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The Effect of the Geometry of the Particle Distribution in Ostwald Ripening

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Abstract: Based on [1], we derive equations for the radii and the centers that we relate to the Lifshitz-Slyozov-Wagner theory.

Introduction

In this note, based on the estimates in [1], we derive rigorously corrections to the equations of the Lifshitz-Slyozov-Wagner theory of coarsening (cf. (1.2) in [1]). Specifically we correct the equations for the radii by taking into account the distance and the size of the neighboring particles. We also provide equations for the motion of the centers of the particles. These corrections amount to carrying out (rigorously) the asymptotic expansion to a higher order. Because all this can be achieved by elaborating the main estimates in [1], for keeping the size of the present paper under control, we did not opt for a self contained exposition; instead we took the liberty of referring to the various formulae in [1] with a minimum of explanation.

We establish the following refinement of the main theorem in [1]:

Theorem 1. *Let* $\Omega \subset \mathbb{R}^3$ *be a bounded, smooth and connected domain. Assume that* $\Gamma^0 = \cup_{i=1}^N \Gamma_i^0,\ N \geq 2$, *where* Γ_i^0 *satisfies* $\Gamma_i = \{x \backslash x = \xi_i + \epsilon\rho_i(1+\epsilon r_i(u))u,\ u \in S^2\}$ *with* $\xi_{i0} \in \Omega$, $\rho_{i0} > 0$, *and* $r_{i0} \in C^{3+\alpha}(S^2)$ *satisfying* $\int_{S^2} r_i(u)du = 0$, $\int_{S^2} r_i(u)\langle u, e_j\rangle du = 0,\ j = 1, 2, 3$. *Assume* $\xi_{i0} \neq \xi_{j0}$ *for* $i \neq j$, $\rho_{10} < \ldots < \rho_{N0}$. *There is* $\bar{\epsilon} > 0$ *such that, if* $0 < \epsilon < \bar{\epsilon}$, *then the solution* $t \to \Gamma(t)$ *of the Mullins-Sekerka problem for this class of initial conditions* Γ^0 *can be represented in* ξ, ρ *and* r *coordinates and exists globally as a weak solution. There exist constants* $C_\rho, C_\xi > 0$

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such that

$$\begin{aligned}
\epsilon\dot{\rho}_i &= \frac{1}{\epsilon\rho_i}\left\{\left(\frac{1}{\epsilon\bar{\rho}}-\frac{1}{\epsilon\rho_i}\right)+\frac{1}{N\epsilon\bar{\rho}}\sum_{h,k}{}' \frac{\epsilon\rho_h}{|\xi_h-\xi_k|}\left(\frac{\epsilon\rho_k}{\epsilon\bar{\rho}}-1\right)\right.\\
&\quad -\sum_j{}' \frac{1}{|\xi_j-\xi_i|}\left(\frac{\epsilon\rho_j}{\epsilon\bar{\rho}}-1\right)+\frac{1}{N}\sum_{i,h}4\pi\,\frac{\epsilon\rho_i}{\epsilon\bar{\rho}}\,\gamma(\xi_i,\xi_h)\left(\frac{\epsilon\rho_h}{\epsilon\bar{\rho}}-1\right)\\
&\quad \left.-\sum_h \gamma(\xi_i,\xi_h)4\pi\left(\frac{\epsilon\rho_h}{\epsilon\bar{\rho}}-1\right)+\frac{g_i}{\epsilon\rho_i}\right\},\ \textit{with } |g_i|<\epsilon^2 C_\rho,\\
\dot{\xi}_i &= -3\sum_k{}' \left(\frac{1}{\bar{\rho}}-\frac{1}{\rho_k}\right)\rho_k\,\frac{\xi_k-\xi_i}{|\xi_k-\xi_i|^3}\\
&\quad -3\sum_h \rho_h\,\frac{\partial\gamma(\xi_i,\xi_h)}{\partial x}\left(\frac{1}{\bar{\rho}}-\frac{1}{\rho_h}\right)+\phi_i,\quad \textit{with } |\phi_i|<\epsilon^3 C_\xi, \qquad (1)
\end{aligned}$$

where $\overline{\rho}=\frac{1}{N-i+1}\sum_{h=1}^{N}\rho_h$ *and* $\|r_i(t)\|_{C^{3+\alpha}(S^2)}<C_r$ *as long as* r_i *is defined.*

