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On the convergence of a fourth order evolution equation to the Allen–Cahn equation

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ABSTRACT

We construct special sequences of solutions to a fourth order nonlinear parabolic equation of Cahn–Hilliard/Allen–Cahn type, converging to the second order Allen–Cahn equation. We consider the evolution equation without boundary, as well as the stationary case on domains with Dirichlet boundary conditions. The proofs exploit the equivalence of the fourth order equation with a system of two second order elliptic equations with "good signs".

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1. Introduction

This article is concerned with the mathematical study of the following mean field partial differential equation which was recently derived and studied in [1–3]:

$$\begin{cases} u_t = -\epsilon^2 D\Delta\left(\Delta u + \dfrac{f(u)}{\epsilon^2}\right) + \Delta u + \dfrac{f(u)}{\epsilon^2} \\ u(0,x) = u_0(x), \end{cases} \tag{1.1}$$

where $f(u) = -W'(u)$, W is a double-well potential with wells ± 1, $D > 0$ is the diffusion constant and $0 < \epsilon \ll 1$ is a small parameter. A typical choice for W is $W(u) = (u^2-1)^2/4$, in which case we have $f(u) = u - u^3$.

From the physical point of view, Eq. (1.1) is associated with the effect of multiple microscopic mechanisms such as surface diffusion and adsorption/desorption which are typically involved in surface processes, on macroscopic cluster interface morphology and evolution. Typically surface processes take place simultaneously and interact. For instance we can consider a combination of Arrhenius adsorption/desorption dynamics, Metropolis surface diffusion and simple unimolecular reaction; the corresponding mesoscopic equation is:

$$u_t - D\nabla\cdot[\nabla u - \beta u(1-u)\nabla J * u] - [k_a p(1-u) - k_d u \exp(-\beta J * u)] + k_r u = 0\,. \tag{1.2}$$

Here D is the diffusion constant, k_r, k_d and k_a denote, respectively the reaction, desorption and adsorption constants and p is the partial pressure of the gaseous species. The partial pressure p is assumed to be a constant, although realistically it is

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given by the fluids equations in the gas phase. However there are no general rigorous results available on the existence of travelling waves for (1.2); some numerical simulations were carried out in [1] indicating the existence of non-monotone travelling waves. Results on existence, uniqueness and stability concerning non local evolution equations can be found in [4–6]. Our model (1.1) in discussion can be obtained from rescalings of (1.2) close to the critical temperature and retains all its fundamental structure.

We note that Eq. (1.1) may be viewed as a combination of the well-known Cahn–Hilliard (CH) equation

$$u_t = -\epsilon^2 \Delta\left(\Delta u + \frac{f(u)}{\epsilon^2}\right), \qquad u(0,x) = u_0(x)$$

and of the Allen–Cahn (AC) equation

$$u_t = \Delta u + \frac{f(u)}{\epsilon^2}, \qquad u(0,x) = u_0(x). \tag{1.3}$$

We recall that the CH model can describe surface diffusion including particle/particle interactions, while the AC describes a simplified model of adsorption to and desorption from the surface. It is worth mentioning that in the model described by (1.1) the mobility is completely different from the one of the AC equation. This implies in particular that the diffusion speeds up the mean curvature flow, [2]. It is well known that the AC and CH equations can serve as diffuse interface models for limiting sharp interface motion. The AC equation serves as a diffuse interface model for antiphase grain boundary coarsening in the sense that the singular limit of the equation yields a geometric problem in which a sharp interface separating two phase variants evolves according to motion by mean curvature ($V = k$), [7–9]. On the other hand, the CH equation was constructed to describe mass conservative phase separation. By considering an appropriate singular limit ($\epsilon \to 0$) it can describe the motion of interphase boundaries separating two phases of differing composition during the later stages of coarsening.

Our aim in this article is to construct suitable sequences (u_n) of solutions to (1.1), such that u_n converges to a solution u of the second order AC equation. The interest of such sequences lies in the fact that, since their limits satisfy a second order equation, certain properties typical to solutions of second order equations, such as the maximum property or some of its consequences, may be extended to u_n for large values of n, despite the fact that u_n satisfies a fourth order equation. It should be mentioned that, unexpectedly, an analogous situation occurs in the quite different context of Maxwell–Chern–Simons (MCS) vortices [10], which was actually the main motivation of our analysis. Indeed, MCS vortices are generally described by a nonlinear fourth order elliptic equation. The fourth order term corresponds to the Maxwell term. When neglecting the Maxwell term, the equation reduces to a second order elliptic equation describing "pure" Chern–Simons multivortices, see, e.g., the monograph [11]. The existence of sequences of MCS vortices converging to a Chern–Simons vortex is a key issue in [10]. It is used to obtain multiplicity of solutions to the MCS equation, by adopting to the corresponding fourth order equation an argument from [12] for second order equations, involving super/subsolutions. On the other hand, the existence of special solutions to (1.1) which are "close" to solutions of the second order AC equation opens the interesting possibility of extending to the fourth order equation some of the many existing powerful techniques developed for the second order AC equation, such as those in [8,13,14]; see also [15].

A relevant mathematical feature of (1.1) which we will use throughout this paper is that it may be formulated as a system of two second order equations with "good signs". Namely, setting $v = \Delta u + f(u)$ in (1.1), we see that (1.1) is equivalent to the following system of second order equations:

$$\begin{cases} u_t = -\epsilon^2 D\Delta v + v \\ v = \Delta u + \epsilon^{-2} f(u) \\ u(0,x) = u_0(x). \end{cases} \tag{1.4}$$

