Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

Website: http://AIMsciences.org

pp. **X–XX**

EXISTENCE OF SOLUTION FOR A GENERALIZED STOCHASTIC CAHN-HILLIARD EQUATION ON CONVEX DOMAINS

Dimitra Antonopoulou

Department of Applied Mathematics, University of Crete, Heraklion, Greece,
and Institute of Applied and Computational Mathematics, FORTH, Greece

Georgia Karali

Department of Applied Mathematics, University of Crete, Heraklion, Greece,
and Institute of Applied and Computational Mathematics, FORTH, Greece

(Communicated by the associate editor name)

Abstract. We consider a generalized Stochastic Cahn-Hilliard equation with multiplicative white noise posed on bounded convex domains in $\mathbb{R}^d$, $d = 1, 2, 3$, with piece-wise smooth boundary, and introduce an additive time dependent white noise term in the chemical potential. Since the Green's function of the problem is induced by a convolution semigroup, we present the equation in a weak stochastic integral formulation and prove existence of solution when $d \leq 2$ for general domains, and for $d = 3$ for domains with minimum eigenfunction growth, without making use of any explicit expression of the spectrum and the eigenfunctions. The analysis is based on stochastic integral calculus, Galerkin approximations and the asymptotic spectral properties of the Neumann Laplacian operator. Existence is also derived for some non-convex cases when the boundary is smooth.

1. Introduction.

1.1. The problem.

We study the generalized stochastic Cahn-Hilliard partial differential equation

$$u_t = \Delta\Big(- \Delta u + f'(u) + F_2(x,t)\dot{V}\Big) + F_1(u)\dot{W}, \quad x \in \mathcal{D}, \quad t > 0, \tag{1}$$

associated with Neumann boundary conditions

$$\frac{\partial u}{\partial n} = \frac{\partial \Delta u}{\partial n} = 0 \quad \text{on} \quad \partial\mathcal{D}, \tag{2}$$

where $\mathcal{D}$ is a convex bounded domain in $\mathbb{R}^d$, $d = 1, 2, 3$, of sufficiently piece-wise smooth boundary. The term $\dot{V}$ appearing additive in the chemical potential is a time dependent white noise; the multiplicative noise $\dot{W}$ is the formal derivative of an one-dimensional l-parameter Wiener process. When the domain is rectangular

2000 *Mathematics Subject Classification.* 35K55, 35K40, 60H30, 60H15.

Key words and phrases. Stochastic Cahn-Hilliard, space-time white noise, Neumann Laplacian eigenfunction basis, convolution semigroup, Green's function, convex domains, Galerkin approximations.

The second author is supported by a Marie Curie International Reintegration Grant within the 7th European Community Framework Programme, MIRG-CT-2007-200526.

we set $l := 1 + d$ and define $\dot{W}$ as a space-time white noise, while in the case of general domains we consider $l := 1$ and define $\dot{W}$ as $\dot{V}$ i.e. as a time dependent white noise. The function F_2 is real and smooth on $\mathcal{D}$ for any $t > 0$, and F_1 is Lipschitz and bounded as a function of u. We consider $f'(u) := \partial_u f(u) = u(u^2-1)$, where $f := \frac{1}{4}(u^2-1)^2$ is a double equal-well potential taking its global minimum value 0 at $u = \pm 1$ [2] and models the tendency of a two species homogeneous alloy to return in a two separated phases equilibrium [29].

The stochastic Cahn-Hilliard equation is one of the important cases of the non-linear Langevin equations. It is based on a field-theoretic approach to the non-equilibrium dynamics of metastable states [17, 35, 40]. When $F_1 = F_2 = 0$, equation (1) becomes the deterministic Cahn-Hilliard equation. The originally proposed equation by Cahn and Hilliard contains logarithmic poles in the potential ([14, 13]) and is a model for phase separation of a binary alloy at fixed temperature, where $u(x,t)$ defines the mass concentration of one of the phases at a point x of a vessel $\mathcal{D}$ at time t. The evolution of the concentration u undergoes two phases called phase separation and phase coarsening. For more physical background, derivation and discussion of the deterministic Cahn-Hilliard equation and related equations we refer to [7, 13, 14, 26, 28] and the references therein.

1.2. **The effect of noise.** The standard Cahn-Hilliard model was extended by Cook [17] (see also [40]) in order to incorporate thermal fluctuations as additive noise:

$$\partial_t u = \Delta\Big(-\Delta u + f'(u)\Big) + \xi(x,t). \tag{3}$$

This equation is usually called Cahn-Hilliard-Cook, and ξ is in general a Gaussian noise. In the theory of Critical Dynamics, a Cahn-Hilliard equation of the form (1) is described as Model B [35]. Such a generalized type Cahn-Hilliard model [32], is based on the balance law for microforces; in this case the term F_2 of (1) is the external field [35, 32]. In [38], the Kawasaki exchange dynamics are applied, and a modified Cahn-Hilliard equation is proposed, where F_2 is the external gravity field. The F_1 term stands for the Gaussian noise $\xi(x,\tau)$ in Model B of [35] in accordance with the Cahn-Hilliard-Cook model, while, following [32], the quantity F_1 is the external mass supply. Such model appears in [4] where spinodal decomposition is analyzed as a mechanism for the formation of Liesengang bands. A generalized Cahn-Hilliard equation appears also as a mesoscopic model for surface reactions. In [34], a combination of Arrhenius absorption/desorption dynamics, Metropolis surface diffusion and simple unimolecular reaction is considered. A special case of this model, where the external force field enters the equation as a multiplicative term, is the following generalized Cahn-Hilliard equation [5, 37, 36]:

$$\partial_t u = \Delta\Big(-\Delta u + f'(u) + F_2(x,t)\Big) + F_1(x,t)(1-u),$$

here obviously the function $1-u$ is Lipschitz in u.

The stochastic Cahn-Hilliard with f' polynomial of odd degree when $F_2 := 0$ and $F_1 := 1$ posed on multi-dimensional rectangular domains, was analyzed by Da Prato and Debussche in [19]. In this case, an additive noise more regular than white noise was defined as an infinite linear combination of the $L^2(\mathcal{D})$ orthonormal basis with coefficients consisting of independent, $t-$dependent Brownian motions. When the trace of the Wiener process is finite, existence was analyzed in [25]. Results for the noisy Cahn-Hilliard equation are of great interest for the studying of coarsening (Ostwald ripening) [3] and nucleation [8]. For a survey, including numerical results and conjectures concerning the nucleation problem, see [11]. In the stochastic case the polynomial nonlinearity has been analyzed in [10, 11, 15, 16, 19, 25], while in [22, 21, 31] a stochastic Cahn-Hilliard with reflection is considered. Numerical results for the Cahn-Hilliard equation on the unit square has been presented in [41].

1.2.1. *Motivation for the noise in the chemical potential.* At the proposed model (1) we split the noise into two terms. The chemical potential noise stands for external fields while the free-energy independent noise may describe thermal fluctuations or external mass supply. This presentation indicates the different physical meaning of each term and seems to be important in an equivalent stochastic system formulation.

1.3. **Existence and domain's geometry.** The aim of the present paper is to study existence of solution for the generalized stochastic Cahn-Hilliard equation in general convex domains. In [15], Cardon-Weber used an appropriate convolution semigroup and established existence of solution in $\mathcal{D} := (0,\pi)^d$ for the case $F_2 = 0$ by using the explicit formulae for the spectrum and eigenfuctions of Neumann Laplacian in this cube. Motivated by [15], in our proofs we avoid completely any explicit formula in order to derive analogous results for general domains. We first remark that if $\mathcal{D}$ is a cube of edge a, then by the change of variables $x \to \frac{x}{a/\pi}$, $t \to \frac{t}{a^2/\pi^2}$ we can always consider the equivalent Cahn-Hilliard in $(0,\pi)^d$. Several existence results for various formulations concerning the stochastic Cahn-Hilliard equation are referred to cubic or rectangular space domains, and are based on the explicit formula for the spectrum and eigenfunctions of the Neumann Laplacian operator which is well- known for rectangles, [18]. In our approach, are used instead only the asymptotic spectral properties of the Neumann Laplacian in domains of general convex geometry with piece-wise smooth boundary, where the spectrum and eigenfunctions are unknown. By convexity we derive Lipschitz estimates for the eigenfunctions.

We extend in rectangles the existence result appeared in [15], while for convex or Lipschitz domains (not necessarily convex) in $\mathbb{R}^d$ we study a model with time white noise and prove existence for general domains when $d \leq 2$. In three dimensions, existence is proved under the assumption of minimum eigenfunction growth. Usually in numerical simulations smooth domains are approximated by polyhedra. A case of interest also considered in this paper is when $\mathcal{D}$ is a convex polyhedron. In addition, our proof is valid for the standard ε-dependent Stochastic Cahn-Hilliard, where ε is a measure for the inner interfaces length during spinodal decomposition.

Existence is proved for various simply connected $\mathcal{D}$:

1. $d = 1$: if $\mathcal{D}$ is an open interval.
2. $d = 2$: (a) if $\mathcal{D}$ is convex and of smooth boundary, (b) if $\mathcal{D}$ is a convex polyhedron, (c) if $\mathcal{D}$ is Lipschitz and of smooth boundary (here convexity is not necessary).
3. $d = 3$: for the same cases (a), (b), (c) as in $d = 2$, plus the property of minimum eigenfunctions growth.

Our title refers only to convexity while existence is proved for various domains not necessarily convex, because our proof does not cover the non-convex polyhedral case. Existence is also derived for the standard ε-dependent stochastic Cahn-Hilliard equation.

The paper is organized as follows: in Section 2 we write the problem (1)-(2) in a weak stochastic integral formulation and describe in details the main results. A weaker stochastic partial differential equation formulation is analyzed in Section 3, while existence for the general problem is proved in Section 4. Finally, the last section stands as an appendix for basic definitions from stochastic calculus.

2. Main results.

2.1. Preliminaries.

We consider the Neumann Laplacian operator $\mathcal{T}_N := -\Delta$ defined on $\mathcal{D}(\mathcal{T}_N) = \{u \in H^2(\mathcal{D}) : \frac{\partial u}{\partial n} = 0 \text{ on } \partial\mathcal{D}\}$, where $\mathcal{D}$ is a bounded, domain (open, simply connected set) in $\mathbb{R}^d$, $d = 1, 2, 3$. The eigenvalue problem

$$\mathcal{T}_N v = \mu v \text{ in } \mathcal{D}, \qquad \frac{\partial v}{\partial n} = 0 \text{ on } \partial\mathcal{D}, \tag{4}$$

admits a countable set of eigenvalues as $\mathcal{D}$ is open, bounded and connected. The cases of interest in this paper are when $\mathcal{D}$ is convex or Lipschitz and $\partial\mathcal{D}$ is C^2, or when $\mathcal{D}$ is a convex polyhedron (in this case $\partial\mathcal{D}$ is piece-wise C^∞).

In $L^2(\mathcal{D})$, we consider the usual inner product $< w, v >:= \int_{\mathcal{D}} wv dx$ and the induced norm $\|w\|_2 :=< w, w >^{\frac{1}{2}}$. Any eigenvalue μ is real and non-negative because $< \nabla v, \nabla v >= \mu\|v\|_2^2$ for any eigenfunction v corresponding to μ. There exists an orthonormal basis in $L^2(\mathcal{D})$ consisting of eigenfunctions $\{w_0,\ w_1,\ w_2, \cdots\}$ corresponding to the eigenvalues $0 = \mu_0 < \mu_1 \leq \mu_2 \leq \cdots$ of (4) [39]. w_0 related to $\mu_0 = 0$ is obviously a constant function with $w_0 = (M(\mathcal{D}))^{-1/2}$. From now on, we will consider the $L^2(\mathcal{D})$ together with the orthonormal eigenfunction basis $\{w_0,\ w_1,\ w_2, \cdots\}$. We remark that $\mu_k \to \infty$ as $k \to \infty$ [20].

