

ASYMPTOTICS FOR A GENERALIZED CAHN-HILLIARD EQUATION WITH FORCING TERMS

D.C. ANTONOPOULOU

Department of Applied Mathematics, University of Crete, 714 09 Heraklion, Greece,
and Institute of Applied and Computational Mathematics, FORTH, Greece

G.D. KARALI

Department of Applied Mathematics, University of Crete, 714 09 Heraklion, Greece,
and Institute of Applied and Computational Mathematics, FORTH, Greece

G.T. KOSSIORIS

Department of Mathematics, University of Crete, Greece,
and Institute of Applied and Computational Mathematics, FORTH, Greece

(Communicated by the associate editor name)

ABSTRACT. Motivated by the physical theory of Critical Dynamics the Cahn-Hilliard equation on a bounded space domain is considered and forcing terms of general type are introduced. For such a rescaled equation the limiting interface problem is studied and the following are derived: (i) asymptotic results indicating that the forcing terms may slow down the equilibrium locally or globally, (ii) the sharp interface limit problem in the multidimensional case demonstrating a local influence in phase transitions of terms that stem from the chemical potential, while free energy independent terms act on the rest of the domain, (iii) a limiting non-homogeneous linear diffusion equation for the one-dimensional problem in the case of deterministic forcing term that follows the white noise scaling.

1. Introduction.

1.1. Generalized Cahn-Hilliard models. Non-equilibrium physical systems are systems of unstable configuration and of no detailed balance, abundantly met in nature. These systems are not yet in an equilibrium state because of the presence of slow relaxation and of possible external driving current. As a result, slowing down of the dynamics and phase transitions between non-equilibrium stationary states may be observed. The phase separation is described by a quantity, called order parameter, which takes non zero values in the ordered phases whereas it vanishes in the disordered phase, see [33]. For example, for a ferromagnetic system undergoing a phase transition, the order parameter is the net magnetization. For solid/liquid or liquid/gas transitions, it is the density. In the classical theory of phase transitions, generalized equations of Cahn-Hilliard type are of conserved order parameter and are described as Model B according to the universality classification of Hohenberg

2000 *Mathematics Subject Classification.* Primary: 35K55, 35K57.

Key words and phrases. Generalized Cahn-Hilliard equation, asymptotics, Hele-Shaw problem.

The second author is supported by a Marie Curie International Reintegration Grant within the 7th European Community Framework Programme, MIRG-CT-2007-200526.

and Halperin [25]. Such equations should be considered as mesoscopic models since they are obtained after taking averages of physical observables at a microscopic level (see e.g. [34], [29]).

In the present paper, we consider generalized Cahn-Hilliard equations of the form

$$\partial_\tau U = \Delta(-\epsilon^2 \Delta U + f'(U) + F_2(x, \tau; \epsilon)) + F_1(x, \tau; \epsilon), \quad x \in \Omega, \quad \tau > 0. \quad (1.1)$$

Herein, U is normalized taking values in $[-1, 1]$. The parameter $\epsilon > 0$ is a measure of the width of inner interfaces that may be developed along phase transitions during time evolution, $f'(U) = \partial_U f(U) = \partial_U(\frac{1}{4}(U^2 - 1)^2) = U(U^2 - 1)$, F_1 and F_2 are real functions defined on a bounded domain $\Omega \subset \mathbb{R}^n$, $n \leq 3$, for any $\tau > 0$, and $f = \frac{1}{4}(U^2 - 1)^2$ is a double equal-well potential taking its global minimum value 0 at $U = \pm 1$, [8], [1].

Setting $F_1 = F_2 = 0$, equation (1.1) becomes the standard Cahn-Hilliard equation, proposed by Cahn and Hilliard in [8], [9] as a simple phase separation model of a binary alloy at a fixed temperature. The order parameter U is the normalized local concentration of one of the alloy components at time t . The values $U = \pm 1$ correspond to the equilibrium concentrations. The function $f'(U)$ is nonlinear and models the tendency of a two species alloy, forced to homogenization, to return in a two separated phases equilibrium. The gradient dynamics of the equation are generated by the free energy functional

$$F_E[U] := \int_{\Omega} \left(\frac{1}{2} \epsilon^2 |\nabla U|^2 + f(U) \right) dx. \quad (1.2)$$

The standard Cahn-Hilliard model was extended by Cook [11] (see also [34]) to incorporate thermal fluctuations in the form of an additive noise. The resulting equation

$$\partial_\tau U = \Delta(-\epsilon^2 \Delta U + f'(U)) + \xi(x, \tau) \quad (1.3)$$

is usually called Cahn-Hilliard-Cook, where ξ is in general a Gaussian noise.

The general Cahn-Hilliard equation in the form (1.1) was also derived in [23] based on the balance law for microforces. The term F_2 appearing in (1.1) stands for the external fields, see [25], [23]. In [30], for example, the authors apply the Kawasaki exchange dynamics to derive a modified Cahn-Hilliard equation where F_2 describes the external gravity field. In Model B of [25] the F_1 term is the conservative white noise in accordance with the Cahn-Hilliard-Cook model. On the other hand, following [23], the quantity F_1 describes the external mass supply. Such a model is described for example in [3], in order to model spinodal decomposition in the presence of a moving particle source, as a mechanism for the formation of Liesegang bands. The source term F_1 is a deterministic Gaussian function.

As far as the stochastic Cahn-Hilliard equations are concerned, equations such as (1.3) are primarily studied in the mathematical literature. They are given a rigorous meaning through stochastic integration (see e.g. [12], [42]) when, for example, $\xi(x, \tau)$ is the usual space-time white noise, see e.g. [13], [7] and references cited therein. The authors in [17] study the case when the correlation is a smoother function in the space variables. The case of the derivative noise, in one space dimension, is studied in [14] allowing a certain monotonicity in the nonlinearity. A survey on the stochastic Cahn-Hilliard-Cook equation, including numerical results and conjectures concerning the nucleation problem, is presented in [7]. Results for the noisy Cahn-Hilliard equation can be also of great interest for the studying of coarsening (Ostwald ripening) where the displacement of bubbles is generally neglected, but may increase the frequency of collisions, ([4], [2], [27]).

A different example of generalized Cahn-Hilliard equations appears as a mesoscopic model for surface reactions. For instance, the authors in [24, 29, 28] consider a combination of Arrhenius absorption/desorption dynamics, Metropolis surface diffusion and simple unimolecular reaction to obtain a mesoscopic Cahn-Hilliard equation. Motivated by this model, where the external force field described by the pressure function $p(\tau, x)$ acting on the surface of the reaction enters the equation as a multiplicative term, we also consider the following generalized Cahn-Hilliard equation:

$$\partial_\tau U = \Delta(-\epsilon^2 \Delta U + f'(U) + F_2(x, \tau; \epsilon)) + F_1(x, \tau; \epsilon)(1 - U). \quad (1.4)$$

1.2. Main Results. A well-known result formally obtained by Pego [37] and rigorously derived by Alikakos, Bates and Chen [1] shows that the sharp interface problem for the multidimensional homogeneous Cahn-Hilliard equation is the homogeneous Hele-Shaw problem. Dynamics for the one-dimensional Cahn-Hilliard equation in a neighborhood of an equilibrium having $N + 1$ transition layers have been studied among others by Bates and Xun in [5], [6]. Motivated by the homogeneous problem we will analyze the sharp interface motion for nonhomogeneous Cahn-Hilliard equations of the form (1.1) and (1.4). We will employ formal asymptotics for the derivation of the equations of motion of the interfaces separating the $U = \pm 1$ phases of the alloy. Rescaling time, we set $t := \epsilon\tau$ and $u(x, t) := U(x, \tau)$. The domain Ω gets the decomposition

$$\Omega = \Omega_\epsilon^+(t) \cup \Omega_\epsilon^-(t) \cup \Omega_\epsilon^I(t)$$

such that

$$u \approx 1 \text{ for } x \in \Omega_\epsilon^+(t) \text{ and } u \approx -1 \text{ for } x \in \Omega_\epsilon^-(t),$$

where $\Omega_\epsilon^I(t)$ is a narrow interfacial region of thickness ϵ .

1. We first study the asymptotic behavior of (1.1) as $\epsilon \rightarrow 0^+$ for the case $n \geq 2$ in the t scale, i.e. we consider

$$\partial_t u = \Delta(-\epsilon \Delta u + \epsilon^{-1} f'(u) - G_2(x, t; \epsilon)) + G_1(x, t; \epsilon) \quad (1.5)$$

where $G_1(x, t; \epsilon) := \epsilon^{-1} F_1$, $G_2(x, t; \epsilon) := -\epsilon^{-1} F_2$, and F_1, F_2 must be of order $\mathcal{O}(\epsilon^{1+\gamma})$ with $\gamma \geq 0$. We assume that G_1, G_2 are smooth and uniformly bounded for any $x \in \Omega, t > 0$ as $\epsilon \rightarrow 0^+$, and show how they slow down the equilibrium by deriving their local and global contribution to a non-homogeneous Hele-Shaw limit problem.