This paper is organized as follows. In Section 2 we first construct solutions to (1.1) in the case where f is a general nonlinearity satisfying $\|f\|_{C^2} < +\infty$, for any fixed value of $D > 0$, see Theorem 2.1. To this end, we use a Galerkin approximation following some ideas from [16]. Then, in Theorem 2.2 we show that in the case where $D \to 0$, the corresponding solutions converge to a solution for the AC equation. In Section 3 we obtain an energy estimate for the physically significant case where $f(u) = u - u^3$. We show that, consequently, in the one-dimensional case the Galerkin approximation yields solutions to (1.1) with $f(u) = u - u^3$, see Theorem 3.1. Moreover, as $D \to 0$, such solutions converge to an AC solution, see Theorem 3.2. In Section 4 we consider the stationary problem under Dirichlet boundary conditions, namely:

$$-\epsilon^2 D\Delta\left(\Delta u + \frac{f(u)}{\epsilon^2}\right) + \Delta u + \frac{f(u)}{\epsilon^2} = 0 \quad \text{in } \Omega, \qquad u = \varphi \quad \text{on } \partial\Omega, \tag{1.5}$$

where $\Omega \subset \mathbb{R}^d$ is a smooth bounded domain and φ is a bounded function. In Theorem 4.1 we will show that for any *fixed* value of D a suitable sequence of solutions u_n to (1.5) may be constructed, such that u_n converges to a solution u to the stationary AC equation

$$\Delta u + \frac{f(u)}{\epsilon^2} = 0 \quad \text{in } \Omega, \qquad u = \varphi \quad \text{on } \partial\Omega.$$

2. The evolution case: A Galerkin approximation

In this section we assume that Ω is without boundary and $f \in C^2(\mathbb{R})$ is a general nonlinearity such that $\|f\|_{C^2} < \infty$. We set $\Omega_T = \Omega \times (0,T)$. Our aim in this section is to prove the global existence of solutions to (1.1) which tend to an AC solution as $D \to 0$ by a Galerkin approximation. By the rescaling $t = \epsilon^2 t'$, $x = \epsilon x'$, we may assume that $\epsilon = 1$. In this case, Eq. (1.1) takes the form

$$\begin{cases} u_t = -D\Delta(\Delta u + f(u)) + \Delta u + f(u) \\ u(0,x) = u_0(x), \end{cases} \tag{2.6}$$

and (1.4) takes the form

$$\begin{cases} u_t = -D\Delta v + v \\ v = \Delta u + f(u). \end{cases} \tag{2.7}$$

We first prove an existence result.

Theorem 2.1. *Let $T > 0$, $\|f\|_{C^2} < \infty$ and suppose that $u_0 \in H^1(\Omega)$. There exists a pair of functions (u,v) such that $u \in L^\infty(0,T;H^1(\Omega)) \cap C([0,T];H^\lambda)$, $\lambda < 1$, $v \in L^2(0,T;H^1(\Omega))$, $u_t \in L^2(0,T;H^{-1}(\Omega))$, $u(0) = u_0$ in $L^2(\Omega)$, and (u,v) satisfies (1.4) in the following weak sense:*

$$\begin{cases} \iint_{\Omega_T} v\varphi = -\iint_{\Omega_T} \nabla u \nabla\varphi + \iint_{\Omega_T} f(u)\varphi \\ \iint_{\Omega_T} u_t\varphi = D\iint_{\Omega_T} \nabla v\nabla\varphi + \iint_{\Omega_T} v\varphi \end{cases} \tag{2.8}$$

for all $\varphi \in L^2(0,T;H^1(\Omega))$.

Now we show that the solutions obtained in Theorem 2.1 converge to an AC solution as $D \to 0$, in the following sense.

Theorem 2.2. *Let $D_n \to 0$ and denote by (u_n, v_n) the solutions to (2.8) as obtained in Theorem 2.1. Then, there exists a solution u to the AC equation (1.3) such that $u_n \to u$ weakly-* in $L^\infty(0,T;H^1(\Omega))$.*

In order to define the Galerkin approximation, let ψ_i, $i \in \mathbb{N}$ denote the eigenfunction of $-\Delta$ on Ω corresponding to the eigenvalue λ_i, namely

$$-\Delta\psi_i = \lambda_i\psi_i \quad \text{in } \Omega.$$

We assume the normalization condition $\int_\Omega \psi_i\psi_j = \delta_{ij}$ for $0 = \lambda_1 < \lambda_2 \le \cdots$. For every $N \in \mathbb{N}$ we consider the pair of functions (u^N, v^N) defined by the Galerkin ansatz

$$u^N(x,t) = \sum_{i=1}^N a_i^N(t)\psi_i(x), \qquad v^N(x,t) = \sum_{i=1}^N b_i^N(t)\psi_i(x), \tag{2.9}$$

and subject to the following conditions related to (2.7):

$$\begin{cases} \int_\Omega u_t^N\psi_j = -D\int_\Omega \Delta v^N\psi_j + \int_\Omega v^N\psi_j, & j = 1,2,\dots,N \\ \int_\Omega v^N\psi_j = \int_\Omega \Delta u^N\psi_j + \int_\Omega f(u^N)\psi_j, & j = 1,2,\dots,N \\ \int_\Omega u^N(x,0)\psi_j = \int_\Omega u_0\psi_j, & j = 1,2,\dots,N. \end{cases} \tag{2.10}$$

System (2.10) yields the following initial value problem for $a_j^N(t)$, $j = 1,2,\dots,N$:

$$\begin{cases} \dfrac{\mathrm{d}a_j^N(t)}{\mathrm{d}t} = (D\lambda_j + 1)\left[-\lambda_j a_j^N(t) + \int_\Omega f(u^N)\psi_j\right] \\ a_j^N(0) = \int_\Omega u_0\psi_j, \end{cases} \tag{2.11}$$

while b_j^N is determined by a_j^N, $j = 1,2,\dots,N$, by the equation

$$b_j^N(t) = -\lambda_j a_j^N(t) + \int_\Omega f(u^N)\psi_j. \tag{2.12}$$

By standard arguments, it is readily seen that problem (2.11) has a local solution. We want to show that a global solution $(a_j^N)_{j=1,2,\dots,N}$ exists on $(0,T)$ for any $T > 0$. Namely, we have the following.

Proposition 2.1. *Let $T > 0$. For every $N \in \mathbb{N}$ there exists a solution $(a_j^N, b_j^N)_{j=1,2,\dots,N}$ to (2.11)–(2.12) globally defined on $(0, T)$.*

We begin by obtaining some estimates for u^N.