The symbol $\sum'$ means summation avoiding equal indices. Here γ is the smooth part of the Green's function (cf. (3.8)). Notice that ρ, and ξ form a closed system of equations if the highest order terms are ignored. Equations (1) are derived formally in [2] for the case when the boundary $\partial\Omega$ is further away ($\gamma=0$). Special cases of (1) have appeared in the literature before. The ρ-equations with $\gamma=0$ are derived in [5]. The ξ-equations for 2 particles were derived in [4] by the method of images (that does not extend to more particles). Also in two dimensions and for two particles a simpler analog of system (1) can be found in [3]. It is worth mentioning that Eqs. (1) provide a correction to the classical Lifshitz-Slyozov-Wagner theory of coarsening by taking into account the size, the distance of the neighboring particles and the effect of the boundary. If we consider the ρ equations, we observe that the main term is $\frac{1}{\epsilon\rho_i}(\frac{1}{\epsilon\bar{\rho}}-\frac{1}{\epsilon\rho_i})$ and the rest of the terms are smaller in relation to the main term which is like $\frac{1}{\epsilon}$. Moreover, the centers do move but in general slower than the radii.

In what follows we denote by V the normal velocity which is taken positive for a shrinking sphere, H the mean curvature, $\overline{H}=\frac{1}{|\Gamma|}\int_\Gamma H dS_y$ the average mean curvature, $|\Gamma|$ the surface area, $W_i=H(x)-E$ with $E=\overline{H}-\frac{1}{|\Gamma|}\int_\Gamma\int_\Gamma g(x,y)V(y)dydx$ and T the Dirichlet-Neumann operator (cf Sect. 6 in [1]). We take $\Gamma=\cup_{i=1}^N\Gamma_i$ with $\Gamma_i=\{x\backslash x=X^i(u):=\xi_i+\epsilon\rho_i(1+\epsilon r_i(u))u, u\in S^2\}$. If $\epsilon>0$ is small, the map $X^i: S^2\to\Gamma_i$ is a diffeomorphism with the same regularity as r_i. We let $u^i:\Gamma\to S^2$ be the inverse of X^i. Under the above assumption Eq. (3.3) in [1] is written in the form

$$\sum_{h=1}^{N}\int_{\Gamma_h} g(x,y)V_h(u^h(y))dy=H(x)-E,\qquad x\in\Gamma_i, i=1,\ldots,N.$$

We begin by presenting a refinement of Proposition 8.1 in [1] from which the proof of Theorem 1 follows.

Proposition 1. *Let $\xi_i \in \Omega$, $\rho_i > 0$, $r_i \in C^{1+\alpha}(S^2)$, $W_i \in C^{1+\alpha}(S^2)$, $i = 1, \dots, N$, be given and assume that $\xi_i \neq \xi_j$ for $i \neq j$. Then the system*

$$\sum_{h=1}^{N} \int_{\Gamma_h} g(x,y) V_h(u^h(y))dy = W_i(u^i(x)), x \in \Gamma_i, i = 1, \dots, N \tag{2}$$

has a unique solution $V_i \in C^\alpha(S^2)$. Moreover the following estimate holds true:

