar{R}} - \dfrac{1}{R_1}\right), \\ \dot{R}_2 = \dfrac{2}{|\log \varphi|}\dfrac{1}{R_2}\left(\dfrac{1}{\bar{R}} - \dfrac{1}{R_2}\right) \end{cases}$$

$$\Rightarrow \dot{R}_1(t) < 0, \quad \dot{R}_2(t) > 0$$

$$\begin{cases} \dot{\xi}_1 = -\dfrac{4}{|\log \varphi|}\left(\dfrac{1}{\bar{R}} - \dfrac{1}{R_2}\right)\dfrac{\xi_2 - \xi_1}{|\xi_2 - \xi_1|^2}, \\ \dot{\xi}_2 = -\dfrac{4}{|\log \varphi|}\left(\dfrac{1}{\bar{R}} - \dfrac{1}{R_1}\right)\dfrac{\xi_1 - \xi_2}{|\xi_2 - \xi_1|^2}. \end{cases}$$

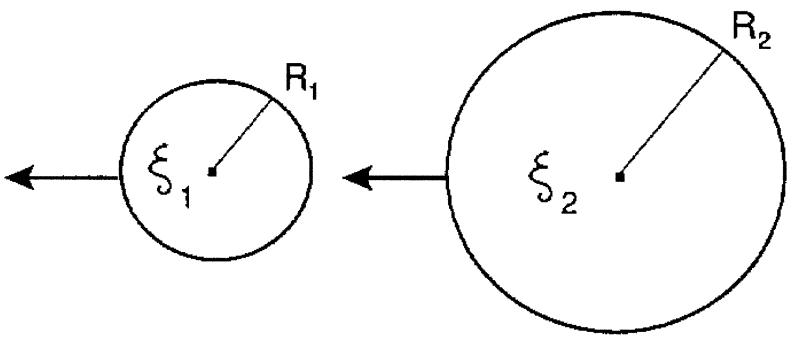

Fig. 1. The two-particle case for $R_1 < R_2$.

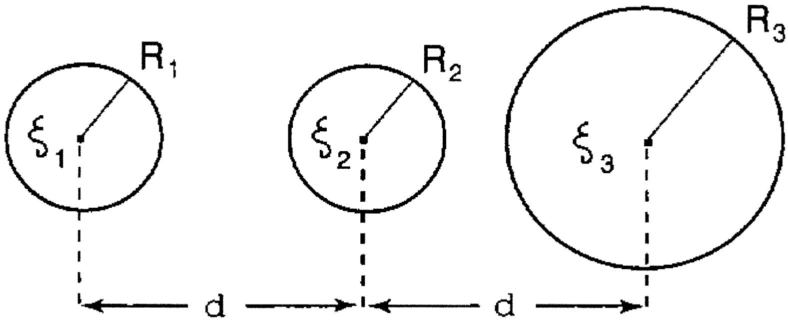

Fig. 2. The three-particle case for $R_1 = R_2 < R_3$.

Notice that if $\tilde{u} = \frac{\xi_2 - \xi_1}{|\xi_2 - \xi_1|}$ is the unit vector along ξ_1, ξ_2 then $\langle \dot{\xi}_1, \tilde{u} \rangle > 0$, $\langle \dot{\xi}_2, \tilde{u} \rangle > 0$. Moreover, it holds that $\langle \dot{\xi}_1(t), \tilde{u} \rangle = \langle \dot{\xi}_2(t), \tilde{u} \rangle$. In the two-particle system, we see that the smaller particle becomes smaller and the bigger becomes bigger. The two unequal particles move in the same direction, in the direction of the smaller particle with equal speeds.

The three-particle case (Fig. 2).

We consider the above arrangement and we calculate

$$\dot{R}_1 = \frac{2}{|\log \varphi|} \frac{1}{R_1} \Bigg\{ \left(\frac{1}{\bar{R}} - \frac{1}{R_1} \right) + \frac{2}{|\log \varphi|} \Bigg[\log \frac{R_1}{\varphi^{\frac{1}{2}}} \left(\frac{1}{\bar{R}} - \frac{1}{R_1} \right) + \log d \left(\frac{1}{\bar{R}} - \frac{1}{R_2} \right) + \log 2d \left(\frac{1}{\bar{R}} - \frac{1}{R_3} \right) - E \Bigg] \Bigg\},$$

$$\dot{R}_2 = \frac{2}{|\log \varphi|} \frac{1}{R_2} \Bigg\{ \left(\frac{1}{\bar{R}} - \frac{1}{R_2} \right) + \frac{2}{|\log \varphi|} \Bigg[\log \frac{R_2}{\varphi^{\frac{1}{2}}} \left(\frac{1}{\bar{R}} - \frac{1}{R_2} \right) + \log d \left(\frac{1}{\bar{R}} - \frac{1}{R_1} \right) + \log d \left(\frac{1}{\bar{R}} - \frac{1}{R_3} \right) - E \Bigg] \Bigg\}.$$

By subtracting the above equations we obtain

$$\dot{R}_1 - \dot{R}_2 = \log 2 \left(\frac{1}{\bar{R}} - \frac{1}{R_3} \right) > 0.$$

So, the proximity to particle 3 impedes the growth of particle 2 and as a result particle 1 grows faster than particle 2 (see Fig. 3).

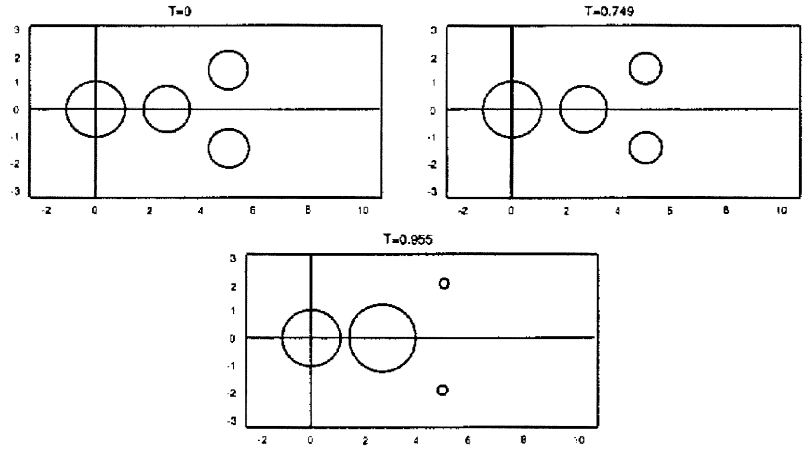

Fig. 3. Four-particle system with the particles numbered from left to right, the leftmost particle denoted as particle 1, the center particle as particle 2 and the rightmost particles as particles 3 and 4. Particles 3 and 4 are close to particle 2. We observe that despite particle 2 being smaller than particle 1, particle 2 grows at the expense of 3 and 4.

Four-particle case. In systems with more than two particles, the competition is complicated. Although location of a particle between particles of smaller and larger sizes is a necessary condition for migration, calculations show that it is not a sufficient condition. Small changes in the locations of the particles relative to one another can have large effects on the evolution. With the permission of the authors [43] we reproduce below some of their numerical results on four-particle systems (see Figs. 3–5). A similar problem concerning numerical studies can be found in [39] where the Laplace equation is solved only inside the region of the interface.

The paper is organized as follows: In Section 2, we present the statement of the main result. In Section 3, we analyze the system of ODEs and present the integral formulation of the equation. In Section 4, we present the operators T, L and the operator $A = TL$. Moreover, we express the mean curvature H in ξ, ρ, r coordinates. In Section 5, we study the coordinate system [2,5,11,42]. Given an interface Γ close to circular, we would like to associate to it in a unique way a circle and view the interface as a small perturbation of that circle such that

$$\Gamma(t) = \xi + \varepsilon\rho(1 + \varepsilon r(u))u, \quad u \in S^1,$$

$$\int_{S^1} r(u)\, du = 0, \quad \int_{S^1} r(u) \langle u, e_i \rangle \, du = 0, \quad i = 1, 2.$$

In Section 6, we solve the linear equation $S(V) = H - \bar{H}$, assuming that H is known and we obtain a system for ξ, ρ and r equations with estimates for the higher order terms. In Section 7, we obtain bounds on r by analyzing the r-equation. The control on r is accomplished by utilizing the maximal regularity theory of da Prato and

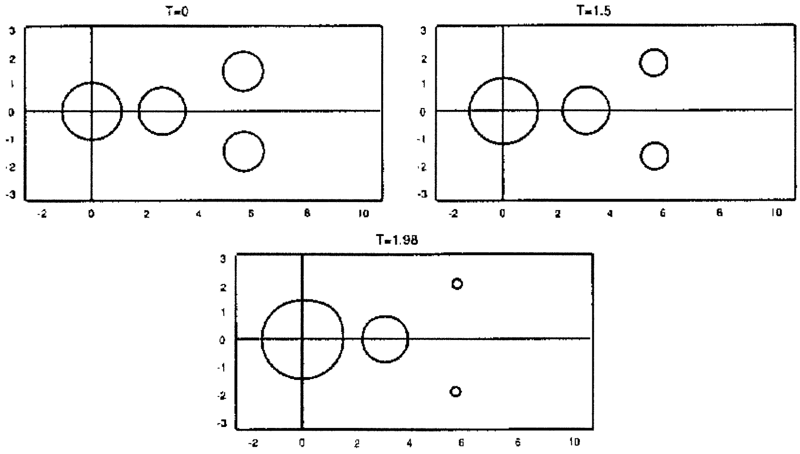

Fig. 4. A four-particle system with the same radius as in Fig. 3 with the difference that the two smaller particles are away from particle 2.

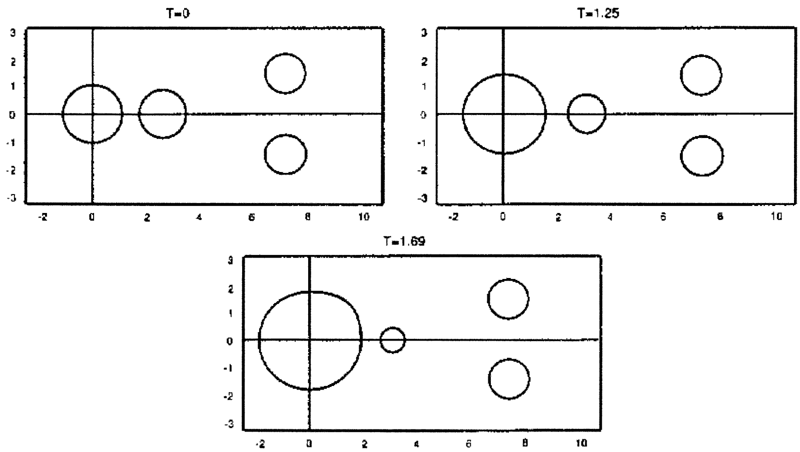

Fig. 5. A four-particle system with the same radius as in Fig. 3 but now the distance between particle 2 and particles 3 and 4 is larger than those in Fig. 3 or Fig. 4. As a result particle 2 disappears before particles 3 and 4.

Grisvard [18]. In the previous sections we assume that $\varepsilon||r||_{C^{1+\alpha}(S^1)} < \delta$ for $\delta > 0$, a fixed small number. In Section 7, a uniform bound on $||r||$ is established. The uniform bound on $||r||$ is one of the harder analytic results. We should mention that the bound on r implies the robustness of the spherical shape. Chen [14] and independently Constantin and Pugh [17] established the stability of a single circle equilibrium in two space dimensions. Later [21] this was reestablished in a more general way. In our case, we first note that a configuration of two or more circles is unstable and we show for arbitrary initial data of unequal circles that the distortion away from circularity is small globally in time. Although the general layout is quite close to [5] there are differences between two and three dimensions also on the technical level with the two dimensional case tending in general to be harder. These differences can always be traced back to the fundamental solution of the Laplacian in two dimensions.

2. Statement of the main result

In Section 5, following [5], we will show that given an interface Γ close to circular, we associate to it in a unique way a circle and view the interface as a small perturbation of this circle. So, each interface will have unique $\xi \in \mathbb{R}^2, \rho > 0$ and $r \in C^1(S^1)$ satisfying

$$\Gamma(t) = \{x/x = \xi + \varepsilon\rho(1 + \varepsilon r(u))u, u \in S^1\}, \tag{2.1}$$

$$\int_{S^1} r(u)\, du = 0, \quad \int_{S^1} r(u)\langle u, e_i \rangle\, du = 0, \quad i = 1, 2. \tag{2.2}$$

Theorem 2.1. *Let* $\Omega \subset \mathbb{R}^2$ *be a bounded, connected and smooth domain. Assume that* $\Gamma(0) = \bigcup_{i=1}^N \Gamma_i(0)$, $N \geqslant 2$, *with* $\Gamma_i(0)$ *of the form* (2.1), $\xi_i(0) \in \Omega$, $R_i(0) > 0$ *and* $r_i(0) \in C^{3+\alpha}(S^1)$ *satisfying* (2.2). *Assume that*

$$\xi_i(0) \neq \xi_j(0) \quad \text{for} \quad i \neq j,$$

$$R_1(0) < R_2(0) < \cdots < R_N(0)$$

then there is $\bar{\varepsilon} > 0$ *such that*

$$\varphi = \frac{\sum_{i=1}^N \pi R_i^2(0)}{|\Omega|} < \bar{\varepsilon}^2,$$

$$||r_i(0)||_{C^{3+\alpha}(S^1)} < \bar{\varepsilon},$$

imply that the solution $t \to \Gamma(t)$ *of the Mullins–Sekerka problem* (1.7) *satisfies* $\Gamma(t) = \bigcup_{i=1}^N \Gamma_i(t)$ *with* $\Gamma_i(t)$ *of the form* (2.1) *with* $\xi_i(t)$, $R_i(t) > 0$, $r_i(t) \in C^{3+\alpha}(S^1)$ *satisfying* (2.2) *and exists globally as a weak solution in the sense of* [15,37]. *Moreover*,

(i) *There exist times* $\hat{T}_1 < \cdots < \hat{T}_{N-1}$ *such that* $\lim_{t \to \hat{T}_i} R_i(t) = 0$, $i = 1, \ldots, N-1$. *The solution is classical except at that times.*

(ii) *There are constants* C_r, $C_R > 0$ *depending on* $\frac{R_i(0)}{R_1(0)}$ $i = 2, \dots, N$ *such that*

$$\dot{R}_i = \frac{2}{|\log \varphi|} \frac{1}{R_i} \left(\frac{1}{\bar{R}} - \frac{1}{R_i} \right) + C_R g_i, \tag{2.3}$$

where $\bar{R}$ *is the harmonic mean of* R_i *defined by*

$$\frac{1}{\bar{R}} = \frac{1}{N} \sum_{j=1}^{N} \frac{1}{R_j}$$

and $g_i(R, \varepsilon, r)$ *a smooth function satisfying*

$$|g_i| < \frac{1}{|\log \varphi|^2}.$$

The above expression holds for $\hat{T}_{i-1} < t < \hat{T}_i$ *where for* $t > \hat{T}_i$ $i = 1, \dots, N$ *we have that* $R_i(t) = 0$.