2.2. Weak formulation.

In the present paper we write the generalized stochastic Cahn-Hilliard equation (1) in a rigorous integral representation utilizing the Green's function induced by the operator proposed by Da Prato and Debussche [19, 15]. More specifically, let $S(t) := e^{-A^2 t}$ be the semi-group generated by the operator $A^2 u := \sum_{i=1}^{\infty} \mu_i^2 u_i w_i$ where $u := \sum_{i=0}^{\infty} u_i w_i$. Then the convolution semi-group [15], is defined by $S(t)U(x) = \sum_{i=1}^{\infty} e^{-\mu_i^2 t}(U, w_i)w_i(x)$ for any $U(x)$ in $L^2(\mathcal{D})$, with the associated Green's function

$$G(t, x, y) = \sum_{i=0}^{\infty} e^{-\mu_i^2 t} w_i(x) w_i(y). \tag{5}$$

As in [15] we let $\varphi \in C^4(\overline{\mathcal{D}})$ such that $\frac{\partial \varphi}{\partial n} = \frac{\partial \Delta\varphi}{\partial n} = 0$ on $\partial\mathcal{D}$. Let u_0 be the initial value at $t = 0$, then for any u satisfying the boundary conditions (2) equation (1) is written in the weak form:

$$\begin{aligned}\int_{\mathcal{D}} \Big(u(x,t) - u_0(x)\Big)\varphi(x)\, dx = -\int_0^t \int_{\mathcal{D}} (\Delta^2\varphi) u \, dxds + \int_0^t \int_{\mathcal{D}} (\Delta\varphi) f'(u)\, dxds \\ + \int_0^t \int_{\mathcal{D}} \Delta F_2(x,t)\varphi(x)\, dx V(ds) + \int_0^t \int_{\mathcal{D}} F_1(u)\varphi(x) W(dx, ds).\end{aligned} \tag{6}$$

Here, $V(ds)$ is the Wiener measure induced by the one-dimensional one-parameter Wiener process V (with respect to time variable), i.e. $V := \{V(t), t \in [0, T]\}$ defined on the probability space $(\Omega, \mathcal{F}, P)$ is $\mathcal{F}_t$-adapted for any $s \leq t$, where $\mathcal{F}_t$, $t \geq 0$ is an increasing family of σ-algebras. The function F_2 is considered smooth, hence, the time integral $\int_0^t \Delta F_2(x,t)\varphi(x)\, V(ds)$ is a martingale.

We use a unified notation $\int_0^t \int_{\mathcal{D}} \cdots W(dx, ds)$ for representing stochastic integration in time and in space, for the sake of a generalized symbolism. In the case of general domain $\mathcal{D}$, we consider $W(dx, ds) := dxW_1(ds)$, with W_1 one-dimensional, one-parameter Wiener process defined as V. The measure $W(dx, ds)$ when the domain is rectangular is induced by the one-dimensional $d+1$-parameter Wiener process (d for space variables, 1 for the time variable) $W := \{W(x,t), t \in [0,T]; x \in \mathcal{D}\}$ in the set of the $\mathcal{F}_t$-adapted processes $\{W(x,s); s \le t, x \in \mathcal{D}\}$ [15, 46]. In this case too, since F_1 is Lipschitz and bounded, the appearing stochastic space-time integral is a martingale. We note that the analysis appearing in the present paper is valid for both cases, and thus we proceed by keeping the general notation $W(dx, ds)$ for the space-time measure.

Remark 1. In the Appendix, we present detailed definitions concerning stochastic process, Wiener process and $\mathcal{F}_t$-adaptive processes.

Following J.B. Walsh formulation for parabolic problems [46], and C. Cardon-Weber [15], a function u is considered a weak solution to the Stochastic Cahn-Hilliard (1)-(2) or equivalently solution of (6) if and only if for any $x \in \mathcal{D}$ and $t \in [0,T]$ satisfies the Stochastic Partial Differential Equation (SPDE)

$$\begin{aligned} u(x,t) &= \int_{\mathcal{D}} G(t,x,y)u_0(y)\,dy + \int_0^t \int_{\mathcal{D}} \Delta G(t-s,x,y) f'(u)\,dyds \\ &+ \int_0^t \int_{\mathcal{D}} G(t-s,x,y)\Delta F_2(y,s)\,dyV(ds) + \int_0^t \int_{\mathcal{D}} G(t-s,x,y)F_1(u)\,W(dy,ds), \end{aligned} \tag{7}$$

where G is the Green's function defined by (5).

We remind that F_1 is Lipschitz in u and bounded, while F_2 is smooth. In order to study existence of solution for the generalized Cahn-Hilliard equation, we take $u_0 \in L^q(\mathcal{D})$, $q \ge 4$ and consider the following cut-off SPDE system

$$\begin{aligned} u_n(x,t) = &\int_{\mathcal{D}} G(t,x,y)u_0(y)dy \\ &+ \int_0^t \int_{\mathcal{D}} \Delta G(t-s,x,y)\chi_n\Big(\|u_n(\cdot,s)\|_q\Big) f'\Big(u_n(y,s)\Big)dyds \\ &+ \int_0^t \int_{\mathcal{D}} G(t-s,x,y)\Delta F_2(y,s)dyV(ds) \\ &+ \int_0^t \int_{\mathcal{D}} G(t-s,x,y)F_1\Big(u_n(y,s)\Big)W(dy,ds), \end{aligned} \tag{8}$$

where $\chi_n : \mathbb{R}^+ \to \mathbb{R}^+$ are C^1 functions, such that $|\chi_n| \le 2$, $|\chi_n'| \le 2$, for any $n > 0$, and satisfy $\chi_n(x) = \begin{cases} 1 & \text{for} \quad x < n \\ 0 & \text{for} \quad x \ge n+1. \end{cases}$

2.3. **Existence of solution for the Stochastic Cahn-Hilliard.** The general procedure that we follow in order to establish existence of solution for the SPDE (7) for any $t \ge 0$ is the following:

1. We first prove existence and uniqueness for the solution u_n of the SPDE (8) in an appropriate set $\mathcal{W}$ of $\mathcal{F}_t$-adapted random processes, for any initial value u_0, in every interval $[0,T]$.
2. By uniqueness of the process u_n for any $t \le T_n$, where T_n is a stopping time defined by
$$T_n := \inf\Big\{t \ge 0 : \ \|u_n(t,\cdot)\|_q \ge n\Big\},$$

follows that $u_m(\cdot,t) = u_n(\cdot,t)$ for any $m > n$, and the process $u(\cdot,t) := u_n(\cdot,t)$ is well defined for all $t \leq T_n$.

3. If $t < T_n$, then $\|u_n(t,\cdot)\|_q < n$ and thus it holds that $\chi_n(\|u_n\|_q) = 1$. So the process u is a solution of (7) (see (7) and (8) for $\chi_n = 1$) in the interval $[0, T_n)$ for any $n \geq 1$. Existence of solution for (7) for all $t \geq 0$ a.s. is established by proving that $\lim_{n\to\infty} p[T_n \leq T] = 0$ for any $T > 0$, i.e. $\lim_{n\to\infty} T_n = \infty$ a.s.

In details, eq. (8) is written as a sum of operators:

$$u_n(x,t) = \int_{\mathcal{D}} G(t,x,y)u_0(y)\,dy + \mathcal{M}_n(u_n)(x,t) + \mathcal{L}(u_n)(x,t),$$

where the operator $\mathcal{L}$ involves the stochastic integrals. By fix point arguments, since F_1 is Lipschitz and F_2 is smooth and independent of u, we prove that a unique solution of (8) exists if T is sufficiently small, where T is independent of the initial condition u_0. Therefore, we extend each time the interval of solution in t by setting the solution $u_n(\cdot,T)$ as u_0.

We prove Hölder type inequalities related to the Green's function G. More specifically, we first prove Lipschitz inequalities for the Neumann eigenfunctions in $\overline{\mathcal{D}}$ (for this, the domain's geometry is crucial). We then use the asymptotic properties of the eigenfunctions and eigenvalues of the Neumann Laplacian operator and prove space-time Hölder type inequalities for the Green's function in $\overline{\mathcal{D}}$.

Further, we define $v_n := u_n - \mathcal{L}(u_n)$ and estimate u_n in the $L^4(\mathcal{D})$ norm and v_n in $L^r(\mathcal{D})$ for any $r \geq 2$; the estimates of v_n are derived by constructing Galerkin approximations. Combining these estimates together with the Hölder estimates, and using a generalization of Kolmogorov's Theorem, we prove existence of solution for the SPDE (7) in an appropriate set of processes a.s. for any $t \geq 0$.

Remark 2. In our initial and boundary value problem, we may consider in place of (1) the following ε-dependent generalized stochastic Cahn-Hilliard equation

$$u_t = \Delta\Big(-\varepsilon^2\Delta u + f'(u) + F_2(x,t;\varepsilon)\dot{V}\Big) + F_1(x,t;\varepsilon)\dot{W},\ x \in \mathcal{D}, \tag{9}$$

where the noise terms are defined as in (1). Here $\varepsilon > 0$ is a measure of the width of inner interfaces that may be developed along phase transitions during time evolution in spinodal decomposition. In order to keep the same Green's function for our fourth order evolutionary problem, we apply the simple rescaling $t \to t\varepsilon^2$. Thus, equation (9) is transformed into an equivalent one of the form (1) where existence of solution is proved.

Each noise term has a different physical meaning. $F_1\dot{W}$ is in general a Gaussian noise (thermal fluctuations or external mass supply, [17, 40, 35]), while $F_2\dot{V}$ is an external field noise [38, 32, 35]. In our analysis the noise $\dot{V}$ appeared into the chemical potential of (1) or (9) is defined as a time noise, since it is coupled with the Laplacian.

The proposed model (1) or (9) separating chemical potential noise and free-energy independent noise is just emphatic and indicates an equivalent stochastic system formulation. In particular, the rescaled (9)

$$\partial_t u = \Delta(-\varepsilon\Delta u + \varepsilon^{-1}f'(u) - G_2\dot{V}) + G_1\dot{W} \tag{10}$$

where $G_1 := \varepsilon^{-1}F_1$, $G_2 := -\varepsilon^{-1}F_2$, is written as the stochastic system

$$\begin{cases} \partial_t u &= -\Delta v + G_1\dot{W} \\ \\ v &= -\dfrac{f'(u)}{\varepsilon} + \varepsilon\Delta u + G_2\dot{V}, \end{cases} \tag{11}$$

where v is the chemical potential. The previous, may be useful for a rigorous asymptotic analysis of the stochastic equation (9) as $\varepsilon \to 0^+$ which is a very interesting open problem, or for the construction of numerical approximations.

In [5], the asymptotic behaviour of the deterministic (9) or (11) as $\varepsilon \to 0^+$ has been analyzed (i.e. for $\dot{V} = \dot{W} = 1$). The sharp interface limit problem in the multidimensional case demonstrated a local influence in phase transitions of forcing terms that stem from the chemical potential, while free energy independent terms act on the rest of the domain. In addition, the forcing may slow down the equilibrium. Note that the case $G_1 = G_2 = 0$ has been analyzed in [2, 43].

In our existence analysis the system representation is not used, thus the mathematical treatment of the noise term $F_2\dot{V}$ is the same as if it was additive in the form $(\Delta F_2)\dot{V}$.

3. **Existence and uniqueness of solution for the SPDE (8).** Letting $q \geq 1$, we define $\|\cdot\|_q$ as the usual norm in $L^q(\mathcal{D})$. Our aim in this section is to first prove that (8) admits a unique solution in an interval $[0,T]$, in the set

$$\mathcal{W} = \Big\{u(\cdot,t) \in L^q(\mathcal{D}) : \ u \text{ is } \mathcal{F}_t - \text{adapted random process and } \|u\|_{\mathcal{W}} < \infty\Big\},$$

where $\|u\|_{\mathcal{W}} := \sup\limits_{0\leq t\leq T} E(\|u(\cdot,t)\|_q^\beta)^{1/\beta}$, $\beta \geq q$ if $d = 1, 2$, and $\frac{6q}{(6-q)^+} > \beta \geq q$ if $d = 3$. Further, we will prove that solution exists for any $T > 0$.