Lemma 2.1. *Let u^N be defined by (2.9)–(2.11). Then, the following identity holds:*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla u^N|^2 + D\int_\Omega |\nabla \Delta u^N|^2 + \int_\Omega (\Delta u^N)^2 = D\int_\Omega f(u^N)\Delta^2 u^N - \int_\Omega f(u^N)\Delta u^N. \tag{2.13}$$

In particular, we have the following estimates:

(i) $\sup_{t\in(0,T)} \int_\Omega |\nabla u^N|^2 \le \mathrm{e}^{2C_0T}\int_\Omega |\nabla u_0|^2$;

(ii) $D\int_0^T\int_\Omega |\nabla\Delta u^N|^2 + 2\int_0^T\int_\Omega (\Delta u^N)^2 \le (2C_0T\mathrm{e}^{2C_0T}+1)\int_\Omega |\nabla u_0|^2$,

where $C_0 = \|f\|_{C_1}(1+\frac{D}{2}\|f\|_{C_1})$.

We collect in the following lemma some well-known identities which will be used repeatedly.

Lemma 2.2. *The following identities hold, for all $N \in \mathbb{N}$, $i = 1, 2, \dots, N$:*

(i) $$\int_\Omega |\nabla\psi_i|^2 = \lambda_i;$$

(ii) $$\int_\Omega |\nabla u^N|^2 = \sum_{i=1}^N \lambda_i (a_i^N)^2;$$

(iii) $$\int_\Omega (\Delta u^N)^2 = \sum_{i=1}^N \lambda_i^2 (a_i^N)^2;$$

(iv) $$\int_\Omega |\nabla\Delta u^N|^2 = \sum_{i=1}^N \lambda_i^3 (a_i^N)^2;$$

(v) $$\sum_{i=1}^N \lambda_i a_i^N\psi_i = -\Delta u^N, \qquad \sum_{i=1}^N \lambda_i^2 a_i^N\psi_i = \Delta^2 u^N.$$

Proof. (i) We readily have that

$$\int_\Omega |\nabla\psi_i|^2 = -\int_\Omega \psi_i\Delta\psi_i = \lambda_i\int_\Omega \psi_i^2 = \lambda_i.$$

(ii) Using the orthogonality conditions on ψ_i and (i), we have

$$\int_\Omega |\nabla u^N|^2 = \int_\Omega \left|\sum_{i=1}^N a_i^N\nabla\psi_i\right|^2 = \sum_{i=1}^N (a_i^N)^2\int_\Omega |\nabla\psi_i|^2 = \sum_{i=1}^N \lambda_i(a_i^N)^2.$$

(iii) Similarly as above, we have

$$\int_\Omega (\Delta u^N)^2 = \int_\Omega \left(\sum_{i=1}^N a_i^N\Delta\psi_i\right)^2 = \int_\Omega \sum_{i=1}^N \lambda_i^2(a_i^N)^2\psi_i^2 = \sum_{i=1}^N \lambda_i^2(a_i^N)^2.$$

(iv) We note that

$$\nabla\Delta u^N = \sum_{i=1}^N a_i^N\nabla\Delta\psi_i = -\sum_{i=1}^N \lambda_i a_i^N\nabla\psi_i.$$

Therefore, recalling (i) and the orthogonality conditions we obtain

$$\int_\Omega |\nabla\Delta u^N|^2 = \int_\Omega \left|-\sum_{i=1}^N \lambda_i a_i^N\nabla\psi_i\right|^2 = \sum_{i=1}^N \int_\Omega \lambda_i^2(a_i^N)^2|\nabla\psi_i|^2 = \sum_{i=1}^N \lambda_i^3(a_i^N)^2.$$

(v) We have

$$\sum_{i=1}^N \lambda_i a_i^N\psi_i = \sum_{i=1}^N a_i^N(-\Delta\psi_i) = -\Delta\sum_{i=1}^N a_i^N\psi_i = -\Delta u^N.$$

Note that $\Delta^2\psi_i = -\lambda_i \Delta\psi_i = \lambda_i^2\psi_i$. Therefore,

$$\sum_{i=1}^{N}(\lambda_i)^2 a_i^N \psi_i = \sum_{i=1}^{N} a_i^N \Delta^2\psi_i = \Delta^2\left(\sum_{i=1}^{N} a_i^N\psi_i\right) = \Delta^2 u^N. \quad \square$$

Proof of Lemma 2.1. Multiplying (2.11) by $-\lambda_j a_j^N(t)$ and adding over $j = 1, 2, \ldots, N$, we have

$$-\sum_{j=1}^{N}\lambda_j \frac{\mathrm{d}a_j^N}{\mathrm{d}t}a_j^N = \sum_{j=1}^{N}(D\lambda_j+1)\lambda_j^2(a_j^N)^2 - \sum_{j=1}^{N}\lambda_j a_j^N(D\lambda_j+1)\int_\Omega f(u^N)\psi_j.$$

We have in view of Lemma 2.2-(ii) that

$$-\sum_{j=1}^{N}\lambda_j \frac{\mathrm{d}a_j^N}{\mathrm{d}t}a_j^N = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{j=1}^{N}\lambda_j(a_j^N)^2 = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla u^N|^2.$$

By making use of Lemma 2.2-(iii)–(iv) we have:

$$\sum_{j=1}^{N}(D\lambda_j+1)\lambda_j^2(a_j^N)^2 = D\sum_{j=1}^{N}\lambda_j^3(a_j^N)^2 + \sum_{j=1}^{N}\lambda_j^2(a_j^N)^2 = D\int_\Omega |\nabla\Delta u^N|^2 + \int_\Omega (\Delta u^N)^2.$$

Also by Lemma 2.2-(v) we have:

$$\begin{aligned}\sum_{j=1}^{N}\lambda_j a_j^N(D\lambda_j+1)\int_\Omega f(u^N)\psi_j &= D\sum_{j=1}^{N}\lambda_j^2 a_j^N\int_\Omega f(u^N)\psi_j + \sum_{j=1}^{N}\lambda_j a_j^N\int_\Omega f(u^N)\psi_j \\ &= D\int_\Omega \Delta^2 u^N f(u^N) - \int_\Omega f(u^N)\Delta u^N.\end{aligned}$$

Hence, we obtain that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla u^N|^2 + D\int_\Omega |\nabla\Delta u^N|^2 + \int_\Omega (\Delta u^N)^2 = D\int_\Omega f(u^N)\Delta^2 u^N - \int_\Omega f(u^N)\Delta u^N$$

and (2.13) is established.