$$\begin{aligned}
&\Bigg\| \epsilon\rho_i V_i - T_0 W_i + \epsilon T_0 \int_{S^2} \frac{\frac{3}{2}\, r_i(v) - \frac{1}{2}\, r_i(\cdot)}{4\pi|\cdot - v|} (T_0 W_i)(v)dv + \sum_h \epsilon\rho_h \gamma(\xi_i, \xi_h) \int_{S^2} T_0 W_h \\
&+ {\sum_{h,i}}' \frac{\epsilon\rho_h}{4\pi|\xi_i - \xi_h|} \int_{S^2} T_0 W_h - \epsilon^2 T_0 \int_{S^2} \frac{r_i(\cdot) + r_i(v)}{4\pi|\cdot - v|}\, r_i(v)(T_0 W_i)(v)dv \\
&+ {\sum_h}' \frac{\epsilon^2 \rho_h^2}{|\xi_i - \xi_h|^2} T_0 \int_{S^2} \left\langle v, \frac{\xi_i - \xi_h}{|\xi_i - \xi_h|} \right\rangle (T_0 W_h)(v) \\
&- {\sum_h}' \frac{\epsilon^2 \rho_h \rho_i}{|\xi_h - \xi_i|^2}\, 3 \left\langle u, \frac{\xi_i - \xi_h}{|\xi_i - \xi_h|} \right\rangle \int_{S^2} T_0 W_h + \sum_h 2\epsilon^2 \rho_h r_h \gamma(\xi_i, \xi_h) \int_{S^2} T_0 W_h \\
&+ \sum_h \epsilon^2 \rho_h^2\, 3 \left\langle u, \frac{\partial \gamma(\xi_i, \xi_h)}{\partial x} \right\rangle \int_{S^2} T_0 W_h + \sum_h \epsilon^2 \rho_h^2 \int_{S^2} \left\langle v, \frac{\partial \gamma(\xi_i, \xi_h)}{\partial y} \right\rangle (T_0 W_h) \Bigg\|_{C^\alpha(S^2)} \\
&\quad \leq C \Bigg\{ \sum_h \Bigg[\sum_k \Bigg(\sum_\mu \left(\frac{\epsilon\rho_\mu}{|\xi_k - \xi_\mu|} + \frac{\epsilon\rho_\mu}{l} + \epsilon \|r_i\|_{C^{1+\alpha}(S^2)} + \epsilon \|r_k\|_{C^{1+\alpha}(S^2)} \right) \\
&\qquad \left(\frac{\epsilon\rho_k}{|\xi_h - \xi_k|} + \frac{\epsilon\rho_k}{l} + \epsilon \|r_i\|_{C^{1+\alpha}(S^2)} + \epsilon \|r_h\|_{C^{1+\alpha}(S^2)} \right) \Bigg] \\
&\qquad \left(\frac{\epsilon\rho_h}{|\xi_i - \xi_h|} + \frac{\epsilon\rho_h}{l} \right) \|T_0 W_i\|_{C^\alpha(S^2)} + \epsilon^3 \|r_i\|^3_{C^{1+\alpha}(S^2)} \|T_0 W_i\|_{C^\alpha(S^2)} \Bigg\}.
\end{aligned} \tag{3}$$

Proof. We have that

$$\int_{\Gamma_h} g(x,y) V_h(u^h(y))dy = \int_{\Gamma_h} \frac{1}{4\pi|x-y|}\, V_h(u^h(y))dy + \int_{\Gamma_h} \gamma(x,y) V_h(u^h(y))dy$$

and arguing as in [1] we obtain

$$\begin{aligned}
(JV_i)(u) = &\int_{S^2} \frac{1}{4\pi|u-v|} \left(\frac{|u-v| - |u - v + \epsilon(r_i(u)u - r_i(v)v)|}{\epsilon|u - v + \epsilon(r_i(u)u - r_i(v)v)|} \right) V_i(v)dv \\
&+ \int_{S^2} \frac{1}{4\pi|u-v|} \left(\frac{|u-v| - |u - v + \epsilon(r_i(u)u - r_i(v)v)|}{\epsilon|u - v + \epsilon(r_i(u)u - r_i(v)v)|} \right) 2\epsilon r_i V_i(v)dv \\
&+ \int_{S^2} \frac{1}{4\pi|u-v|} \left(\frac{|u-v| - |u - v + \epsilon(r_i(u)u - r_i(v)v)|}{\epsilon|u - v + \epsilon(r_i(u)u - r_i(v)v)|} \right) \\
&O\left(\epsilon^2 \|r_i\|^2_{C^{1+\alpha}(S^2)}\right) V_i(v)dv \\
= &(I_1 V_i)(u) + (I_2 V_i)(u) + (I_3 V_i)(u).
\end{aligned} \tag{4}$$