Let us now consider the non-linear operators $\mathcal{M}_n$ defined on $\mathcal{W}$ by

$$\mathcal{M}_n(u)(x,t) := \int_0^t \int_{\mathcal{D}} \Delta G(t-s,x,y)\chi_n\Big(\|u(\cdot,s)\|_q\Big) f'\Big(u(y,s)\Big)\, dyds,$$

and let $\mathcal{L}$ be the operator on $\mathcal{W}$ given by

$$\begin{aligned} \mathcal{L}(u)(x,t) : \ &= \int_0^t \int_{\mathcal{D}} G(t-s,x,y)F_1\Big(u(y,s)\Big)W(dy,ds) \\ &\quad + \int_0^t \int_{\mathcal{D}} G(t-s,x,y)\Delta F_2(y,t)\, dyV(ds). \end{aligned}$$

By the formulation of (8) we derive that

$$u_n(x,t) = \int_{\mathcal{D}} G(t,x,y)u_0(y)\, dy + \mathcal{M}_n(u_n)(x,t) + \mathcal{L}(u_n)(x,t). \tag{12}$$

Remark 3. According to [24], for $\mathcal{D}$ smooth, there exist positive constants c_1, c_2 such that for $t \in (0,T]$ and for any $x, y \in \mathcal{D}$

$$|G(t,x,y)| \leq c_1 t^{-d/4} \exp\Big(- c_2|x-y|^{4/3}|t|^{-1/3}\Big), \tag{13}$$

$$|\Delta G(t,x,y)| \leq c_1 t^{-(d+2)/4} \exp\Big(- c_2|x-y|^{4/3}|t|^{-1/3}\Big). \tag{14}$$

In the case of convex polyhedra, the same estimates follow on the limit by a standard technique of approximating the polyhedral domain by a sequence $\mathcal{D}_n$ of convex smooth subdomains, where $\overline{\mathcal{D}_1} \subset \overline{\mathcal{D}_2} \subset \cdots \subset \overline{\mathcal{D}_n} \subset \cdots \overline{\mathcal{D}}$, and $\partial\mathcal{D}_n \cap \partial\mathcal{D} \subseteq \partial\mathcal{D}_{n+1} \cap \partial\mathcal{D} \subset \partial\mathcal{D}$ for any n. Here, $\partial\mathcal{D}_n \cap \partial\mathcal{D}$ is an increasing sequence of finite unions of planar surfaces and linear segments in $\partial\mathcal{D}$ (a detailed proof for the cube can be found in [15]). By use of relations (13)-(14) it follows that if $v \in L^1([0,T], L^\rho(\mathcal{D}))$ with $1 \leq \rho \leq \infty$, then there exists positive constant c such that for any $\rho \leq q \leq \infty$ and any $x \in \mathcal{D}$

$$\Big\|\int_{t_0}^t\int_{\mathcal{D}}\Delta G(t-s,x,y)v(s,y)dyds\Big\|_q \leq c\int_{t_0}^t (t-s)^{-\frac{d+2}{4}+\frac{d}{4r}}\|v(\cdot,s)\|_\rho ds, \tag{15}$$

$$\Big\|\int_{t_0}^t\int_{\mathcal{D}}G^2(t-s,x,y)v(s,y)dyds\Big\|_q \leq c\int_{t_0}^t (t-s)^{-\frac{d}{2}+\frac{d}{4r}}\|v(\cdot,s)\|_\rho ds, \tag{16}$$

where $t \geq t_0 \geq 0$ and $r > 1$ such that $\frac{1}{r} = \frac{1}{q} - \frac{1}{\rho} + 1$. In the case $d = 3$ for (15) we need $r < 3$ while in (16) r is less than $3/2$.

By a fixed point argument we prove existence and uniqueness for the SPDE (8) for any initial value u_0.

Theorem 3.1. *The SPDE (8) admits a unique solution u_n in every interval* $[0,T]$.

Proof. In [15], for the general case where $f'(u)$ is a third degree polynomial with positive dominant coefficient when (8) is posed on cubic domains for $F_2 = 0$ and F_1 Lipschitz, by use of (13)-(16) (which have been extended for the cubic case) is proved that the operator $\mathcal{M}_n(\mathcal{W}), \mathcal{L}(\mathcal{W}) \subseteq \mathcal{W}$, and $\mathcal{M}_n + \mathcal{L}$ is a contraction in $\mathcal{W}$ if $T \leq T_1$ for some T_1 small and independent from u_0.

In our case, the proof is similar with some slight differences. (a) We have an extra term $\int_0^t\int_{\mathcal{D}} G(t-s,x,y)\Delta F_2(y,s)V(ds)$, appearing at the operator $\mathcal{L}$ which is a martingale with continuous representative because F_2 is smooth, and is u-independent. (b) The non-linearity $f' = u(u^2-1)$ is a third degree polynomial with positive dominant coefficient. (c) In addition, since (13)-(16) hold in C^2 domains ([24]) or in convex polyhedra (see Remark 3), we obtain as in [15] that $\mathcal{M}_n(\mathcal{W}), \mathcal{L}(\mathcal{W}) \subseteq \mathcal{W}$, $\mathcal{M}_n + \mathcal{L}$ is a contraction in $\mathcal{W}$ and thus admits a unique fixed point in the set $\{u \in \mathcal{W} : \ u(\cdot,0) = u_0\}$ if $T \leq T^*$, with T^* small and independent from u_0.

Consequently, in $[0,T]$, for $T \leq T^*$, a unique solution u_n exists for the SPDE (8). Setting $t_0 := T$ and initial value $u_0(x) = u(x,T)$, we extend every time the interval of existence and thus the solution exists on every interval $[0,T]$. □

4. **Existence and uniqueness of solution for the SPDE (7).**

4.1. **Hölder estimates.** In this paragraph we prove Hölder-type estimates for the Green's function G in time and space also valid for the boundary $\partial\mathcal{D}$. We consider that $d = 1,2,3$; generally if $\mathcal{D}$ is open and bounded, then the eigenvector basis $\{w_0,\ w_1,\ w_2,\cdots\}$ is in $C^\infty(\mathcal{D})$. If the open bounded domain $\mathcal{D}$ is convex then convexity supplies the eigenvector basis $\{w_0,\ w_1,\ w_2,\cdots\}$ with Lipschitz regularity in $\overline{\mathcal{D}}$ (the same is proved for smooth Lipschitz domains also). More specifically, following Evans [27], we shall first prove that the basis is Lipschitz in $\overline{\mathcal{D}}$ and for each eigenfunction w_i we shall calculate the Lipschitz coefficient depending on eigenvalue μ_i. Furthermore, we shall derive upper L^∞ bounds for the eigenfunction basis in $\overline{\mathcal{D}}$.

Consider two points x and y in $\overline{\mathcal{D}}$ identified by their position vectors and define for $0 < \lambda < 1$ the convex linear combination $xy := \lambda x + (1-\lambda)y$. As $\mathcal{D}$ is convex, only the following two cases appear (for $d = 2$ see Fig. 1):

1. xy lies in $\mathcal{D}^\circ = \mathcal{D}$.

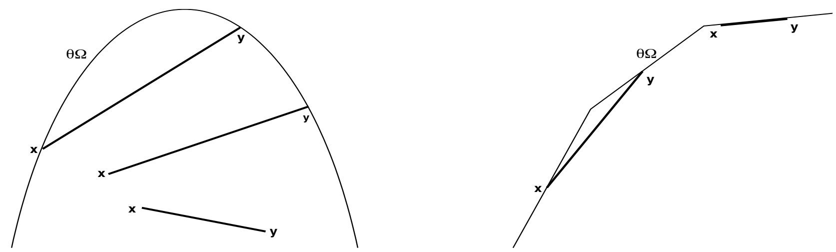

FIGURE 1. Various cases for the linear segment xy in convex domains.

2. xy lies on the boundary $\partial\mathcal{D}$ (this case implies that the boundary $\partial\mathcal{D}$ includes linear segments or planar surfaces).

Consider now the general case: $\mathcal{D}$ is in $\mathbb{R}^n$ and define by $|x-y|$ the usual metric in $\mathbb{R}^n$. By convexity and Cauchy-Schwarz inequality we obtain the next elementary result.

Lemma 4.1. *Let $\mathcal{D}$ be a convex bounded, domain (open and simply connected) in $\mathbb{R}^n$, and u is smooth, then for any $x=(x_1,\cdots,x_n)$, $y=(y_1,\cdots,y_n)$ in $\mathcal{D}$, and $\widetilde{x}(t):=tx+(1-t)y$, for any $0<t<1$, it holds that*

$$|u(x)-u(y)| \leq |x-y| \int_0^1 \Big|\nabla u(\widetilde{x}(t))\Big| dt. \tag{17}$$

Remark 4. The estimate (17) is true even if $x,y\in\partial\mathcal{D}$ under the assumption that $\widetilde{x}(t)$ lies in $\mathcal{D}$ where u is regular.

We now extend Lemma 4.1 in order to obtain an analogous estimate in the case where on the boundary exist linear segments or planar surfaces.

Lemma 4.2. *Let $\mathcal{D}$ be a convex bounded domain of $\mathbb{R}^n$. If u is smooth, $x=(x_1,\cdots,x_n)$, $y=(y_1,\cdots,y_n)$ are in $\partial\mathcal{D}$, and $\widetilde{x}(t):=tx+(1-t)y$, $0<t<1$, lies in $\partial\mathcal{D}$, then there exists a positive constant c, independent from u, x, y, t, such that*

$$|u(x)-u(y)| \leq c|x-y| \int_0^1 \Big(\Big|\nabla u(\widetilde{x_1}(t))\Big| + \Big|\nabla u(\widetilde{x_2}(t))\Big|\Big) dt, \tag{18}$$

where $\widetilde{x_1}(t):=tx+(1-t)z$, $\widetilde{x_2}(t):=ty+(1-t)z$, lie in $\mathcal{D}$ for any $t\in[0,1)$.

Proof. $\mathcal{D}$ is convex, hence we can always construct a triangle xzy of vertex $z\in\mathcal{D}$ and of sufficiently small height h (measured from z) such that the edges $\widetilde{x_1}(t):=tx+(1-t)z$, $\widetilde{x_2}(t):=ty+(1-t)z$ are equal and in $\mathcal{D}$ for $t\in[0,1)$, and $h\leq|x-y|$ (see Fig. 2 for the case $d=2,3$). By Lemma 4.1 and Remark 3 we obtain

$$\begin{aligned} |u(x)-u(y)| &\leq |u(x)-u(z)|+|u(z)-u(y)| \\ &\leq |x-z|\int_0^1 |\nabla u(\widetilde{x_1}(t))|dt + |y-z|\int_0^1 |\nabla u(\widetilde{x_2}(t))|dt, \end{aligned} \tag{19}$$

obviously $|x-z|^2=|y-z|^2=h^2+|x-y|^2/4\leq\frac{5}{4}|x-y|^2$; thus by (19) we get the result. $\square$

In the following, we show that the basis $\{w_0,\ w_1,\ w_2,\cdots\}$ is Lipschitz in $\overline{\mathcal{D}}$ and estimate the Lipschitz constants by the eigenvalues μ_i.

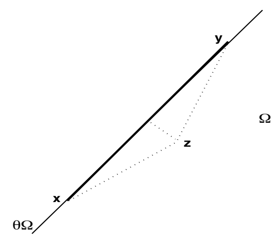

FIGURE 2. Lipschitz on the boundary: the inner triangle.

Proposition 4.3. *There exists a positive constant c such that*

$$|w_i(x) - w_i(y)| \leq c\mu_i^{\frac{a}{2}}|x-y|, \tag{20}$$

for any x, y in $\overline{\mathcal{D}}$, and any $i \geq 1$, where $\mathcal{D}$ is a bounded convex domain of $\mathbb{R}^d$, $d = 1, 2, 3$; $a = 1$ for $d = 1$ while $a = 3$ for $d = 2, 3$.

Proof. By Lemmas 4.1-4.2 in order to estimate the Lipschitz coefficient appearing in (20) we need L^2 or L^∞ estimates for the gradient of w_i. For the one-dimensional case ($d = 1$) it holds that

$$\int_0^t |\nabla w_i(\tilde{z}(t))|dt \leq \int_{\mathcal{D}} |\nabla w_i(x)|dx \leq c\|\nabla w_i\|_{L^2(\mathcal{D})}$$

for any linear segment $\tilde{z}(t)$ in $\mathcal{D} \subset \mathbb{R}$ (in this case the integral along $\tilde{z}(t)$, for $t \in (0,1)$ coincides to Lebesgue measure in $\mathbb{R}$ restricted in $\tilde{z}(t)$, for $t \in (0,1)$). The Neumann Laplacian boundary conditions give $\|\nabla w_i\|_{L^2(\mathcal{D})} = \mu_i^{1/2}$, therefore, by (17) and (18) we get that $|w_i(x) - w_i(y)| \leq c\mu_i^{1/2}|x-y|$.