We supplement the Cahn-Hilliard equation with an initial condition u_0 and assume that the interface has been formed initially, that is there exists a smooth closed $n - 1$ dimensional hypersurface $\Gamma_0 \subset\subset \Omega$ such that $u_0 \approx -1$ in the open region enclosed by Γ_0 and $u_0 \approx 1$ elsewhere in Ω . We formally show that $v := \lim_{\epsilon \rightarrow 0^+} (\epsilon \Delta u - \epsilon^{-1} f'(u) + G_2)$ satisfies the Hele-Shaw free boundary problem

$$\begin{cases} \Delta v = \lim_{\epsilon \rightarrow 0^+} G_1 \text{ in } \Omega \setminus \Gamma(t), \quad t > 0 \\ \partial_n v = 0 \text{ on } \partial\Omega \\ v = \lambda H + \lim_{\epsilon \rightarrow 0^+} G_2 \text{ on } \Gamma(t) \\ V = \frac{1}{2}(\partial_n v^+ - \partial_n v^-) \text{ on } \Gamma(t) \\ \Gamma(0) = \Gamma_0, \end{cases} \quad (1.6)$$

where Γ is the zero level surface of u contained in the region Ω_ϵ^I ; Γ is a closed $n-1$ dimensional hypersurface of mean curvature $H = H(t)$ and of velocity $V = V(t)$ that divides Ω in two open sets $\Omega^+(t)$, $\Omega^-(t)$. The constant λ is positive, and n is the unit outward normal vector at the inner and outer boundaries $\Gamma(t)$, $\partial\Omega$.

Furthermore, we demonstrate that $\Delta v \approx 0 + \mathcal{O}(\epsilon^2) + G_1(x, t; \epsilon)$ in $\Omega \setminus \Gamma(t)$, while $v \approx \lambda H + \mathcal{O}(\epsilon) + G_2(x, t; \epsilon)$ on $\Gamma(t)$. Considering the homogeneous Cahn-Hilliard, i.e. setting $G_1 = G_2 = 0$ we observe that the limit problem is the homogeneous Hele-Shaw problem with order of convergence equal to $\mathcal{O}(\epsilon^1)$. Let us now assume the non-homogeneous Cahn-Hilliard case where for some $\lambda_1, \lambda_2 > 0$ the forcing terms are of the form $G_1 = c_1(x, t)\epsilon^{\lambda_1}$ and $G_2 = c_2(x, t)\epsilon^{\lambda_2}$ for c_1, c_2 smooth, converging uniformly to zero functions, as $\epsilon \rightarrow 0^+$, with orders of convergence $\mathcal{O}(\epsilon^{\lambda_1})$ and $\mathcal{O}(\epsilon^{\lambda_2})$, then obviously the limit problem is again the homogeneous Hele-Shaw and the order of convergence for the new problem is equal to $\min\{\mathcal{O}(\epsilon^1), \mathcal{O}(\epsilon^{\lambda_1}), \mathcal{O}(\epsilon^{\lambda_2})\}$. Therefore, considering the aforementioned non-homogeneous Cahn-Hilliard equation the following cases hold true:

- If $\min\{\lambda_1, \lambda_2\} \geq 1$, then the order of convergence of the homogeneous and the non-homogeneous Cahn-Hilliard equation to the homogeneous Hele-Shaw problem is $\mathcal{O}(\epsilon^1)$ for both cases.
- If $0 < \lambda := \min\{\lambda_1, \lambda_2\} < 1$, then the convergence of the non-homogeneous equation to the homogeneous Hele-Shaw in comparison with the homogeneous Cahn-Hilliard is slowed down, since it is of order $\mathcal{O}(\epsilon^\lambda) < \mathcal{O}(\epsilon^1)$ and an analogous delay during spinodal decomposition is expected for the equilibrium.

We also study (1.4), which in t -scale is transformed to

$$\partial_t u = \Delta(-\epsilon \Delta u + f'(u) - G_2(x, t; \epsilon)) + G_1(x, t; \epsilon)(1 - u), \quad (1.7)$$

where $G_1 := \epsilon^{-1}F_1$ and $G_2 := -\epsilon^{-1}F_2$. If G_1 and G_2 are smooth and uniformly bounded as $\epsilon \rightarrow 0^+$ then we derive the following sharp interface problem:

$$\begin{cases} \Delta v = 2 \lim_{\epsilon \rightarrow 0^+} G_1 \text{ in } \Omega^- & \text{and } \Delta v = 0 \text{ in } \Omega^+ \\ \partial_n v = 0 & \text{on } \partial\Omega \\ v = \lambda H + \lim_{\epsilon \rightarrow 0^+} G_2 & \text{on } \Gamma(t) \\ V = \frac{1}{2}(\partial_n v^+ - \partial_n v^-) & \text{on } \Gamma(t) \\ \Gamma(0) = \Gamma_0. \end{cases} \quad (1.8)$$

Here, the forcing term G_1 appears in the limit only in the -1 phase.

2. Let $t = \epsilon\tau$, given the time scaling $\beta(t) = \sqrt{\epsilon}\beta(\frac{t}{\epsilon})$ of a standard Brownian motion $\beta(t)$, the time white noise $\dot{\beta}(t)$ which can be considered as the derivative of $\beta(t)$ in the sense of distributions scales as follows, [36]:

$$\dot{\beta}(t) = \frac{1}{\sqrt{\epsilon}}\dot{\beta}(\tau). \quad (1.9)$$

In an analogous way the space-time noise could be considered as the time derivative, in the sense of distributions, of an H -valued process

$$W(t) = \sum_{l=1}^{\infty} \gamma_l^{1/2} e_l \beta_l(t), \quad t \geq 0, \quad (1.10)$$

with covariance operator Q , where H is a properly defined Hilbert space, $\{e_l(x)\}$ is an orthonormal basis of H , $\{\beta_l\}$ is a family of real-valued independent Brownian motions, Q is of trace class and γ_l are the eigenvalues of Q (see e.g. [12], [26]). For example, in the case of equation (1.3) defined on $[0, 1]$, $H = L^2([0, 1])$ and the family $\{e_l(x)\}$ is the orthonormal system of eigenfunctions for the Laplace operator defined on $[0, 1]$ with the proper boundary conditions. The representation (1.10) is also given a meaning in the case where Q has an infinite trace by defining a cylindrical Wiener process, see [12], [26]. This is the case for example of the white noise where Q is the identity operator with $\gamma_l = 1$.

The spectral approximation of the noise obtained by truncating (1.10) is defined as follows:

$$\xi_N(x, t) = \sum_{l=1}^N \gamma_l^{1/2} e_l(x) \dot{\beta}_l(t), \quad t \geq 0. \quad (1.11)$$

It is apparent from (1.10), (1.11) that the space-time white noise as well as the truncated one follow the scaling (1.9). Using the truncated noise (1.11) e.g. in (1.3), instead of the full noise, we come up with an approximate equation that can be used for approximating the solution of (1.3), cf. [38], [35].

Moreover, following [21], $\dot{\beta}_l$ can be approximated by smooth independent stochastic processes $\sigma^{-\gamma} \zeta_l(\sigma^{-2\gamma} t)$, $t > 0$, $0 < \gamma < 1/3$, as $\sigma \rightarrow 0$. Herein, $\zeta_l \in C^1$ in t (a.s.) and $|\zeta_l|, |\dot{\zeta}_l| \leq M$, for some non-random $M > 0$, $E[\zeta_l(t)] = 0$ and ζ_l are stationary and satisfy a strongly mixing condition. In this way making use of (1.11) we can obtain a smooth approximation of the space-time white noise. Note that, this is not the only way to approximate the noise and preserve the time scaling (1.9). The standard analytical way is by mollifying the noise term, cf. [20]. One can also obtain different approximations of the space-time white noise that preserve the time scaling (1.9) by projecting the noise on different finite dimensional subspaces. These approximations can be used for the construction of efficient numerical schemes for the fully stochastic problem, see e.g. [15], [40], [41], [31], [32].

In the t -scale in one dimension, if $F_2 = 0$ and $F_1 = F_1(x, \tau)$ then (1.1) is transformed to

$$\partial_t u = \Delta(-\epsilon \Delta u + \epsilon^{-1} f'(u)) + \xi(x, t) \text{ in } \Omega, \quad (1.12)$$

where $\xi(x, t) := \epsilon^{-1} F_1$. Smooth approximations of a white noise preserving the white noise scaling are suitable for the analytic study of stochastic models from a deterministic point of view. Motivated by the Cahn-Hilliard-Cook model (1.3) for which an analysis of the interface motion for a given realization (in the underlying probability space) is a challenging open problem, we are lead to the study, as a first step, of a nonhomogeneous Cahn-Hilliard equation. Driven by (1.9), (1.10) and (1.11), we consider a deterministic ξ satisfying the white noise scaling

$$\xi(x, t) = \frac{1}{\sqrt{\epsilon}} \xi(x, \tau).$$

Replacing $\xi(x, t)$ in (1.12) and using that $\tau = t/\epsilon$ we arrive at

$$\partial_t u = \Delta(-\epsilon \Delta u + \epsilon^{-1} f'(u)) + \frac{1}{\sqrt{\epsilon}} \xi\left(x, \frac{t}{\epsilon}\right), \quad x \in \Omega, \quad t > 0, \quad (1.13)$$

where Ω is a bounded domain in \mathbb{R} , while the function $\xi(x, \frac{t}{\epsilon})$ is assumed to be uniformly bounded for any x as a function of $\tau = \frac{t}{\epsilon}$.