In addition,

$$||r_i||_{C^{3+\alpha}(S^1)} < C_r.$$

3. The integral equation formulation and the equations of Ostwald Ripening

We would like to formulate the Mullins–Sekerka problem as an integral equation in the class of $C^{3+\alpha}$ interfaces. The immediate advantage of this approach is that the space dimension of the problem will be reduced by one. We will only need to solve the integral equations along the boundaries of the evolving domains instead of solving the two space dimension PDE problem. This approach has been used in many problems such as Ostwald ripening [39], water waves [10], etc. It is known [43] that we have the following integral formulation for the Mullins–Sekerka problem (1.7):

$$\int_{\Gamma(t)} g(x,y) V(y)\, dS_y - \frac{1}{|\Gamma(t)|} \int_{\Gamma(t)} \int_{\Gamma(t)} g(x,y) V(y)\, dS_y\, dS_x = H(x) - \bar{H} \tag{3.1}$$

with

$$\bar{H} = \frac{1}{|\Gamma(t)|} \int_{\Gamma(t)} H\, dS_y.$$

Then problem (1.7) takes the form $S(V) = H - \bar{H}$ where S is a linear operator, V the normal velocity, H the mean curvature, $\bar{H}$ the average value of H over $\Gamma(t)$ and $|\Gamma(t)|$ the surface area of $\Gamma(t)$. Moreover, we have that the Green's function in two dimensions has the form

$$g(x,y) = -\frac{1}{2\pi}\log|x-y| + \gamma(x,y), \tag{3.2}$$

where g is the function associated to the problem 5

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ \dfrac{\partial u}{\partial v} = 0, & \text{on } \partial\Omega, \\ \int_\Omega u\,dx = \int_\Omega f\,dx = 0 \end{cases} \tag{3.3}$$

and satisfies

$$\begin{cases} -\Delta_y g(x,y) = \delta_x(y) - \frac{1}{|\Omega|} & \text{in } \Omega, \\ \dfrac{\partial g}{\partial v_y} = 0 & \text{on } \partial\Omega, \\ \int_\Omega g(x,y)\,dy = 0. \end{cases} \tag{3.4}$$

On the other hand, γ is the smooth part of the Green's function and satisfies

$$\begin{cases} -\Delta_y \gamma(x,y) = -\dfrac{1}{|\Omega|}, & x\in\Omega, \quad y\in\Omega, \\ \dfrac{\partial\gamma(x,y)}{\partial v_y} = \dfrac{\partial}{\partial v_y}\left(\dfrac{1}{2\pi}\log|x-y|\right), & x\in\Omega, \quad y\in\partial\Omega, \\ \int_\Omega \gamma(x,y)\,dy = \int_\Omega \dfrac{1}{2\pi}\log|x-y|dy, \end{cases} \tag{3.5}$$

where Ω is an open, bounded, connected, smooth set in $\mathbb{R}^2$ (the container of the mixture) and $\delta_x(y)$ is the Dirac δ supported at $x\in\Omega$.

From classical elliptic theory [24] one has the estimates

$$|\gamma(x,y)| \leqslant C\log(dist(x,\partial\Omega)), \qquad \left|\frac{\partial\gamma(x,y)}{\partial y}\right| \leqslant C\log^2(dist(x,\partial\Omega)). \tag{3.6}$$

Proposition 3.1. *We consider the system of ODEs*

$$\begin{cases} \dfrac{d\mathsf{R}_i}{dt} = \dfrac{2}{|\log\varphi|}\dfrac{1}{\mathsf{R}_i}\left(\dfrac{1}{\bar{\mathsf{R}}} - \dfrac{1}{\mathsf{R}_i}\right), & T_{i-1} < t < T_i, \quad i = 1,\dots,N, \\ \dfrac{d\xi_i}{dt} = -\dfrac{4}{|\log\varphi|}\sum_{h\neq i}\left(\dfrac{1}{\bar{\mathsf{R}}} - \dfrac{1}{\mathsf{R}_h}\right)\dfrac{\xi_h - \xi_i}{|\xi_h - \xi_i|^2}, \end{cases} \tag{3.7}$$

where

$$\frac{1}{\bar{\mathsf{R}}} = \frac{1}{N}\sum_{j=1}^{N}\frac{1}{\mathsf{R}_j}$$

and we assume that $\mathsf{R}_1(0) < \mathsf{R}_2(0) < \cdots < \mathsf{R}_N(0)$.

For system (3.7) *the following properties hold true*:

(i) *If* $\mathsf{R}_i(0) < \mathsf{R}_j(0)$ *then* $\mathsf{R}_i(t) < \mathsf{R}_j(t)$ *on their common domain of existence.*

(ii) $\frac{d}{dt}\left(\sum_{i=1}^{N}\mathsf{R}_i^2(t)\right) = 0$.

(iii) $\frac{d}{dt}\left(\sum_{i=1}^{N}\mathsf{R}_i(t)\right) \leqslant 0$.

(iv) $\mathsf{R}_1(t)$ *is nonincreasing in time and* $\mathsf{R}_N(t)$ *is nondecreasing.*

(v) *If we assume* $\mathsf{R}_{N-1}(0) < \mathsf{R}_N(0)$ *then all except the Nth particle get extinct in finite times* $T_1 < \cdots < T_{N-1}$ *and we have the estimate*

$$\frac{|\log\varphi|}{6}\mathsf{R}_1^3(0) \leqslant T_1 \leqslant \frac{|\log\varphi|}{6}\mathsf{R}_1^3(0)\frac{N\mathsf{R}_N(0)}{\mathsf{R}_N(0) - \mathsf{R}_1(0)}$$

(vi) *System* (3.7) *has a scale invariance compatible with the* $t^{\frac{1}{3}}$ *law, that is if* $\mathsf{R}(t), \xi(t)$ *is a solution then so is* $\mu\mathsf{R}(\frac{t}{\mu^3}),\ \mu\xi(\frac{t}{\mu^3}),\ \mu > 0$.

Proof. (i) Suppose that there exists $t^* > 0$ such that $\mathsf{R}_i(t^*) = \mathsf{R}_j(t^*) > 0$. Then we observe that $\mathsf{R}_i(t)$ and $\mathsf{R}_j(t)$ satisfy the same equation. By uniqueness, we can conclude that $\mathsf{R}_i(0) = \mathsf{R}_j(0)$, contradicting the assumption.

(ii) Between extinction times we have

$$\begin{aligned}\frac{d}{dt}\left(\sum_{i=1}^{N}\mathsf{R}_i^2(t)\right) &= \sum_{i=1}^{N}2\mathsf{R}_i\dot{\mathsf{R}}_i = \sum_{i=1}^{N}2\mathsf{R}_i\frac{1}{\mathsf{R}_i}\frac{2}{|\log\varphi|}\left(\frac{1}{\bar{\mathsf{R}}} - \frac{1}{\mathsf{R}_i}\right)\\ &= \frac{4}{|\log\varphi|}\sum_{i=1}^{N}\left(\frac{1}{\bar{\mathsf{R}}} - \frac{1}{\mathsf{R}_i}\right) = \frac{4}{|\log\varphi|}\left(\frac{N}{\bar{\mathsf{R}}} - \sum_{i=1}^{N}\frac{1}{\mathsf{R}_i}\right) = 0.\end{aligned}$$

(iii) Between extinction times we have

$$\begin{aligned}\frac{d}{dt}\sum_{i=1}^{N}\mathsf{R}_i(t) &= \sum_{i=1}^{N}\dot{\mathsf{R}}_i(t) = \sum_{i=1}^{N}\frac{1}{\mathsf{R}_i}\frac{2}{|\log\varphi|}\left(\frac{1}{\bar{\mathsf{R}}} - \frac{1}{\mathsf{R}_i}\right)\\ &= \frac{2}{|\log\varphi|}\left(\frac{1}{\bar{\mathsf{R}}}\sum_{i=1}^{N}\frac{1}{\mathsf{R}_i} - \sum_{i=1}^{N}\frac{1}{\mathsf{R}_i^2}\right) = \frac{2}{|\log\varphi|}\left(\frac{N}{\bar{\mathsf{R}}^2} - \sum_{i=1}^{N}\frac{1}{\mathsf{R}_i^2}\right) \leqslant 0.\end{aligned}$$

(iv) We have that

$$\frac{1}{\bar{\mathsf{R}}} = \frac{1}{N}\sum_{j=1}^{N}\frac{1}{\mathsf{R}_j} \leqslant \frac{1}{N}\frac{N}{\mathsf{R}_1} = \frac{1}{\mathsf{R}_1}.$$

Analogously,

$$\frac{1}{\bar{\mathsf{R}}} \geqslant \frac{1}{\mathsf{R}_N}.$$

So,

$$\frac{1}{\mathsf{R}_N} \leqslant \frac{1}{\bar{\mathsf{R}}} \leqslant \frac{1}{\mathsf{R}_1} \Leftrightarrow \mathsf{R}_1(t) \leqslant \bar{\mathsf{R}}(t) \leqslant \mathsf{R}_N(t).$$

The result follows immediately from the above expression and the use of system (3.7).

(v) We have that

$$\frac{d\mathsf{R}_1(t)}{dt} = \frac{2}{|\log\varphi|}\frac{1}{\mathsf{R}_1}\left(\frac{1}{\bar{\mathsf{R}}} - \frac{1}{\mathsf{R}_1}\right) = \frac{2}{|\log\varphi|}\frac{1}{\mathsf{R}_1^2}\left(\frac{\mathsf{R}_1}{\bar{\mathsf{R}}} - 1\right) \geqslant -\frac{2}{|\log\varphi|}\frac{1}{\mathsf{R}_1^2}.$$

By integration, we obtain the inequality on the left

$$\mathsf{R}_1^3(t) \geqslant -\frac{6}{|\log\varphi|}t + \mathsf{R}_1^3(0).$$

For the inequality on the right, we have the estimate

$$\begin{aligned}\frac{1}{\bar{\mathsf{R}}} - \frac{1}{\mathsf{R}_1} &= \frac{1}{N}\left(\frac{1}{\mathsf{R}_1} + \cdots + \frac{1}{\mathsf{R}_N}\right) - \frac{N}{N\mathsf{R}_1} = \frac{1}{N}\left[\left(\frac{1}{\mathsf{R}_2} - \frac{1}{\mathsf{R}_1}\right) + \cdots + \left(\frac{1}{\mathsf{R}_N} - \frac{1}{\mathsf{R}_1}\right)\right]\\ &\leqslant \frac{1}{N}\left(\frac{1}{\mathsf{R}_N} - \frac{1}{\mathsf{R}_1}\right) = \frac{1}{N}\left(\frac{\mathsf{R}_1 - \mathsf{R}_N}{\mathsf{R}_1\mathsf{R}_N}\right).\end{aligned}$$

By making use of this estimate, we compute

$$\begin{aligned}\frac{d\mathsf{R}_1(t)}{dt} &= \frac{2}{|\log\varphi|}\frac{1}{\mathsf{R}_1}\left(\frac{1}{\bar{\mathsf{R}}} - \frac{1}{\mathsf{R}_1}\right) \leqslant \frac{2}{|\log\varphi|}\frac{1}{\mathsf{R}_1}\frac{1}{N}\left(\frac{\mathsf{R}_1 - \mathsf{R}_N}{\mathsf{R}_1\mathsf{R}_N}\right)\\ &\leqslant \frac{2}{|\log\varphi|}\frac{1}{N}\frac{1}{\mathsf{R}_1^2}\left(\frac{\mathsf{R}_1 - \mathsf{R}_N}{\mathsf{R}_N}\right) \leqslant \frac{2}{|\log\varphi|}\frac{1}{N}\frac{1}{\mathsf{R}_1^2}\left(\frac{\mathsf{R}_1(0) - \mathsf{R}_N(0)}{\mathsf{R}_N(0)}\right).\end{aligned}$$

Integrating we obtain the right-hand side in (v).

(vi) Straightforward verification.

The proof of Proposition 3.1 is complete. □

4. The operators T, L and A and the mean curvature in ξ, ρ, r coordinates

The operator T: Consider $\Omega \subset \mathbb{R}^2$ a bounded, smooth, connected set in $\mathbb{R}^2$ and Γ a $C^{1+\alpha}$ closed, orientable surface in Ω, Ω^- represents the part of Ω enclosed by Γ and $\Omega^+ = \Omega \backslash \bar{\Omega}^-$. Set

$$T\phi = \frac{\partial u^-}{\partial n^-} + \frac{\partial u^+}{\partial n^+} =: \left[\left[\frac{\partial u}{\partial n}\right]\right]_\Gamma, \quad x \in \Gamma. \tag{4.1}$$

For $\Gamma \in C^{1+\alpha}$, $\phi : \Gamma \to \mathbb{R}$ a sufficiently regular function satisfying $\int_\Gamma \phi = 0$ and u^-, u^+ are the solutions to the Dirichlet problems

$$\begin{cases} -\Delta u^- = 0, & x \in \Omega^-, \\ u^- = \phi, & x \in \Gamma, \end{cases}$$

$$\begin{cases} -\Delta u^+ = 0, & x \in \Omega^+, \\ u^+ = \phi, & x \in \Gamma, \\ \dfrac{\partial u^-}{\partial v} = 0, & x \in \partial\Omega, \end{cases}$$

where n^-, n^+ are the outward normals to $\partial\Omega^-$, $\partial\Omega^+$ and $\vec{v}$ is the outward normal to $\partial\Omega$. T is the Dirichlet–Neumann operator

$$T\phi = \left[\left[\frac{\partial u}{\partial n}\right]\right]_\Gamma = \psi \tag{4.2}$$

and is invertible in the class

$$\int_\Gamma \psi = \int_\Gamma \phi = 0,$$

where the inverse is given by

$$(S\psi)(x) = \int_\Gamma g(x,y)\psi(y)\, dy - \frac{1}{|\Gamma|}\int_\Gamma \int_\Gamma g(x,y)\psi(y)\, dy\, dx. \tag{4.3}$$

The operator S can be interpreted as the inverse of the restriction T to the set of functions satisfying $\int_\Gamma \phi = 0$. In fact if

$$u(x) = \int_\Gamma g(x,y)\psi(y)\, dy - \frac{1}{|\Gamma|}\int_\Gamma \int_\Gamma g(x,y)\psi(y)\, dy\, dx, \quad x \in \Omega, \tag{4.4}$$

then u is harmonic in Ω^-, Ω^+, $\left[\left[\frac{\partial u}{\partial n}\right]\right]_\Gamma = \psi$ and so S is the Neumann–Dirichlet operator.

Let $X = \{\phi \in L^2(\Gamma), \int_\Gamma \phi = 0\}$. We denote by $X^{-\frac{1}{2}}, X^{\frac{1}{2}}$ the closures of $\{\phi \in C^1(\Gamma),\ \int_\Gamma \phi = 0\}$ under the norms

$$||\cdot||_{-\frac{1}{2}} = \sqrt{\langle S\cdot, \cdot\rangle_{L^2(\Gamma)}}, \quad ||\cdot||_{\frac{1}{2}} = \sqrt{\langle T\cdot, \cdot\rangle_{L^2(\Gamma)}}, \tag{4.5}$$

where $\langle , \rangle_{L^2(\Gamma)}$ denotes the standard inner product in $L^2(\Gamma)$, $X^{-\frac{1}{2}}$, $X^{\frac{1}{2}}$ are Hilbert spaces and Eq. (4.5) implies that

$$||\phi||_{\frac{1}{2}} = ||\psi||_{-\frac{1}{2}} \tag{4.6}$$

and so T, S are isometries,

$$T : X^{\frac{1}{2}} \to X^{-\frac{1}{2}},$$

$$S : X^{-\frac{1}{2}} \to X^{\frac{1}{2}}$$

with

$$ST = id_{X^{\frac{1}{2}}}, \quad TS = id_{X^{-\frac{1}{2}}}. \tag{4.7}$$

The operator L: L is the classical Jacobi operator

$$L = \Delta_{\Gamma_0} + k^2, \tag{4.8}$$

where $\Gamma_0 = S^1$, Δ_{Γ_0} is the Laplace–Beltrami operator on Γ_0 and k^2 is the principal curvature of Γ_0.