When $d = 2, 3$, by general Sobolev inequality [1], valid for Lipschitz domains of $\mathbb{R}^n$ [12], i.e. valid if $\mathcal{D}$ is C^2 and convex or if $\mathcal{D}$ is a convex polyhedron, follows that

$$|\nabla w_i| \leq c\sum_{j=1}^d \|\partial_{x_j} w_i\|_{H^2(\mathcal{D})} \leq c\sum_{j=1}^d \Big(\|\Delta\partial_{x_j} w_i\|^2_{L^2(\mathcal{D})} + \sum_{j=1}^d < w_0, \partial_{x_j} w_i >^2\Big)^{1/2}.$$

Using that $\|\Delta\partial_{x_j} w_i\|^2_{L^2(\mathcal{D})} = \|\mu_i\partial_{x_j} w_i\|^2_{L^2(\mathcal{D})} = \mu_i^3$, the estimate $|(w_0, \partial_{x_j} w_i)| \leq c\|\partial_{x_j} w_i\|_{L^2(\mathcal{D})} \leq c\mu_i^{1/2}$ (where w_0 is the constant function), observing that $\mu_i \to \infty$ and using (17), (18), we finally obtain: $|w_i(x) - w_i(y)| \leq c\mu_i^{3/2}|x-y|$. $\square$

Remark 5. Without geometric assumptions for $\mathcal{D}$, an upper bound in the L^∞ norm for the eigenfunction basis is: $\sup_{\mathcal{D}}|w_i(x)| \leq c\mu_i^{\frac{d-1}{4}}$, which according to Duistermaat-Guillemin Theorem is sharp if the set of periodic geodesics at $\mathcal{D}$ is of zero measure [23, 33]. Indeed, in the multi-dimensional case $d \geq 2$ when $\mathcal{D} \subset \mathbb{R}^d$ is rectangle, this estimate is not the best as the basis is uniformly bounded in any i [9, 18], while $\mu_i^{\frac{d-1}{4}} \to \infty$.

Given i, when $x \in \partial\mathcal{D}$, then, as $\mathcal{D}$ is convex we choose $y_i \in \mathcal{D}$ such that $|x - y_i| = h_i$ (for sufficiently small h_i), thus by Lipschitz condition and the bound in L^∞ norm, we get

$$|w_i(x)| \le |w_i(x) - w_i(y_i)| + |w_i(y_i)| \le \mu_i^{a/2}|x - y_i| + c_0\mu_i^{\frac{d-1}{4}} = h_i\mu_i^{a/2} + c_0\mu_i^{\frac{d-1}{4}},$$

as $\mu_i \to \infty$ it is sufficient to choose $h_i := \epsilon\mu_i^{-a/2}$, ϵ bounded, to get that there exists a positive constant c such that $|w_i(x)| \le c\mu_i^{\frac{d-1}{4}}$ for any $x \in \overline{\mathcal{D}}$, and any $i \ge 0$, and obviously, follows that $|w_i(x) - w_i(y)| \le c\mu_i^{\frac{d-1}{4}}$ for any $x \in \overline{\mathcal{D}}$ and any $i \ge 0$.

If $d = 3$, a case of interest is when the manifold of solutions for the Neumann Laplacian has the minimum eigenfunction growth, i.e. by definition when the eigenfunction basis is uniformly bounded for any i in $\mathcal{D}$. In this case if $\mathcal{D}$ is convex, then the analogous calculations yield: $|w_i(x)| \le c$ for any $x \in \overline{\mathcal{D}}$ and any i.

Now we are able to prove the next crucial for our analysis space-time Hölder type estimate, valid in $\overline{\mathcal{D}}$, involving Green's function G, for general convex domains $\mathcal{D}$ in $\mathbb{R}^d$ for $d \le 2$, or, in the case $d = 3$ if the Neumann Laplacian has the minimum eigenfunction growth in the convex domain $\mathcal{D}$.

Theorem 4.4. *Let G be the Green's function defined by (5). There exist $c > 0$ and positive γ, γ', such that for any $t > s$ and any $x, y \in \overline{\mathcal{D}}$ holds that*

$$\int_0^t \int_{\mathcal{D}} |G(t-r,x,z) - G(t-r,y,z)|^2 dzdr \le c|x-y|^{\gamma}, \tag{21}$$

$$\int_0^t \int_{\mathcal{D}} |G(t-r,x,z) - G(s-r,x,z)|^2 dzdr \le c|t-s|^{\gamma'}, \tag{22}$$

$$\int_s^t \int_{\mathcal{D}} |G(t-r,x,z)|^2 dzdr \le c(|t-s| + |t-s|^{\gamma'}), \tag{23}$$

for $d \le 2$, or when $d = 3$ if the manifold of solutions for the Neumann Laplacian in $\mathcal{D}$ has the minimum eigenfunction growth.

Proof. Using $|w_0(x) - w_0(y)| = 0$, $\int_{\mathcal{D}} |w_i(z)|^2 dz = 1$ and $\mu_i \to \infty$, we get

$$\begin{aligned}\int_0^t \int_{\mathcal{D}} |G(t-r,x,z) - G(t-r,y,z)|^2 dzdr &\le \sum_{i=1}^{\infty} \Big[\frac{e^{-2\mu_i^2(t-r)}}{2\mu_i^2}\Big]_0^t |w_i(x) - w_i(y)|^2 \\ = \sum_{i=1}^{\infty} \frac{1}{2\mu_i^2}[1 - e^{-2\mu_i^2 t}]|w_i(x) - w_i(y)|^2 &\le c\sum_{i=1}^{\infty} \mu_i^{-2}|w_i(x) - w_i(y)|^2.\end{aligned}$$

As $|w_i(x) - w_i(y)|$ is of order $O\Big(\mu_i^{\frac{d-1}{4}}\Big)$ in $\overline{\mathcal{D}}$ for any i, we may define a positive constant c such that: $\Big(c\mu_i^{\frac{d-1}{4}}\Big)^{-1}|w_i(x) - w_i(y)| \le 1$, hence, $|w_i(x) - w_i(y)|^2 \le c\Big(\mu_i^{\frac{d-1}{4}}\Big)^2 |w_i(x) - w_i(y)|^{2l}$, for $l \in (0, 1]$ because $\mu_i \to \infty$. Therefore, by making use of Proposition 4.3 we obtain

$$\int_0^t \int_{\mathcal{D}} |G(t-r,x,z) - G(t-r,y,z)|^2 dzdr \le c\sum_{i=1}^{\infty} \mu_i^{-2}|w_i(x) - w_i(y)|^2$$

$$\le c\sum_{i=1}^{\infty} \mu_i^{-2+\frac{d-1}{2}}|w_i(x) - w_i(y)|^{2l} \le c|x-y|^{2l}\sum_{i=1}^{\infty} \mu_i^{-2+\frac{d-1}{2}}\mu_i^{2la/2} = c|x-y|^{2l}\sum_{i=1}^{\infty} \mu_i^{\sigma},$$

for $\sigma = -2 + \frac{d-1}{2} + la$. The asymptotic behavior of eigenvalues for large i is $\mu_i = O\Big((i-1)^{2/d}\Big)$ [47], therefore, the appearing series in the previous inequality converges if $\frac{2}{d}(-2 + \frac{d-1}{2} + la) < -1$ or equivalently for $l < \frac{5-2d}{2a}$. Thus, for $0 < l < \min\{1, \frac{5-2d}{2a}\}$ we get (21) for $0 < \gamma := 2l < \min\{2, \frac{5-2d}{a}\}$ (we note that the dimension is $d = 1, 2$). If $d = 3$ and the manifold of solutions for the Neumann Laplacian has the minimum eigenfunction growth, then by the analogous calculations we obtain:

$$\int_0^t \int_{\mathcal{D}} |G(t-r,x,z) - G(t-r,y,z)|^2 dzdr \leq c|x-y|^{2l} \sum_{i=1}^{\infty} \mu_i^{-2+la}$$

hence, the series converges if $\frac{2}{d}(-2 + la) < -1$, i.e. when $l < 1/(2a)$. In this case we set $0 < \gamma := 2l < 1/a$.

By simple calculations, using the basis upper bound and $\mu_0 = 0$, we obtain

$$\int_0^t \int_{\mathcal{D}} |G(t-r,x,z) - G(s-r,x,z)|^2 dzdr \leq c \sum_{i=1}^{\infty} \mu_i^{\frac{d-1}{2}} \frac{1}{2\mu_i^2}(e^{-\mu_i^2(t-s)} - 1)^2(1 - e^{-2\mu_i^2 s}).$$

But $t > s$ and $\mu_i \to \infty$, therefore $|e^{-\mu_i^2(t-s)} - 1|$ is uniformly bounded, and also $|e^{-\mu_i^2(t-s)} - 1| \leq c\mu_i^2|t-s|$ as the exponential is Lipschitz. Therefore for $l \in (0,1)$ we obtain

$$\int_0^t \int_{\mathcal{D}} |G(t-r,x,z) - G(s-r,x,z)|^2 dzdr \leq c|t-s|^l \sum_{i=1}^{\infty} \mu_i^{-2+\frac{d-1}{2}+2l}.$$

The series in the above inequality converges if $\frac{2}{d}(-2+\frac{d-1}{2}+2l) < -1$ or equivalently if $l < \frac{5-2d}{4}$. Consequently the estimate (22) follows for $0 < \gamma' := l < \min\{1, \frac{5-2d}{4}\}$ for $d = 1, 2$. In the case when $d = 3$, we get

$$\int_0^t \int_{\mathcal{D}} |G(t-r,x,z) - G(s-r,x,z)|^2 dzdr \leq c|t-s|^l \sum_{i=1}^{\infty} \mu_i^{-2+2l},$$

and the result follows for $0 < \gamma' := l < 1/4$.

Finally, by simple calculations we get that

$$\int_s^t \int_{\mathcal{D}} |G(t-r,x,z)|^2 dzdr \leq c|t-s| + c|t-s|^l \sum_{i=1}^{\infty} \mu_i^{-2+c_0\frac{d-1}{2}+2l},$$

for $l \in (0,1)$. The estimate (23) follows for $0 < \gamma' := l < \min\{1, \frac{5-2d}{4}\}$ if $d = 1, 2$ (where $c_0 = 1$) or for $0 < \gamma' := l < 1/4$ when $d = 3$ (in this case $c_0 = 0$). □

Remark 6. The minimum eigenfunction growth holds for example in rectangular domains or torus [33]. We also note that the results (21), (22), (23) of Theorem 4.4 can be extended for some cases where $\mathcal{D}$ is simply connected but non-convex, under the assumption of Lipschitz boundary (if $d \leq 3$), and minimum eigenfunction growth (if $d = 3$). More specifically, Lemma 4.1 is generalized in the following convexity-independent result:

Lemma 4.5. *Let $\mathcal{D}$ be a simply connected Lipschitz domain in $\mathbb{R}^n$ of smooth boundary, and u is smooth, then there exists a positive constant k such that for any $x = (x_1, \cdots, x_n)$, $y = (y_1, \cdots, y_n)$ in $\mathcal{D}$ it holds that*

$$|u(x) - u(y)| \leq k|x-y| \sup_{z\in\mathcal{D}} |\nabla u(z)|. \tag{24}$$

Proof. By the definition of $\mathcal{D}$, there exists a positive constant k such that for any choice of points x, y in $\mathcal{D}$ there exists a smooth curve $\widetilde{X}$ in $\mathcal{D}$ of length $l(\widetilde{X})$ less or equal $k|x-y|$ connecting the two points (see for example [45]). Let us consider $\widetilde{X} = \widetilde{X}(t) = (\widetilde{X}_1(t), \cdots, \widetilde{X}_n(t)) \in \mathbb{R}^n$ for $t \in [0,1]$, where $\widetilde{X}(0) = x$, $\widetilde{X}(1) = y$. Defining $\dot{\widetilde{X}}_i(t) := \frac{d}{dt}\widetilde{X}_i(t)$, it follows that

$$u(y) - u(x) = \int_0^1 \frac{d}{dt} u(\widetilde{X}(t))dt \leq \int_0^1 |\nabla u(\widetilde{X}(t))||\dot{\widetilde{X}}(t)|dt$$

$$\leq \sup_{z\in\mathcal{D}} |\nabla u(z)| \int_0^1 |\dot{\widetilde{X}}(t)|dt = \sup_{z\in\mathcal{D}} |\nabla u(z)| l(\widetilde{X}) \leq k|x-y| \sup_{z\in\mathcal{D}} |\nabla u(z)|. \quad \square$$

Therefore, Proposition 4.3 holds true (in dimensions $d = 2, 3$ where the non-convex simply connected case may appear, we estimate $|\nabla w_i|$). So, the space-time Hölder estimates (21), (22), (23) follow.

4.2. L^4 **estimates.** We define $v_n := u_n - \mathcal{L}(u_n)$, then v_n is the weak solution in $[0, T]$ of the SPDE

$$\begin{aligned} &\partial_t v_n + \Delta^2 v_n - \Delta\Big(\chi_n(\|u_n\|_q) f'(u_n)\Big) = 0 \text{ in } \mathcal{D}, \\ &v_n(x,0) = u_0(x), \quad \frac{\partial v_n}{\partial n} = \frac{\partial \Delta v_n}{\partial n} \text{ on } \partial\mathcal{D}, \end{aligned} \tag{25}$$

by the sense of the weak formulation:

$$\begin{aligned} \int_{\mathcal{D}} \Big(v_n(x,t) - u_0(x)\Big)\phi(x)dx = &-\int_0^t \int_{\mathcal{D}} \Delta^2\phi(x) v_n(x,s)dxds \\ &+ \int_0^t \int_{\mathcal{D}} \Delta\phi(x)\chi_n\Big(\|u_n(\cdot,s)\|_q\Big) f'\Big(u_n(x,s)\Big)dxds, \end{aligned} \tag{26}$$

for any $\phi \in C^4(\overline{\mathcal{D}})$ such that $\frac{\partial \phi}{\partial n} = \frac{\partial \Delta\phi}{\partial n}$ on $\partial\mathcal{D}$ [15].