We study the limit problem for (1.13) by applying a novel asymptotic expansion consisting of four variables (due to the scaling of the ξ term) and show that with this time scaling we are at a time regime where the linearized non-homogeneous Cahn-Hilliard equation is dominant. We actually prove in τ variable that

$$\partial_\tau u_{1/2}(x, \tau) = f''(1)\Delta u_{1/2}(x, \tau) + \xi(x, \tau) + \mathcal{O}(\sqrt{\epsilon}) \text{ on } \Omega \setminus \Omega_\epsilon^I, \quad (1.14)$$

for $u = 1 + \sqrt{\epsilon} u_{1/2}(x, \tau) + \mathcal{O}(\epsilon)$. The absence of inner boundary conditions on the limit observed in formal analysis is a result of the lower order approximation $\mathcal{O}(\sqrt{\epsilon})$ which simulates a slower time scale. Since oscillations are observed only in $\Omega \setminus \Gamma$ while the inner boundary remains fixed this implies that noise-like forcing slows down the evolution.

The proposed deterministic asymptotic expansion gives new insights on treating the stochastic case. In fact, equation (1.13) and the resulting limit problem may serve as toy models for a rigorous asymptotic analysis of the stochastic Cahn-Hilliard with additive noise. Deterministic forcing strength may model an analogous noise strength of a certain order in ϵ . The asymptotic results presented in this paper for the deterministic case agree with stochastic numerical simulations. In [39] the authors demonstrate experimentally that increasing of the noise strength during spinodal decomposition leads to more diffuse interfaces and slower transitions.

The paper is organized as follows: some preliminaries are given in Section 2; the multidimensional case is presented in Section 3, and the case of the time-scaled non-homogeneous term in one-dimension is studied in Section 4.

2. Preliminaries. Let us consider the homogeneous Cahn-Hilliard equation stemming from (1.1) by setting $F_1 = F_2 = 0$. If ∂_n is the derivative along the normal vector n of $\partial\Omega$, then integrating (1.1) over Ω we get

$$\partial_\tau m(\tau) = \int_{\partial\Omega} \partial_n(-\epsilon^2 \Delta U + f'(U)) ds, \quad (2.15)$$

where $m(\tau) = \int_\Omega U(x, \tau) dx$. Obviously, (2.15) gives mass conservation if the following boundary condition holds

$$\partial_n(-\epsilon^2 \Delta U + f'(U)) = 0 \text{ on } \partial\Omega. \quad (2.16)$$

Differentiating (1.2) with respect to τ and by making use of (1.1) we arrive at

$$\partial_\tau(F_E[U](\tau)) = - \int_\Omega |\nabla(-\epsilon^2 \Delta v + f'(U))|^2 dx, \quad (2.17)$$

which is a total free energy decreasing condition. A complete mathematical analysis of the initial boundary value problem for the homogeneous Cahn-Hilliard equation with boundary conditions $\partial_n U = 0$ and (2.16) was given in [16].

Let us introduce now the ϵ -dependent smooth forcing terms \widetilde{F}_1 , and F_2 as follows:

$$\partial_\tau U = \Delta(-\epsilon^2 \Delta U + f'(U) + F_2(x, \tau; \epsilon)) + \widetilde{F}_1(x, \tau, U(x, \tau); \epsilon), \quad x \in \Omega, \quad \tau > 0. \quad (2.18)$$

Integrating (2.18) over Ω , we arrive at

$$\partial_\tau m(\tau) = \int_{\partial\Omega} \partial_n(-\epsilon^2 \Delta U + f'(U) + F_2(x, \tau; \epsilon)) ds + \int_\Omega \widetilde{F}_1(x, \tau, U(x, \tau); \epsilon) dx. \quad (2.19)$$

Obviously, a Neumann boundary condition on the chemical potential of the form

$$\partial_n(-\epsilon^2 \Delta U + f'(U) + F_2(x, \tau; \epsilon)) = 0 \text{ on } \partial\Omega, \quad (2.20)$$

when $\widetilde{F}_1 = 0$ or when $\widetilde{F}_1(x, \tau, U(x, \tau); \epsilon) := F_1(x, \tau; \epsilon)$ for $\int_{\Omega} F_1(x, \tau; \epsilon) dx = 0$, gives mass conservation for the model (2.18). In general, due to the presence of the external force field F_2 and the external mass supply, a free energy decreasing is not expected. For example, when $F_2 = 0$, assuming (2.16), the free energy decreases when the rate of change of the free energy $\int_{\Omega} |\nabla(-\epsilon^2 \Delta v + f'(U))|^2 dx$ dominates the rate $\int_{\Omega} (-\epsilon^2 \Delta v + f'(U)) \widetilde{F}_1 dx$ at which the free energy contains mass supply \widetilde{F}_1 . The mathematical analysis of the problem when $F_2 = 0$ and F_1 is in L^2 is given in [43], [16].

The asymptotic analysis of the rescaled homogeneous Cahn-Hilliard has attracted a lot of attention. In [1, 37] the authors derived that the homogeneous Cahn-Hilliard equation on the limit ($\epsilon \rightarrow 0^+$) is related to a Hele-Shaw problem that describes evolution process after spinodal decomposition during coarsening stage. More specifically, for $\epsilon > 0$, the following conservative and free energy decreasing initial, and boundary value problem is considered:

$$\begin{cases} \partial_t u(x, t; \epsilon) = \Delta(-\epsilon \Delta u + \epsilon^{-1} f'(u)), & x \in \Omega, t > 0 \\ u(x, 0) = u_0(x) \\ \partial_n u = \partial_n(-\epsilon \Delta u + \epsilon^{-1} f'(u)) = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.21)$$

Here Ω is a bounded domain in \mathbb{R}^n , $n \geq 2$. There exists a smooth closed $n - 1$ dimensional hypersurface Γ_0 in Ω such that, [1], $v := \lim_{\epsilon \rightarrow 0^+} (\epsilon \Delta u - \epsilon^{-1} f'(u))$ satisfies the following perimeter shortening and volume preserving Hele-Shaw free boundary problem

$$\begin{cases} \Delta v = 0 & \text{in } \Omega \setminus \Gamma(t), t > 0, \\ \partial_n v = 0 & \text{on } \partial\Omega, \\ v = \lambda H & \text{on } \Gamma(t), \\ V = \frac{1}{2}(\partial_n v^+ - \partial_n v^-) & \text{on } \Gamma(t), \\ \Gamma(0) = \Gamma_0, \end{cases} \quad (2.22)$$

where Γ , H , V , Ω^+ , Ω^- and λ are defined as in (1.6).

3. Asymptotics-multidimensional case.

3.1. The $\mathcal{O}(\epsilon)$ limit problems. Our aim in this section is the study of the asymptotic behavior of the non-homogeneous Cahn-Hilliard equation (1.5) as $\epsilon \rightarrow 0^+$. Let $v(\epsilon) := \epsilon \Delta u - \epsilon^{-1} f'(u) + G_2$ (chemical potential term). We define v_0 such that

$$v(\epsilon) = v_0 + \epsilon v_1 + \mathcal{O}(\epsilon^2).$$

Let $d = d(x, t)$ be the signed distance function from the inner boundary. We set $z := \frac{d}{\epsilon}$, assume that v_1 is of the form $\widetilde{Q}(z) := v_1$, and show that v_0 which is the $\mathcal{O}(\epsilon)$ -approximation of $v(\epsilon)$ satisfies the following problem

$$\begin{cases} \Delta v_0 = G_1 + \mathcal{O}(\epsilon^2) & \text{in } \Omega \setminus \Omega_{\epsilon}^I(t), t > 0 \\ v_0 = \lambda H + G_2 + \mathcal{O}(\epsilon) & \text{on } \Omega_{\epsilon}^I(t) \\ V = \frac{1}{2}(\widetilde{Q}_z(+\infty) - \widetilde{Q}_z(-\infty)) + \mathcal{O}(\epsilon^2) & \text{on } \Omega_{\epsilon}^I(t), \end{cases} \quad (3.23)$$

where H and V are the mean curvature and velocity respectively of the zero level surface $\Gamma(t)$ contained in the interfacial region $\Omega_{\epsilon}^I(t)$. This implies formally on the limit the non-homogeneous Hele-Shaw problem (1.6).