The operator A: The linearized Mullins–Sekerka operator A at $\Gamma_0 = S^1$ is given by

$$A = TL \tag{4.9}$$

considering conservative perturbations along the normal direction and where T, L are defined as above.

Remarks.

- In what follows, we use the operator T_0 defined by

$$T_0\mathcal{X} = \frac{\partial u^-}{\partial n^-} + \frac{\partial u^+}{\partial n^+} \quad \text{in } S^1, \quad \mathcal{X} \in C^{1+\alpha}(S^1),$$

where u^-, u^+ are the harmonic functions determined by

$$\begin{cases} -\Delta u^- = 0 & \text{on } B_1 = \{x \in \mathbb{R}^2 / |x| < 1\}, \\ u^- = \mathcal{X} & \text{on } S^1, \end{cases}$$

$$\begin{cases} -\Delta u^+ = 0 & \text{on } \mathbb{R}^2 \backslash \bar{B}_1, \\ u^+ = \mathcal{X} & \text{on } S^1, \\ \lim_{x \to \infty} u^+ = 0. \end{cases}$$

T_0 is the analog of T for $\Gamma = S^1$ and Ω replaced by $\mathbb{R}^2$ and it holds that

$$T_0 Y_n = \frac{\partial u^-}{\partial n^-} + \frac{\partial u^+}{\partial n^+} = \frac{n}{2} Y_n,$$

where Y_n are the spherical harmonics of degree n [23].

- The linearization of the one phase Mullins–Sekerka operator for general interfaces Γ has been determined by Kimura [26], for the sphere it can be found in [21]. More information about T, L, A operators and their spectrum can be found in [5].

In what follows, we prefer to rescale R and r according to $R = \varepsilon\rho$, $\mathsf{r} = \varepsilon r$ and introduce the quantities ρ, r which are of order 1.

Eq. (2.1) reads

$$\Gamma(t) = \{x/x = \xi + \varepsilon\rho(1 + \varepsilon r(u))u, \quad u \in S^1\}. \tag{$*$}$$

Instead of $(*)$, we could more precisely scale $R = \varepsilon\rho$, $\mathsf{r} = \varepsilon r$ with two independent small parameters ε_1, ε_2 and write

$$\Gamma(t) = \{x/x = \xi + \varepsilon_1\rho(1 + \varepsilon_2 r(u))u, \quad u \in S^1\}.$$

Going through the various steps of the proof of Proposition 6.1, it appears that the whole argument applies to this more general situation. We limit ourselves to case $(*)$ where $\varepsilon_1 = \varepsilon_2 = \varepsilon$ because all our estimates were originally done under this assumption and also to keep the notation of already complicated computations into reasonable limits.

The mean curvature in ξ, ρ, r coordinates: We would like to derive an expression for the mean curvature $H(X(u))$ of Γ at a point $X(u)$ assuming that $\Gamma = \{x/x = \xi + \varepsilon\rho(1 + \varepsilon r(u))u,\ u \in S^1\}$ with $r \in C^{1+\alpha}(S^1)$.

Proposition 4.1. *The mean curvature $H(X(u))$ of Γ has the form*

$$H(X(u)) = \frac{1}{\varepsilon\rho}(1 - \varepsilon L r + B), \tag{4.10}$$

where L is the classical Jacobi operator on S^1 as described above and B is an operator of the form $B = b(\varepsilon r, \varepsilon G r, \varepsilon G^2 r)$ with $G = \left((1 + \varepsilon r)^2 + \varepsilon^2 r_\vartheta^2\right)^{\frac{1}{2}}$ and $b(z, p, P)$ is a linear *function in P that satisfies the estimate*

$$|b(z,p,P)| \leqslant C\Big(|z|^2 + |p|^2 + (|z| + |p|)|P|\Big) \quad \textit{for } |z| < \delta.$$

Proof. H as a function of r, ϑ (r, ϑ polar coordinates) has the following expression:

$$H = \frac{1}{\varepsilon\rho(1+\varepsilon r)G^{\frac{1}{2}}}\left\{1 + \varepsilon r - \varepsilon r_{\vartheta\vartheta} - \varepsilon r_\vartheta \frac{\cos\vartheta}{\sin\vartheta} + \frac{1}{G}\{\varepsilon r_\vartheta[\varepsilon(1+\varepsilon r)r_\vartheta + \varepsilon^2 r_\vartheta r_{\vartheta\vartheta}]\}\right\}, \tag{4.11}$$

where

$$G = \Big((1+\varepsilon r)^2 + \varepsilon^2 r_\vartheta^2\Big)^{\frac{1}{2}}.$$

The desired result is obtained by (4.11) and also by using the expression of the Laplace–Beltrami operator $\Delta^s r$ on S^1 which takes the form $\frac{1}{\sin\vartheta}((r_\vartheta \sin\vartheta)_\vartheta)$ [19]. □

5. The coordinate system

The definition of coordinate system is very important. Given an interface close to circular, we associate to it in a unique way a circle and view the interface as a small perturbation of that circle. Specifically, if the interface is already circular, the procedure associates the same circle. So, to each interface in a certain class we will give a center and a radius. There are many different coordinate systems that can be used to accomplish this. The important fact about the coordinate system comes from the way we intend to utilize it which is for studying the global stability properties of the circular shape for a class of geometric operators related to the mean curvature. By $S_{\xi,\rho} \subset \mathbb{R}^2$ we denote the circle of center ξ and radius ρ.

Proposition 5.1. *Given an interface Γ in a sufficiently small C^1 neighborhood of a circle $S_{\tilde{\xi},\tilde{\rho}}$, there are unique $\xi \in \mathbb{R}^2$, $\rho > 0$, $r \in C^1(S^1)$ such that*

$$\Gamma = \{x/x = \xi + \varepsilon\rho(1 + \varepsilon r(u))u, \quad u \in S^1\}, \tag{5.1}$$

$$\int_{S^1} r(u)\, du = 0, \tag{5.2}$$

$$\int_{S^1} r(u)\langle u, e_i\rangle \, du = 0, \quad i = 1, 2, \tag{5.3}$$

where $e_1 = (1,0)$, $e_2 = (0,1)$ *and* $\langle \cdot, \cdot \rangle$ *is the euclidian inner product in* $\mathbb{R}^2$.

Proof. The representation of all Γ's in a C^1 neighborhood of $S_{\tilde{\xi},\tilde{\rho}}$ is given by the form

$$\Gamma = \{x/x = \tilde{\xi} + \varepsilon\tilde{\rho}(1 + \varepsilon\tilde{r}(\tilde{u}))\tilde{u}, \quad \tilde{u} \in S^1\}.$$

Choosing the new origin at ξ and for all ξ's in the neighborhood of $\tilde{\xi}$ we have an alternative expression

$$\Gamma = \{x/x = \xi + \varepsilon\rho(1 + \varepsilon r(u))u, \quad u \in S^1\},$$

where ξ, ρ, r and $\tilde{\xi}, \tilde{\rho}, \tilde{r}$ are related through

$$\tilde{\xi} + \varepsilon\tilde{\rho}(1 + \varepsilon\tilde{r}(\tilde{u}))\tilde{u} = \xi + \varepsilon\rho(1 + \varepsilon r(u))u. \tag{5.4}$$

From the above equation we have

$$u = \frac{\tilde{\xi} - \xi + \varepsilon\tilde{\rho}(1 + \varepsilon\tilde{r}(\tilde{u}))\tilde{u}}{|\tilde{\xi} - \xi + \varepsilon\tilde{\rho}(1 + \varepsilon\tilde{r}(\tilde{u}))\tilde{u}|} \tag{5.5}$$

and

$$\varepsilon\rho(1 + \varepsilon r(u)) = |\tilde{\xi} - \xi + \varepsilon\tilde{\rho}(1 + \varepsilon\tilde{r}(\tilde{u}))\tilde{u}|. \tag{5.6}$$

Condition (5.2) is equivalent to taking

$$\varepsilon\rho = \frac{1}{2\pi}\int_{S^1} |\tilde{\xi} - \xi + \varepsilon\tilde{\rho}(1 + \varepsilon\tilde{r}(\tilde{u}(u)))\tilde{u}(u) \, du \tag{5.7}$$

and ξ is the remaining free variable while the map $\tilde{u} \to u$ for $|\tilde{\xi} - \xi|$ small is a C^1 diffeomorphism. What we would like to show next is to choose ξ such that Eq. (5.3) is satisfied. In other words,

$$\int_{S^1} |\tilde{\xi} - \xi + \varepsilon\tilde{\rho}(1 + \varepsilon\tilde{r}(\tilde{u}))\tilde{u}|\langle u, e_i\rangle du = 0, \quad i = 1, 2. \tag{5.8}$$

Hence, (5.8) is equivalent to solving the system

$$0 = F_i(\xi, \varepsilon\tilde{\rho}, \varepsilon\tilde{r}) := \int_{S^1} |\tilde{\xi} - \xi + \varepsilon\tilde{\rho}(1 + \varepsilon\tilde{r}(\tilde{u}))\tilde{u}|\langle u, e_i\rangle du = 0, \quad i = 1, 2, \tag{5.9}$$

where $\tilde{u} = \tilde{u}(u, \xi, \varepsilon\tilde{\rho}, \varepsilon\tilde{r})$ is implicitly defined by (5.5). So, we seek to solve

$$F_i(\xi, \varepsilon\tilde{\rho}, \varepsilon\tilde{r}) = 0, \quad i = 1, 2$$

and we observe that

$$F_i(\tilde{\xi}, \varepsilon\tilde{\rho}, 0) = 0.$$

We would like to employ the implicit function theorem. For this purpose we need to calculate

$$\begin{aligned} D_\varepsilon F_i(\tilde{\xi}, \varepsilon\tilde{\rho}, 0)\hat{\xi} &= D_\xi \int_{S^1} |\tilde{\xi} + \varepsilon\tilde{\rho}(1 + \varepsilon\tilde{r}(\tilde{u}))\tilde{u}| \langle u, e_i \rangle du(\hat{\xi}) \\ &= D_\xi \int_{S^1} \langle \tilde{\xi} + \varepsilon\tilde{\rho}(1 + \varepsilon\tilde{r}(\tilde{u}))\tilde{u}, u \rangle \langle u, e_i \rangle du(\hat{\xi}) \\ &= \int_{S^1} \langle -\hat{\xi} + \varepsilon\tilde{\rho} D_\varepsilon \tilde{u}\hat{\xi}, u \rangle \langle u, e_i \rangle du, \quad \hat{\xi} \in \mathbb{R}^2, \end{aligned} \tag{5.10}$$

where D_ξ is the gradient of F_i with respect to the first entry and we have set

$$D_\xi \tilde{u} = D_\xi \tilde{u}(u, \tilde{\xi}, \varepsilon\tilde{\rho}, 0).$$

Moreover, by differentiation of (5.5) with $\tilde{r} = 0$, for $\xi = \tilde{\xi}$ we get

$$-\hat{\xi} + \varepsilon\tilde{\rho} D_\xi \tilde{u}\hat{\xi} - \langle -\hat{\xi} + \varepsilon\tilde{\rho} D_\xi \tilde{u}\hat{\xi},\ u \rangle u = 0. \tag{5.11}$$

By using the fact that $\langle D_\xi \tilde{u}\hat{\xi}, u \rangle = 0$ we obtain from (5.11) that

$$\varepsilon\tilde{\rho} D_\xi \tilde{u}\hat{\xi} = \hat{\xi} - \langle \hat{\xi}, u \rangle u. \tag{5.12}$$

From (5.12) and (5.10) we conclude

$$D_\xi F_i(\tilde{\xi}, \varepsilon\tilde{\rho}, 0)\hat{\xi} = -\int_{S^1} \langle u, \hat{\xi} \rangle \langle u, e_i \rangle du = -\pi\hat{\xi}_i. \tag{5.13}$$

So, the implicit function theorem applies for $||\varepsilon\tilde{r}||_{C^1(S^1)} < \delta$, for $\delta > 0$ and claims that Eq. (5.9) has a unique solution $\xi = \xi(\varepsilon\tilde{r}, \tilde{\xi}, \varepsilon\tilde{\rho})$ such that $\xi = \xi(0, \tilde{\xi}, \varepsilon\tilde{\rho}) = \tilde{\xi}$. □

Remarks.

- It is important to note that $r(u)$ is the distortion from circularity and we would like it to say under control during the evolution. By imposing condition (5.3) we remove from it the element corresponding to translation and so we have r under control. The spectrum of the restricted operator is also stable, something which reflects the stability of the circular shape and suggests that the coordinate system will be preserved along evolution. In order to see the meaning of condition (5.3), we observe that the translate $S_{\xi+\tilde{\delta}e_i,\bar{\rho}}$ of $S_{\bar{\xi},\bar{\rho}}$ by $\tilde{\delta}$, where $\tilde{\delta} > 0$ is a small number, in

the direction e_i is given by $S_{\xi+\bar{\delta}e_i,\bar{\rho}} = \{x/x = \bar{\xi} + (\bar{\rho} + \tilde{\delta}\langle u, e_i\rangle + O(\tilde{\delta}^2))u, u\in S^1\}$. Therefore if Q is an operator which has $S_{\bar{\xi},\bar{\rho}}$ and all its translates as equilibria then $\langle u, e_i\rangle$ has to be a zero eigenfunction of the linearization DQ_0 of Q at $S_{\bar{\xi},\bar{\rho}}$. The Mullins–Sekerka operator has this property and so (5.3) are orthogonality conditions. Condition (5.2) implies that, to principal order, ρ is the average radius of Γ.
- Conditions (5.2) and (5.3) introduce 3 constraints but we also have 3 parameters $\xi\in\mathbb{R}^2$ and $\rho>0$.
- We can find similar coordinate systems in [2,11] concerning three and two dimensions, respectively.

6. Solving the linear equation $S(V) = H - \bar{H}$ for given H

As it was mentioned in Section 3, instead of solving the two-space dimension Mullins–Sekerka problem, we will solve the integral equations along the boundaries of the evolving domains. We have the integral formulation

$$\int_{\Gamma(t)} g(x,y)V(y)\, dS_y - \frac{1}{|\Gamma(t)|}\int_{\Gamma(t)}\int_{\Gamma(t)} g(x,y)V(y)\, dS_y\, dS_x = H(x) - \bar{H}. \quad (6.1)$$

Throughout this section, we write Γ instead of $\Gamma(t)$ and we take $\Gamma = \bigcup_{i=1}^N \Gamma_i$ with $\Gamma_i = \{x/x = X^i(u) := \xi_i + \varepsilon\rho_i(1+\varepsilon r_i(u))u,\ u\in S^1\}$. For $\varepsilon>0$ small, the map $X^i : S^1\to\Gamma_i$ is a diffeomorphism with the same regularity as r_i. We let $u^i : \Gamma_i\to S^1$ be the inverse of X^i. Eq. (6.1) can be written in the form

$$\sum_{h=1}^N \int_{\Gamma_h} g(x,y)V_h(u^h(y))\, dy = H(x) - E, \quad x\in\Gamma_i, \quad i = 1,\dots,N,$$

where

$$E = \bar{H} - \frac{1}{|\Gamma|}\int_\Gamma\int_\Gamma g(x,y)V(y)\, dy\, dx$$

and $V_h(u^h(\cdot))$ is the restriction of V to Γ_h. We can see Eq. (6.1) as an equation in V which is nonlinear in Γ due to the nonlinear dependence of H on the representation of the interface. We are going to solve the above linear equation for H given and then we are going to derive an expression for V in terms of ξ, ρ and r.