From (8.16), (8.17), (8.18) in [1] we obtain

$$\|I_3 V_i\|_{C^{1+\alpha}(S^2)} \leq C\epsilon^2 \|r_i\|^3_{C^{1+\alpha}(S^2)} \|V_i\|_{C^{1+\alpha}(S^2)}. \tag{5}$$

So,

$$
\begin{aligned}
JV_i &= -\frac{1}{2}\int_{S^2} \frac{r_i(\cdot)+r_i(v)}{4\pi|\cdot - v|}\, V_i(v)dv - \epsilon\int_{S^2}\frac{r_i(\cdot)+r_i(v)}{4\pi|\cdot - v|}\, r_i(v)V_i(v)dv \\
&\quad + \epsilon^2 O\left(\|r_i\|^3_{C^{1+\alpha}(S^2)}\|V_i\|_{C^\alpha(S^2)}\right), \qquad (6)
\end{aligned}
$$

$$
\begin{aligned}
\gamma(X^i(u),\epsilon\rho_i z) &= \gamma(\xi_i,\xi_i) + \epsilon\rho_i\left\langle u, \frac{\partial\gamma(\xi_i,\xi_i)}{\partial x}\right\rangle + \epsilon\rho_i\left\langle v, \frac{\partial\gamma(\xi_i,\xi_i)}{\partial y}\right\rangle \\
&\quad + O\left(\epsilon^2\left(\rho_i^2\frac{1}{l^3} + \rho_i\|r_i\|_{C^{1+\alpha}(S^2)}\frac{1}{l^2}\right)\right). \qquad (7)
\end{aligned}
$$

From the above, it follows that

$$
\begin{aligned}
I_2^i V_i &= \gamma(\xi_i,\xi_i)\int_{S^2} V_i + 2\epsilon r_i\gamma(\xi_i,\xi_i)\int_{S^2} V_i \\
&\quad + \epsilon\rho_i\left\langle u, \frac{\partial\gamma(\xi_i,\xi_i)}{\partial x}\right\rangle\int_{S^2} V_i + \epsilon\rho_i\int_{S^2}\left\langle v, \frac{\partial\gamma(\xi_i,\xi_i)}{\partial y}\right\rangle V_i \\
&\quad + \epsilon\rho_i O\left(\frac{\epsilon^2\|r_i\|^2_{C^{1+\alpha}(S^2)}}{l} + \frac{\epsilon^2\rho_i\|r_i\|_{C^{1+\alpha}(S^2)}}{l^2}\right. \\
&\quad \left. + \epsilon^2\left(\rho_i^2\frac{1}{l^3} + \rho_i\|r_i\|_{C^{1+\alpha}(S^2)}\frac{1}{l^2}\right)\|V_i\|_{C^\alpha(S^2)}\right). \qquad (8)
\end{aligned}
$$

From the analysis above it follows that

$$
\int_{\Gamma_i} g(X^i(u),y)V_i(u^i(y))dy = \epsilon\rho_i\int_{S^2}\frac{1}{4\pi|u-v|}V_i(v)dv + \epsilon\rho_i(I^{ii}V_i)(v), \qquad (9)
$$

where I^{ii} is a linear operator that satisfies

$$
\|I^{ii}V_i\|_{C^{1+\alpha}(S^2)} \le C\|V_i\|_{C^\alpha(S^2)}, \qquad (10)
$$