In order to obtain the asserted estimates (i)–(ii) we use a Gronwall argument. Integrating by parts, we may write:

$$\int_\Omega f(u^N)\Delta^2 u^N = -\int_\Omega f'(u^N)\nabla u^N\cdot\nabla\Delta u^N$$

and

$$\int_\Omega f(u^N)\Delta u^N = -\int_\Omega f'(u^N)|\nabla u^N|^2.$$

Hence, for any $m \neq 0$ we have:

$$\left|\int_\Omega f(u^N)\Delta^2 u^N\right| \le \|f\|_{C^1}\left[\frac{m^2}{2}\int_\Omega |\nabla u^N|^2 + \frac{1}{2m^2}\int_\Omega |\nabla\Delta u^N|^2\right]$$

and consequently we derive from (2.13) that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla u^N|^2 + D\int_\Omega |\nabla\Delta u^N|^2 + \int_\Omega (\Delta u^N)^2 \le D\|f\|_{C^1}\frac{m^2}{2}\int_\Omega |\nabla u^N|^2 + D\frac{\|f\|_{C^1}}{2m^2}\int_\Omega |\nabla\Delta u^N|^2 + \|f\|_{C^1}\int_\Omega |\nabla u^N|^2.$$

Choosing $m^2 = \|f\|_{C^1}$, we derive

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla u^N|^2 + \frac{D}{2}\int_\Omega |\nabla\Delta u^N|^2 + \int_\Omega (\Delta u^N)^2 \le \|f\|_{C^1}\left(\frac{D}{2}\|f\|_{C^1}+1\right)\int_\Omega |\nabla u^N|^2. \tag{2.14}$$

At this point a standard Gronwall argument yields, for every $N \in \mathbb{N}$:

$$\sup_{t\in(0,T)}\int_\Omega |\nabla u^N(x,t)|^2 \le \mathrm{e}^{2C_0T}\int_\Omega |\nabla u^N(x,0)|^2.$$

We claim that

$$\int_\Omega |\nabla u^N(x,0)|^2 \le \int_\Omega |\nabla u_0|^2.$$

Indeed, in view of Lemma 2.2 and the definition of $a_i^N(0)$ in (2.11), we have:

$$\int_\Omega |\nabla u^N(x,0)|^2 = \sum_{i=1}^N \lambda_i (a_i^N(0))^2 = \sum_{i=1}^N \lambda_i \left(\int_\Omega u_0\psi_i\right)^2 \le \sum_{i=1}^\infty \lambda_i \left(\int_\Omega u_0\psi_i\right)^2 = \int_\Omega |\nabla u_0|^2.$$

Hence, estimate (i) is established. Integrating (2.14) we derive (ii). □

Proof of Proposition 2.1. Now we observe that, since $\lambda_1 = 0$ and $\psi_1 = |\Omega|^{-1/2}$, the initial value problem (2.11) for $a_1^N(t)$ takes the form

$$\dot a_1^N(t) = |\Omega|^{-1/2}\int_\Omega f(u^N), \qquad a_1^N(0) = |\Omega|^{-1/2}\int_\Omega u_0.$$

In particular,

$$|\dot a_1^N(t)| \le \|f\|_{C^0}|\Omega|^{1/2} =: C_1$$

and we derive that

$$|a_1^N(t)| \le |\Omega|^{-1/2}\left|\int_\Omega u_0\right| + C_1 T$$

for all $t \in (0,T)$. On the other hand, since $\int_\Omega u^N = a_1^N(t)|\Omega|^{1/2}$, we have that

$$\left|\int_\Omega u^N\right| = |a_1^N(t)|\,|\Omega|^{1/2} \le |\Omega|^{1/2}\left(|\Omega|^{-1/2}\left|\int_\Omega u_0\right| + C_1 T\right).$$

In view of Lemma 2.1-(i) and the Poincaré inequality, we conclude that

$$\sup_{t\in(0,T)} \|u^N\|_{H^1(\Omega)} \le C_2 e^{2C_0 T},$$

for some $C_2 > 0$ independent of N. In view of Lemma 2.2-(ii), we conclude in particular that $\|a_j^N\|_{L^\infty(0,T)} \le C_3 e^{2C_0T}$. Consequently, $a_j^N(t)$ exists globally in $(0,T)$. In turn, in view of (2.12), $b_j^N(t)$ also exists globally in $(0,T)$. □

In order to prove Theorem 2.1 we need the following estimates for v^N and u_t^N.

Lemma 2.3. *Suppose that $\|f\|_{C^2} < +\infty$. Let (u^N, v^N) be defined by (2.9)–(2.11)–(2.12). Then, the following estimates hold:*
(i) $\int_0^T\int_\Omega |\nabla v^N|^2 + \int_0^T\int_\Omega (v^N)^2 \le C$
(ii) $\|u_t^N\|_{L^2(0,T;H^{-1}(\Omega))} \le C$
where $C = C(T)$ does not depend on N.