We note that for t fixed and for any $w(t,\cdot) \in H^2(\mathcal{D}) \subset L^2(\mathcal{D})$ it holds that $-\Delta w = \sum_{k=0}^{\infty} \mu_k < w_k, w > w_k = \sum_{k=1}^{\infty} \mu_k < w_k, w > w_k$, as $\{w_0,\ w_1,\ w_2, \cdots\}$ is an orthonormal eigenfunction basis of $L^2(\mathcal{D})$ and $\mu_0 = 0$, while $-\Delta w_k = \mu_k w_k$. We define

$$\|\mathcal{L}(u_n)\|_\infty := \sup_{t\in[0,T]} \sup_{x\in\mathcal{D}} |\mathcal{L}(u_n)(x,t)|. \tag{27}$$

At the following lemma we make use of the Hölder estimates of the previous paragraph.

Lemma 4.6. *If u_n is the solution of the SPDE (8) then for any ρ, $\delta > 1$ holds*

$$\sup_{n\in N} E\Big(\|\mathcal{L}(u_n)\|_\infty^{2\rho\delta}\Big) < \infty. \tag{28}$$

Proof. Let $a > 1$, $T > 0$ and consider t, $t' \in [0,T]$ and x, $x' \in \overline{\mathcal{D}}$, then

$$\begin{aligned} E\Big(|\mathcal{L}(u_n)(x,t) - \mathcal{L}(u_n)(x',t')|^{2a}\Big) = E\Big(\Big|&\int_0^t \int_{\mathcal{D}} \mathcal{A}F_1(u_n(y,s))W(dy,ds) \\ &+ \int_0^t \int_{\mathcal{D}} \mathcal{A}(\Delta F_2(y,s))dyV(ds) - \int_t^{t'} \int_{\mathcal{D}} \mathcal{B}F_1(u_n(y,s))W(dy,ds) \\ &- \int_t^{t'} \int_{\mathcal{D}} \mathcal{B}(\Delta F_2(y,s))dyV(ds)\Big|^{2a}\Big), \end{aligned}$$

where $\mathcal{A} := G(t-s,x,y) - G(t'-s,x',y')$ and $\mathcal{B} := G(t'-s,x',y)$. So, by making use of the Burkholder inequality we get

$$\begin{aligned}
&E\Big(|\mathcal{L}(u_n)(x,t) - \mathcal{L}(u_n)(x',t')|^{2a}\Big) \leq \\
&c\Big\{E\Big(|\int_0^t\int_{\mathcal{D}}\mathcal{A}F_1(u_n(y,s))W(dy,ds)|^{2a}\Big) + E\Big(|\int_0^t\int_{\mathcal{D}}\mathcal{A}(\Delta F_2(y,s))dyV(ds))|^{2a}\Big)\\
&+E\Big(|\int_t^{t'}\int_{\mathcal{D}}\mathcal{B}F_1(u_n(y,s))W(dy,ds)|^{2a}\Big) + E\Big(|\int_t^{t'}\int_{\mathcal{D}}\mathcal{B}(\Delta F_2(y,s))dyV(ds)|^{2a}\Big)\Big\}\\
&\leq c\Big\{E\Big(\int_0^t\int_{\mathcal{D}}|\mathcal{A}|^2|F_1(u_n(y,s))|^2dyds\Big)^a + E\Big(\int_0^t\int_{\mathcal{D}}|\mathcal{A}|^2|\Delta F_2(y,s)|^2dyds\Big)^a\\
&+E\Big(\int_t^{t'}\int_{\mathcal{D}}|\mathcal{B}|^2|F_1(u_n(y,s))|^2dyds\Big)^a + E\Big(\int_t^{t'}\int_{\mathcal{D}}|\mathcal{B}|^2|\Delta F_2(y,s)|^2dyds\Big)^a\Big\}.
\end{aligned}$$

Thus, by Theorem (4.4) the next Hölder-type space-time estimate follows: there exist γ, $\gamma'>0$ such that

$$E\Big(|\mathcal{L}(u_n)(x,t) - \mathcal{L}(u_n)(x',t')|^{2a}\Big) \leq c(|x-x'|^{\gamma})^a + c(|t-t'| + |t-t'|^{\gamma'})^a. \quad (29)$$

Since F_1, F_2 are uniformly bounded on u_n, the definition of $\mathcal{L}$ and Burkholder inequality yield

$$\sup_{n\in N}\sup_{t\in[0,T]}\sup_{x\in\mathcal{D}} E\Big(|\mathcal{L}(u_n)(x,t)|^{2\rho\delta}\Big) < \infty. \quad (30)$$

By (29) and (30) according to Garsia's Lemma (generalization of Kolmogorov's Theorem [30, 15, 46]) the estimate (28) follows. □

The Green's function is symmetric in space variables and satisfies (13). As in [19, 15], we prove *a priori* estimates for $v_n = u_n - \mathcal{L}(u_n)$.

Theorem 4.7. *There exists positive constant $\widetilde{c}$, depending only on the measure of $\mathcal{D}$, such that for the solution u_n of the SPDE (8) the next estimate holds true*

$$\begin{aligned}
\int_0^t \chi_n\Big(\|u_n\|_q\Big)\|u_n\|_4^4 ds \leq& \widetilde{c}\int_0^t\Big(1 + |<u_0,w_0>w_0|^4 + \|\mathcal{L}(u_n)\|_\infty^4\Big)ds\\
&+\frac{1}{2}\Big\|\sum_{k=1}^\infty \mu_k^{-1/2}<w_k,u_0>w_k\Big\|_2^2.
\end{aligned} \quad (31)$$

Proof. Using the orthonormal basis for representing $v_n\in L^2(\mathcal{D})$ we write $v_n = \sum_{j=0}^\infty \rho_j w_j$. After some computations we get

$$\sum_{k=1}^\infty \mu_k^{-1}<w_k,\partial_t v_n>w_k = \sum_{k=1}^\infty \mu_k^{-1}<w_k,(\partial_t\rho_k)w_k>. \quad (32)$$

Using the boundary conditions of (25) and the Neumann condition for w_k, since Δ is symmetric and w_k are eigenfunctions we obtain

$$\sum_{k=1}^\infty \mu_k^{-1}<w_k,\Delta^2 v_n>w_k = \sum_{k=1}^\infty \mu_k<w_k,v_n>w_k. \quad (33)$$

Finally, for $L^2(\mathcal{D})\ni Q := \chi_n(\|u_n\|_q)f'(u_n) = \sum_{j=0}^\infty l_j w_j$ it follows

$$\sum_{k=1}^\infty \mu_k^{-1}<w_k,\Delta Q>w_k = -Q + l_0 w_0. \quad (34)$$

Replacing (32)-(34) in equation (25) and taking the inner product with $Rv_n := \sum_{j=1}^{\infty}\rho_j w_j = v_n - \rho_0 w_0$, the next statement follows

$$\begin{aligned} <\sum_{k=1}^{\infty}\mu_k^{-1}(\partial_t\rho_k)w_k,\sum_{j=1}^{\infty}\rho_j w_j> + <\sum_{k=1}^{\infty}\mu_k<w_k,\sum_{j=1}^{\infty}\rho_j w_j>w_k,\sum_{j=1}^{\infty}\rho_j w_j> \\ +<Q,Rv_n> - <l_0w_0,Rv_n>=0. \end{aligned} \tag{35}$$

We note that $<l_0w_0,Rv_n>=(\int_{\mathcal{D}}Qdx)(\int_{\mathcal{D}}Rv_ndx)=0$ as $Rv_n=\sum_{j=1}^{\infty}\rho_jw_j$, and obviously $\int_{\mathcal{D}}w_jw_0=0$ for any $j=1,\cdots$ for w_0 the constant function. Therefore, by orthonormality, (35) takes the form $\sum_{k=1}^{\infty}\mu_k^{-1}(\partial_t\rho_k)\rho_k+\sum_{k=1}^{\infty}\mu_k\rho_k^2+<Q,Rv_n>=0$, and equivalently

$$\frac{1}{2}\partial_t(\sum_{k=1}^{\infty}\mu_k^{-1}\rho_k^2)+\sum_{k=1}^{\infty}\mu_k\rho_k^2+<Q,Rv_n>=0. \tag{36}$$

Integrating (36) in time yields

$$\begin{aligned} \frac{1}{2}\sum_{k=1}^{\infty}\mu_k^{-1}\rho_k^2(t)-\frac{1}{2}\sum_{k=1}^{\infty}\mu_k^{-1}\rho_k^2(0)+\int_0^t\sum_{k=1}^{\infty}\mu_k\rho_k^2(s)ds \\ +\int_0^t<Q(s),Rv_n(s)>ds=0. \end{aligned} \tag{37}$$

The equality (37) is well defined because all appearing series converge. More specifically for t fixed, v_n is in $L^2(\mathcal{D})$ thus $v_n=\sum_{j=0}^{\infty}\rho_jw_j$ converges and the same holds for $\sum_{j=1}^{\infty}\mu_j^{-1}\rho_jw_j$ because $\mu_j\to\infty$ as $j\to\infty$. Let define $\mathcal{B}(u_0):=\|\sum_{k=1}^{\infty}\mu_k^{-1/2}<w_k,u_0>w_k\|_2^2$, we note that $v_n(0)=u_n(0)-\mathcal{L}(u_n)(0)=u_n(0)=u_0$, hence

$$\sum_{k=1}^{\infty}\mu_k^{-1}\rho_k^2(0)=\|\sum_{k=1}^{\infty}\mu_k^{-1/2}<w_k,v_n(0)>w_k\|_2^2=\|\sum_{k=1}^{\infty}\mu_k^{-1/2}<w_k,u_0>w_k\|_2^2.$$

We replace Q, $\mathcal{B}(u_0)$ in (37) and get the following inequality

$$\int_0^t\chi_n(\|u_n\|_q)\int_{\mathcal{D}}f'(u_n)Rv_ndxds\leq\frac{1}{2}\mathcal{B}(u_0). \tag{38}$$

We write (38) in the equivalent form

$$\begin{aligned} \int_0^t\chi_n(\|u_n\|_q)\int_{\mathcal{D}}f'(u_n)u_ndxds\leq \\ \int_0^t\chi_n(\|u_n\|_q)\int_{\mathcal{D}}f'(u_n)(u_n-Rv_n)dxds+\frac{1}{2}\mathcal{B}(u_0). \end{aligned} \tag{39}$$

We note that by definition $u_n(s)=v_n(s)+\mathcal{L}(u_n)(s)$, and $Rv_n(s)=v_n(s)-\rho_0(s)w_0$, and thus $u_n(s)-Rv_n(s)=\mathcal{L}(u_n)(s)+\rho_0(s)w_0$. We also remark by (26) that $\int_{\mathcal{D}}[v_n(s,x)-u_0(x)]w_0dx=0$ as w_0 is the constant function, consequently $\rho_0(s)=<u_0,w_0>$ and thus $u_n(s)-Rv_n(s)=\mathcal{L}(u_n)(s)+<u_0,w_0>w_0$. We replace this in (39) and obtain

$$\begin{aligned} \int_0^t\chi_n(\|u_n\|_q)\int_{\mathcal{D}}f'(u_n)u_ndxds\leq \\ \int_0^t\chi_n(\|u_n\|_q)\int_{\mathcal{D}}f'(u_n)(\mathcal{L}(u_n)+<u_0,w_0>w_0)dxds+\frac{1}{2}\mathcal{B}(u_0). \end{aligned} \tag{40}$$

We use $f'(u_n)u_n = u_n^4 - u_n^2$ and Young's inequality to get that $|f'(u_n)(\mathcal{L}(u_n)+ < u_0, w_0 > w_0)| \leq \widetilde{c_0}|u_n|^4 + \widetilde{c_1}|\mathcal{L}(u_n)|^4 + \widetilde{c_2}| < u_0, w_0 > w_0|^4 + \widetilde{c_3}$, where $\widetilde{c_0}, \widetilde{c_1}, \widetilde{c_2}, \widetilde{c_3}$ are positive constants (independent of u_n, $\mathcal{L}(u_n)$, $< u_0, w_0 > w_0$) and $\widetilde{c_0}$ is as small as we want. By replacing $\mathcal{B}(u_0)$, as χ_n is non-negative and bounded function, relation (40) gives for $\mathcal{D}$ bounded the desired estimate (31). □

4.3. **Galerkin approximation of v_n.** We approximate v_n by functions v_n^m belonging to finite dimensional subspaces of $L^2(\mathcal{D})$ produced by the m first eigenfunctions of the orthonormal basis of $L^2(\mathcal{D})$ in order to derive on the weak limit as $m \to \infty$ the properties of v_n. Let $S^m :=< w_0, w_2, \cdots, w_m >$ be the finite dimensional subspace of $L^2(\mathcal{D})$ produced by the m first orthonormal eigenfunctions of the $L^2(\mathcal{D})$ basis. Let define $P^m : \ L^2(\mathcal{D}) \to S^m$ such that $< w, \phi >=< P^m w, \phi >$ for any $\phi \in S^m$; obviously P^m is the L^2 projection of $L^2(\mathcal{D})$ into S^m.