where

$$
\begin{aligned}
C &= \frac{\epsilon}{2}\int_{S^2}\frac{3r_i(v) - r_i(\cdot)}{4\pi|\cdot - v|}dv - \epsilon^2\int_{S^2}\frac{r_i(\cdot)+r_i(v)}{4\pi|\cdot - v|}\, r_i(v)dv + \epsilon^3 O\left(\|r_i\|^3_{C^{1+\alpha}(S^2)}\right) \\
&\quad + \epsilon\rho_i\gamma(\xi_i,\xi_i) + 2\epsilon^2 r_i\rho_i\gamma(\xi_i,\xi_i) + \epsilon^2\rho_i^2\left\langle u, \frac{\partial\gamma(\xi_i,\xi_i)}{\partial x}\right\rangle \\
&\quad + O\left(\frac{\epsilon^3\rho_i\|r_i\|^2_{C^{1+\alpha}(S^2)}}{l} + \frac{\epsilon^3\rho_i^2\|r_i\|_{C^{1+\alpha}(S^2)}}{l^2} + \epsilon^3\rho_i\left(\rho_i^2\frac{1}{l^3} + \rho_i\|r_i\|_{C^{1+\alpha}(S^2)}\frac{1}{l^2}\right)\right),
\end{aligned}
$$

$$
\begin{aligned}
&\int_{\Gamma_h}\frac{1}{4\pi|X^i(u)-y|}\, V_h(u^h(y))dy \\
&\quad = \frac{\epsilon^2\rho_h^2}{4\pi|\xi_i-\xi_h|}\int_{S^2} V_h + \frac{2\epsilon^3\rho_h^2 r_h}{|\xi_i-\xi_h|}\int_{S^2} V_h - \frac{\epsilon^3\rho_h^2}{|\xi_i-\xi_h|^2}\left\langle\rho_i u, \frac{\xi_i-\xi_h}{|\xi_i-\xi_h|}\right\rangle\int_{S^2} V_h \\
&\qquad + \frac{\epsilon^3\rho_h^2}{|\xi_i-\xi_h|^2}\int_{S^2}\left\langle\rho_h v, \frac{\xi_i-\xi_h}{|\xi_i-\xi_h|}\right\rangle V_h(v)dv \\
&\qquad + \epsilon^2\rho_h^2 O\left(\frac{\epsilon^2\|r_h\|^2_{C^{1+\alpha}(S^2)}}{|\xi_i-\xi_h|} + \frac{\epsilon^2\|r_h\|_{C^{1+\alpha}(S^2)}}{|\xi_i-\xi_h|^2}\,(\rho_i+\rho_h)\right) \\
&\qquad + \epsilon^2\left(\frac{\rho_i^2+\rho_h^2}{|\xi_i-\xi_h|^3} + \frac{\rho_i\|r_i\|_{C^{1+\alpha}(S^2)} + \rho_h\|r_h\|_{C^{1+\alpha}(S^2)}}{|\xi_i-\xi_h|^2}\right)\|V_h\|_{C^\alpha(S^2)}. \qquad (11)
\end{aligned}
$$

From the definition of X^h, $h = 1, \ldots, N$, that implies

$$\gamma(X^i(u), X^h(v)) = \gamma(\xi_i, \xi_h) + \epsilon\rho_i \left\langle u, \frac{\partial\gamma(\xi_i, \xi_h)}{\partial x} \right\rangle + \epsilon\rho_h \left\langle v, \frac{\partial\gamma(\xi_i, \xi_h)}{\partial y} \right\rangle$$
$$+2\epsilon r_h \gamma(\xi_i, \xi_h) + O\left(\frac{\epsilon^2 \|r_h\|^2_{C^{1+\alpha}(S^2)}}{l} + \epsilon^2 \left(\rho_i^2 + \rho_h^2\right) \frac{1}{l^3} \right.$$
$$\left. +\epsilon^2 (\rho_i \|r_i\|_{C^{1+\alpha}(S^2)} + \rho_h \|r_h\|_{C^{1+\alpha}(S^2)}) \frac{1}{l^2} \right).$$