Proof. Throughout this proof, we denote by $C = C(T)$ a general constant independent of N, whose actual value may vary from line to line. We recall that $v^N = \sum_{j=1}^N b_j^N(t)\psi_j(x)$, where $b_j^N, j = 1,2,\ldots,N$ is defined by

$$b_j^N = -\lambda_j a_j^N + \int_\Omega f(u^N)\psi_j.$$

Moreover, by similar arguments as in Lemma 2.2, we have

$$\int_\Omega (v^N)^2 = \sum_{j=1}^N (b_j^N)^2, \qquad \int_\Omega |\nabla v^N|^2 = \sum_{j=1}^N \lambda_j (b_j^N)^2, \qquad \int_\Omega v_N = b_1^N(t) = |\Omega|^{-1/2}\int_\Omega f(u^N).$$

Therefore, we may write

$$\int_\Omega |\nabla v^N|^2 = -\sum_{j=1}^N \lambda_j^2 a_j^N b_j^N + \sum_{j=1}^N \lambda_j b_j^N \int_\Omega f(u^N)\psi_j.$$

In view of Lemma 2.1, we estimate:

$$\begin{aligned}\int_0^T \left|\sum_{j=1}^N \lambda_j^2 a_j^N b_j^N\right| &\le \int_0^T \left(\sum_{j=1}^N \lambda_j (b_j^N)^2\right)^{1/2}\left(\sum_{j=1}^N \lambda_j^3 (a_j^N)^2\right)^{1/2}\\ &\le \left(\int_0^T \sum_{j=1}^N \lambda_j (b_j^N)^2\right)^{1/2}\left(\int_0^T \sum_{j=1}^N \lambda_j^3 (a_j^N)^2\right)^{1/2}\\ &= \left(\int_0^T\int_\Omega |\nabla v^N|^2\right)^{1/2}\left(\int_0^T\int_\Omega |\nabla\Delta u^N|^2\right)^{1/2} \le C\left(\int_0^T\int_\Omega |\nabla v^N|^2\right)^{1/2}.\end{aligned}$$

Similarly, we have

$$\left|\sum_{j=1}^{N}\lambda_j b_j\int_\Omega f(u^N)\psi_j\right|\le\left(\sum_{j=1}^{N}\lambda_j (b_j^N)^2\right)^{1/2}\left(\sum_{j=1}^{N}\lambda_j\left(\int_\Omega f(u^N)\psi_j\right)^2\right)^{1/2}.$$

We note that $|(\int_\Omega f(u^N)\psi_j)|\le C$ and therefore we may estimate

$$\sum_{j=1}^{N}\lambda_j\left(\int_\Omega f(u^N)\psi_j\right)^2\le C\sum_{j=1}^{N}\lambda_j\left|\int_\Omega f(u^N)\psi_j\right|.$$

Integration by parts yields

$$\lambda_j\int_\Omega f(u^N)\psi_j=-\int_\Omega f(u^N)\Delta\psi_j=-\int_\Omega f''(u^N)|\nabla u^N|^2\psi_j-\int_\Omega f'(u^N)\Delta u^N\psi_j.$$

Consequently, recalling Lemma 2.1,

$$\left|\int_0^T\lambda_j\int_\Omega f(u^N)\psi_j\right|\le\|f\|_{C^2}\left(\int_0^T\int_\Omega|\nabla u^N|^2+\int_0^T\int_\Omega|\Delta u^N|\right)\le C\|f\|_{C^2}.$$

It follows that

$$\left|\int_0^T\sum_{j=1}^{N}\lambda_j b_j^N\int_\Omega f(u^N)\psi_j\right|\le C\int_0^T\left(\sum_{j=1}^{N}\lambda_j(b_j^N)^2\right)^{1/2}\le C\left(\int_0^T\sum_{j=1}^{N}\lambda_j(b_j^N)^2\right)^{1/2}$$
$$=C\left(\int_0^T\int_\Omega|\nabla v^N|^2\right)^{1/2}.$$

We have obtained that

$$\int_0^T\int_\Omega|\nabla v^N|^2\le C\left(1+\left(\int_0^T\int_\Omega|\nabla v^N|^2\right)^{1/2}\right)$$

and hence $\int_0^T\int_\Omega|\nabla v^N|^2\le C$. Now we observe that since $\lambda_1=0$ we have

$$\left|\int_\Omega v^N\right|=\left|b_1\int_\Omega\psi_1\right|=\left|\int_\Omega f(u^N)\psi_1\right|\left|\int_\Omega\psi_1\right|\le|\Omega|\|f\|_{L^\infty}.$$

Hence, we may estimate

$$\int_\Omega(v^N)^2=\sum_{j=1}^{N}(b_j^N)^2\le(b_1^N)^2+\sum_{j=1}^{N}\lambda_j(b_j^N)^2.$$

It follows that

$$\int_0^T\int_\Omega(v^N)^2\le C\left(1+\int_0^T\int_\Omega|\nabla v^N|^2\right)\le C$$

and hence (i) is established.

In order to prove (ii), we denote by $\Pi_N:L^2(\Omega)\to\mathrm{span}\{\psi_1,\psi_2,\dots,\psi_N\}$ the projection operator. Let $\psi\in L^2(0,T;H^1(\Omega))$. Then, we have:

$$\int_0^T\int_\Omega u_t^N\psi=D\int_0^T\int_\Omega\nabla v^N\cdot\nabla(\Pi_N\psi)+\int_0^T\int_\Omega v^N\Pi_N\psi.$$

Therefore, in view of Lemma 2.1, we conclude that

$$\left|\int_0^T\int_\Omega u_t^N\psi\right|\le C(T)\|\psi\|_{L^2(0,T;H^1(\Omega))}.$$

Hence, (ii) is also established. □

Proof of Theorem 2.1. In view of the estimates in Lemmas 2.1 and 2.3, the proof of Theorem 2.1 readily follows by standard compactness results, as may be found, e.g., in [17,18]. □

Proof of Theorem 2.2. For $0<D_n\le 1$, the estimates in Lemmas 2.1 and 2.3 imply that $\sup_{t\in(0,T)}\int_\Omega|\nabla u_n|^2\le C$, $\int_0^T\int_\Omega(\Delta u_n)^2\le C$, $\int_0^T\int_\Omega|\nabla v_n|^2+\int_0^T\int_\Omega v_n^2\le C$, $\|u_t\|_{L^2(0,T;H^{-1}(\Omega))}\le C$, where $C>0$ depends on f and u_0 only. Now the claim follows by standard compactness results. □

3. An energy estimate and the one-dimensional case

In this section we assume that f has the specific, physically relevant form $f(u) = u - u^3$. Our aim is to derive some global estimates for this case, that ensure global existence for (1.1) in the one-dimensional case. Namely, we shall prove the following.