Proposition 6.1. *Let $\xi_i\in\Omega$, $\rho_i>0$, $r_i\in C^{1+\alpha}(S^1)$, $i = 1,\dots,N$ be given and assume $\xi_i\neq\xi_j$ for $i\neq j$. Then the system*

$$\sum_{h=1}^N \int_{\Gamma_h} g(x,y)V_h(u^h(y))\, dy = H(x) - E, \quad x\in\Gamma_i, \quad i = 1,\dots,N \quad (6.2)$$

has a unique solution $V_i \in C^\alpha(S^1)$ *and*

$$||\varepsilon\rho_i V_i - \varepsilon\rho_i \bar{V}_i + K||_{C^\alpha(S^1)} \leqslant ||F||, \tag{6.3}$$

where

$$\bar{V}_i = -\frac{1}{\varepsilon\rho_i}\frac{1}{|\log \varepsilon|}\left(\frac{1}{\varepsilon\rho_i} - E\right) + O\left(\sup_h \frac{||V_h||}{|\log \varepsilon|^2}\right)$$

and

$$\begin{aligned} K = & \sum_{h\neq i,h=1}^{N} \frac{1}{2\pi}\varepsilon^2\rho_h^2 \log|\xi_i - \xi_h| \\ & \times \int_{S^1} V_h(v)\, dv + \sum_{h=1}^{N} \varepsilon^2\rho_h^2\gamma(\xi_i,\xi_h) \int_{S^1} V_h(v)\, dv - \frac{1}{\varepsilon\rho_i}T_0 L r_i + \varepsilon^2 T_0 \\ & \times \int_{S^1} \frac{1}{2\pi}\log|\varepsilon\rho_i u - \varepsilon\rho_i v|(3r_i(u) - r_i(v))\bar{V}_i\, dv \end{aligned}$$

and F includes precise estimates for higher order terms

$$\begin{aligned} ||F|| = & O_{C^{1+\alpha}(S^1)}\left(\varepsilon^2||r_i||_{C^{1+\alpha}(S^1)}\rho_h\gamma(\xi_i,\xi_i) + \varepsilon^2\rho_i^2\frac{\partial\gamma(\xi_i,\xi_i)}{\partial x} + \varepsilon^2||r_i||_{C^{1+\alpha}2(S^1)}\varepsilon\rho_h\right)||V_i||_{C^\alpha(S^1)} \\ & + \varepsilon\rho_h O_{C^{1+\alpha}(S^1)}\left(\varepsilon||r_h||_{C^{1+\alpha}(S^1)}\log|\xi_i - \xi_h| + \frac{\varepsilon\rho_i + \varepsilon\rho_h}{|\xi_i - \xi_h|}\right)||V_h||_{C^\alpha(S^1)} \\ & + \varepsilon\rho_h O_{C^{1+\alpha}(S^1)}\left(\varepsilon\rho_h\frac{\partial\gamma(\xi_i,\xi_h)}{\partial y} + \varepsilon||r_h||_{C^{1+\alpha}(S^1)}\gamma(\xi_i,\xi_h)\right)||V_h||_{C^\alpha(S^1)}. \end{aligned}$$

Proof. We have mentioned that the Green's function has the form

$$g(x,y) = -\frac{1}{2\pi}\log|x-y| + \gamma(x,y)$$

and we have

$$\begin{aligned} \int_{\Gamma_h} g(x,y)V_h(u^h(y))\, dy = & \int_{\Gamma_h} -\frac{1}{2\pi}\log|x-y|V_h(u^h(y))\, dy \\ & + \int_{\Gamma_h} \gamma(x,y)V_h(u^h(y))\, dy. \end{aligned} \tag{6.4}$$

In what follows, we employ the notation $f = O_{C^\alpha(S^1)}(||r||_{C^{3+\alpha}(S^1)})$ meaning that the $C^\alpha(S^1)$ norm of f is $O_{C^\alpha(S^1)}(||r||_{C^{3+\alpha}(S^1)})$.

Step 1: We consider the case $h = i$, $x \in \Gamma_i$ and we are interested in the analysis of the two integrals on the right-hand side of Eq. (6.4).

(a) We start our analysis with the study of the first integral on the right-hand side of Eq. (6.4).

Consider the case $h = i$, $x \in \Gamma_i$. Let $\Omega_i = \{z/z = \lambda u, 0 \leqslant \lambda < 1 + \varepsilon r(u),\ u \in S^1\}$ and consider the function $U^i : \Omega_i \to \mathbb{R}$ defined by

$$\begin{aligned} U^i(z) &:= \int_{\Gamma_i} -\frac{1}{2\pi} \log|\xi_i + \varepsilon\rho_i z - y| V_i(u^i(y))\, dy \\ &= \varepsilon\rho_i \int_{\partial\Omega_i} -\frac{1}{2\pi} \log|\varepsilon\rho_i z - \varepsilon\rho_i z'| V_i(u^i(\xi_i + \varepsilon\rho_i z'))\, dz'. \end{aligned} \tag{6.5}$$

As soon as $\partial\Omega_i$ is a surface of class $C^{1+\alpha}(S^1)$, $r_i \in C^{1+\alpha}(S^1)$, we have by Theorem 2.I, p. 307 in [29] applied to the derivatives of U^i, that U^i can be extended as a $C^{1+\alpha}$ function to the closure $\bar{\Omega}_i$ of Ω_i and we have the estimate

$$||U^i(\cdot)||_{C^{1+\alpha}(\bar{\Omega}_i)} \leqslant \varepsilon\rho_i C||V_i(u^i(\xi_i + \varepsilon\rho_i\cdot))||_{C^{\alpha}(\partial\Omega_i)}, \tag{6.6}$$

where C is a constant independent of r under the assumption $||r_i|| < \frac{\delta}{\varepsilon}$. Moreover, the map $z \in \partial\Omega_i \to u^i(\xi_i + \varepsilon\rho_i z) \in S^1$ is a $C^{1+\alpha}$ diffeomorphism and

$$||u^i(\xi_i + \varepsilon\rho_i\cdot)||_{C^{1+\alpha}(\partial\Omega_i)} < \tilde{C}(1 + \varepsilon||r_i||_{C^{1+\alpha}(S^1)}) < C$$

while we have a similar estimate for the inverse map $u \to z$.

$$||V_i(u^i(\xi_i + \varepsilon\rho_i\cdot))||_{C^{1+\alpha}(\partial\Omega_i)} \leqslant C||V_i||_{C^{\alpha}(S^1)}. \tag{6.7}$$

From (6.5)–(6.7), we obtain that we have a map $V_i \in C^{\alpha}(S^1) \to U^i|\partial\Omega_i \in C^{1+\alpha}(S^1)$ and together with the above properties of the diffeomorphism $u \to z(u) := X^i(u) - \xi_i$, we can define a map $I_1^i : C^{\alpha}(S^1) \to C^{1+\alpha}(S^1)$ as follows:

$$\varepsilon\rho_i(I_1^i V_i)(u) = U^i(X^i(u) - \xi_i). \tag{6.8}$$

Moreover, from Eqs. (6.6) and (6.8) we obtain the estimate

$$||I_1^i V_i||_{C^{1+\alpha}(S^1)} \leqslant C||V_i||_{C^{\alpha}(S^1)} \tag{6.9}$$

Our purpose is to compute the main term in $I_1^i V_i$. From (6.5) and

$$dz' = (1 + 2\varepsilon r_i + O_{C^{\alpha}(S^1)}(\varepsilon^2||r_i||^2_{C^{1+\alpha}(S^1)}))\, du \tag{6.10}$$

we have that

$$
\begin{aligned}
(I_1^i V_i)(u) &= \int_{S^1} -\frac{1}{2\pi}\log|X^i(u)-X^i(v)|(1+2\varepsilon r_i(v)+O_{C^\alpha(S^1)}(\varepsilon^2\|r_i\|^2_{C^{1+\alpha}(S^1)}))(v)V_i(v)\,dv \\
&= \int_{S^1} -\frac{1}{2\pi}\log|\varepsilon\rho_i u-\varepsilon\rho_i v|V_i(v)\,dv - \varepsilon\int_{S^1}\frac{1}{2\pi}\log|\varepsilon\rho_i u-\varepsilon\rho_i v|2r_i(v)V_i(v)\,dv \\
&\quad -\varepsilon^2\int_{S^1}\frac{1}{2\pi}\log|\varepsilon\rho_i u-\varepsilon\rho_i v|O_{C^\alpha(S^1)}(\|r_i\|^2_{C^{1+\alpha}(S^1)})(v)V_i(v)\,dv \\
&\quad -\varepsilon\int_{S^1}\frac{1}{2\pi}\log|\varepsilon\rho_i u-\varepsilon\rho_i v|\left(\frac{\log|\varepsilon\rho_i u-\varepsilon\rho_i v|-\log|X^i(u)-X^i(v)|}{\log|X^i(u)-X^i(v)|}\right) \\
&\quad \cdot(1+2\varepsilon r_i+O_{C^\alpha(S^1)}(\varepsilon^2\|r_i\|^2_{C^{1+\alpha}(S^1)}))V_i(v)\,dv.
\end{aligned}
\tag{6.11}
$$

By Mirandd [29], $I_1^i V_i$ and the first 3 integrals on the right-hand side of (6.11) are $C^{1+\alpha}(S^1)$ functions. Also the last integral on the right-hand side of (6.11) belongs to $C^{1+\alpha}(S^1)$.

Let $\mathcal{I}V_i$ be the last integral. We have

$$
\begin{aligned}
(\mathcal{I}V_i)(u) &= \int_{S^1} -\frac{1}{2\pi}\log|\varepsilon\rho_i u-\varepsilon\rho_i v| \\
&\quad\times\left(\frac{\log|\varepsilon\rho_i u-\varepsilon\rho_i v|-\log|\varepsilon\rho_i u-\varepsilon\rho_i v+\varepsilon^2(\rho_i r_i(u)u-\rho_i r_i(v)v|}{\log|\varepsilon\rho_i u-\varepsilon\rho_i v+\varepsilon^2(\rho_i r_i(u)u-\rho_i r_i(v)v|}\right) \\
&\quad\times(1+2\varepsilon r_i+O_{C^\alpha(S^1)}(\varepsilon^2\|r_i\|^2_{C^{1+\alpha}(S^1)}))(v)V_i(v)\,dv \\
&=: (\mathcal{I}_1V_i)(u)+(\mathcal{I}_2V_i)(u)+(\mathcal{I}_3V_i)(u).
\end{aligned}
\tag{6.12}
$$

After estimating $(\mathcal{I}_1V_i)$, $(\mathcal{I}_2V_i)$, $(\mathcal{I}_3V_i)$, we conclude that

$$
\begin{aligned}
(\mathcal{I}V_i)(u) = &-\int_{S^1}\frac{1}{2\pi}\log|\varepsilon\rho_i u-\varepsilon\rho_i v|(r_i(u)+r_i(v))V_i(v)\,dv \\
&+\varepsilon O_{C^{1+\alpha}(S^1)}(\|r_i\|^2_{C^{1+\alpha}(S^1)})\|V_i\|_{c^\alpha(S^1)}.
\end{aligned}
\tag{6.13}
$$

(b) We now study the second integral on the right-hand side of Eq. (6.4)

$$
\begin{aligned}
&\int_{\Gamma_i}\gamma(X^i(u),y)V_i(u^i(y))\,dy \\
&\quad=\varepsilon\rho_i\int_{\partial\Omega_i}\gamma(X^i(u),\xi_i+\varepsilon\rho_i z)V_i(u^i(\xi_i+\varepsilon\rho_i z))\,dz =: \varepsilon\rho_i(I_2^iV_i)(u).
\end{aligned}
\tag{6.14}
$$

The definition of $X^i(u)$ implies that

$$
\gamma(X^i(u),\varepsilon\rho_i z)=\gamma(\xi_i,\xi_i)+O_{C^{1+\alpha}(S^1)}\left(\varepsilon\rho_i\frac{\partial\gamma(\xi_i,\xi_i)}{\partial x}\right).
\tag{6.15}
$$

From (6.14) and (6.15) it follows

$$I_2^i V_i = \gamma(\xi_i, \xi_i) \int_{S^1} V_i(v)dv + O_{C^{1+\alpha}(S^1)}\left(\varepsilon ||r_i||_{C^{1+\alpha}(S^1)} \gamma(\xi_i, \xi_i) + \varepsilon\rho_i \frac{\partial \gamma(\xi_i, \xi_i)}{\partial x}\right) \times ||V_i||_{C^\alpha(S^1)}. \tag{6.16}$$

Eqs. (6.16) and (6.7) imply the estimate

$$||I_2^i V_i||_{C^{1+\alpha}(S^1)} \leqslant C||V_i||_{C^\alpha(S^1)}. \tag{6.17}$$

So, from Eqs. (6.4), (6.11), (6.13), (6.16) we conclude

$$\int_{\Gamma_i} g(X^i(u), y) V_i(u^i(y))\, dy = \varepsilon\rho_i \int_{S^1} \left(-\frac{1}{2\pi} \log|\varepsilon\rho_i u - \varepsilon\rho_i v| + \gamma(\xi_i, \xi_i) \right) V_i(v)\, dv + \varepsilon\rho_i (I^{ii} V_i)(u), \tag{6.18}$$

where I^{ii} satisfies

$$||I^{ii} V_i||_{C^{1+\alpha}(S^1)} \leqslant C||V_i||_{C^\alpha(S^1)} \tag{6.19}$$

and

$$C = O_{C^{1+\alpha}(S^1)}\left(\varepsilon ||r_i||_{C^{1+\alpha}(S^1)} \gamma(\xi_i, \xi_i) + \varepsilon\rho_i \frac{\partial \gamma(\xi_i, \xi_i)}{\partial x}\right). \tag{6.20}$$

Step 2: We now consider the case $h \neq i$, where $x \in \Gamma_i$ and $y \in \Gamma_h$. Again our aim is to analyze the two integrals on the right-hand side of Eq. (6.4).