It follows that

$$\int_{\Gamma_h} \gamma(X^i(u), y) V_h(u^h(h)) dy$$
$$= \epsilon^2 \rho_h^2 \gamma(\xi_i, \xi_h) \int_{S^2} V_h + \epsilon^3 \rho_i \rho_h^2 \left\langle u, \frac{\partial\gamma(\xi_i, \xi_h)}{\partial x} \right\rangle \int_{S^2} V_h$$
$$+ \epsilon^3 \rho_h^3 \int_{S^2} \left\langle v, \frac{\partial\gamma(\xi_i, \xi_h)}{\partial y} \right\rangle V_h + 2\epsilon^3 \rho_h^2 r_h \gamma(\xi_i, \xi_h) \int_{S^2} V_h$$
$$+ \epsilon^2 \rho_h^2 O\left(\epsilon^2 \left(\rho_i^2 + \rho_h^2\right) \frac{1}{l^3} + \epsilon^2 \left(\rho_i \|r_i\|_{C^{1+\alpha}(S^2)} + \rho_h \|r_h\|_{C^{1+\alpha}(S^2)} \right.\right.$$
$$\left.\left. + \rho_i \|r_h\|_{C^{1+\alpha}(S^2)}\right) \frac{1}{l^2} + \epsilon^2 \|r_h\|^2_{C^{1+\alpha}(S^2)} \frac{1}{l} \right) \|V_h\|_{C^\alpha(S^2)}. \quad (12)$$

Also we have the estimate

$$\|I^{ih} V_h\|_{C^{1+\alpha}(S^2)} \le C \|V_h\|_{C^\alpha(S^2)}, \quad h \neq i, \quad (13)$$

where

$$C = \frac{\epsilon\rho_h}{|\xi_i - \xi_h|} + \frac{2\epsilon^2 \rho_h r_h}{|\xi_i - \xi_h|} - \frac{\epsilon^2 \rho_h \rho_i}{|\xi_i - \xi_h|^2} \left\langle u, \frac{\xi_i - \xi_h}{|\xi_i - \xi_h|} \right\rangle$$
$$+ \frac{\epsilon^2 \rho_h^2}{|\xi_i - \xi_h|^2} + \epsilon\rho_h \gamma(\xi_i, \xi_h) + 2\epsilon^2 \rho_h r_h \gamma(\xi_i, \xi_h)$$
$$+ \epsilon^2 \rho_i \rho_h \left\langle u, \frac{\partial\gamma(\xi_i, \xi_h)}{\partial x} \right\rangle + \epsilon^2 \rho_h^2 \left\langle v, \frac{\partial\gamma(\xi_i, \xi_h)}{\partial y} \right\rangle$$
$$+ \epsilon\rho_h O\left(\frac{\epsilon^2 \|r_h\|^2_{C^{1+\alpha}(S^2)}}{|\xi_i - \xi_h|} + \frac{\epsilon^2 \|r_h\|_{C^{1+\alpha}(S^2)}}{|\xi_i - \xi_h|^2} (\rho_i + \rho_h) \right.$$
$$+ \epsilon^2 \left(\frac{\rho_i^2 + \rho_h^2}{|\xi_i - \xi_h|^3} + \frac{\rho_i \|r_i\|_{C^{1+\alpha}(S^2)} + \rho_h \|r_h\|_{C^{1+\alpha}(S^2)}}{|\xi_i - \xi_h|^2} \right)$$
$$+ \epsilon^2 (\rho_i^2 + \rho_h^2) \frac{1}{l^3} + \epsilon^2 (\rho_i \|r_i\|_{C^{1+\alpha}(S^2)} + \rho_h \|r_h\|_{C^{1+\alpha}(S^2)} + \rho_i \|r_h\|_{C^{1+\alpha}(S^2)}) \frac{1}{l^2}$$
$$\left. + \epsilon^2 \|r_h\|^2_{C^{1+\alpha}(S^2)} \frac{1}{l} \right).$$

System, (2) is equivalent to

$$\epsilon\rho_i V_i = T_0 W_i - \sum_{h=1}^{N} \epsilon\rho_h T_0 I^{ih} V_h. \quad (14)$$