We consider the following initial and boundary value problem: we seek a function $v_n^m \in S^m$ satisfying

$$\begin{aligned}
&\partial_t v_n^m + \Delta^2 v_n^m - \chi_n\Big(\|u_n^m\|_q\Big)\Delta\Big[P^m\Big(f'(u_n^m)\Big)\Big], \ \text{ in } \mathcal{D},\ 0 < t \leq T,\\
&v_n^m(0,x) = P^m(u_0)(x), \ \text{ in } \mathcal{D},\\
&\frac{\partial v_n^m}{\partial n} = \frac{\partial \Delta v_n^m}{\partial n} = 0, \ \text{ on } \partial\mathcal{D},\ 0 < t \leq T,
\end{aligned} \tag{41}$$

where $u_n^m := v_n^m + \mathcal{L}(u_n)$. We multiply the partial differential equation (41) by v_n^m and integrate in $\mathcal{D}$. The boundary conditions of (41) and the definition of the projection P^m yield

$$\frac{1}{2}\frac{d}{dt}\|v_n^m\|_2^2 + \|\Delta v_n^m\|_2^2 - \chi_n(\|u_n^m\|_q) < f'(u_n^m), \Delta v_n^m >= 0, \tag{42}$$

since Δv_n^m is in S^m.

Remark 7. As in [15], our purpose is to estimate v_n in the $L^2([0,T], H^2(\mathcal{D}))$ norm. We establish first the analogous estimates for the Galerkin approximation v_n^m; on the weak limit as $m \to \infty$ the estimates for v_n will follow.

By use of (42), since $f'(w) = w^3 - w$, then the next lemma follows.

Lemma 4.8. *If $w \in H^2(\mathcal{D})$ and $\frac{\partial w}{\partial n} = 0$ on $\partial\mathcal{D}$, then*

$$< w^3, \Delta w >\leq 0, \tag{43}$$

$$< f'(w), \Delta w >\leq c_0\|\Delta w\|_2^2 + c\Big(1 + \|w\|_4^4\Big), \tag{44}$$

where c_0, c are positive constants independent of w, and c_0 is arbitrary small.

By Lemma 4.8 and the definition of f' we derive the following result.

Lemma 4.9. *For the Galerkin approximation v_n^m holds that*

$$\begin{aligned}
< f'(u_n^m), \Delta v_n^m >\leq & c\|\mathcal{L}(u_n)\|_\infty^2\Big(1 + \|\mathcal{L}(u_n)\|_\infty^4 + \|v_n^m\|_4^4\Big)\\
&+ c\Big(1 + \|v_n^m\|_4^4\Big) + \widetilde{c_0}\|\Delta v_n^m\|_2^2,
\end{aligned}$$

for $c, \widetilde{c_0}$ positive constants independent from v_n^m and u_n and $\widetilde{c_0}$ arbitrary small.

Using Lemma 4.9 we prove the next estimate.

Theorem 4.10. *Let u_n be the solution of SPDE (8); then for the Galerkin approximation v_n^m the next estimate holds*

$$\begin{aligned}&\frac{1}{2}\|v_n^m(t,\cdot)\|_2^2 + c\int_0^t \Big[\|\Delta v_n^m(s,\cdot)\|_2^2 + \Big| < v_n^m(s,\cdot), w_0 > w_0\Big|^2\Big] ds \\ &\leq \frac{1}{2}\|u_0\|_2^2 + cT\Big(1 + \|\mathcal{L}(u_n)\|_\infty^6 + \Big| < u_0, w_0 > w_0\Big|^2\Big) + c\Big(1 + \|\mathcal{L}(u_n)\|_\infty^2\Big) X^m \\ &c\Big(1 + \|\mathcal{L}(u_n)\|_\infty^2\Big)\int_0^t \chi_n\Big(\|v_n^m(s,\cdot) + \mathcal{L}(u_n)(s,\cdot)\|_q\Big)\|\mathcal{L}(u_n)(s,\cdot)\|_4^4 ds,\end{aligned} \tag{45}$$

where

$$X^m := \widetilde{c}\int_0^t \Big(1 + | < u_0, w_0 > w_0|^4 + \|\mathcal{L}(u_n)\|_\infty^4\Big) ds + \frac{1}{2}\|\sum_{k=1}^m \mu_k^{-1/2} < w_k, u_0 > w_k\|_2^2.$$

Proof. We use Lemma 4.9 in (42); χ_n is nonnegative and bounded function, hence

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|v_n^m\|_2^2 + c\|\Delta v_n^m\|_2^2 \leq c\chi_n(\|v_n^m + \mathcal{L}(u_n)\|_q)\cdot \\ &\cdot\Big\{\|\mathcal{L}(u_n)\|_\infty^2(1 + \|\mathcal{L}(u_n)\|_\infty^4 + \|v_n^m\|_4^4) + c(1 + \|v_n^m\|_4^4)\Big\}\end{aligned} \tag{46}$$

for c positive constant. We integrate (46) in $[0,t] \subseteq [0,T]$, we use that $v_n^m(0,x) = P^m(u_0)(x)$ and the fact that by definition $\|P^m(u_0)\|_2 = \|u_0\|_2$ to obtain

$$\begin{aligned}&\frac{1}{2}\|v_n^m(t,\cdot)\|_2^2 + c\int_0^t \|\Delta v_n^m(s,\cdot)\|_2^2 ds \leq \frac{1}{2}\|u_0\|_2^2 + cT(1 + \|\mathcal{L}(u_n)\|_\infty^6) \\ &+c(1 + \|\mathcal{L}(u_n)\|_\infty^2)\int_0^t \chi_n(\|v_n^m(s,\cdot) + \mathcal{L}(u_n)(s,\cdot)\|_q)\|v_n^m(s,\cdot)\|_4^4 ds.\end{aligned} \tag{47}$$

Using the orthonormal basis for representing v_n^m we write $v_n^m = \sum_{j=0}^m \rho_j^m w_j$. In equation (41) we take the inner product with $Rv_n^m := \sum_{j=1}^m \rho_j^m w_j = v_n^m - \rho_0^m w_0$, use the boundary conditions of v_n^m and the Neumann condition for w_k, to obtain for $S^m \ni Q^m := \chi_n(\|u_n^m\|_q)P^m(f'(u_n^m)) = \sum_{j=0}^m l_j^m w_j$ that

$$\frac{1}{2}\partial_t(\sum_{k=1}^m \mu_k^{-1}(\rho_k^m)^2) + \sum_{k=1}^m \mu_k(\rho_k^m)^2 + < Q^m, Rv_n^m >= 0.$$

Integration in time yields

$$\begin{aligned}\frac{1}{2}\sum_{k=1}^m \mu_k^{-1}(\rho_k^m)^2(t) - \frac{1}{2}\sum_{k=1}^m \mu_k^{-1}(\rho_k^m)^2(0) + \int_0^t \sum_{k=1}^m \mu_k(\rho_k^m)^2(s) ds \\ + \int_0^t < Q^m(s), Rv_n^m(s) > ds = 0.\end{aligned} \tag{48}$$

By the definition of the L^2 projection and Young's inequality we get finally

$$\begin{aligned}\int_0^t \chi_n(\|u_n^m\|_q)\|u_n^m\|_4^4 ds \leq &\widetilde{c}\int_0^t (1 + |\rho_0^m(0)w_0|^4 + \|\mathcal{L}(u_n)\|_\infty^4) ds \\ &+ \frac{1}{2}\|\sum_{k=1}^m \mu_k^{-1/2} < w_k, u_0 > w_k\|_2^2,\end{aligned} \tag{49}$$

where $\widetilde{c}$ is a positive constant depending only on the measure of $\mathcal{D}$ and t. We note that as $v_n^m(0) = P^m(u_0)$ then $\rho_0^m(0) =< v_n^m(0), w_0 >=< P^m(u_0), w_0 >=< u_0, w_0 >$

and consequently $\rho_0^m(0)w_0 =< u_0, w_0 > w_0$, replacing in (49) we arrive at the estimate

$$\begin{aligned}&\int_0^t \chi_n(\|v_n^m + \mathcal{L}(u_n)\|_q)\|v_n^m + \mathcal{L}(u_n)\|_4^4 ds \leq \\ &\widetilde{c}\int_0^t (1 + |< u_0, w_0 > w_0|^4 + \|\mathcal{L}(u_n)\|_\infty^4)ds + \frac{1}{2}\|\sum_{k=1}^m \mu_k^{-1/2} < w_k, u_0 > w_k\|_2^2,\end{aligned} \tag{50}$$

We note that $< v_n, w_0 > w_0 =< v_n^m, w_0 > w_0 =< u_0, w_0 > w_0$, hence, by (47) we get

$$\begin{aligned}&\frac{1}{2}\|v_n^m(t,\cdot)\|_2^2 + c\int_0^t [\|\Delta v_n^m(s,\cdot)\|_2^2 + |< v_n^m(s,\cdot), w_0 > w_0|^2]ds \leq \\ &\frac{1}{2}\|u_0\|_2^2 + cT(1 + \|\mathcal{L}(u_n)\|_\infty^6 + |< u_0, w_0 > w_0|^2) \\ &+c(1 + \|\mathcal{L}(u_n)\|_\infty^2)\int_0^t \chi_n(\|v_n^m(s,\cdot) + \mathcal{L}(u_n)(s,\cdot)\|_q)\|v_n^m(s,\cdot)\|_4^4 ds.\end{aligned} \tag{51}$$

The function χ_n is nonnegative; we use (50) in (51) and obtain the desired estimate (45). □

Theorem 4.11. *Let u_n be the solution of the SPDE (8), then for v_n it holds that*

$$\begin{aligned}&\frac{1}{2}\|v_n(t,\cdot)\|_2^2 + c\int_0^t \Big[\|\Delta v_n(s,\cdot)\|_2^2 + \Big|< v_n(s,\cdot), w_0 > w_0\Big|^2\Big]ds \leq \\ &\frac{1}{2}\|u_0\|_2^2 + cT\Big(1 + \|\mathcal{L}(u_n)\|_\infty^6 + |< u_0, w_0 > w_0|^2\Big) + c\Big(1 + \|\mathcal{L}(u_n)\|_\infty^2\Big)X \\ &c\Big(1 + \|\mathcal{L}(u_n)\|_\infty^2\Big)\int_0^t \chi_n\Big(\|v_n(s,\cdot) + \mathcal{L}(u_n)(s,\cdot)\|_q\Big)\|\mathcal{L}(u_n)(s,\cdot)\|_4^4 ds,\end{aligned} \tag{52}$$

where

$$X := \widetilde{c}\int_0^t \Big(1 + |< u_0, w_0 > w_0|^4 + \|\mathcal{L}(u_n)\|_\infty^4\Big)ds + \frac{1}{2}\|\sum_{k=1}^\infty \mu_k^{-1/2} < w_k, u_0 > w_k\|_2^2.$$

Proof. From the estimate (45), using that χ_n is bounded, we get that $v_n^m \in L^2(\mathcal{D}) \cap L^2([0,T], H^2(\mathcal{D}))$ because the norm $\|w\| := \|\Delta w(s,\cdot)\|_2^2 + |< w(s,\cdot), w_0 >|^2$ is equivalent to the $H^2(\mathcal{D})$ norm [19]. Thus, the sequence $(v_n^m)_{m\in\mathbb{N}}$ is bounded in $L^2([0,T], H^2(\mathcal{D}))$ and converges in the weak* topology of $L^2([0,T], H^2(\mathcal{D}))$ as $m \to \infty$ [15]. Hence its weak limit is the weak solution of (25), and thus equals v_n. Consequently, we obtain that $v_n \in L^2([0,T], H^2(\mathcal{D}))$. As (31) holds, by repeating the same computations as in the proof of (45) for v_n in place of v_n^m, the estimate (52) follows. □

4.4. **Existence of solution a.s. for any t.** Our aim is to establish existence of the solution of (7) in any time interval a.s. Now by Lemma 4.6, using the estimate (52) and the Sobolev inequality we prove the following lemma.