For $h \neq i$ and $X = X^i(u)$ both integrals on the right-hand side of Eq. (6.4) have as functions of $u \in S^1$ the same smoothness as X^i. The first integral on the right-hand side of Eq. (6.4) gives

$$\begin{aligned}
&\int_{\Gamma_h} -\frac{1}{2\pi} \log|X^i(u) - y| V_h(u^h(y))dy \\
&= \varepsilon\rho_h \int_{S^1} -\frac{1}{2\pi} \log|X^i(u) - X^h(v)|(1 + 2\,\varepsilon r_h(v) + O_{C^\alpha(S^1)}(\varepsilon^2 ||r_h||^2_{C^{1+\alpha}(S^1)})) V_h(v)\, dv \\
&= \varepsilon\rho_h \int_{S^1} -\frac{1}{2\pi} \log|\xi_i - \xi_h| V_h(v)\, dv \\
&\quad + \varepsilon\rho_h O_{C^{1+\alpha}(S^1)}\left(\varepsilon ||r_h||_{C^{1+\alpha}(S^1)} \log|\xi_i - \xi_h| + \frac{\xi\rho_i + \varepsilon\rho_h}{|\xi_i - \xi_h|}\right) ||V_h||_{C^\alpha(S^1)}.
\end{aligned} \tag{6.21}$$

The second integral on the right-hand side of Eq. (6.4) implies

$$\begin{aligned}
&\int_{\Gamma_h} \gamma(X^i(u), y) V_h(u^h(y))\, dy \\
&\quad = \varepsilon\rho_h \int_{S^1} \gamma(X^i(u), X^h(v))(1 + 2\varepsilon r_h(v) + O_{C^\alpha(S^1)}(\varepsilon^2 ||r_h||^2_{C^{1+\alpha}(S^1)})) V_h(v)\, dv \\
&\quad = \varepsilon\rho_h \gamma(\xi_i, \xi_h) \int_{S^1} V_h(v) dv + \varepsilon\rho_h O_{C^{1+\alpha}(S^1)}\bigg(\varepsilon\rho_i \frac{\partial\gamma(\xi_i, \xi_h)}{\partial x} + \varepsilon\rho_h \frac{\partial\gamma(\xi_i, \xi_h)}{\partial y} \\
&\qquad + \varepsilon ||r_h||_{C^{1+\alpha}(S^1)} \gamma(\xi_i, \xi_h)\bigg) ||V_h||_{C^\alpha(S^1)},
\end{aligned} \tag{6.22}$$

where the following expansion has been used:

$$\gamma(X^i(u), X^h(v)) = \gamma(\xi_i, \xi_h) + O_{C^\alpha(S^1)}\left(\varepsilon\rho_i \frac{\partial\gamma(\xi_i, \xi_h)}{\partial x} + \varepsilon\rho_h \frac{\partial\gamma(\xi_i, \xi_h)}{\partial y}\right). \tag{6.23}$$

From (6.21), (6.22) we conclude that (6.4) takes the form

$$\begin{aligned}
&\int_{\Gamma_h} g(X^i(u), y) V_h(u^h(y))\, dy \\
&\quad \int_{\Gamma_h} -\frac{1}{2\pi} \log |X^i(u) - y| V_h(u^h(y))\, dy + \int_{\Gamma_h} \gamma(X^i(u), y) V_h(u^h(y))\, dy \\
&\quad = \varepsilon\rho_h \int_{S^1} \left(-\frac{1}{2\pi} \log |\xi_i - \xi_h| + \gamma(\xi_i, \xi_h)\right) V_h(v)\, dv + \varepsilon\rho_h (I^{ih} V_h)(v),
\end{aligned} \tag{6.24}$$

where

$$\begin{aligned}
I^{ih} V_h = {} & O_{C^{1+\alpha}(S^1)}\left(\varepsilon ||r_h||_{C^{1+\alpha}(S^1)} \log |\xi_i - \xi_h| + \frac{\varepsilon\rho_i + \varepsilon\rho_h}{|\xi_i - \xi_h|}\right) ||V_h||_{C^\alpha(S^1)} \\
& + O_{C^{1+\alpha}(S^1)}\left(\varepsilon\rho_h \frac{\partial\gamma(\xi_i, \xi_h)}{\partial y} + \varepsilon ||r_h||_{C^{1+\alpha}(S^1)} \gamma(\xi_i, \xi_h)\right)
\end{aligned}$$

and I^{ih} is a linear operator that satisfies

$$||I^{ih} V_h|| \leqslant C ||V_h||_{C^\alpha(S^1)}, \quad h \neq i. \tag{6.25}$$

Step 3: We compute $\bar{V}_i$: the average of V_i on the sphere S^1, by utilizing the integral representation (6.2) and the expressions obtained in Steps 1 and 2.

We saw that Eq. (6.1) can be written in the form

$$\sum_{h=1}^{N} \int_{\Gamma_h} g(x, y) V_h(u^h(y))\, dy = H(x) - E, \quad x \in \Gamma_i, \ i = 1, \ldots, N$$

equivalently

$$\sum_{h=1}^{N} \int_{\Gamma_h} g(X^i(u), y) V_h(u^h(y))\, dy = H_i(x) - E, \quad i = 1, \ldots, N.$$

From Eqs. (6.18) and (6.24), we have

$$\begin{aligned} &-\frac{1}{2\pi}\int_{S^1} (\log|\varepsilon\rho_i u - \varepsilon\rho_i v| + 2\pi\gamma(\xi_i, \xi_i))\, \varepsilon\rho_i V_i(v)\, dv \\ &-\frac{1}{2\pi}\sum_{h\neq i} \int_{S^1} (\log|\xi_i - \xi_h| + 2\pi\gamma(\xi_i, \xi_h))\varepsilon\rho_h V_h(v)\, dv \\ &+ \varepsilon\rho_i (I^{ii} V_i)(u) + \sum_{h\neq i} \varepsilon\rho_h (I^{ih} V_h)(u) = H_i - E. \end{aligned} \tag{6.26}$$

From (6.26) we obtain

$$\begin{aligned} \frac{1}{2\pi}\int_{S^1} \log|\varepsilon\rho_i u - \varepsilon\rho_i v|\varepsilon\rho_i V_i(v)\, dv &= -\int_{S^1} \gamma(\xi_i, \xi_i)\, \varepsilon\rho_i V_i(v)\, dv \\ &-\frac{1}{2\pi}\sum_{h\neq i} \int_{S^1} (\log|\xi_i - \xi_h| + 2\pi\gamma(\xi_i, \xi_h))\, \varepsilon\rho_h V_h(v)\, dv \\ &+ \sum_{h=1}^{N} \varepsilon\rho_h (I^{ih} V_h)(u) - H_i + E. \end{aligned} \tag{6.27}$$

Let $\bar{V}_i$ denotes the average of V_i on the circle S^1 then by utilizing $\bar{V}_i$ and $\int_{S^1}(V_i - \bar{V}_i) = 0$, Eq. (6.27) takes the form

$$\begin{aligned} \frac{1}{2\pi}\int_{S^1} \log|\varepsilon\rho_i u - \varepsilon\rho_i v|\varepsilon\rho_i (V_i - \bar{V}_i)\, dv = &-\frac{1}{2\pi}\int_{S^1} \log|\varepsilon\rho_i u - \varepsilon\rho_i v|\varepsilon\rho_i \bar{V}_i dv \\ &- \gamma(\xi_i, \xi_i)\varepsilon\rho_i \bar{V}_i \\ &- \frac{1}{2\pi}\sum_{h\neq i} \log|\xi_i - \xi_h|\varepsilon\rho_h \bar{V}_h \\ &- \sum_{h\neq i} \gamma(\xi_i, \xi_h)\varepsilon\rho_h \bar{V}_h - H_i + E \\ &+ \varepsilon\rho_h \sum_{h=1}^{N} (I^{ih} V_h)(u). \end{aligned} \tag{6.28}$$

Then by the theory of single layer potential, we know that the left-hand side defines an harmonic function which is bounded at infinity. We extend the right-hand side of

Eq. (6.28) in order to have also an harmonic bounded function at infinity

$$\begin{aligned}\frac{1}{2\pi}\int_{S^1} &\log|\varepsilon\rho_i u - \varepsilon\rho_i v|\varepsilon\rho_i(V_i - \bar{V}_i)\,dv\\ &= -\varepsilon\rho_i\bar{V}_i\log r - \gamma(\xi_i,\xi_i)\varepsilon\rho_i\bar{V}_i - \frac{1}{2\pi}\sum_{h\neq i}\log|\xi_i-\xi_h|\varepsilon\rho_h\bar{V}_h\\ &\quad -\sum_{h\neq i}\gamma(\xi_i,\xi_h)\varepsilon\rho_h\bar{V}_h + c_1\log r + c_2 + \varepsilon\rho_h\sum_{h=1}^{N}(I^{ih}V_h)(u). \qquad (6.29)\end{aligned}$$

Our aim is to determine $\bar{V}_i$. We will accomplish it by the following procedure: On the boundary of the ith particle we have the leading order to condition

$$c_1\log\varepsilon\rho_i + c_2 = -\frac{1}{\varepsilon\rho_i} + E. \qquad (6.30)$$

Moreover, in order to obtain that the right-hand side is bounded, we need to impose the following conditions:

$$-\varepsilon\rho_i\bar{V}_i = -c_1, \qquad (6.31)$$

$$-\varepsilon\rho_i\bar{V}_i\gamma(\xi_i,\xi_i) - \frac{1}{2\pi}\sum_{h\neq i}\log|\xi_i-\xi_h|\varepsilon\rho_h\bar{V}_h - \sum_{h\neq i}\gamma(\xi_i,\xi_h)\varepsilon\rho_h\bar{V}_h = -c_2. \qquad (6.32)$$

The above equations (6.30)–(6.32) form a system of $3N$ equations with $3N$ unknowns. By substituting (6.31), (6.32) into (6.30) we obtain that

$$\begin{aligned}(\log|\varepsilon\rho_i| + \gamma(\xi_i,\xi_i))\varepsilon\rho_i\bar{V}_i - \left(-\frac{1}{2\pi}\sum_{h\neq i}\log|\xi_i-\xi_h|\varepsilon\rho_h\bar{V}_h - \sum_{h\neq i}\gamma(\xi_i,\xi_h)\varepsilon\rho_h\bar{V}_h\right)\\ = E - \frac{1}{\varepsilon\rho_i}. \qquad (6.33)\end{aligned}$$

Eq. (6.33) forms a diagonal dominant system for $0<\varepsilon\ll 1$ and it has a unique solution which is given to principal order by the formula

$$\bar{V}_i = -\frac{1}{|\log\varepsilon|}\frac{1}{\varepsilon\rho_i}\left(\frac{1}{\varepsilon\rho_i} - E\right) + O_{C^{1+\alpha}(S^1)}\left(\frac{1}{|\log\varepsilon|^2}\right). \qquad (6.34)$$

Step 4: As soon as the right-hand side of Eq. (6.4) has been analyzed and $\bar{V}_i$ has been determined, we utilize the integral representation (6.2) in order to conclude on to the desired result which is (6.3).

Using again theorem 2.I in [29], we observe that the function $I^{ih}V_h, h = 1,\dots,N$ has a harmonic extension both to the interior $B_i = \{x/|x|<1\}$ and to the exterior $\mathbb{R}^2\backslash\bar{B}_1$ of S^1. These harmonic functions can be extended as a $C^{1+\alpha}$ functions to B_1

and $\mathbb{R}^2\backslash\bar{B}_1$ with the estimate

$$||(I^{ih}V_h)^-||_{C^{1+\alpha}(\bar{B}_1)} \leqslant C||I^{ih}V_h||_{C^{1+\alpha}(S^1)},$$

$$||(I^{ih}V_h)^+||_{C^{1+\alpha}(\mathbb{R}^2\backslash B_1)} \leqslant C||I^{ih}V_h||_{C^{1+\alpha}(S^1)},$$

where the subscripts $\pm$ denote the extensions, and C is a universal constant. From these inequalities, it follows that

$$||T_0 I^{ih}V_h||_{C^{\alpha}(S^1)} \leqslant C||I^{ih}V_h||_{C^{1+\alpha}(S^1)}.$$

After applying T_0 to Eq. (6.28) and utilizing the expression for H given in Proposition 4.1, Eq. (6.28) takes the form

$$\varepsilon\rho_i V_i = \varepsilon\rho_i \bar{V}_i - \sum_{h=1}^{N} \varepsilon\rho_h (T_0 I^{ih}\bar{V}_h) + \frac{1}{\varepsilon\rho_i} O_{C^{1+\alpha}(S^1)}(\varepsilon||r_i||_{C^{1+\alpha}(S^1)}) - T_0 W_i, \tag{6.35}$$

where $W_i = H|\Gamma_i - E = \frac{1}{\varepsilon\rho_i}(1 - \varepsilon L r_i + B_i) - E,\ i = 1,2,\dots,N$ and $T_0 W_i = \frac{1}{\varepsilon\rho_i} T_0 L r_i \frac{\varepsilon}{\varepsilon\rho_i} O_{C^{\alpha}(S^1)}(||r_i||_{C^{1+\alpha}(S^1)}||r_i||_{C^{3+\alpha}(S^1)})$.

For $0<\varepsilon\ll 1$, system (6.35) has a unique solution that can be computed by iteration. The approximate solution $(\rho V)^{(n)} = (\rho_1 V_1^{(n)},\dots,\rho_N V_N^{(n)})$ at step n is computed by solving Eq. (6.35) with $\rho V = (\rho V)^{(n-1)}$ in the right-hand side starting with $(\rho V)^{(0)} = 0$, Moreover, Eqs. (6.11), (6.13), (6.14), (6.18), (6.19) imply

$$\begin{aligned} &||I^{ii}V_i - \int_{S^1} \frac{1}{2\pi}\log|\varepsilon\rho_i u - \varepsilon\rho_i v|(3r_i(v) - r_i(\cdot))V_i(v)\,dv - \varepsilon\rho_i\gamma(\xi_i,\xi_i)\bar{V}_i||_{C^{1+\alpha}(S^1)} \\ &\quad = O_{C^{1+\alpha}(S^1)}\left(\varepsilon||r_i||_{C^{1+\alpha}(S^1)}\gamma(\xi_i,\xi_i) + \varepsilon\rho_i\frac{\partial\gamma(\xi_i,\xi_i)}{\partial x} + \varepsilon^2||r_i||_{C^{1+\alpha}(S^1)}\right)||V_i||_{C^{\alpha}(S^1)} \end{aligned} \tag{6.36}$$

and from Eqs. (6.21), (6.22), (6.24), (6.25)

$$\begin{aligned} &\left|\left|I^{ih}V_h + \varepsilon\rho_h\left(\frac{1}{2\pi}\log|\xi_i - \xi_h| - \gamma(\xi_i,\xi_h)\right)\int_{S^1} V_h(v)\,dv\right|\right|_{C^{1+\alpha}(S^1)} \\ &\quad = O_{C^{1+\alpha}(S^1)}\left(\varepsilon||r_h||_{C^{1+\alpha}(S^1)}\log|\xi_i - \xi_h| + \frac{\varepsilon\rho_i + \varepsilon\rho_h}{|\xi_i - \xi_h|}\right)||V_h||_{C^{\alpha}(S^1)} \\ &\qquad + O_{C^{1+\alpha}(S^1)}\left(\varepsilon\rho_h\frac{\partial\gamma(\xi_i,\xi_h)}{\partial y} + \varepsilon||r_h||_{C^{1+\alpha}(S^1)}\gamma(\xi_i,\xi_h)\right) \\ &\qquad \times ||V_h||_{C^{\alpha}(S^1)}. \end{aligned} \tag{6.37}$$

Inserting these expressions of $I^{ii}V_i, I^{ih}V_h$ into Eq. (6.29) we have the desired result which is Eq. (6.3). $\square$

As soon as $\bar{V}_i$ is known, we notice that the previous proposition reduces the determination of V_i to a system that, to principle order is coupled only through E. The coupling constant E can be determined by the conservation requirement

$$\int_{\Gamma} V = \sum_{h=1}^{N} \int_{\Gamma_h} V_h = 0. \tag{6.38}$$

Proposition 6.2. *Condition* (6.39) *uniquely determines the constant* E. *Moreover*,

$$E = \frac{1}{N} \sum_{j=1}^{N} \frac{1}{\varepsilon \rho_j} + O_{C^{1+\alpha}(S^1)} \left(1 + \frac{||r||_{C^{1+\alpha}(S^1)}}{\bar{\rho}} \right), \tag{6.39}$$

where

$$\frac{1}{\varepsilon \bar{\rho}} = \frac{1}{N} \sum_{j=1}^{N} \frac{1}{\varepsilon \rho_j}$$

and $||r||_{C^{1+\alpha}(S^1)}$ *is the norm of* $r = (r_1, \ldots, r_N)$.