The non-homogeneous terms may lead to a slower equilibrium. To clarify our point, let us assume for $\lambda_1, \lambda_2 > 0$ that $G_1 := \epsilon^{\lambda_1} c_1(x, t)$ and $G_2 := \epsilon^{\lambda_2} c_2(x, t)$, for c_1, c_2 bounded. In this case (3.23) converges to the homogeneous Hele-Shaw problem with order

$$\min\{\mathcal{O}(\epsilon), \mathcal{O}(\epsilon^{\lambda_1}), \mathcal{O}(\epsilon^{\lambda_2})\}.$$

Obviously, if $\lambda := \min\{\lambda_1, \lambda_2\} \geq 1$ then the order of convergence for the homogeneous and non-homogeneous Cahn-Hilliard equation to a homogeneous Hele-Shaw problem is $\mathcal{O}(\epsilon)$, while if $0 < \lambda < 1$ the convergence for the non-homogeneous Cahn-Hilliard is of lower order $\mathcal{O}(\epsilon^\lambda)$. If G_1, G_2 are independent of ϵ , then the limit problem for the non-homogeneous Cahn-Hilliard is a non-homogeneous Hele-Shaw problem.

Keeping the same notation as in (3.23) we now consider equation (1.7). After linearizing $G_1(1-u)$ we arrive at the following problem

$$\begin{cases} \Delta v_0 = 2G_1 + \mathcal{O}(\epsilon^2) & \text{in } \Omega_\epsilon^-, \quad t > 0 \\ \Delta v_0 = 0 + \mathcal{O}(\epsilon^2) & \text{in } \Omega_\epsilon^+, \quad t > 0 \\ v_0 = \lambda H + G_2 + \mathcal{O}(\epsilon) & \text{on } \Omega_\epsilon^I(t) \\ V = \frac{1}{2}(\tilde{Q}_z(+\infty) - \tilde{Q}_z(-\infty)) + \mathcal{O}(\epsilon^2) & \text{on } \Omega_\epsilon^I(t) \end{cases} \quad (3.24)$$

and derive the limit problem (1.8). In this case, the inner and outer problem satisfy a different equation as the term G_1 seems to contribute only in the -1 -phase.

3.2. Formal Analysis. Let $v(\epsilon) := \epsilon \Delta u - \epsilon^{-1} f'(u) + G_2$. Then (1.5) can be written in a system form

$$\begin{cases} \partial_t u &= -\Delta v + G_1 \\ v &= -\frac{f'(u)}{\epsilon} + \epsilon \Delta u + G_2. \end{cases} \quad (3.25)$$

We construct an inner solution close to the interface, an outer solution away from the interface, and with the appropriate matching we pass to the limit and derive the corresponding free boundary problem. We also assume that the interface Γ does not intersect the boundary.

Outer expansion. We consider that the inner interface is known and seek the outer expansion far from the interface, i.e. seek an expansion in the form

$$\begin{aligned} u &= u_0 + \epsilon u_1 + \dots, \\ v &= v_0 + \epsilon v_1 + \dots, \end{aligned}$$

where ‘...’ denote higher order terms and u_0, \dots, v_0, \dots are smooth functions. We insert the outer expansion into (3.25)₂ and obtain

$$\begin{aligned} v_0 + \epsilon v_1 + \mathcal{O}(\epsilon^2) &= -\frac{1}{\epsilon}(f'(u_0) + \epsilon f''(u_0)u_1 + \mathcal{O}(\epsilon^2)) \\ &\quad + \epsilon \Delta(u_0 + \epsilon u_1 + \mathcal{O}(\epsilon^2)) + G_2 + \mathcal{O}(\epsilon^2). \end{aligned} \quad (3.26)$$

Collecting the $\mathcal{O}(\frac{1}{\epsilon})$ terms in (3.26) we arrive at $f'(u_0) = 0$, since $f'(u) = u^3 - u = \frac{1}{4} \frac{d}{dt}(u^2 - 1)^2$ is the derivative of a double equal-well potential taking its global minimum value 0 at $u = \pm 1$, we get $u_0 = \pm 1$, as in Remark 4.1. (1) of [1]. Further, we collect the $\mathcal{O}(1)$ terms in (3.26) and get $v_0 = -f''(u_0)u_1 + G_2 + \mathcal{O}(\epsilon^2)$ which

implies $u_1 = \frac{G_2 - v_0}{f''(u_0)} + \mathcal{O}(\epsilon^2)$. As u_0 is a constant it is implied that $\partial_t u_0 = 0$. We plug now the outer expansion into (3.25)₁ and have

$$\partial_t(u_0 + \epsilon u_1 + \mathcal{O}(\epsilon^2)) = -\Delta(v_0 + \epsilon v_1 + \mathcal{O}(\epsilon^2)) + G_1,$$

which after collecting the $\mathcal{O}(1)$ terms gives

$$-\Delta v_0 + G_1 = 0 + \mathcal{O}(\epsilon^2).$$

Collecting finally the $\mathcal{O}(\epsilon)$ terms we get

$$\partial_t u_1 = -\Delta v_1.$$

Inner expansion. Let x be a point in Ω that at time t is near the interface $\Gamma(t)$. Introduce the stretched normal distance to the interface, $z := \frac{d}{\epsilon}$, where $d(x, t)$ is the signed distance from the point x in Ω to the interface $\Gamma(t)$, such that $d(x, t) > 0$ in Ω_ϵ^+ , $d(x, t) < 0$ in Ω_ϵ^- . Obviously Γ has the representation

$$\Gamma(t) = \{x \in \Omega : d(x, t) = 0\}.$$

Near Γ , if Γ is smooth, then d is smooth and $|\nabla d| = 1$ in a neighborhood of Γ . Following [1] and [37], we seek for an inner expansion valid for x near Γ of the form

$$\begin{aligned} u &= q\left(\frac{d(x, t)}{\epsilon}, x, t\right) + \epsilon Q\left(\frac{d(x, t)}{\epsilon}, x, t\right) + \dots, \\ v &= \tilde{q}\left(\frac{d(x, t)}{\epsilon}, x, t\right) + \epsilon \tilde{Q}\left(\frac{d(x, t)}{\epsilon}, x, t\right) + \dots, \end{aligned}$$

where ‘ \dots ’ denote higher order terms and $q, Q, \dots, \tilde{q}, \tilde{Q}, \dots$ are smooth. It will be convenient to require that the quantities depending on (z, x, t) are defined for x in a full neighborhood of Γ but do not change when x varies normal to Γ with z held fixed, [37]. We insert the inner expansion into (3.25)₂, utilize that $|\nabla d|^2 = 1$ and obtain the following expression

$$\tilde{q} + \epsilon \tilde{Q} + \mathcal{O}(\epsilon^2) = -\frac{1}{\epsilon}(f'(q) + \epsilon f''(q)Q) + \epsilon \left(\frac{\partial_z q}{\epsilon} \Delta d + \frac{\partial_{zz} q}{\epsilon^2} + \partial_z Q \Delta d + \frac{\partial_{zz} Q}{\epsilon} \right) + G_2. \quad (3.27)$$

We collect the order $\mathcal{O}\left(\frac{1}{\epsilon}\right)$ and get

$$\partial_{zz} q - f'(q) = 0.$$

By matching now the terms of order $\mathcal{O}(1)$ in (3.27), we obtain

$$\tilde{q} + \mathcal{O}(\epsilon) = -f''(q)Q + \partial_{zz} Q + G_2 + \partial_z q \Delta d,$$

or equivalently

$$\tilde{q} - \partial_z q \Delta d = \partial_{zz} Q - f''(q)Q + G_2 + \mathcal{O}(\epsilon). \quad (3.28)$$

We define for $Q(z)$ the linear operator

$$\mathcal{L}Q = \partial_{zz} Q - f''(q)Q$$

which is the linearized Allen-Cahn operator. Then (3.28) is written as

$$\tilde{q} - \partial_z q \Delta d = \mathcal{L}Q + G_2 + \mathcal{O}(\epsilon). \quad (3.29)$$

This equation is solvable if for any $\chi \in \text{Ker } \mathcal{L}^*$ holds that

$$\chi \perp (\tilde{q} - \partial_z q \Delta d - G_2),$$

or equivalently if

$$\int_{-\infty}^{\infty} (\tilde{q} - \partial_z q \Delta d - G_2) \chi dz = 0. \quad (3.30)$$

Obviously for any x on Γ holds that $d(x, t) = 0$, $\Delta d(x, t) = H$. Replacing in (3.30) we obtain the sufficient condition on the interface

$$\tilde{q} = \lambda H + G_2 + \mathcal{O}(\epsilon). \quad (3.31)$$

Plugging the inner expansion into (3.25)₁ we get

$$\frac{\partial_z q}{\epsilon} d_t + \partial_z Q d_t + \mathcal{O}(\epsilon) = - \left(\frac{\partial_z \tilde{q}}{\epsilon} \Delta d + \frac{\partial_{zz} \tilde{q}}{\epsilon^2} + \partial_z \tilde{Q} \Delta d + \frac{\partial_{zz} \tilde{Q}}{\epsilon} \right) + G_1. \quad (3.32)$$

We collect the $\mathcal{O}\left(\frac{1}{\epsilon^2}\right)$ terms and arrive at

$$\partial_{zz} \tilde{q} = 0$$

which implies that

$$\tilde{q} = a(x, t)z + b(x, t).$$

For further progress, the matching condition for the inner and outer expansions must be developed. In general, these are obtained by the following procedure ([10]). Fixing x on Γ , we seek to match the expansions by requiring formally

$$\begin{aligned} \tilde{q} + \epsilon \tilde{Q} + \mathcal{O}(\epsilon^2) &= v_0 + \epsilon v_1 + \mathcal{O}(\epsilon^2), \\ v_0 &= \lim_{z \rightarrow \infty} \tilde{q} = \lim_{z \rightarrow \infty} (a(x, t)z + b(x, t)). \end{aligned}$$

Matching as $z \rightarrow \infty$, we obtain $a = 0$ and thus $\tilde{q} = b$. So, utilizing (3.31) we have that on the interface

$$v_0 = \lambda H + G_2 + \mathcal{O}(\epsilon).$$

What is missing is the evolution law, which should come from the inner expansion.