Lemma 4.12. *If u_n is the solution of the SPDE (8) then*

$$\sup_{n\in N} E\Big(\sup_{0\leq t\leq T}\Big(\int_0^t \|u_n\|_q^a\Big)^\beta\Big) < \infty, \tag{53}$$

for any $q \in [2,\infty)$, $a \in [q,\infty)$.

Proof. We remind that by Lemma 4.6

$$\sup_{n\in N} E(\|\mathcal{L}(u_n)\|_\infty^{2p\delta}) < \infty,$$

while

$$\|\mathcal{L}(u_n)\|_\infty := \sup_{t\in[0,T]} \sup_{x\in\mathcal{D}} |\mathcal{L}(u_n)(t,x)|,$$

thus

$$\int_0^t \|\mathcal{L}(u_n)(s,\cdot)\|_4^4 ds \leq T\|\mathcal{L}(u_n)\|_\infty^4.$$

By (52) we get

$$\sup_{t\in[0,T]} \|v_n\|_2^{2\beta} \leq c(1+\|\mathcal{L}(u_n)\|_\infty^{6\beta}) \quad \text{and} \quad E(\sup_{t\in[0,T]} \|v_n\|_2^{2\beta}) \leq c(1+E(\|\mathcal{L}(u_n)\|_\infty^{6\beta})).$$

Taking the supremum over n, we arrive at

$$\sup_n E(\sup_{t\in[0,T]} \|v_n\|_2^{2\beta}) \leq c(1+\sup_n E(\|\mathcal{L}(u_n)\|_\infty^{6\beta})) < \infty. \tag{54}$$

If $\partial\mathcal{D}$ is C^1 and if $w \in H^2(\mathcal{D})$ then by Sobolev inequality [27], $\|w\|_r \leq c\|w\|_{H^2(\mathcal{D})}$ for any $r \geq 2$ (this inequality holds also in convex polyhedra as the L^∞ norm is bounded by the H^2 norm [12]), hence (52) gives

$$\int_0^t \|v_n(s,\cdot)\|_r^2 ds \leq c\int_0^t \|v_n(s,\cdot)\|_{H^2(\mathcal{D})}^2 ds \leq c(1+\|\mathcal{L}(u_n)\|_\infty^6).$$

We set $t = T$ and get $E[(\int_0^T \|v_n(s,\cdot)\|_r^2 ds)^\beta] \leq c(1+E(\|\mathcal{L}(u_n)\|_\infty^{6\beta}))$. Taking the supremum over n the next estimate follows

$$\sup_{n\in N} E[(\int_0^T \|v_n(s,\cdot)\|_r^2 ds)^\beta] \leq c(1+\sup_{n\in N} E(\|\mathcal{L}(u_n)\|_\infty^{6\beta})) < \infty. \tag{55}$$

By definition $u_n = v_n + \mathcal{L}(u_n)$, and thus

$$\|u_n\|_r^{2\beta} \leq c(\|v_n\|_r^{2\beta} + \|\mathcal{L}(u_n)\|_r^{2\beta}) \leq c(\|v_n\|_r^{2\beta} + \|\mathcal{L}(u_n)\|_\infty^{2\beta}).$$

Therefore, by (54) and (55) follows for $r \geq 2$

$$\sup_n E(\sup_{0\leq t\leq T} \|u_n\|_2^{2\beta}) < \infty, \quad \text{and} \quad \sup_{n\in N} E[(\int_0^T \|u_n(s,\cdot)\|_r^2 ds)^\beta] < \infty. \tag{56}$$

Hölder inequality gives $\|u_n\|_q^a \leq \|u_n\|_2^{2a(1-\lambda)/q}\|u_n\|_r^2$ for $\lambda \in [0,1]$. The previous yields: $\int_0^T \|u_n\|_q^a ds \leq c \sup_{0\leq t\leq T} \|u_n\|_2^{2a(1-\lambda)/q} \int_0^T \|u_n\|_r^2 ds$. Consequently

$$(\int_0^t \|u_n\|_q^a ds)^\beta \leq c(\sup_{0\leq t\leq T} \|u_n\|_2^{2a(1-\lambda)/q})^\beta (\int_0^T \|u_n\|_r^2 ds)^\beta,$$

and by (56), the relation (53) follows for any $q \in [2,\infty)$, $a \in [q,\infty)$. □

Assume that the initial condition u_0 of (8) is in $L^q(\mathcal{D})$; by Theorem 3.1 $u_n(\cdot,t)$ is in $L^q(\mathcal{D})$ for any $t \leq T$. We consider $q \geq 4$ and prove the next theorems.

Theorem 4.13. *If u_n solves the SPDE (8) and $q \geq 4$ then*

$$\sup_n E\Big(\sup_{0\leq t\leq T} \|\mathcal{M}_n(u_n)\|_q^\beta\Big) < \infty. \tag{57}$$

Proof. Let $q \geq 4$ and set $\rho := q/3$; then by Young's inequality and by the definition of f' it follows that

$$\|f'\|_\rho = \|u_n^3 - u_n\|_\rho \leq c\|u_n\|_q^3 + c. \tag{58}$$

Using (15) for $\rho := q/3$ and (58) we obtain for $t_0 := 0$

$$\|\mathcal{M}_n(u_n)\|_q \leq c\int_0^t (t-s)^{-\frac{d+2}{4}+\frac{d}{4r}}(\|u_n\|_q^3 + 1)ds. \tag{59}$$

Hölder inequality for $\gamma' \in (1,\infty)$ and (59) give

$$\|\mathcal{M}_n(u_n)\|_q^\beta \leq c + c(\int_0^T \|u_n\|_q^{3\gamma'}ds)^{\beta/\gamma'},$$

and thus $E(\|\mathcal{M}_n(u_n)\|_q^\beta) \leq c + cE((\int_0^T \|u_n\|_q^{3\gamma'})^\beta)^{1/\gamma'}$. Taking supremum over $t \in [0,T]$ and n we arrive at

$$\sup_n E(\sup_{0\leq t\leq T}\|\mathcal{M}_n(u_n)\|_q^\beta) \leq c + c\sup_n E(\sup_{0\leq t\leq T}(\int_0^T \|u_n\|_q^{3\gamma'}ds)^\beta)^{1/\gamma'}. \tag{60}$$

By use of Lemma 4.12 and (60) for $3\gamma' \geq q \geq 2$ we obtain (57). □

We define the stopping time

$$T_n := \inf\Big\{t \geq 0 : \ \|u_n(t,\cdot)\|_q \geq n\Big\}. \tag{61}$$

Then the process $u(\cdot,t) := u_n(\cdot,t)$ is well defined on any $t \leq T_n$ and constitutes a solution for (7) in the interval $[0,T_n)$ for any $n \geq 1$. In the next theorem we will show that $\lim_{n\to\infty} T_n = \infty$ a.s, i.e. the solution u exists in $[0,T]$ for any $T > 0$ a.s.

Theorem 4.14. *The solution u of the SPDE (7) exists in the interval $[0,T]$ for any $T > 0$ a.s. or equivalently*

$$\lim_{n\to\infty} p[T_n \leq T] = 0 \ \textit{ for any } \ T > 0.$$

Proof. We recall the definition (61) of T_n, that is if $T_n \leq T$, then for any $t : T_n \leq t \leq T$ follows that $\|u_n(t,\cdot)\|_q^{2\beta} \geq n^{2\beta}$, and thus $\sup_{T_n\leq t\leq T}\|u_n(t,\cdot)\|_q^{2\beta} \geq n^{2\beta}$. Hence, the next inequality follows for $n > 0$

$$p[T_n \leq T] \leq p\Big[\sup_{T_n\leq t\leq T}\|u_n(t,\cdot)\|_q^{2\beta} \geq n^{2\beta}\Big]. \tag{62}$$

But for the density $\mathrm{f} \geq 0$ of a probability measure in $[0,\infty)$ it holds that

$$p[y \geq 1] = \int_1^\infty \mathrm{f}(y)dy \leq \int_1^\infty y\mathrm{f}(y)dy \leq \int_0^\infty y\mathrm{f}(y)dy = E(y).$$

Setting $y := n^{-2\beta}\sup_{T_n\leq t\leq T}\|u_n(t,\cdot)\|_q^{2\beta}$ in (62), we obtain

$$p[T_n \leq T] \leq E(y) = \frac{1}{n^{2\beta}}E\Big(\sup_{T_n\leq t\leq T}\|u_n(t,\cdot)\|_q^{2\beta}\Big). \tag{63}$$

Obviously $\sup_{[T_n,T]}\|u_n\|_q^{2\beta} \leq \sup_{[0,T]}\|u_n\|_q^{2\beta}$, therefore, (63) yields

$$p[T_n \leq T] \leq \frac{1}{n^{2\beta}}E\Big(\sup_{0\leq t\leq T}\|u_n(t,\cdot)\|_q^{2\beta}\Big) \to 0 \ \text{ as } \ n \to \infty, \tag{64}$$

because by (57) of Theorem 4.13, Lemma 4.6, by (12), and the definition (27) of $\|\mathcal{L}(u_n)\|_\infty$ holds that $\sup_{n\in N} E\Big(\sup_{0\le t\le T}\|u_n(t,\cdot)\|_q^{2\beta}\Big) < \infty$. Consequently the result follows, i.e. $\lim_{n\to\infty} p[T_n \le T] = 0$. $\Box$

5. Conclusions-Generalizations.

5.1. A geometry based existence proof for the Stochastic Cahn-Hilliard.

Inspired by Cardon-Weber's paper [15] for cubic domains, we relate in our work the role of domain's geometry in an existence proof for a stochastic equation. As in [15, 19] we use an eigenvalue-dependent convolution semi-group. It is well known that the spectrum defines exactly the domain. For general geometry, the computation of exact formulae for the eigenvalues and spectral analysis is one of the most difficult open problems.

A. Avoiding any explicit eigenvalue formula, we derive the space-time Hölder estimates (21), (22), (23) for the Green's kernel for various simply connected $\mathcal{D}$:

1. $d = 1$: if $\mathcal{D}$ is an open interval.
2. $d = 2$:
   (a) if $\mathcal{D}$ is convex and of smooth boundary,
   (b) if $\mathcal{D}$ is a convex polyhedron,
   (c) if $\mathcal{D}$ is Lipschitz and of smooth boundary (see Remark 6, here convexity is not necessary).
3. $d = 3$: for the same cases (a), (b), (c) as in $d = 2$, plus the property of minimum eigenfunctions growth.

These estimates are independent results useful for the analysis of fourth order stochastic equations with various types of noise (It$\widehat{o}$, Stratonovich, in the sense of Walsh).

B. In our existence proof we also use the Green's estimates (13), (14) of [24] and the resulting (15), (16) for domains of smooth boundary that can be extended in convex polyhedra. Therefore, since all other arguments in this paper hold true, *existence for the generalized stochastic C-H equation (1) is valid for any case presented in* ***A***. The title refers only to convexity while existence is proved for various domains not necessarily convex, because our proof does not cover the non-convex polyhedral case. Convexity seems to be an important issue for the proof of (13), (14) for piece-wise smooth boundaries, since only in the convex case the approximating smooth domain sequence is in $\overline{\mathcal{D}}$ (see Remark 3, or the cubic case in [15]).

C. In [15] the case $\mathcal{D} = (0,\pi)^d$ for $F_2 = 0$ was analyzed and the explicit eigenvalue formulae for this cube was used. As mentioned in the introduction, if $\mathcal{D}$ is a cube of edge a, then by using the C-H scale i.e. the change of variables $x \to \frac{x}{a/\pi}$, $t \to \frac{t}{a^2/\pi^2}$ we can always consider the equivalent C-H in $(0,\pi)^d$, therefore the result of [15] is extended for any cube.

For a general rectangular domain of $\mathbb{R}^d$, $d = 2, 3$ with edges a_i one could transform $\mathcal{D}$ into the cube $(0,\pi)^d$ by applying a weighted change of space variables $x_i \to \frac{x_i}{a_i/\pi}$ in every direction, but this would change the fourth order operator Δ^2 at the right-hand side of C-H i.e. the Green's function and the weak formulation, so the result of [15] is not directly applicable. In our eigenvalue formulae-free proof this case is considered as a special case of convex polyhedron; we denote that in rectangles the minimum eigenfunction growth holds (this property is needed in our analysis only if $d = 3$).