Proof. By inserting the expression of $\bar{V}_i$ obtained from (6.34) into (6.3) under the standing assumption that for $\varepsilon \, ||r_h||_{C^{1+\alpha}(S^1)} < \delta$ for $\delta > 0$ and by [29] we obtain

$$\begin{aligned} \varepsilon \rho_i V_i = {} & \varepsilon \rho_i \bar{V}_i + \frac{1}{\rho_i} O_{C^{\alpha}(S^1)}(||r_i||_{C^{3+\alpha}(S^1)}) + O_{C^{\alpha}(S^1)}(\varepsilon^2 ||r_i||^2_{C^{1+\alpha}(S^1)}) \\ & + \frac{1}{\varepsilon \rho_i} O(\varepsilon ||r_i||_{C^{1+\alpha}(S^1)}). \end{aligned} \tag{6.40}$$

By utilizing the conservation requirement (6.38) we have

$$0 = \sum_{i=1}^{N} \int_{\Gamma_i} V_i = \sum_{i=1}^{N} \varepsilon \rho_i \int_{S^1} V_i (1 + 2\varepsilon r_i + O_{C^{1+\alpha}(S^1)} \, (\varepsilon^2 ||r_i||^2_{C^{1+\alpha}(S^1)})) \, du. \tag{6.41}$$

Substituting Eq. (6.41) into (6.40) and after some calculations we conclude

$$E = \frac{1}{N} \sum_{j=1}^{N} \frac{1}{\varepsilon \rho_j} + O_{C^{1+\alpha}(S^1)} \left(\frac{||r||_{C^{1+\alpha}(S^1)}}{\bar{\rho}} + 1 + \varepsilon (||r||_{C^{1+\alpha}(S^1)} + \bar{\rho}) E \right).$$

If $\varepsilon > 0$ is smaller than some $\bar{\varepsilon} > 0$ this equation can be solved for E yielding estimate (6.39). □

We now give a decomposition result for a general V in terms of $\dot{p}, \dot{\xi}, \rho r_t$ for interfaces with representation (5.1). We let $V = V(u, t)$ to be the speed of $\Gamma(t)$ in the orthogonal direction to $\Gamma(t)$ at the point $x \in \Gamma(t)$ and we study the relationship between V and ρ, ρr_t and $\dot{\xi}$.

Proposition 6.3. *Assume that* $\varepsilon||r||_{C^{1+\alpha}(S^1)} < \delta$ *for* $\delta > 0$ *a small fixed number, so that Proposition* 5.1 *holds. Then* V *is a linear combination of* $\varepsilon\dot{\rho}, \varepsilon^2\rho r_t, \dot{\xi}$ *and the equation* $V = Z$ *with* $Z \in C^\alpha(S^2)$ *a given function, determines uniquely* $\varepsilon\dot{\rho}, \varepsilon^2\rho r_t, \dot{\xi}$. *Moreover, the following estimates hold true*:

$$\begin{cases} |2\sqrt{\pi}\varepsilon\dot{\rho} + \langle Z, w_0 \rangle_{L^2(S^2)}| \leqslant C\varepsilon\Big(\varepsilon||r||^2_{C^{1+\alpha}(S^2)}||Z||_{C^\alpha(S^2)} \\ \quad +||r||_{C^{1+\alpha}(S^2)}\sum_{h=1}^2 |\langle Z, w_h \rangle_{L^2(S^2)}|\Big), \\ |2\sqrt{\frac{\pi}{3}}\dot{\xi}_j + \langle Z, w_j \rangle_{L^2(S^2)}| \leqslant C\varepsilon\Big(\varepsilon||r||^2_{C^{1+\alpha}(S^2)}||Z||_{C^\alpha(S^2)} \\ \quad +||r||_{C^{1+\alpha}(S^2)}\sum_{h=1}^2 |\langle Z, w_h \rangle_{L^2(S^2)}|\Big), \\ ||\varepsilon^2\rho r_t + Z - \sum_{j=0}^2 \langle Z, w_j \rangle_{L^2(S^2)} w_j - \varepsilon\langle Z, w_0 \rangle_{L^2(S^2)} w_0 r||_{C^\alpha(S^2)} \\ \quad \leqslant C\varepsilon\Big(\varepsilon||r||^2_{C^{1+\alpha}(S^2)}||Z||_{C^\alpha(S^2)} + ||r||^2_{C^\alpha(S^2)}\sum_{h=1}^2 |\langle Z, w_h \rangle_{L^2(S^2)}|\Big) \end{cases} \tag{6.42}$$

for some constant $C > 0$ *and* w_j, $j = 0, 1, 2$ *defined as follows*: $w_0 = \frac{1}{\sqrt{2\pi}}, w_1 = \frac{1}{\sqrt{2\pi}}\langle u, e_1 \rangle = \frac{1}{\sqrt{2\pi}}\cos\vartheta, w_2 = \frac{1}{\sqrt{2\pi}}\langle u, e_2 \rangle = \frac{1}{\sqrt{2\pi}}\sin\vartheta.$

Proof. We refer the reader to [5] for a detailed proof which can be easily adapted to two dimensions. □

Proposition 6.4. *There is* $\bar{\varepsilon} > 0$ *such that for* $\varphi < \bar{\varepsilon}^2$ Eq. (6.1) *is equivalent to the following system of evolution equations*:

$$\begin{cases} \varepsilon\dfrac{d\rho_i}{dt} = \dfrac{2}{|\log\varepsilon|}\dfrac{1}{\varepsilon\rho_i}\left(\dfrac{1}{\varepsilon\bar{\rho}} - \dfrac{1}{\varepsilon\rho_i}\right) + f_i^\rho(\rho, \xi, r), \\ \dfrac{dr_i}{dt} = \dfrac{2}{|\log\varepsilon|}\left[\dfrac{1}{\varepsilon^3\rho_i^3}\tilde{A}r_i - \dfrac{1}{\varepsilon^3\rho_i^2\bar{\rho}}(3r_i - T_0 r_i)\right] + f_i^r(\rho, \xi, r), \\ \dfrac{d\xi_i}{dt} = f_i^\xi(\rho, \xi, r), \end{cases} \tag{6.43}$$

where

$$\tilde{A} = T_0(\Delta^s - I) + 3I$$

and $f_i^\rho(\rho, \xi, r), f_i^r(\rho, \xi, r), f_i^\xi(\rho, \xi, r)$ *are smooth, uniformly bounded in* ε *functions of* $\rho = (\rho_1, \ldots, \rho_N), \xi = (\xi_1, \ldots, \xi_N)$ *and* $r = (r_1, \ldots, r_N)$ *and* $\varepsilon\bar{\rho}$ *is the harmonic mean of* $\varepsilon\rho_i$ *defined by*

$$\frac{1}{\varepsilon\bar{\rho}} = \frac{1}{N}\sum_{j=1}^N \frac{1}{\varepsilon\rho_j}.$$

Proof. Eqs. (6.43) follow from Eqs. (6.42) and Proposition 6.1. More details can be found in [5]. □

7. The R, r estimates

Now, we turn back to the old notation $R = \varepsilon\rho, \mathsf{r} = \varepsilon r$ and our aim is to provide estimates for the R, r equations. Especially, the control on r is a very important factor. In the previous sections we assume that $||\mathsf{r}||_{C^{1+\alpha}(S^1)} < \delta$ for $\delta > 0$ a fixed small number. In this section, we are going to establish a uniform bound on $||\mathsf{r}||_{C^{3+\alpha}(S^1)}$. By recalling that $\varphi = \frac{\sum_{i=1}^N \pi R_i^2}{|\Omega|}$, we have from Proposition 6.5 the following system of evolution equations:

$$\begin{cases} \dfrac{dR_i}{dt} = \dfrac{2}{|\log\varphi|}\dfrac{1}{R_i}\left(\dfrac{1}{\bar{R}} - \dfrac{1}{R_i}\right) + f_i^R(R, \xi, \mathsf{r}), \\ \dfrac{d\mathsf{r}_i}{dt} = \dfrac{2}{|\log\varphi|}\left[\dfrac{1}{R_i^3}\tilde{A}\mathsf{r}_i - \dfrac{1}{R_i^2\bar{R}}\ (3\mathsf{r}_i - T_0\mathsf{r}_i)\right] + f_i^{\mathsf{r}}(R, \xi, \mathsf{r}), \\ \dfrac{d\xi_i}{dt} = f_i^{\xi}(R, \xi, \mathsf{r}). \end{cases} \tag{7.1}$$

In [16,22] the local (in time) existence of classical solutions is established for the Mullins–Sekerka model for arbitrary space dimensions. From Eqs. (7.1) with initial conditions $R_i(0), \mathsf{r}_i(0), \xi_i(0)$ we have a classical solution $R_i = R_i(t), \mathsf{r}_i = \mathsf{r}_i(t), \xi_i = \xi_i(t)$ in some maximal interval of existence $[0, \hat{T})$. We start the analysis of this section by providing information which will be used in the proof of subsequent theorems.

Lemma 7.1. *Let $\tilde{A}$ be the operator defined in Eq.* (6.43). *Then $\tilde{A}$ is a self-adjoint operator on $X^{-\frac{1}{2}}$ and the eigenvalues of $\tilde{A}$ are given by*

$$\mu_n = 2n(1 - n^2) + 3, \quad n = 1, 2, \ldots, \tag{7.2}$$

where μ_n has multiplicity $2n$ and the corresponding eigenspace is spanned by the $2n$ spherical harmonics Y_n of degree n. Additionally, the eigenvalues of $\tilde{A}$ restricted to the subspace

$$X = \{r \in X^{-\frac{1}{2}}, \quad \langle r, w_i \rangle_{L^2} = 0, \quad i = 0, 1, 2, 3\}$$

satisfy

$$\mu \leqslant -9. \tag{7.3}$$

Proof. The spherical harmonics of degree n are eigenfunctions of the operator T_0.

$$T_0 Y_n = \frac{\partial u_i}{\partial n_i} + \frac{\partial u_e}{\partial n_e} = 2n Y_n. \tag{7.4}$$

Moreover, we have

$$\Delta^s Y_n = -n^2 Y_n, \tag{7.5}$$

where the eigenvalue $-n^2$ has multiplicity n^2. From the definition of $\tilde{A}$ and the completeness of the set of spherical harmonics we have the desired result. □

Remarks. We will need to obtain estimates on r for

$$\mathsf{r}_{(t)} = T_0 L\mathsf{r} + f = \tilde{A}\mathsf{r} + f(\mathsf{r}(t)). \tag{7.6}$$

If r is a solution of $\mathsf{r}_t = \tilde{A}\mathsf{r} + f(\mathsf{r}(t))$ then r satisfies the "variation of constants formula"

$$\mathsf{r}(t) = e^{-\tilde{A}t}\mathsf{r}(0) + \int_0^t e^{-\tilde{A}(t-s)} f(\mathsf{r}(s))\, ds. \tag{7.7}$$

Moreover, from well known properties of analytic semigroups [25] we have: If $\tilde{A}$ is a self-adjoint densely defined operator and if $\tilde{A}$ is bounded below, then $\tilde{A}$ is a sectorial and if $\tilde{A}$ is a sectorial operator then it is the infinitesimal generator of an analytic semigroup, while the following estimates hold:

$$||e^{\tilde{A}s}\varphi||_{C^{3+\alpha}(S^1)} \leqslant Me^{-\mu s}||\varphi||_{C^{3+\alpha}(S^1)}, \quad \varphi \in E_1, \tag{7.8}$$

$$||e^{\tilde{A}s}\varphi||_{C^{3+\alpha}(S^1)} \leqslant \frac{M}{s^\beta} Me^{-\mu s}\varphi||_{C^{2+\alpha}(S^1)}, \quad \varphi \in C^{2+\alpha}(S^1) \cap E_0, \quad \beta = \tfrac{1}{3}, \tag{7.9}$$

where $\mu, M > 0$ constants. We can assume that the constant M satisfies $M > 1$. We will be utilizing the semigroup setting of maximal regularity due to da Prato and Grisvard, [8,9,28]. The result of this theory is:

$A \in M_1(E_0, E_1)$ if the following estimate holds:

$$\sup_{0 \leqslant s \leqslant \bar{s}} \left|\left| \int_0^s e^{\tilde{A}(s-\sigma)} \varphi(\sigma)\, d\sigma \right|\right|_{C^{3+\alpha}(S^1)} \leqslant \bar{C} \sup_{0 \leqslant s \leqslant \bar{s}} ||\varphi(s)||_{C^\alpha(S^1)}, \tag{7.10}$$

where $\bar{C} > 0$ is a constant independent of $\bar{s}$. In general, $\bar{C}$ is dependent of s but here from Lemma 7.1 the spectrum of $\tilde{A}$ is bounded above by a negative number. Let $h^{k+\alpha}(S^1)$ be the "little" Hölder space and let

$$E_0 = h^\alpha(S^1) \cap X, \quad E_1 = h^{\alpha+3}(S^1) \cap X. \tag{7.11}$$

The "little" Hölder spaces $h^\alpha, 0<\alpha<1$ are obtained by completing the C^∞ functions in the C^α norm. More generally, the spaces of the little Hölder continuous functions are defined by

$$h^\alpha(I,X) = \left\{ f \in C^\alpha(I,X) : \lim_{\delta\to 0} \sup_{|t-s|<\delta} \frac{||f(t)-f(s)||}{|t-s|^\alpha} = 0 \right\},$$

$$h^{k+\alpha}(I,X) = \{ f \in C_b^k(I,X) : f^{(k)} \in h^\alpha(I,X) \}.$$

One checks immediately that $C^\theta(I,X) \subset h^\alpha(I,X)$ for $\theta>\alpha$. Moreover, if $0<\alpha<1$ and $\theta>\alpha$ then $h^\alpha(I,X)$ is the closure of $C^\theta(I,X)$ in $C^\alpha(I,X)$ and the following interpolation fact is true:

$$(h^\alpha, h^{1+\alpha})_\theta = h^{(1-\theta)\alpha+\theta(1-\alpha)}.$$

Moreover, if $A \in M_1(E_0, E_1)$ we have a relationship between the fractional spaces and the interpolation spaces E_θ:

$$||A^\theta x||_0 \leqslant ||x||_{E_\theta},$$

where

$$||x||_{E_\theta} = \sup_{\mathsf{m}>0} \mathsf{m}^{1-\theta} ||Ae^{\mathsf{A}}x||_0$$

and E_θ is defined as

$$E_\theta = \left\{ x \in E_0 : \lim_{\mathsf{m}\to 0} \mathsf{m}^{1-\theta} ||Ae^{\mathsf{m}A}x||_0 = 0 \right\}, \quad 0<\theta<1.$$

The following proposition justifies Eqs. (7.1) by showing that R_i can be approximated well by R_i, the solutions of (3.7). We focus on the first extinction interval $[0, \hat{T}_1)$. By repeating the argument, we obtain analogous results in $(\hat{T}_1, \hat{T}_2), \ldots, (\hat{T}_{N-2}, \hat{T}_{N-1})$.