From (3.32), collect the $\mathcal{O}\left(\frac{1}{\epsilon}\right)$ terms and get

$$\partial_z q d_t = -\partial_z \tilde{q} \Delta d - \partial_{zz} \tilde{Q} + \mathcal{O}(\epsilon^2).$$

Recall that $-d_t = V$, while $\Delta d = H$ [1], and integrate over z from $-\infty$ to ∞ to get

$$- \int_{-\infty}^{\infty} \partial_z q V dz = - \int_{-\infty}^{\infty} \partial_{zz} \tilde{Q} dz + \mathcal{O}(\epsilon^2).$$

From the matching conditions we get

$$q(+\infty) = 1 + \mathcal{O}(\epsilon^3), \quad q(-\infty) = -1 + \mathcal{O}(\epsilon^3),$$

hence

$$V = \frac{1}{2} [\partial_z \tilde{Q}(+\infty) - \partial_z \tilde{Q}(-\infty)] + \mathcal{O}(\epsilon^2).$$

Thus, by (3.23) we notice that the forcing G_1 influences globally the evolution process, while G_2 has a local contribution on the interface.

For the mesoscopic model case, ([24, 29, 28]), we consider equation (1.7). Let $v(\epsilon) := \epsilon \Delta u - \frac{f(u)}{\epsilon} + G_2$; then (1.7) is written in a system form as follows:

$$\begin{cases} \partial_t u &= -\Delta v + G_1(1 - u), \\ v &= -\frac{f'(u)}{\epsilon} + \epsilon \Delta u + G_2. \end{cases} \quad (3.33)$$

Keeping the same notation and following the same derivation as above, we derive (3.24) by linearizing $G_1(1-u)$. In particular, if $u_0 \approx -1$, then $\Delta v_0 = 2G_1 + \mathcal{O}(\epsilon^2)$, while if $u_0 \approx 1$, then $\Delta v_0 = 0 + \mathcal{O}(\epsilon^2)$. We note that $u_0 \approx -1$ in Ω_ϵ^- , while $u_0 \approx 1$ in Ω_ϵ^+ .

Remark: Keeping the same definitions as in (1.6), we consider for $\lambda > 0$ a non-homogeneous Hele-Shaw free boundary problem

$$\begin{cases} \Delta v = A(x, t) & \text{in } \Omega \setminus \Gamma(t), \quad t > 0 \\ \partial_n v = 0 & \text{on } \partial\Omega \\ v = \lambda H + B(x, t) & \text{on } \Gamma(t) \\ V = \frac{1}{2}(\partial_n v^+ - \partial_n v^-) & \text{on } \Gamma(t) \\ \Gamma(0) = \Gamma_0, \end{cases} \quad (3.34)$$

where A and B are smooth functions. By making use of the mean curvature and velocity definitions ([18]) and replacing the boundary conditions of v on Γ and $\partial\Omega$ we obtain the expressions

$$\begin{aligned} \frac{1}{n-1} \frac{d}{dt} \text{Per}(\Gamma(t)) &:= - \int_{\Gamma} HV = - \frac{1}{2\lambda} \int_{\Omega \setminus \Gamma} Av + \frac{1}{\lambda} \int_{\Gamma} BV - \frac{1}{2\lambda} \int_{\Omega \setminus \Gamma} |\nabla v|^2 \\ \frac{d}{dt} \text{Vol}(\Omega^-(t)) &:= - \int_{\Gamma} V = - \frac{1}{2} \int_{\Omega \setminus \Gamma} A. \end{aligned} \quad (3.35)$$

Obviously, the condition

$$\frac{d}{dt} \text{Per}(\Gamma(t)) \leq 0$$

gives minimal surface decreasing while

$$\frac{d}{dt} \text{Vol}(\Omega^-(t)) = 0$$

is the volume conservation condition. If $A = B = 0$, then the evolution for any t is driven under volume conservation and surface decreasing. This is not the case for the non-homogeneous problem for any t .

The term A models a forcing acting almost everywhere in Ω affecting the two phases of the binary alloy, while B appears in phase transitions. Let us consider the case where $A(x, t)$ is the limit as $\epsilon \rightarrow 0^+$ of $G_1(x, t; \epsilon)$ of the Cahn-Hilliard equation, describing for example thermal fluctuations during coarsening. We assume that fluctuations stop and A eventually vanishes after $t = t_A$; then for any $t > t_A$ volume conservation holds while

$$\frac{1}{n-1} \frac{d}{dt} \text{Per}(\Gamma(t)) = \frac{1}{\lambda} \int_{\Gamma} BV - \frac{1}{2\lambda} \int_{\Omega \setminus \Gamma} |\nabla v|^2.$$

This demonstrates that if $B \neq 0$ in an interval (t_A, t_B) then a minimal surface decreasing (i.e. $\frac{d}{dt} \text{Per}(\Gamma(t)) \leq 0$) is generally not reached until $|B|$ is very small on $\Gamma(t)$.

4. Asymptotics-one-dimensional case.

4.1. Preliminaries. We consider the non-homogeneous Cahn-Hilliard equation (1.13)

$$\partial_t u = \Delta(-\epsilon \Delta u + \epsilon^{-1} f'(u)) + \frac{1}{\sqrt{\epsilon}} \xi\left(x, \frac{t}{\epsilon}\right), \quad x \in \Omega \subset \mathbb{R}, \quad t > 0,$$

where $\xi(x, \frac{t}{\epsilon})$ does not stem from the chemical potential. The deterministic term $\frac{1}{\sqrt{\epsilon}} \xi(x, \frac{t}{\epsilon})$ scales like space-time white noise. Indeed, if we consider ξ to be a gaussian space-time white noise with $E[\xi(x, \tau)] = 0$, $E[\xi(x, \tau) \xi(x', \tau')] = \delta(x - x') \delta(\tau - \tau')$, then $\xi(x, t) = \frac{1}{\sqrt{\epsilon}} \xi(x, \frac{t}{\epsilon})$, see e.g. [42], where we have set $\tau = \frac{t}{\epsilon}$. As it was pointed out in the introduction we can regularize the noise in different ways preserving the scaling property. Noise terms in stochastic reaction-diffusion equations with scaling parameter $\epsilon > 0$ and bistable reaction term are usually considered to be of order $\mathcal{O}(\epsilon^\gamma)$ (c.f. [20], [22]).

4.2. Formal analysis. There are two time scales, t and $\tau = \frac{t}{\epsilon}$, so functions should depend on both. Moreover, there are two spatial scales: x and $z = \frac{x}{\epsilon}$. Thus, in general we expect u to be asymptotically approximated by functions depending on t, τ, x, z as follows:

$$u = u_0(t, \tau, x, z) + \sqrt{\epsilon} u_1(t, \tau, x, z) + \epsilon u_2(t, \tau, x, z) + \dots, \quad (4.36)$$

where ‘ \dots ’ denotes the higher order terms. By definition, *outer* expansion means to seek a solution u away from the interface which does not depend on z , while for the *inner* expansion we need in principle, all variables since we are close to the interface.

The following novel asymptotic expansions consist of four variables due to the scaling of the extra term. Pego in [37] uses two scaling variables to derive the limit problem for the standard Cahn-Hilliard equation.