D. Existence is also derived for the standard ε-dependent stochastic C-H equation (9) (see Remark 2).

5.2. **Noise in the chemical potential.** The noise in (1) or the ε-dependent (9) is splited in two terms. Every term has a different physical meaning. The chemical potential noise describes external fields while the free-energy independent noise may describe thermal fluctuations or external mass supply. This presentation is important in an equivalent stochastic system formulation (see Remark 2).

Acknowledgment. The second author is supported by a Marie Curie International Reintegration Grant within the 7th European Community Framework Programme, MIRG-CT-2007-200526, and partially supported by the FP7-REGPOT-2009-1 project “Archimedes Center for Modeling, Analysis and Computation”. The authors would like to thank Prof. S. Luckhaus and Prof. E. Presutti for stimulating discussions. The authors also wish to thank the two anonymous referees for their valuable comments and suggestions.

6. **Appendix.** In this appendix, we provide detailed definitions concerning stochastic processes, Wiener processes and $\mathcal{F}_t-$ adaptive processes.

6.1. **Definitions.** We proceed by presenting the following basic definitions [42, 6, 44, 46]:

- Let Ω, $\mathcal{F}$ a $\sigma-$algebra on Ω, and $P:\mathcal{F}\rightarrow[0,1]$ a probability measure on $(\Omega,\mathcal{F})$. A function $Y:\Omega\rightarrow\mathbb{R}^n$ is called $\mathcal{F}-$measurable if $Y^{-1}(U):=\{\omega\in\Omega; Y(\omega)\in U\}\in\mathcal{F}$, for any Borel set $U\subset\mathbb{R}^n$.
- A random variable X is an $\mathcal{F}-$measurable function $X:\Omega\rightarrow\mathbb{R}^n$, and induces a probability measure μ_X on $\mathbb{R}^n$, defined by $\mu_X(B):=P(X^{-1}(B))$. The measure μ_X is called distribution of X.
- Expectation of X: $E[X]:=\int_\Omega X(\omega)dP(\omega):=\int_{\mathbb{R}^n}xd\mu_X(x)$.
- For $p\in[1,\infty)$ and $X:\Omega\rightarrow\mathbb{R}^n$,we define the L^p norm by $|||X|||_p=(\int_\Omega|X(\omega)|^pdP(\omega))^{1/p}$. The space $L^p(P):=\{X:\Omega\rightarrow\mathbb{R}^n; |||X|||_p<\infty\}$ is a Banach space with the norm $|||\cdot|||_p$, and for $p=2$ is Hilbert space with the inner product $(X,Y):=E[XY]$.
- A family $\{X_t, t\in I\}$ of $\mathbb{R}^n$-valued random variables is called a stochastic process with index set I and state space $\mathbb{R}^n$.
- Let $\{\mathcal{F}_t\}_{t\geq0}$ be an increasing family of $\sigma-$algebras of subsets of Ω. The process $\sigma(t,\omega):[0,\infty)\times\Omega\rightarrow\mathbb{R}$ is called $\mathcal{F}_t-$adapted, if for any $t\geq0$ the function $\omega\rightarrow\sigma(t,\omega)$ is $\mathcal{F}_t-$measurable.
- A stochastic process $\{X_t, t\in[t_0,T]\}$, $t_0\geq0$ defined on the probability space $(\Omega,\mathcal{F},P)$ with index set $I=[t_0,T]\subset[0,\infty)$ and state space $\mathbb{R}^n$, where $\mathcal{F}=(\mathcal{B}^n)[t_0,T]$ is the product σ-algebra generated by the Borel sets of $\mathbb{R}^n$ in $\mathcal{B}^n$ for $t\in[t_0,T]$, is called Markov process if for any $t_0\leq s\leq t\leq T$ and any $B\in\mathcal{B}^n$ then $P(X_t\in B|\mathcal{F}([t_0,s]))=P(X_t\in B|X_s)$. Given a Markov process the past and future are statistically independent when the present is known.
- The Wiener process is a mathematical model of the Brownian motion of a free particle in the absence of friction, and is defined as a homogeneous n-dimensional Markov process W_t on $[0,\infty)$ with stationary transition probability $P(W_{t+s}\in B|W_s=x)=\int_B(2\pi t)^{-n/2}e^{-|x-y|^2/2t}dy$.

- Burkholder inequality for stochastic integrals, [44], [46]:

$$E[|\int_y f\ W(dy)|^p] \leq c\,E[(\int_y f^2 dy)^{p/2}],$$

for any $p \geq 2$ when $\int_y f\ W(dy)$ is a local martingale; here c is a positive constant. We note that in It$\hat{o}$ calculus this inequality appears as identity for $p = 2$ and $c = 1$ and is called It$\hat{o}$ isometry [42].

REFERENCES

[1] R. A. Adams, *Sobolev spaces*, Academic Press, 1993.

[2] N. D. Alikakos, P. W. Bates and X. Chen, *Convergence of the Cahn-Hilliard Equation to the Hele-Shaw Model*, Arch. Rat. Mech. Anal., 128 (1994), pp. 165–205.

[3] N. D. Alikakos, G. Fusco and G. Karali, *Ostwald Ripening in two dimensions - The rigorous derivation of the equations from the Mullins-Sekerka dynamics*, J. Differential Equations 205, 1 (2004), pp. 1-49.

[4] T. Antal, M. Droz, J. Magnin and Z. Rácz, *Formation of Liesengang Patterns: A Spinodal Decomposition Scenarion*, Phys. Rev. Lett. 83 (1999), pp. 2880–2883.

[5] D. C. Antonopoulou, G. D. Karali, G. T. Kossioris, *Asymptotics for a generalized Cahn-Hilliard equation with forcing terms*, to appear in Discrete and Cont. Dyn. Syst. A.

[6] L. Arnold, *Stochastic Differential Equations: Theory and Applications*, Wiley and Sons, 1974.

[7] P. W. Bates and P. C. Fife, *The dynamics of nucleation for the Cahn-Hilliard*, SIAM J. Appl. Math. 53 (1993), pp. 990–1008.

[8] G. Belletini, M.S. Gelli, S. Luckhaus, M. Novaga, *Deterministic equivalent for the Allen Cahn energy of a scaling law in the Ising model*, Calc. Var. 26, (2006), pp. 429–445.

[9] J. M. Berezanskii, *Expansions in Eigenfunctions of Selfadjoint Operators*, Translations of Mathematical Monographs, American Mathematical Society, 1968.

[10] D. Blömker, S. Maier-Paape, and T. Wanner, *Spinodal decomposition for the Cahn-Hilliard-Cook equation*, Communications in Mathematical Physics, 223(3) (2001) pp. 553-582.

[11] D. Blömker, S. Maier-Paape, and T. Wanner, *Phase separation in stochastic Cahn-Hilliard models*, Mathematical Methods and Models in Phase Transitions (2005), pp. 1-41.

[12] S. C. Brenner, L. R. Scott, *The Mathematical Theory of Finite Element Methods*, Springer-Verlag, 1994.

[13] J. W. Cahn, *On spinodal decomposition*, Acta Metallurgica, 9(9) (1961), pp. 795801.

[14] J. W. Cahn, J. E. Hilliard, *Free Energy of a Nonuniform System. I. Interfacial Free Energy*, J. Chem. Phys., 28 (1958), pp. 258–267.

[15] C. Cardon-Weber, *Cahn-Hilliard stochastic equation: existence of the solution and of its density*, Bernoulli, 7(5) (2001), pp. 777–816.

[16] C. Cardon-Weber, *Cahn-Hilliard stochastic equation: strict positivity of the density*, Stoch. Stoch. Rep., 72(3-4) (2002), pp. 191-227.

[17] H. Cook, *Brownian motion in spinodal decomposition*, Acta Metallurgica, 18 (1970), pp. 297-306.

[18] R. Courant, D. Hilbert, *Methods of Mathematical Physics*, Vol.1, Interscience Publishers, 1953.

[19] G. Da Prato, A. Debussche, *Stochastic Cahn-Hilliard Equation*, Nonlin. Anal. Th. Meth. Appl., 26(2) (1996), pp. 241–263.

[20] A. Debussche, L. Dettori, *On the Cahn-Hilliard Equation with logarithmic free energy*, Nonlin. Anal. Th. Meth. Appl., 24(16) (1995), pp. 1491–1514.

[21] A. Debussche, L. Goudenege, *Stochastic Cahn-Hilliard equation with double singular nonlinearities and two reflections*, Prepublication - IRMAR 2009-29 (2009).

[22] A. Debussche and L. Zambotti, *Conservative stochastic Cahn-Hilliard equation with reflection*, Ann. Probab., 35(5) (2007), pp. 1706-1739.

[23] J. J. Duistermaat and V. W. Guillemin, *The Spectrum of Positive Elliptic Operators and Periodic Bicharacteristics*, Invent. Math., 29 (1975), pp. 39–79.

[24] S. D. Eidelman, N. V. Ivasisen, *Investigation of the Green Matrix for homogeneous parabolic boundary value problem*, Trans. Moscow. Math. Soc., 23 (1970), pp. 179–242.

[25] N. Elezovic and A. Mikelic, *On the stochastic Cahn-Hilliard equation*, Nonlinear Anal., 16(12) (1991), pp. 1169-1200.

[26] C. M. Elliott and S. Zheng, *On the Cahn-Hilliard equation*, Arch. Rational Mech. Anal. 96 (1986), pp. 339–357

[27] L. C. Evans, *Partial Differential Equations*, American Mathematical Society, 1998.

[28] P. C. Fife, *Dynamical Aspects of the Cahn-Hilliard equations*, Barret Lectures, University of Tennessee, Spring 1991.

[29] P. C. Fife, *Models for phase separation and their mathematics*, El. Journ. Diff. E., 48 (2000), pp. 1–26.

[30] A. Garsia, *Continuity properties of Gaussian process with multi-dimensional time parameter*, Proc. Sixth Berkeley Symp. Math. Stat. Probab. (1972), pp. 369–374.

[31] L. Goudenege, *Stochastic CahnHilliard equation with singular nonlinearity and reflection*, Stochastic Processes and their Applications 119(10) (2009), pp. 3516–3548.

[32] M. E. Gurtin, *Generalized Ginzburg-Landau and Cahn-Hilliard equations based on a micro-force balance*, Physica D 92 (1996), pp. 178–192.

[33] A. Hassell, *Eigenvalues and eigenfunctions of the Laplacian*, Lecture notes, Department of Mathematics, Australian National University.

[34] M. Hildebrand and A. S. Mikhailov, *Mesoscopic modeling in the kinetic theory of adsorbates*, J. Phys. Chem. 100 (1996), pp. 19089–19101.

[35] P. C. Hohenberg and B. I. Halperin, *Theory of dynamic critical phenomena*, Rev. mod. Phys. 49 (1977), pp. 435–479.

[36] G. Karali and M. A. Katsoulakis, *The role of multiple microscopic mechanisms in cluster interface evolution*, J. Differential Equations, 235 (2007), pp. 418–438.

[37] M. A. Katsoulakis and D. G. Vlachos, *From microscopic interactions to macroscopic laws of cluster evolution*, Phys. Rev. Letters, 84 (2000), pp. 1511–1514.

[38] K. Kitahara, Y. Oono and D. Jasnow, *Phase separation dynamics and external force field*, Mod. Phys. Letters B 2 (1988), pp. 765–771.

[39] P. Kröger, *Upper bounds for the Neumann eigenvalues on a bounded domain in Euclidean Space*, J. Funct. Anal. 106 (1992) pp. 353-357.

[40] J. S. Langer, *Theory of spinodal decomposition in alloys*, Annals of Physics, 65 (1971), pp. 53-86.

[41] S. Maier-Paape, U. Miller, K. Mischaikow, T. Wanner, *Rigorous numerics for the Cahn-Hilliard equation on the unit square*, Rev. Mat. Complut. 21(2) (2008), pp. 351–426.

[42] B. ∅ksendal, *Stochastic Differential Equations*, Springer, New York, 2003.

[43] R. L. Pego,*Front migration in the non-linear Cahn-Hilliard equation*, Proc. R. Soc. Lond. A(422) (1989), pp. 261–278.

[44] P. E. Protter, *Stochastic Integration and Differential Equations*, Springer-Verlag Berlin Heidelberg, 2005.

[45] S. Semmes,*Some aspects of calculus on non-smooth sets*, arXiv: 0709.2508v3 (2007).

[46] J. B. Walsh, *An introduction to stochastic partial differential equations*, pp. 265–439, Lecture Notes in Math., 1180, Springer, Berlin, 1986.

[47] H. Weyl, *Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen*, Math. Ann., 71 (1911), pp. 441-479.

E-mail address: danton@tem.uoc.gr
E-mail address: gkarali@tem.uoc.gr