Theorem 3.1. *Suppose that $\Omega = \mathbb{R}/\mathbb{Z}$, $f(u) = u - u^3$ and $0 < D \le 1$. Then, for every $T > 0$ there exists a solution to (1.1) on $\Omega \times (0, T)$.*

Similarly as in Section 2, we consequently derive:

Theorem 3.2. *Suppose that $\Omega = \mathbb{R}/\mathbb{Z}$, $f(u) = u - u^3$, $0 < D_n \le 1$ and $D_n \to 0$. Let u_n be the solution to (1.1) with $D = D_n$ as obtained in Theorem 3.1. Then, there exists a solution to the AC equation (1.3) such that, up to subsequences, u_n converges to u.*

We begin by proving the following estimates, which hold in arbitrary dimension d.

Proposition 3.1. *Suppose that $f(u) = u - u^3$, $0 < D \le 1$ and suppose that $u(x, t)$ satisfies (2.6). Then, the following estimate holds:*

$$\sup_{t\in(0,T)} \left[\int_\Omega |\nabla u|^2 + \frac{1}{2}\int_\Omega u^4\right] \le \int_\Omega |\nabla u_0|^2 + \frac{1}{2}\int_\Omega u_0^4 + C(T)\int_\Omega u_0^2. \tag{3.15}$$

In order to establish Proposition 3.1 we first prove an L^2-estimate.

Lemma 3.1. *Suppose that $0 < D \le 1$ and suppose that $u(x, t)$ satisfies (2.6). Then, there exists $C = C(T) > 0$ such that*

$$\sup_{t\in(0,T)} \int_\Omega u^2(x, t) \le C(T)\int_\Omega u_0^2.$$

Proof. Multiplying (2.6) by u and integrating over Ω, we have

$$\begin{aligned}\int_\Omega uu_t &= -D\int_\Omega u\Delta(\Delta u + u - u^3) + \int_\Omega u(\Delta u + u - u^3)\\ &= -D\int_\Omega \Delta u(\Delta u + u - u^3) + \int_\Omega u(\Delta u + u - u^3)\\ &= -D\int_\Omega (\Delta u)^2 + (1-D)\int_\Omega u\Delta u + D\int_\Omega u^3\Delta u + \int_\Omega u^2 - \int_\Omega u^4.\end{aligned}$$

Integration by parts yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^2 = -D\int_\Omega (\Delta u)^2 - (1-D)\int_\Omega |\nabla u|^2 - 3D\int_\Omega u^2|\nabla u|^2 + \int_\Omega u^2 - \int_\Omega u^4.$$

In particular, since $1 - D \ge 0$, we obtain that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^2 \le \int_\Omega u^2.$$

Now the asserted estimate follows by a Gronwall argument. □

Proof of Proposition 3.1. Setting $v = \Delta u + u - u^3$, we derive the equivalent system

$$\begin{cases} u_t = -D\Delta v + v\\ v = \Delta u + u - u^3\\ u(x, 0) = u_0(x).\end{cases} \tag{3.16}$$

Multiplying the first equation in (3.16) by v and integrating, we have:

$$\int_\Omega vu_t = D\int_\Omega |\nabla v|^2 + \int_\Omega v^2. \tag{3.17}$$

Multiplying the second equation in (3.16) by u_t and integrating, we have:

$$\int_\Omega vu_t = \int_\Omega u_t\Delta u + \int_\Omega u_t(u - u^3) = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_\Omega |\nabla u|^2 - \int_\Omega u^2 + \frac{1}{2}\int_\Omega u^4\right]. \tag{3.18}$$

From (3.17)–(3.18) we derive:

$$-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_\Omega |\nabla u|^2 - \int_\Omega u^2 + \frac{1}{2}\int_\Omega u^4\right] = D\int_\Omega |\nabla v|^2 + \int_\Omega v^2 \ge 0.$$

It follows that

$$\int_\Omega |\nabla u|^2 - \int_\Omega u^2 + \frac{1}{2}\int_\Omega u^4 \le \int_\Omega |\nabla u_0|^2 - \int_\Omega u_0^2 + \frac{1}{2}\int_\Omega u_0^4$$

and, consequently:

$$\int_\Omega |\nabla u|^2 + \frac{1}{2}\int_\Omega u^4 \le \int_\Omega |\nabla u_0|^2 + \frac{1}{2}\int_\Omega u_0^4 + \int_\Omega u^2.$$

Now the asserted estimate follows in view of Lemma 3.1. □

Now we can prove Theorem 3.1.

Proof of Theorem 3.1. Proposition 3.1 together with Lemma 3.1 and the Sobolev embedding implies that any solution to (1.1) satisfies the estimate:

$$\sup_{t\in(0,T)} \|u(x,t)\|_{L^\infty(\Omega)} \le C(T, u_0).$$

Therefore, the nonlinearity $f(u) = u - u^3$ may be truncated. Now existence follows by Theorem 2.1. □

Proof of Theorem 3.2. The proof is analogous to that of Theorem 2.2. □

4. The stationary case

It is readily seen that under doubly periodic or Neumann boundary conditions there is an "order reduction", in the sense that the stationary solutions to the fourth order CH/AC equation are exactly the stationary solutions to the second order AC equation obtained by taking $D = 0$. Indeed, stationary solutions to (1.4) satisfy

$$\begin{cases} -\epsilon^2 D\Delta v + v = 0 \\ v = \Delta u + \epsilon^{-2} f(u). \end{cases}$$

Multiplying by v and integrating, under periodic or Neumann boundary conditions we have that $\epsilon^2 D\int_\Omega |\nabla v|^2 + \int_\Omega v^2 = 0$. It follows that $v = 0$ and u satisfies the stationary AC equation $\Delta u + \epsilon^{-2} f(u) = 0$.