Outer expansion. First, consider the outer expansion far from the interface

$$u = 1 + \sqrt{\epsilon} u_{1/2}(x, \tau) + \epsilon v_1(t, x) + \epsilon u_1(\tau, x) + \epsilon^{3/2} v_{3/2}(t, x) + \epsilon^{3/2} u_{3/2}(\tau, x) + \epsilon^2 v_2(t, x) + \epsilon^2 u_2(\tau, x) + \dots$$

where $u_{1/2}, v_1, u_1, v_{3/2}, \dots$ are smooth functions. Taking the derivative of u on t by making use of outer solution, the following expression holds:

$$\partial_t u = \frac{1}{\sqrt{\epsilon}} \partial_\tau u_{1/2} + \epsilon \partial_t v_1 + \partial_\tau u_1 + \epsilon^{3/2} \partial_t v_{3/2} + \sqrt{\epsilon} \partial_\tau u_{3/2} + \epsilon^2 \partial_t v_2 + \epsilon \partial_\tau u_2 + \dots$$

We also compute that

$$\begin{aligned}
-\epsilon \Delta u + \frac{f'(u)}{\epsilon} &= \frac{1}{\epsilon} f'(1) + \frac{1}{\sqrt{\epsilon}} f''(1) u_{1/2} + f''(1)(v_1 + u_1) + \frac{1}{2} f'''(1) u_{1/2}^2 \\
&+ \sqrt{\epsilon} \left(u_{1/2}(v_1 + u_1) f'''(1) + f''(1)(u_{3/2} + v_{3/2}) \right) \\
&+ \epsilon \left(f''(1)(v_2 + u_2) + \frac{1}{2} (v_1 + u_1)^2 f'''(1) + f'''(1) u_{1/2}(v_{3/2} + u_{3/2}) \right) \\
&+ \epsilon \sqrt{\epsilon} \left(-\Delta u_{1/2} + f'''(1) u_{1/2}(v_2 + u_2) + f'''(1)(v_1 + u_1)(v_{3/2} + u_{3/2}) \right) \quad (4.37) \\
&+ \epsilon^2 \left(-\Delta v_1 - \Delta u_1 + \frac{1}{2} f'''(1)(v_{3/2} + u_{3/2})^2 + f'''(1)(v_1 + u_1)(v_2 + u_2) \right) \\
&+ \epsilon^{5/2} \left(-\Delta v_{3/2} - \Delta u_{3/2} + f'''(1)(v_{3/2} + u_{3/2})(v_2 + u_2) \right) \\
&+ \epsilon^3 \left(-\Delta v_2 - \Delta u_2 + \frac{1}{2} f'''(1)(v_2 + u_2)^2 \right) + \dots
\end{aligned}$$

We collect first the $\mathcal{O}\left(\frac{1}{\sqrt{\epsilon}}\right)$ terms to obtain

$$\partial_\tau u_{1/2} = f''(1) \Delta u_{1/2} + \xi(x, \tau) \text{ on } \Omega \setminus \Gamma.$$

The partial differential equation involving $u_{1/2}$ admits a smooth solution if Γ is smooth. Collecting now the $\mathcal{O}(1)$ terms, we obtain the following expressions for v_1 and u_1

$$\begin{aligned}
v_1 &= 0, \\
\partial_\tau u_1 &= \Delta u_1 f''(1) + \frac{f'''(1)}{2} \Delta(u_{1/2}^2(x, \tau)) \text{ on } \Omega \setminus \Gamma.
\end{aligned}$$

By collecting the $\mathcal{O}(\sqrt{\epsilon})$ terms we get

$$\partial_\tau u_{3/2} = \Delta[u_{1/2}(v_1 + u_1)] f'''(1) + f''(1) \Delta[(u_{3/2} + v_{3/2})] \text{ on } \Omega \setminus \Gamma.$$

Finally, the $\mathcal{O}(\epsilon)$ terms give

$$\begin{aligned}
\partial_\tau u_2 &= f''(1)(\Delta v_2 + \Delta u_2) + \frac{1}{2} \Delta[(v_1 + u_1)^2] f'''(1) + \\
&f'''(1) \Delta[u_{1/2}(v_{3/2} + u_{3/2})] \text{ on } \Omega \setminus \Gamma.
\end{aligned}$$

We note that the necessity of inner interfaces boundary conditions which are not expected to be derived from outer expansion, becomes clear when studying the inner expansion.

Inner expansion. Keeping all definitions of Section 3, let $d(x, t)$ be the signed distance from the interface. In one dimension, d can be expanded as follows:

$$d(x, t, \tau) = x - \left(x_0(t) + \sqrt{\epsilon} y_{1/2}(\tau) + \epsilon(x_1(t) + y_1(\tau)) + \dots \right). \quad (4.38)$$

We consider the inner expansion near the interface

$$\begin{aligned}
u &= \bar{m} \left(\frac{d(x, t, \tau)}{\epsilon} \right) + \sqrt{\epsilon} m_{1/2} \left(\frac{d(x, t, \tau)}{\epsilon} \right) + \sqrt{\epsilon} u_{1/2}(x, \tau) + \\
&\epsilon m_1 \left(\frac{d(x, t, \tau)}{\epsilon} \right) + \epsilon v_1(t, x) + \epsilon u_1(\tau, x) + \epsilon^{3/2} m_{3/2} \left(\frac{d(x, t, \tau)}{\epsilon} \right) + \epsilon^{3/2} v_{3/2}(t, x) + \\
&\epsilon^{3/2} u_{3/2}(\tau, x) + \epsilon^2 m_2 \left(\frac{d(x, t, \tau)}{\epsilon} \right) + \epsilon^2 v_2(t, x) + \epsilon^2 u_2(\tau, x) + \dots
\end{aligned}$$

Here, function \bar{m} is the unique solution of the Euler-Lagrange equation

$$\begin{aligned} -\bar{m}''(z) + f'(\bar{m}(z)) &= 0, \quad \text{for } z \in \mathbf{R}, \\ \lim_{z \rightarrow \pm\infty} \bar{m}(z) &= \pm 1, \quad \bar{m}(0) = 0, \end{aligned}$$

where $z = \frac{d}{\epsilon}$ and $m_{1/2}, u_{1/2}, m_1, v_1 \dots$ are smooth functions.

It is easy to see that since $f(m) = m^3 - m$, then it holds that

$$\bar{m}(z) = \tanh\left(\frac{z}{\sqrt{2}}\right),$$

and thus

$$\bar{m}'(z) > 0, \quad |\bar{m}(z) \pm 1| \leq c_0 e^{-a|z|}, \quad \left| \frac{d^l}{dz^l} \bar{m}(z) \right| \leq c_l e^{-a|z|}, \quad l = 1, 2 \dots,$$

where all $z \in \mathbf{R}$, $a = \sqrt{2}$ and $c_l, l = 0, 1, \dots$ are positive real constants. We also notice that $f(\bar{m})$ is even, indeed, $f''(\bar{m}) = 3\bar{m}^2 - 1$.

Now, with $f(\bar{m}) = \frac{1}{4}(\bar{m}^2 - 1)^2$, we have $f'(\bar{m}) = \bar{m}^3 - \bar{m}$, $f''(\bar{m}) = 3\bar{m}^2 - 1$, $f'''(\bar{m}) = 6\bar{m}$, $f''''(\bar{m}) = 6$, $f^{(n)}(\bar{m}) = 0$ for $n \geq 5$, $f'(1) = 0$, $f''(1) = 2$, and $f'''(1) = 6$.

For this potential, it follows that

$$\bar{m}(z) = \tanh\left(\frac{z}{\sqrt{2}}\right), \quad \bar{m}' = \frac{1}{\sqrt{2}}(1 - \bar{m}^2).$$

Hence, it holds that

$$f''(\bar{m}) - f''(1) = -3\sqrt{2}\bar{m}', \quad \int_{\mathbf{R}} (\bar{m}'(z))^2 dz = \frac{2\sqrt{2}}{3}.$$

Furthermore, we notice that \bar{m} is an odd function and as a result we get that

$$\int_{\mathbf{R}} (\bar{m}'(z))^2 \bar{m}(z) dz = 0,$$

since $(\bar{m}'(z))^2 \bar{m}(z)$ is also an odd function.