Proposition 7.2. *Assume $N \geqslant 2$. Assume there exists $\zeta>0$ such that*

$$||\mathsf{r}_i(t)||_{C^{3+\alpha}(S^1)} < \zeta, \quad t \in [0, \hat{T}). \tag{7.12}$$

Assume also

$$R_1(0) < R_2(0) < \cdots < R_{N-1}(0), \quad R_i(0) = \mathsf{R}_i(0), \quad i = 1, \ldots, N$$

and let T_1 be the extinction time of R_1 characterised by $\mathsf{R}_1(T_1) = 0$ and $\hat{T}_1 = \hat{T}$ be the extinction time of R_1 characterized by $R_1(\hat{T}_1) = 0$. Then there exists $\bar{\varepsilon}>0$ such that

$\varphi < \bar{\varepsilon}^2$ and positive constants $C_T, c_1, C_1, C_R, C_{\mathsf{R}}$, a depending on $R_i(0), i = 1, \dots, N$ such that the following estimates hold true:

(i) $|\hat{T} - T_1| < C_T \frac{1}{|\log \varphi|^2}$,
(ii) $c_1(\hat{T} - t)^{\frac{1}{3}} < R_1(t) < C_1(\hat{T} - t)^{\frac{1}{3}}$,
(iii) $c_R < R_i(t) < C_R, \quad t \in [0, \hat{T}), \quad i > 1$,
(iv) $|R_i(t) - \mathsf{R}_i(t)| < C_{\mathsf{R}} \frac{1}{|\log \varphi|^{\frac{4}{3}}}, \quad t \in [0, T_1 - C_T \frac{1}{|\log \varphi|^2}]$,
(v) $1 - \frac{R_1(t)}{\bar{R}(t)} > a, \quad t \in [0, \hat{T})$.

Proof. (A) Eq. $(7.1)_1$ can be written in the form

$$\frac{dR_i}{dt} = \frac{2}{|\log \varphi|} \frac{1}{R_i^2}\left[\left(\frac{R_i}{\bar{R}} - 1\right) + R_i O\left(\frac{1}{|\log \varphi|}\right)\right], \quad i = 1, \dots, N. \tag{7.13}$$

(B) The following estimates hold:

(a) $\sum_{i=1}^N (R_i^2(t) - R_i^2(0)) = O(\frac{1}{|\log \varphi|^2})$,
(b) $\sum_{i=1}^N (R_i^2(t) - \mathsf{R}_i^2(t)) = O(\frac{1}{|\log \varphi|^2})$,
(c) R_i: uniformly bounded in φ, equicontinuous in φ and uniformly continuous on $[0, \hat{T})$.

Verification: (a) $\frac{d}{dt}\left(\frac{1}{2}\sum_{i=1}^N R_i^2(t)\right) = \sum_{i=1}^N R_i(t)\dot{R}_i(t) = \sum_{i=1}^N \frac{2}{|\log \varphi|}\frac{1}{R_i}(\frac{R_i}{\bar{R}} - 1) + O(\frac{1}{|\log \varphi|^2}) = O(\frac{1}{|\log \varphi|^2})$, by recalling Proposition 3.1(ii).

Integrating we obtain $\sum_{i=1}^N (R_i^2(t) - R_i^2(0)) \leqslant O(\frac{2}{|\log \varphi|^2})$.

From this by Schwartz and Gronwall we obtain a bound on $\sum_{i=1}^N R_i^2(t)$ hence $\sum_{i=1}^N R_i$ and as a result (a).

Condition (b) follows from (a) by utilizing the conservation of $\sum_{i=1}^N \mathsf{R}_i^2(t)$.

Condition (c) follows from (a).

(C) Let $\hat{T}^* > T_1$ arbitrary otherwise. Then

$$\lim_{\varphi \to 0} \min_{[T_1, \hat{T}^*]} R_1 = 0. \tag{7.14}$$

We argue by contradiction. Assume that $\underline{\lim}_{\varphi \to 0} \inf_{[T_1, \hat{T}^*]} R_1 > 0$. Then $\underline{\lim}_{\varphi \to 0} \inf_{[T_1, \hat{T}^*]} R_j \geqslant c > 0, j = 1, \dots, N$. By continuous dependence as long as there is no singularity in $[T_1, \hat{T}^*]$ we can pass to the limit in (7.13) and deduce that $\mathsf{R}_1(t) \geqslant c > 0$ in $[T_1, \hat{T}^*]$ contradicting that $T_1 < \hat{T}^*$. Finally the $\underline{\lim}$ can be replaced by lim by the equicontinuity of R.

(D) We show that $\hat{T}_1 < \infty$, $R_1 > 0$ on $[0, \hat{T}_1)$, $\hat{T}_1 < \hat{T}_2, \dots, \hat{T}_{N-1}$. We argue by contradiction. Assume that $\hat{T}_1 = \infty$, so $\hat{T} = \infty$. By (7.13) for $i = 1$, we have

$$\frac{d}{dt}\left(\frac{1}{3} R_1^3\right) \leqslant \frac{2}{|\log \varphi|}\left(\frac{R_1}{\bar{R}} - 1\right) + O\left(\frac{1}{|\log \varphi|^2}\right). \tag{7.15}$$

We fix a $\hat{T}^* > T_1$. By (C) there exists $t^{**} \in [T_1, \hat{T}^*]$ such that $\frac{R_1(t^{**})}{\bar{R}(t^{**})} < \frac{1}{2}$ and hence over an interval $[\hat{T}^*, \hat{T}^* + \delta]$ which can be chosen uniformly in φ (by the equicontinuity of R_1). It follows that R_1 is decreasing on $[\hat{T}^*, \hat{T}^* + \delta]$ by a fixed amount. By equicontinuity we can repeat the argument over $[\hat{T}^* + \delta, \hat{T}^* + 2\delta], \dots$ to conclude that $\hat{T}_1 < \infty$. Next, we consider Eq. (7.13) for $i = 1, 2$. By subtracting them we have

$$\frac{1}{3}\frac{d}{dt}(R_2^3 - R_1^3) \geqslant \frac{2}{|\log \varphi|}\frac{1}{\bar{R}}(R_2 - R_1) - C\frac{1}{|\log \varphi|^2} \geqslant -C\frac{1}{|\log \varphi|^2} \quad \text{on } [0, \hat{T}_1),$$

where we used that $R_2 > R_1$. Integrating we obtain

$$R_2^3(t) \geqslant R_2^3(0) - R_1^3(0) - C\frac{1}{|\log \varphi|^2}t$$

from which it follows that

$$R_2^3(\hat{T}_1) \geqslant R_2^3(0) - R_1^3(0) - C\frac{1}{|\log \varphi|^2}\hat{T}_1$$

and as a result $\hat{T}_2 > \hat{T}_1$ for φ: small. Similarly we show that $\hat{T}_{N-1} > \hat{T}_{N-2} > \cdots > \hat{T}_2 > \hat{T}_1$. From which (iii) and (v) follow.

(E) We argue near $\hat{T}_1$. Given $\delta > 0$, there exists $\bar{\delta}$ such that for $\hat{T}_1 - \bar{\delta} < t < \hat{T}_1 \Rightarrow |\frac{R_1}{\bar{R}}| < \delta$. Let $t \in [\hat{T}_1 - \bar{\delta}, \hat{T}_1)$, from (7.13) with $i = 1$, we obtain

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{3} R_1^3\right) &= \frac{2}{|\log \varphi|}\left(\frac{R_1}{\bar{R}} - 1\right) + O\left(\frac{1}{|\log \varphi|^2}\right) \leqslant \frac{2}{|\log \varphi|}\left(\frac{R_1}{\bar{R}} - 1\right) + C\frac{1}{|\log \varphi|^2}\\ &\leqslant (-1 + \delta)\frac{2}{|\log \varphi|} + \frac{C}{|\log \varphi|^2} < -\frac{1}{2}.\end{aligned}$$

Integrating this inequality from t to $\hat{T}_1$ we obtain the inequality on the left-hand side of (ii). To obtain the inequality on the right-hand side we utilize an upper bound on $|\frac{R_1}{\bar{R}}|$ on $[\hat{T}_1 - \bar{\delta}, \hat{T}_1)$ and we have

$$\frac{d}{dt}\left(\frac{1}{3} R_1^3\right) \geqslant \frac{2}{|\log \varphi|}\left(\frac{R_1}{\bar{R}} - 1\right) - C\frac{1}{|\log \varphi|^2} \geqslant -C.$$

Integrating from t to $\hat{T}_1$ we obtain the right-hand side of estimate (ii).

(F) We have

$$\frac{1}{3}\frac{d}{dt}(\mathsf{R}_i^3 - R_i^3) = \frac{2}{|\log\varphi|}\left(\frac{\mathsf{R}_i}{\bar{\mathsf{R}}} - \frac{R_i}{\bar{R}}\right) + O\left(\frac{1}{|\log\varphi|^2}\right).$$

By integration

$$|\mathsf{R}_i^3(t) - R_i^3(t)| \leqslant \frac{2}{|\log\varphi|}\int_0^t \left|\frac{\mathsf{R}_i}{\bar{\mathsf{R}}} - \frac{R_i}{\bar{R}}\right| d\tau + C\frac{1}{|\log\varphi|^2}.$$

Utilizing $\mathsf{R}_1 \leqslant \mathsf{R}_i$ we obtain

$$\mathsf{R}_1^2 \sum_{i=1}^N |\mathsf{R}_i - R_i| \leqslant C\frac{2}{|\log\varphi|}\int_0^t \sum_{i=1}^N |\mathsf{R}_i - R_i| d\tau + C\frac{1}{|\log\varphi|^2}.$$

Set

$$y(t) = \mathsf{R}_1^2(t) \sum_{i=1}^N |\mathsf{R}_i - R_i|.$$

Then

$$y(t) \leqslant \frac{2}{|\log\varphi|}\int_0^t \frac{y(s)}{\mathsf{R}_1^2(s)} ds + C\frac{1}{|\log\varphi|^2}.$$

Utilizing that

$$C_1(T_1 - t)^{\frac{1}{3}} \geqslant \mathsf{R}_1(t) \geqslant c_1(T_1 - t)^{\frac{1}{2}},$$

we obtain that $y(t) \leqslant C\frac{1}{|\log\varphi|^2}$ hence

$$\mathsf{R}_1^2(t) \sum_{i=1}^N |\mathsf{R}_i - R_i| \leqslant C\frac{1}{|\log\varphi|^2}.$$

So,

$$\sum_{i=1}^N |\mathsf{R}_i - R_i| \leqslant C\frac{1}{\mathsf{R}_1^2(t)|\log\varphi|^2}.$$

Now,

$$\mathsf{R}_1(t) \geqslant C\left(\frac{1}{|\log\varphi|}\right)^{\frac{2}{3}} \quad \text{if } t \leqslant T_1 - C_T\frac{1}{|\log\varphi|^2}$$

and so

$$\sum_{i=1}^{N} |\mathsf{R}_i - R_i| \leqslant C\left(\frac{1}{|\log \varphi|^{\frac{4}{3}}}\right), \quad t \in \left[0, T_1 - C_T \frac{1}{|\log \varphi|^2}\right].$$

(G) We now prove (i). We first note that (ii) implies a lower bound on $\hat{T}_1$. Indeed, if $\hat{T}_1 \leqslant T_1 - C\frac{1}{|\log \varphi|^2}$, $C \geqslant C_t$ then

$$|R_1(\hat{T}_1) - \mathsf{R}_1(\hat{T}_1)| \leqslant C_{\mathsf{R}} \frac{1}{|\log \varphi|^{\frac{4}{3}}}$$

hence $|\mathsf{R}_1(\hat{T}_1)| \leqslant C_{\mathsf{R}} \frac{1}{|\log \varphi|^{\frac{4}{3}}}$.

On the other hand by the lower bound on R_1

$$c(T_1 - \hat{T}_1)^{\frac{4}{3}} \leqslant C_{\mathsf{R}} \frac{1}{|\log \varphi|^{\frac{4}{3}}}$$

which if $c(c)^{\frac{4}{3}} > c_{\mathsf{R}}$ we arrive at a contradiction. Thus,

$$\hat{T}_1 \geqslant T_1 - c\frac{1}{|\log \varphi|^2}, \quad c: \text{ appropriate.}$$

To obtain an upper bound we argue as follows: From

$$\left| R_1\left(T_1 - \frac{C}{|\log \varphi|^2}\right) - \mathsf{R}_1\left(T_1 - \frac{C}{|\log \varphi|^2}\right)\right| < C_{\mathsf{R}} \frac{1}{|\log \varphi|^{\frac{4}{3}}},$$

we obtain

$$R_1\left(T_1 - \frac{C}{|\log \varphi|^2}\right) < C^* \frac{1}{|\log \varphi|^{\frac{4}{3}}}.$$

For $t \geqslant T_1 - \frac{C}{|\log \varphi|^2}$ we have $\frac{dR_1}{dt} \leqslant (-1 + \delta)\frac{1}{R_1^2}$. Integrating we obtain

$$0 \leqslant \frac{1}{3} R_1^3(t) \leqslant C^{*3} \frac{1}{|\log \varphi|^2} - \left(t - \left(T_1 - \frac{C}{|\log \varphi|^2}\right)\right)$$

from which we obtain an upper bound for t of the form $T_1 + C\frac{1}{|\log \varphi|^2}$. The proof of the proposition is complete. $\square$

For convenience, we utilize the "pseudo-times" s_i in estimating r_i in order to handle the factor $\frac{2}{|\log\varphi|}\frac{1}{R_i^3}$ infront of $\tilde{A}\mathsf{r}_i$ in Eq. $(7.1)_2$. The "pseudo-times" are defined by

$$s_i =: \int_0^t \frac{2}{|\log\varphi|}\frac{1}{R_i^3(\tau)}\,d\tau, \quad i = 1,\dots,N, \ \ t\in[0,\hat{T}). \tag{7.16}$$

Proposition 7.3. *Assume $N \geqslant 2$. Then there exists $\bar{\varepsilon} > 0$ such that $\varphi < \bar{\varepsilon}^2$, $||\mathsf{r}_i(0)||_{C^{3+\alpha}(S^1)} < \bar{\varepsilon}$ and ζ independent of $\bar{\varepsilon}$ such that the following inequality holds true:*

$$||\mathsf{r}_i(t)||_{C^{3+\alpha}(S^1)} < \zeta, \quad t\in[0,\hat{T}) \ \ i = 1,\dots,N. \tag{7.17}$$

Proof. If we introduce the "pseudo-times" s_i defined in Eq. (7.16) then system (7.1) takes the form

$$\begin{cases} \dfrac{dR_i}{ds_i} = R_i\left[\left(\dfrac{R_i}{\bar{R}} - 1\right) + g_i^R(R,\xi,\mathsf{r})\right], \\ \dfrac{d\mathsf{r}_{i_t}}{ds_i} = \tilde{A}\mathsf{r}_i - \dfrac{R_i}{\bar{R}}(3\mathsf{r}_i - T_0\mathsf{r}_i) + \dfrac{2}{|\log\varphi|}g_i^{\mathsf{r}}(R,\xi,\mathsf{r}), \\ \dfrac{d\xi_i}{ds_i} = g_i^{\xi}(R,\xi,\mathsf{r}), \end{cases} \tag{7.18}$$

where

$$g_i^R = R_i f_i^R, \quad g_i^r = R_i^3 f_i^{\mathsf{r}}, \quad g_i^{\xi} = R_i f_i^{\xi}.$$

1. Let $\zeta > 0$ any number that satisfies

$$||\mathsf{r}_i(0)||_{C^{3+\alpha}(S^2)} < \zeta, \quad i = 1,\dots,N. \tag{7.19}$$

Then by continuity there exists $\hat{T}' \leqslant \hat{T}$ such that the inequality $||\mathsf{r}_i(t)||_{C^{3+\alpha}(S^1)} < \zeta$ holds in $[0,\hat{T}')$. Then estimates (iii), (v) in Proposition 7.2 hold in $[0,\hat{T}')$ and we can also assume $|g_i^R| < a$. Therefore Eq. $(7.18)_1$ implies

$$R_1(t(s_1)) \leqslant R_1(0)e^{-as_1}. \tag{7.20}$$

Let $\hat{s}_1$ be chosen large enough so that,

$$M(3 + ||T_0||)N\frac{R_1(0)}{R_N(0)}e^{-a\hat{s}_1}\sup_{s_1\geqslant\hat{s}_1}\int_{\hat{s}_1}^{s_1}\frac{e^{-\mu(s_1-s)}}{(s_1-s)^{\beta}}\,ds < \frac{1}{4} \tag{7.21}$$

for β fixed as in (7.9).