Therefore, in this section we focus on Dirichlet boundary conditions. Let $\Omega \subset \mathbb{R}^d$ be a smooth bounded domain. We consider the Dirichlet problem

$$\begin{cases} -\epsilon^2 D\Delta\left(\Delta u + \dfrac{f(u)}{\epsilon^2}\right) + \Delta u + \dfrac{f(u)}{\epsilon^2} = 0 & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{4.19}$$

where φ is a smooth bounded function. By setting $v = \Delta u + \frac{f(u)}{\epsilon^2}$, we are led to consider the system

$$\begin{cases} -\epsilon^2 D\Delta v + v = 0 & \text{in } \Omega \\ -\Delta u = -v + \dfrac{f(u)}{\epsilon^2} & \text{in } \Omega \\ u = \varphi, \quad v = \psi & \text{on } \partial\Omega, \end{cases} \tag{4.20}$$

where φ, ψ are smooth bounded functions. We assume that f is a continuous function satisfying

$$\lim_{t\to+\infty} f = -\infty, \qquad \lim_{t\to-\infty} f = +\infty. \tag{4.21}$$

The main result in this section is that solutions to (4.19) may be constructed in such a way that they are arbitrarily close to a solution of an AC equation. More precisely, we have the following.

Theorem 4.1. *Suppose that f satisfies (4.21). For any fixed $\epsilon, D > 0$ there exists a sequence of solutions $(u_n)_{n\in\mathbb{N}}$ to the stationary CH/AC equation (4.19) and a solution u to the stationary AC equation*

$$\begin{cases} \Delta u + \dfrac{f(u)}{\epsilon^2} = 0 & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases} \tag{4.22}$$

such that $u_n \to u$ in $C^{1,\beta}(\overline{\Omega})$.

In order to prove Theorem 4.1 we need some lemmas. We begin by obtaining some L^∞ bounds for u, v. We note that such bounds are independent of $D > 0$, and that the bound for v is independent of $\epsilon > 0$.

Lemma 4.1. *Let (u, v) be a solution to system (4.20). Then, $\|v\|_\infty \le \|\psi\|_\infty$ and there exists a constant $C(\epsilon, \|\psi\|_\infty) > 0$ such that $\|u\|_\infty \le \max\{\|\varphi\|_\infty, C(\epsilon, \|\psi\|_\infty)\}$.*

Proof. Let $\bar{y} \in \Omega : v(\bar{y}) = \max_{\bar{\Omega}} v$. Then $-\Delta v(\bar{y}) \ge 0$ which implies

$$0 = -\epsilon^2 D\Delta v(\bar{y}) + v(\bar{y}) \ge v(\bar{y}).$$

Hence, v cannot attain a positive interior maximum and therefore $v \le \|\psi\|_\infty$. Similarly, let $\underline{y} \in \Omega$: $v(\underline{y}) = \min_{\bar{\Omega}} v$. Then $-\Delta v(\underline{y}) \le 0$ which implies

$$0 = -\epsilon^2 D\Delta v(\underline{y}) + v(\bar{y}) \le v(\underline{y}).$$

That is, v cannot attain a negative interior minimum and $v \ge -\|\psi\|_\infty$. Hence, the estimate for v is established.

Now we derive the estimate for u. Similarly as before, suppose that $\bar{x} \in \Omega$ is such that $\max_{\overline{\Omega}} u = u(\bar{x})$. Then, $0 \le -\Delta u(\bar{x}) = -v(\bar{x}) + \epsilon^{-2} f(u(\bar{x}))$ and therefore, $\epsilon^{-2} f(u(\bar{x})) \ge v(\bar{x}) \ge -\|\psi\|_\infty$. In view of (4.21), it follows that there exists $C_1(\epsilon, \|\psi\|_\infty) > 0$ such that $u(\bar{x}) \le C_1(\epsilon, \|\psi\|_\infty)$. By an analogous procedure, if $\underline{x} \in \Omega$ is such that $u(\underline{x}) = \min_{\overline{\Omega}} u$, then $u(\underline{x}) \ge -C_2(\epsilon, \|\psi\|_\infty)$. The proof of the asserted estimate for u now follows. □

In the following lemma we prove the existence of at least one solution to (4.20) for any given smooth boundary data φ, ψ.

Lemma 4.2. *For any fixed $\epsilon, D > 0$ and for any sufficiently smooth boundary data φ, ψ, there exists a solution (u, v) to system (4.20).*

Proof. By standard elliptic theory, there exists a unique solution v to the problem

$$\begin{cases} -\epsilon^2 D\Delta v + v = 0 & \text{in } \Omega \\ v = \psi & \text{on } \partial\Omega. \end{cases} \tag{4.23}$$

Let $W(u) = -\int_0^u f(t)\mathrm{d}t$. Then, we equivalently need to solve

$$\begin{cases} -\Delta u = -v - \dfrac{W'(u)}{\epsilon^2} & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

where v is the unique solution for (4.23). We identify φ with an extension to Ω as an $H^1(\Omega)$ function and we set $w = u - \varphi$. Then, w satisfies

$$\begin{cases} -\Delta w = -v - \dfrac{W'(w+1)}{\epsilon^2} + \Delta\varphi & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

Solutions to the problem above correspond to critical points in $H_0^1(\Omega)$ for the Modica–Mortola type functional:

$$I(w) := \int_\Omega \left\{ \frac{\epsilon}{2}|\nabla w|^2 + \frac{W(w+\varphi)}{\epsilon} + \epsilon(v - \Delta\varphi)w \right\}.$$

In view of (4.21) and the definition of W, there exists $\overline{W}$ such that $W(t) \ge -\overline{W}$ for all $t \in \mathbf{R}$. Hence, it is readily seen that

$$I(w) \ge \frac{\epsilon}{2}\int_\Omega |\nabla w|^2 - \frac{\overline{W}|\Omega|}{\epsilon} - \epsilon\|v - \Delta\varphi\|_2\|w\|_2 \ge a\epsilon\|\nabla w\|^2 - \frac{C}{\epsilon}$$

for some $a, C > 0$. Therefore, I is bounded below and coercive. Hence, I admits a global minimum corresponding to a solution for (4.20). □

Finally, we can prove our main result for the stationary case.