In the *inner* expansion taking Δu and multiplying by ϵ , by simple calculations the next equality follows:

$$\begin{aligned} -\epsilon \Delta u &= -\frac{1}{\epsilon} \bar{m}'' - \frac{1}{\sqrt{\epsilon}} m''_{1/2} - \epsilon \sqrt{\epsilon} \Delta u_{1/2} - m''_1 - \epsilon^2 \Delta v_1 - \epsilon^2 \Delta u_1 \\ &- \sqrt{\epsilon} m''_{3/2} - \epsilon^{5/2} \Delta v_{3/2} - \epsilon^{5/2} \Delta u_{3/2} - \epsilon m''_2 - \epsilon^3 \Delta v_2 - \epsilon^3 \Delta u_2 + \dots \end{aligned} \quad (4.39)$$

Adding in (4.39) the term $\frac{f'(u)}{\epsilon}$ and expanding $f'(u)$ according to the *inner* expansion of u , we seek

$$\begin{aligned}
& - \epsilon \Delta u + \frac{f'(u)}{\epsilon} = \frac{1}{\epsilon}(-\bar{m}'' + f'(\bar{m})) \\
& + \frac{1}{\sqrt{\epsilon}}(-m''_{1/2} + f''(\bar{m})(m_{1/2} + u_{1/2})) \\
& + (-m''_1 + f''(\bar{m})(m_1 + v_1 + u_1) + \frac{1}{2}f'''(\bar{m})(m_{1/2} + u_{1/2})^2) \\
& + \sqrt{\epsilon} \left((m_{1/2} + u_{1/2})(m_1 + v_1 + u_1)f'''(\bar{m}) + \right. \\
& \quad \left. \underbrace{(-m''_{3/2} + f''(\bar{m})m_{3/2}) + f''(\bar{m})(u_{3/2} + v_{3/2})}_{\mathcal{L}_{\bar{m}}m_{3/2}} \right) \\
& + \epsilon \left(-m''_2 + f''(\bar{m})(m_2 + v_2 + u_2) + \frac{1}{2}(m_1 + v_1 + u_1)^2 f'''(\bar{m}) \right. \\
& \quad \left. + f'''(\bar{m})(m_{1/2} + u_{1/2})(m_{3/2} + v_{3/2} + u_{3/2}) \right) \\
& + \epsilon \sqrt{\epsilon} \left(-\Delta u_{1/2} + f'''(\bar{m})(m_{1/2} + u_{1/2})(m_2 + v_2 + u_2) \right. \\
& \quad \left. + f'''(\bar{m})(m_1 + v_1 + u_1)(m_{3/2} + v_{3/2} + u_{3/2}) \right) \\
& + \epsilon^2 \left(-\Delta v_1 - \Delta u_1 + \frac{1}{2}f'''(\bar{m})(m_{3/2} + v_{3/2} + u_{3/2})^2 \right. \\
& \quad \left. + f'''(\bar{m})(m_1 + v_1 + u_1)(m_2 + v_2 + u_2) \right) \\
& + \epsilon^{5/2} \left(-\Delta v_{3/2} - \Delta u_{3/2} + f'''(\bar{m})(m_{3/2} + v_{3/2} + u_{3/2})(m_2 + v_2 + u_2) \right) \\
& + \epsilon^3 \left(-\Delta v_2 - \Delta u_2 + \frac{1}{2}f'''(\bar{m})(m_2 + v_2 + u_2)^2 \right) + \dots
\end{aligned}$$

We observe that most of the terms in the above computation include for example terms of the form

$$\mathcal{L}_{\bar{m}}m_{3/2} = -m''_{3/2} + f''(\bar{m})m_{3/2}, \quad \mathcal{L}_{\bar{m}}m_{1/2} = -m''_{1/2} + f''(\bar{m})m_{1/2}$$

that are linearized operators around \bar{m} . We take in the *inner* expansion the derivative in t and expand $z = \frac{d}{\epsilon}$, by making use of (4.38), to obtain that

$$\begin{aligned}
\partial_t u &= \frac{1}{\epsilon}(-\bar{m}'\partial_t x_0 - m'_{1/2}\partial_\tau y_{1/2} - \bar{m}'\partial_\tau y_1) \\
&+ \frac{1}{\epsilon\sqrt{\epsilon}}(-\bar{m}'\partial_\tau y_{1/2}) \\
&+ \frac{1}{\sqrt{\epsilon}}(-m'_{1/2}\partial_t x_0 + \partial_\tau u_{1/2} - m'_1\partial_\tau y_{1/2} - m'_{1/2}\partial_\tau y_1) \\
&+ \sqrt{\epsilon}(-m'_{1/2}\partial_t x_1 - m'_2\partial_\tau y_{1/2} - m'_{3/2}\partial_t x_0 - m'_{3/2}\partial_\tau y_1 + \partial_\tau u_{3/2}) \\
&+ \epsilon(-m'_1\partial_t x_1 + \partial_t v_1 - m'_2\partial_t x_0 - m'_2\partial_\tau y_1 + \partial_\tau u_2) \\
&+ (-\bar{m}'\partial_t x_1 - m'_1\partial_t x_0 - m'_1\partial_\tau y_1 + \partial_\tau u_1) \\
&+ \epsilon^2(-m'_2\partial_t x_1 - \partial_t v_2) \\
&+ \epsilon^3(-m'_2\partial_t x_1) \\
&+ \epsilon\sqrt{\epsilon}(-m'_{3/2}\partial_t x_1 + \partial_t v_{3/2} - m'_{3/2}\partial_\tau y_{1/2}) + \dots
\end{aligned}$$

Note that the free boundary problem should not depend on ϵ ; thus by further expansion we get:

$$\begin{aligned}
& u_{1/2}(x_0(t) + \sqrt{\epsilon}y_{1/2}(\tau) + \epsilon(x_1(t) + y_1(\tau))) = u_{1/2}(x_0) \\
& + \sqrt{\epsilon}u'_{1/2}(x_0)(y_{1/2} + \sqrt{\epsilon}(x_1 + y_1)) + \frac{1}{2}\epsilon u''_{1/2}(x_0)(y_{1/2} + \sqrt{\epsilon}(x_1 + y_1))^2 + \dots
\end{aligned}$$

In those expressions we utilize that $f''(\bar{m}) - f''(1) = -3\sqrt{2}\bar{m}'$, and collect the various orders of terms. Thus, by collecting the $\mathcal{O}\left(\frac{1}{\epsilon^3}\right)$, we obtain

$$-\bar{m}'' + f'(\bar{m}) = 0,$$

which justifies (and enforces) the choice of the standing wave \bar{m} as the highest order term in the inner expansion. Collecting the order $\mathcal{O}\left(\frac{1}{\epsilon^2\sqrt{\epsilon}}\right)$ terms we get

$$\mathcal{L}_{\bar{m}}m_{1/2} + f''(1)\Delta u_{1/2} + \left(f''(\bar{m}) - f''(1)\right)\Delta u_{1/2} = 0. \quad (4.40)$$

The operator \mathcal{L} is self adjoint on $L^2(\mathbb{R})$, and has a null space spanned by \bar{m}' . Therefore, the condition for solvability of $\mathcal{L}_{\bar{m}}m_{1/2} = g$ is

$$\int_{\mathbf{R}} g(z)\bar{m}' dz = 0,$$

where

$$\mathcal{L}_{\bar{m}}m_{1/2} = -f''(1)\Delta u_{1/2} - \left(f''(\bar{m}) - f''(1)\right)\Delta u_{1/2} = -2\Delta u_{1/2} - c\Delta u_{1/2}.$$

By applying the Fredholm alternative solvability condition, we compute

$$\begin{aligned} \int_{\mathbf{R}} \bar{m}' \left(2\Delta u_{1/2}(x_0) + c\bar{m}'\Delta u_{1/2}(x_0)\right) dz &= \Delta u_{1/2}(x_0) \int_{\mathbf{R}} \bar{m}' f''(\bar{m}) dz = \\ \Delta u_{1/2}(x_0) \int_{\mathbf{R}} \bar{m}' (3\bar{m}^2 - 1) dz &= \Delta u_{1/2}(x_0) \left[4 - 6 \int_{\mathbf{R}} \bar{m}^2 \bar{m}' dz\right] = \\ \Delta u_{1/2}(x_0) \left[4 - 6 \int_{\mathbf{R}} (\bar{m}^3)' dz\right] &= \Delta u_{1/2}(x_0) [4 - 4] = 0. \end{aligned}$$

Thus, expected compatibility conditions hold as identities and we do not reach to inner interfaces boundary conditions on the limit. More specifically, before matching in equation $\partial_t u = \Delta(-\epsilon\Delta u + \epsilon^{-1}f'(u)) + \frac{1}{\sqrt{\epsilon}}\xi\left(x, \frac{t}{\epsilon}\right)$ using inner solution, we may multiply by $\epsilon^2\sqrt{\epsilon}$. Then by the equality $-\bar{m}'' + f'(\bar{m}) = 0$ and the asymptotic result $-m''_{1/2} + f''(\bar{m})(m_{1/2} + u_{1/2}) = 0 + \mathcal{O}(\epsilon^k)$, $k > 0$ of Fredholm alternative, we observe that all the other terms of matching equality, if smooth, are of order $\mathcal{O}(\epsilon^m)$, $m > 0$, and thus converge to zero functions as $\epsilon \rightarrow 0^+$.

Consequently, on the interface all the terms disappear and we end up only to principal term with

$$\partial_\tau u_{1/2} = f''(1)\Delta u_{1/2} + \xi(x, \tau) \text{ on } \Omega \setminus \Gamma \quad (4.41)$$

and all the other higher order terms that we derive from the outer expansion. The absence of inner boundary conditions is a result of the lower order approximation $\mathcal{O}(\sqrt{\epsilon})$ which simulates a slower time scale.

The homogeneous part of the preceding linear diffusion equation is the linearization of the ill-posed problem ([19])

$$\partial_\tau u_{1/2}(x, \tau) = \Delta(f' u_{1/2}(x, \tau)),$$

in τ time scale. This is a strong indication that time and space scales, and asymptotic expansion orders have been correctly chosen. So, the inner boundary remains fixed while oscillations only in $\Omega \setminus \Gamma$ are observed.