2. In the interval $[0, \hat{T}')$ we have from (7.16), (7.20) and estimate (iii) in Proposition 7.2

$$s_i = \int_0^{s_1} \frac{R_1^3}{R_i^3}\, ds_1{}' \leqslant \left(\frac{R_1(0)}{c_R}\right)^3 \int_0^{s_1} e^{-3as_1{}'} ds_1{}' \\ \leqslant \frac{1}{3a}\left(\frac{R_1(0)}{c_R}\right)^3 = \hat{s}, \quad i > 1. \tag{7.22}$$

Let $\bar{s}_1 = \max\{\hat{s}, \hat{s}_1\}$.

3. The map $s_i \to [\frac{R_i}{\bar{R}}(\mathrm{r}_i - T_0 \mathrm{r}_i)](t(s_i))$ is a continuous map from $s_i([0, \hat{T}'))^{-1}$ into $C^{2+\alpha}(S^2)$ and we have

$$\left|\left|\frac{R_i}{\bar{R}}(3\mathrm{r}_i - T_0\mathrm{r}_i)\right|\right|_{C^{3+\alpha}(S^1)} \leqslant \frac{NC_R}{R_N(0)}(3 + ||T_0||)||\mathrm{r}_i||_{C^{3+\alpha}(S^1)}. \tag{7.23}$$

From this and (7.9) it follows that

$$\left|\left|\int_0^{s_i} e^{A(s_i - s)}\left[\frac{R_i}{\bar{R}}(3\mathrm{r}_i - T_0\mathrm{r}_i)\right](t(s))\, ds\right|\right|_{C^{3+\alpha}(S^1)} \\ \leqslant M\frac{NC_R}{R_N(0)}(3 + ||T_0||)s_i^{1-\beta} \sup_{0 < s \leqslant s_i} ||\mathrm{r}_i(t(s))||_{C^{3+\alpha}(S^1)}. \tag{7.24}$$

We fix $\sigma > 0$ small, so that

$$MN\frac{C_R}{R_N(0)}(3 + ||T_0||)\sigma^{1-\beta} < \frac{1}{4}. \tag{7.25}$$

4. Let $\bar{k} = [\frac{\bar{s}_1}{\sigma}] + 1$ where $\bar{s}_v, \sigma$ are defined in 3 and 4 and where [ ] stands for the integer part.

5. Utilizing estimate (iii) in Proposition 7.2 we have for $t \in [0, \hat{T}')$

$$||g_i^{\mathrm{r}}||_{C^{\alpha}(S^2)} \leqslant C_0(C_R\zeta + \zeta||\mathrm{r}_i||_{C^{3+\alpha}(S^1)}).$$

Therefore from (7.10) it follows that

$$\sup_{0 < s \leqslant s_i}\left|\left|\int_0^{s} e^{\tilde{A}(s-\sigma)}\frac{2}{|\log\varphi|}g_i^{\mathrm{r}}\right|\right|_{C^{3+\alpha}(S^1)} \\ = \frac{2}{|\log\varphi|}\bar{C}C_0C_R(C_R + \zeta) + \frac{2}{|\log\varphi|}\bar{C}C_0\zeta \sup_{0 < s \leqslant s_i} ||\mathrm{r}_i(t(s))||_{C^{3+\alpha}(S^1)} \\ \leqslant \frac{2}{|\log\varphi|}C_1(1 + \zeta) + \frac{2}{|\log\varphi|}C_1\zeta \sup_{0 < s \leqslant s_i} ||\mathrm{r}_i(t(s))||_{C^{3+\alpha}(S^1)}, \tag{7.26}$$

where C_1 is a suitably chosen constant.

6. Assume $\bar{\varepsilon}>0$ so small that for $\varphi<\bar{\varepsilon}^2$

$$2\frac{2}{|\log \varphi|}C_1\zeta<\frac{1}{4}, \tag{7.27}$$

$$2\frac{2}{|\log \varphi|}C_1\sum_{k=0}^{\bar{k}-1}(2M)^k<\frac{1}{8M}. \tag{7.28}$$

We now make a definite choice for ζ (cf. 7.19)

$$\zeta = 8M(2M)^{\bar{k}}\mathsf{r}_0+1, \tag{7.29}$$

where $\bar{k}$ is defined in 4 and $\mathsf{r}_0=\max_i ||\mathsf{r}_i(0)||_{C^{3+\alpha}(S^1)}$.

7. From the variation of constants formula applied to Eq. $(7.18)_2$ it follows via (7.24), (7.26), that

$$z_i\leqslant Mz_i(0)+C_2s_i^{1-\beta}z_i+\frac{2}{|\log \varphi|}C_1(1+\zeta)+\frac{2}{|\log \varphi|}C_1\zeta_i, \tag{7.30}$$

where we have set

$$z_i=\sup_{0<s\leqslant s_i}||\mathsf{r}_i(t(s))||_{C^{3+\alpha}(S^1)},\quad z_{i0}=||\mathsf{r}_i(0)||_{C^{3+\alpha}(S^1)} \tag{7.31}$$

and

$$C_2=MN\frac{C_R}{R_N(0)}(3+||T_0||).$$

Eq. (7.30) is valid in the interval $I_i=[0,s_i(\hat{T}'))$. From (7.25) and (7.27) it follows that for $s_i\in I_i\cap[0,\sigma]$ we have

$$z_i<2Mz_i(0)+2\frac{2}{|\log \varphi|}C_1(1+\zeta),\quad s_i\in I_i\cap[0,\sigma]. \tag{7.32}$$

If we replace in (7.30) $z_i(0)$ and $z_i(\sigma)$ and s_i by $(s_i-\sigma)$ and use Eq. (7.34) to estimate $z_i(\sigma)$ we get

$$z_i<(2M)^2z_{i0}+2\frac{2}{|\log \varphi|}C_1(1+\zeta)(2M+1),\quad s_i\in I_i\cap[0,2\sigma]. \tag{7.33}$$

By iterating this procedure we get

$$z_i<(2M)^kz_i(0)+2\frac{2}{|\log \varphi|}C_1(1+\zeta)\sum_{h=0}^{k-1}(2M)^h,\quad s_i\in I_i\cap[0,k\sigma]. \tag{7.34}$$

Eq. (7.34) is one of the basic estimates needed to complete the proof.

8. We now prove that $\hat{s}_i \in I_1$. This follows from Eq. (7.34). In fact for $k \leqslant \bar{k}$ we have

$$\begin{aligned}&(2M)^k z_i(0) + 2\frac{2}{|\log \varphi|} C_1(1+\zeta) \sum_{h=0}^{k-1} (2M)^h \\ &\leqslant (2M)^{\bar{k}} \mathsf{r}_0 + 2\frac{2}{|\log \varphi|} C_1 \sum_{h=0}^{\bar{k}-1} (2M)^h + 2\frac{2}{|\log \varphi|} C_1 \sum_{h=0}^{\bar{k}-1} (2M)\zeta \\ &\leqslant (2M)^{\bar{k}} \mathsf{r}_0 + \frac{1}{8M} + \frac{\zeta}{8M} < \frac{\zeta}{4M},\end{aligned} \tag{7.35}$$

where we have utilized definition (7.29) of ζ and (7.34). This inequality, recalling the definition of $\bar{k}$ in 4, shows that Eq. (7.34) implies

$$z_i(s_i) < \frac{\zeta}{4M} < \zeta, \quad i = 1, \dots, N \tag{7.36}$$

for $s_i \in [0, \bar{s}_i]$, where $\bar{s}_1$ is defined in 2 and

$$\bar{s}_i = \int_0^{\bar{s}_i} \frac{R_1^3}{R_i^3} ds_1 \quad \text{if } i > 1. \tag{7.37}$$

Since, by definition $\bar{s}_i \geqslant \hat{s}_i$, the claim is proved.

9. From Eq. (7.36) it follows that the condition

$$||\mathsf{r}_i(t)||_{C^{3+\alpha}(S^1)} < \zeta = 8M(2M)^{\bar{k}} \mathsf{r}_0 + 1 \tag{7.38}$$

is satisfied in the time interval $[0, \hat{T}')$. Notice that $\hat{T}'$ cannot be infinity because, as we have seen from Proposition 7.2(ii), condition (7.36) implies $R_1(t) \to 0$ in finite time. Thus the maximal interval of existence of the solution of system (7.1) and therefore $\hat{T}' = \hat{T}$. The proof of the proposition is complete. □

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References

[1] N. Alikakos, P. Bates, X. Chen, The convergence of solutions of the Cahn–Hilliard equation to the solution of the Hele–Shaw model, Arch. Rational Mech. Anal. 128 (1994) 165–205.

[2] N. Alikakos, P.W. Bates, X. Chen, G. Fusco, Mullins–Sekerka motion of small droplet on a fixed boundary, J. Geom. Anal. 4 (2000) 575–596.

[3] N.D. Alikakos, G. Fusco, The equations of Ostwald ripening for dilute systems, J. Statist. Phys. 95 (1999) 851–866.

[4] N.D. Alikakos, G. Fusco, The effect of distribution in space in Ostwald ripening, Centre de Recherches Mathematiques CRM Proceedings and Lecture Notes, Vol. 27, 2001.

[5] N.D. Alikakos, G. Fusco, Ostwald ripening for dilute systems under quasistationary dynamics, Comm. Math. Phys., 238 (2003) 429–479.

[7] N.D. Alikakos, G. Fusco, G. Karali, Continuum limits of particles interacting via diffusion, Abstract Appl. Anal., 2004 (3) (2004) 215–237.

[8] H. Amann, Linear and Quasilinear Parabolic. I. Abstract Linear Theory, Monographs in Mathematics, Vol. 89, Birkhäuser, Basel, 1995.

[9] S.B. Angenent, Nonlinear analytic semigroups, Proc. Roy. Soc. Edinburgh A 115 (1990) 91–107.

[10] G. Baker, O. Meiron, S. Orszag, J. Fluid Mech. 123 (1982) 477.

[11] G. Bellettini, G. Fusco, Some aspects of the dynamics of $S(V) = H - \bar{H}$, J. Differential Equations 157 (1999) 206–246.

[12] J.W. Cahn, On spinodal decomposition, Acta Metall. 9 (1961) 795–801.

[13] J.W. Cahn, J.E. Hilliard, Free energy of a nonuniform system I. interfacial free energy, J. Phys. Chem. 28 (1958) 258–267.

[14] X. Chen, The Hele–Shaw problem and area-preserving curve shortening motion, Arch. Rational Mech. Anal. 123 (1993) 117–151.

[15] X. Chen, Global asymptotic limit of solutions of the Cahn–Hilliard equation, J. Differential Geom. 44 (2) (1996) 262–311.

[16] X. Chen, X. Hong, F. Yi, Existence, uniqueness and regularity of solutions of the Mullins–Sekerka problem, Comm. Partial Differential Equations 21 (1996) 1705–1727.

[17] P. Constantin, M. Pugh, Global solutions for small data to the Hele–Shaw problem, Nonlinearity 6 (1993) 393–415.

[18] G. da Prato, P. Grisvard, Equations d'evolution abstraites non lineaires de type parabolique, Ann. Mat. Pura Appl. 4 (120) (1979) 329–396.

[19] M.P. Do Carmo, Differential Geometry of Curves and Surfaces, Prentice-Hall, New York, 1976.

[21] J. Escher, G. Simonett, Classical solutions for Hele Shaw models with surface tension, Adv. Differential Equations 2 (1997) 619–642.

[22] J. Escher, G. Simonett, A center manifold analysis for the Mullins–Sekerka model, J. Differential Equations 143 (1998) 267–292.

[23] G. Folland, Introduction to Partial Differential Equations, Princeton University Press and University of Tokyo Press, Princeton, NJ, 1976.

[24] D. Gilbarg, N. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer, Berlin.

[25] D. Henry, Geometric Theory of Semilinear Parabolic Equations, Lecture Notes in Mathematics, Springer, Berlin, 1981.

[26] M. Kimura, An application of the shape derivative to the quasistationary Stefan problem, preprint.

[27] I.M. Lifschitz, V.V. Slyozov, The kinetics of precipitation from supersaturated solid solutions, J. Phys. Chem. Solids 19 (1961) 35–50.

[28] A. Lunardi, Analytic semigroups and optimal regularity for parabolic problems, Progress of Nonlinear Differential Equations and its Applications, Vol. 16, Birkhäuser, Basel, 1995.

[29] C. Miranda, Sulle proprieta di regolarita di certe trasformazioni integrali, Atti Accad. Naz. Lincei Mem. Cl. Sci. Fis. Mat. Natur. Sez. I 8 (7) (1965) 303–336.

[30] W.W. Mullins, Grain growth of uniform boundaries with scaling, Acta Mater. 46 (1998) 6219–6226.

[31] W.W. Mullins, R.F. Sekerka, Morphological stability of a particle growing by diffusion and heat flow, J. Appl. Phys. 34 (1963) 323–329.

[32] B. Niethammer, Derivation of the LSW—theory for Ostwald ripening by homogenization methods, Arch. Rational Mech. Anal. 147 (2) (1999) 119–178.
[33] B. Niethammer, F. Otto, Domain coarsening in thin films, Comm. Pure Appl. Math. 54 (3) (2001) 361–384.
[34] W. Ostwald, Z. Phys. Chem. 37 (1901) 385.
[35] R.L. Pego, Front migration in the nonlinear Cahn–Hilliard equation, Proc. Roy. Soc. London Ser. A 422 (1989) 261–278.
[36] G.E. Poirier, W.P. Fitts, J.M. White, Two-dimensional phase diagram of decanethiol on Au(111), Langmuir 17 (2001) 1176–1183.
[37] H.M. Soner, Convergence of the phase-field equations to the Mullins–Sekerka problem with kinetic undercooling, Arch. Rational Mech. Anal. 131 (1995) 139–197.
[38] J. Vinals, W.W. Mullins, Self-similarity and coarsening of three dimensional particles on a two dimensional matrix, J. Appl. Phys. 83 (2) (1998).
[39] P.W. Voorhees, G.B. McFadden, D. Boisvert, D.I. Meiron, Numerical simulation of morphological development during Ostwald ripening, Acta Metall. 36 (1) (1988) 207–222.
[41] C. Wagner, Theorie der Alterung von Niederschlagen dursch Umlosen, Z. Elektrochem. 65 (1961).
[42] R. Ye, Foliation by constant mean curvature spheres, Pacific J. Math. 147 (2) (1991) 381–396.
[43] J. Zhu, X. Chen, T.Y. Hou, An efficient boundary integral method for the Mullins–Sekerka problem, J. comput. Phys. 126 (2) (1996) 246–267.

Further reading

N.D. Alikakos, G. Fusco, G. Karali, The effect of the geometry of the particle distribution in Ostwald ripening, Comm. Math. Phys., 238 (2003) 481–488.
J. Duchon, R. Robert, Evolution d'une interface par capillarite et diffusion de volume I, existence locale en temps, Ann. Inst. H.Poincaré Anal. Non Linéaire 1 (1984) 361–378.
P.W. Voorhees, Schaffer, In situ observation of particle motion and diffusion interactions during coarsening, Acta Metall. 33 (1987) 327–339.