Proof of Theorem 4.1. We recall that (4.19) is equivalent to the system

$$\begin{cases} v = \Delta u + \dfrac{f(u)}{\epsilon^2} & \text{in } \Omega \\ -\epsilon^2 D\Delta v + v = 0 & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega. \end{cases} \tag{4.24}$$

In view of Lemma 4.2, there exist solutions (u_n, v_n) to the problem

$$\begin{cases} -\Delta u_n = -v_n + \dfrac{f(u_n)}{\epsilon^2} & \text{in } \Omega \\ -\epsilon^2 D\Delta v_n + v_n = 0 & \text{in } \Omega \\ u_n = \varphi, \quad v_n = \dfrac{1}{n} & \text{on } \partial\Omega. \end{cases} \tag{4.25}$$

Clearly, for every n, (u_n, v_n) satisfies in particular system (4.24). By Lemma 4.1 and elliptic regularity, $v_n \to 0$ in C^k $\forall k \ge 0$. Moreover, we have $\|u_n\|_\infty \le C(\epsilon, \|\varphi\|)$. Consequently, from the first equation in (4.25) we obtain that $\|\Delta u_n\|_\infty \le C(\epsilon, \|\varphi\|, |\Omega|)$. By the Calderón–Zygmund theorem, we have $\|D^2 u_n\|_p \le C_p(\epsilon, \|\varphi\|, |\Omega|)$ for all $p \ge 1$. In turn, Morrey's embeddings (see, e.g., Theorem 7.26, p. 171 in [19]) imply that (u_n) is compact in $C^{1,\beta}(\overline{\Omega})$ for $0 < \beta < 2 - d/p$, where p is sufficiently large. Therefore, we may assume that $u_n \to u$ in $C^{1,\beta}(\overline{\Omega})$. We are left to check that u is a solution to the

AC equation (4.22). To this end, we multiply the first equation in (4.25) by a smooth function ρ and integrate by parts. We obtain:

$$\int_{\Omega} \nabla u_n \cdot \nabla \rho - \int_{\partial\Omega} \rho \frac{\partial u_n}{\partial \nu} = -\int_{\Omega} v_n u_n + \int_{\Omega} \frac{f(u_n)}{\epsilon^2} \rho.$$

Taking limits, we obtain that

$$\int_{\Omega} \nabla u \cdot \nabla \rho - \int_{\partial\Omega} \rho \frac{\partial u}{\partial \nu} = \int_{\Omega} \frac{f(u_n)}{\epsilon^2} \rho,$$

and therefore u satisfies (4.22). □

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References

[1] M. Hildebrand, A.S. Mikhailov, Mesoscopic modeling in the kinetic theory of adsorbates, J. Phys. Chem. 100 (1996) 19089–19101.
[2] G. Karali, M. Katsoulakis, The role of multiple microscopic mechanisms in cluster interface evolution, J. Differential Equations 235 (2007) 418–438.
[3] M.A. Katsoulakis, D.G. Vlachos, From microscopic interactions to macroscopic laws of cluster evolution, Phys. Rev. Lett. 84 (2000) 1511–1514.
[4] P.W. Bates, F. Chen, Spectral analysis of traveling waves for nonlocal evolution equations, SIAM J. Math. Anal. 38 (1) (2006) 116–126.
[5] P.W. Bates, J. Han, G. Zhao, On a nonlocal phase-field system, J. Nonlinear Anal. 64 (16) (2006) 2251–2278.
[6] P.W. Bates, G. Zhao, Existence, uniqueness and stability of the stationary solution to a nonlocal evolution equation arising in population dispersal, J. Math. Anal. Appl. 332 (2007) 428–440.
[7] S. Allen, J.W. Cahn, A microscopic theory for antiphase boundary motion and its application to antiphase domain coarsening, Acta Metall. 27 (1979) 1084–1095.
[8] L.C. Evans, H.M. Soner, P. Souganidis, Phase transitions and generalized mean curvature flow equations, Comm. Pure Appl. Math. 45 (1992) 1097–1123.
[9] J. Rubinstein, P. Sternberg, J. Keller, Fast reaction, slow diffusion and curve shortening, SIAM J. Appl. Math. 49 (1989) 116–133.
[10] T. Ricciardi, Multiplicity for a nonlinear fourth-order elliptic equation in Maxwell–Chern–Simons vortex theory, Differential Integral Eqs. 17 (3–4) (2004) 369–390.
[11] Y. Yang, Solitons in Field Theory and Nonlinear Analysis, in: Springer Monographs in Mathematics, Springer-Verlag, New York, 2001.
[12] H. Brezis, L. Nirenberg, H^1 versus C^1 local minimizers, C. R. Acad. Sci. Paris, Série I 317 (1993) 465–472.
[13] X. Chen, Generation and propagation of interface in reaction–diffusion equations, J. Differential Equations 96 (1992) 116–141.
[14] T. Ilmanen, Convergence of the Allen–Cahn equation to Brakke's motion by mean curvature, J. Differential Geom. 38 (2) (1993) 417–461.
[15] N.D. Alikakos, G. Fusco, The spectrum of the Cahn–Hilliard operator for generic interface in higher space dimensions, Indiana Univ. Math. J. 42 (1993) 637–674.
[16] R. Dal Passo, L. Giacomelli, A. Novick-Cohen, Existence for an Allen–Cahn/Cahn–Hilliard system with degenerate mobility, Interfaces Free Bound. 1 (2) (1999) 199–226.
[17] J.L. Lions, Quelques méthodes de résolution des problèmes aux limites non linéaires, Dunod, Paris, 1969.
[18] J. Simon, Compact sets in the space $L^p(0, T; B)$, Annali Mat. Pura Appl. 146 (1987) 65–97.
[19] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer-Verlag, Berlin Heidelberg, 2001.