Acknowledgment. The second author is supported by a Marie Curie International Reintegration Grant within the 7th European Community Framework Programme, MIRG-CT-2007-200526, a research grant at the University of Crete, ELKE 2740 and partially supported by the FP7-REGPOT-2009-1 project “Archimedes Center for Modeling, Analysis and Computation”. The third author is partially supported by the FP7-REGPOT-2009-1 project “Archimedes Center for Modeling, Analysis and Computation”. The authors would like to thank Prof. M. Katsoulakis for suggesting the mesoscopic model for surface reactions. The authors also wish to thank the two anonymous referees for their valuable comments and suggestions.

REFERENCES

- [1] N. D. Alikakos, P. W. Bates and X. Chen, Convergence of the Cahn-Hilliard Equation to the Hele-Shaw Model. *Arch. Rat. Mech. Anal.* **128** (1994), 165–205.
- [2] N. D. Alikakos, G. Fusco and G. Karali, Ostwald ripening in two dimensions- The rigorous derivation of the equations from Mullins-Sekerka dynamics, *Journ. of Differential Equations* **205** no. 1, (2004), 1–49.
- [3] T. Antal, M. Droz, J. Magnin and Z. Rácz, Formation of Liesegang Patterns: A Spinodal Decomposition Scenarion. *Phys. Rev. Lett.* **83** (1999), 2880–2883.
- [4] G. Belletini, M.S. Gelli, S. Luckhaus, M. Novaga, Deterministic equivalent for the Allen Cahn energy of a scaling law in the Ising model. *Calc. Var.* **26** (2006), 429–445.
- [5] P. W. Bates, J. Xun, Metastable patterns for the Cahn-Hilliard Equation, Part I. *Journ. of Differential Equations* **111** (1994), 421–457.
- [6] P. W. Bates, J. Xun, Metastable patterns for the Cahn-Hilliard Equation, Part II. *Journ. of Differential Equations* **117** (1995), 165–216.
- [7] D. Blömker, S. Maier-Paape, T. Wanner, *Phase separation in stochastic Cahn-Hilliard models*, Mathematical methods and models in phase transitions, edited by A. Miranville, 1–41, Nova Science Publishers, 2005.
- [8] J. W. Cahn, J. E. Hilliard, Free Energy of a Nonuniform System. I. Interfacial Free Energy. *J. Chem. Phys.* **28** (1958), 258–267.
- [9] J. W. Cahn, J. E. Hilliard, Free Energy of a Nonuniform System II, Thermodynamic basis. *J. Chem. Phys.* **30** (1959), 1121–1124.
- [10] G. Cagninalp, P.C. Fife, Dynamics of layered interfaces arising from phase boundaries. *SIAM J. Appl. Math.* **48** (1988), no. 3, 506–518.
- [11] H. Cook, Brownian motion in spinodal decomposition. *Acta Metallurgica*, **18** (1970), 297–306.
- [12] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge University Press, Cambridge, 1992.
- [13] G. Da Prato, A. Debussche, Stochastic Cahn-Hilliard equation. *Nonlinear Anal.* **26** (1996), 241–263.
- [14] A. Debussche, L. Zambotti, Conservative stochastic Cahn-Hilliard equation with reflection. *Ann. Probab.* **35** (2007), 1706–1739.
- [15] Q. Du, T. Zhang, Numerical approximation of some linear stochastic partial differential equations driven by special additive noises. *SIAM J. Num. Anal.* **40** (2002), 1421–1445.
- [16] C. M. Elliott, S. Zheng, On the Cahn-Hilliard equation. *Arch. Rat. Mech. Anal.* **96** (1986), 339–357.
- [17] N. Elezovic, N, A. Mikelic, On the stochastic CahnHilliard equation. *Nonlinear Anal.* **16** (1991), 1169–1200.
- [18] X. Chen, X. Hong and F. Yi, Existence, uniqueness and regularity of solutions of Mullins-Sekerka problem. *Comm. Partial Differential Equations* **21** (1996), 1705–1727.
- [19] P. C. Fife, Models for phase separation and their mathematics. *El. Journ. Diff. E.* **48** (2000), 1–26.
- [20] T. Funaki, The scaling limit for a Stochastic PDE and the separation of phases. *Probab. Theory Relat. Fields.* **102** (1995), 221–288.
- [21] T. Funaki, Singular Limit for Stochastic Reaction-Diffusion Equation and Generation of Random Interfaces. *Acta Math. Sinica, English Series* **15** (1999), 407–438.
- [22] T. Funaki, *Singular limit for reaction-diffusion equation with self-similar Gaussian noise*, In Proceedings of Taniguchi symposium ‘New Trends in Stochastic Analysis’ (eds. Elworthy, Kusuoka and Shigekawa), World Sci. (2000), 132–152.

- [23] M. E. Gurtin, Generalized Ginzburg-Landau and Cahn-Hilliard equations based on a micro-force balance. *Physica D* **92** (1996), 178–192.
- [24] M. Hildebrand and A. S. Mikhailov, Mesoscopic modeling in the kinetic theory of adsorbates. *J. Phys. Chem.* **100**, (1996), 19089–19101.
- [25] P. C. Hohenberg and B.I. Halperin, Theory of dynamic critical phenomena. *J. Rev. Mod. Phys.* **49** (1977), 435–479.
- [26] G. Kallianpur and J. Xiong, *Stochastic Differential Equations in Infinite Dimensional Spaces*, Institute of Mathematical Statistics, Lecture Notes-Monograph Series vol. 26, Hayward, California, 1995.
- [27] G. Karali, Phase boundaries motion preserving the volume of each connected component, *Asymptotic Analysis* **49**, no 1, (2006), 17–37.
- [28] G. Karali, M.A. Katsoulakis, The role of multiple microscopic mechanisms in cluster interface evolution. *J. Differential Equations* **235** (2007), 418–438.
- [29] M.A. Katsoulakis and D.G. Vlachos, From microscopic interactions to macroscopic laws of cluster evolution. *Phys. Rev. Letters* **84** (2000), 1511–1514.
- [30] K. Kitahara, Y. Oono and D. Jasnow, Phase separation dynamics and external force field. *Mod. Phys. Letters B* **2** (1988), 765–771.
- [31] M. Katsoulakis, G. Kossioris and O. Lakkis, Noise regularization and computations for the 1-dimensional stochastic Allen-Cahn problem *Interfaces Free Bound.* **9** (2007), no. 1, 1–30.
- [32] G. Kossioris and G. Zouraris, Fully-discrete finite element approximations for a fourth-order linear stochastic parabolic equation with additive space-time white noise, *ESAIM: Mathematical Modelling and Numerical Analysis*, in press.
- [33] L.D. Landau and E.M. Lifshitz, Course of Theoretical Physics vol. 5, *Statistical Physics Part I*, Pergamon, 3rd Ed. (1994).
- [34] J. S. Langer, Theory of Spinodal Decomposition in Alloys. *Ann. of Phys.* **65** (1971), 53–86.
- [35] Di Liu, Convergence of the spectral method for stochastic Ginzburg-Landau equation driven by space-time white noise. *Comm. Math. Sci.* **1** (2003), 361–375.
- [36] B. Øksendal, *Stochastic Differential Equations*, Springer, 2003, New York.
- [37] R. L. Pego, Front migration in the non-linear Cahn-Hilliard equation. *Proc. R. Soc. Lond. A* **422** (1989), 261–278.
- [38] J. Printems, On the discretization in time of parabolic stochastic partial differential equations. *Mathematical Modelling and Numerical Analysis*, **35** (2001), 1055–1078.
- [39] T. M. Rogers, K. R. Elder, and R. C. Desai, Numerical study of the late stages of spinodal decomposition. *Phys. Rev. B* **37** (1988), 9638–9649.
- [40] Y. Yan, Semidiscrete Galerkin approximation for a linear stochastic parabolic partial differential equation driven by an additive noise. *BIT Numerical Mathematics*, **44** (2004), 829–847, 2004.
- [41] Y. Yan, Galerkin Finite Element Methods for Stochastic Parabolic Partial Differential Equations, *SIAM J. Numer. Anal.*, **43** (2005), 1363–1384.
- [42] J. B. Walsh, *An introduction to stochastic partial differential equations*, 984, volume 1180 of Lecture Notes in Math. **984**, pp. 265-439 Springer, Berlin, 1986.
- [43] Quan-Fang Wang, Shin-ichi Nakagiri, Weak solutions of Cahn-Hilliard equations having forcing terms and optimal control problems. *Mathematical models in functional equations (Kyoto, 1999)* Sūrikaiseikikenkyūsho Kōkyūroku **1128** (2000), 172–180.

E-mail address: danton@tem.uoc.gr

E-mail address: gkarali@tem.uoc.gr

E-mail address: kossioris@math.uoc.gr