From (12), (14) it follows that

$$\left\|\sum_{h=1}^{N}\epsilon\rho_h T_0 I^{ih} V_h\right\|_{C^\alpha(S^2)}$$
$$\leq C\left\{\sum_h\left(\sum_k\left(\frac{\epsilon\rho_k}{|\xi_h-\xi_k|}+\frac{\epsilon\rho_k}{l}\right)\right)\left(\frac{\epsilon\rho_h}{|\xi_i-\xi_h|}+\frac{\epsilon\rho_h}{l}\right)\epsilon\rho_h\|V_h\|_{C^\alpha(S^2)}\right.$$
$$\left.+\epsilon^3\|r_i\|^2_{C^{1+\alpha}(S^2)}\rho_i\|V_i\|_{C^\alpha(S^2)}\right\}. \tag{15}$$

Thus, we have the estimates

$$\left\|I^{ii}V_i-\frac{\epsilon}{2}\int_{S^2}\frac{3r_i(v)-r_i(\cdot)}{4\pi|\cdot-v|}\,V_i(v)dv-\epsilon\rho_i\gamma(\xi_i,\xi_i)\int_{S^2}V_i(v)dv\right.$$
$$+\epsilon^2\int_{S^2}\frac{r_i(\cdot)+r_i(v)}{4\pi|\cdot-v|}\,r_i(v)V_i(v)dv-2\epsilon^2 r_i\rho_i\gamma(\xi_i,\xi_i)\int_{S^2}V_i(v)dv$$
$$\left.-\epsilon^2\rho_i^2\left\langle u,\frac{\partial\gamma(\xi_i,\xi_i)}{\partial x}\right\rangle\int_{S^2}V_i(v)dv-\epsilon^2\rho_i^2\int_{S^2}\left\langle v,\frac{\partial\gamma(\xi_i,\xi_i)}{\partial y}\right\rangle V_i(v)dv\right\|_{C^{1+\alpha}(S^2)}$$
$$=\epsilon^3 O\left(\frac{\rho_i\|r_i\|^2_{C^{1+\alpha}(S^2)}}{l}+\frac{\rho_i^2\|r_i\|_{C^{1+\alpha}(S^2)}}{l^2}+\rho_i^3\frac{1}{l^3}+\rho_i^2\|r_i\|_{C^{1+\alpha}(S^2)}\frac{1}{l^2}\right.$$
$$\left.+\|r_i\|^3_{C^{1+\alpha}(S^2)}\right)\|V_i\|_{C^\alpha(S^2)},\ \text{and} \tag{16}$$

$$\left\|I^{ih}V_h-\epsilon\rho_h\left(\frac{1}{4\pi|\xi_i-\xi_h|}+\gamma(\xi_i,\xi_h)\right)\int_{S^2}V_h(v)dv\right.$$
$$-\frac{2\epsilon^2\rho_h r_h}{|\xi_i-\xi_h|}\int_{S^2}V_h+\frac{\epsilon^2\rho_i\rho_h}{|\xi_i-\xi_h|^2}\left\langle u,\frac{\xi_i-\xi_h}{|\xi_i-\xi_h|}\right\rangle\int_{S^2}V_h(v)dv$$
$$-\frac{\epsilon^2\rho_h^2}{|\xi_i-\xi_h|^2}\int_{S^2}\left\langle v,\frac{\xi_i-\xi_h}{|\xi_i-\xi_h|}\right\rangle V_h(v)dv-2\epsilon^2\rho_h r_h\gamma(\xi_i,\xi_h)\int_{S^2}V_h$$
$$\left.-\epsilon^2\rho_i\rho_h\left\langle u,\frac{\partial\gamma(\xi_i,\xi_h)}{\partial x}\right\rangle\int_{S^2}V_h(v)dv-\epsilon^2\rho_h^2\frac{\partial\gamma}{\partial\xi_h}\int_{S^2}\left\langle v,\frac{\partial\gamma(\xi_i,\xi_h)}{\partial y}\right\rangle V_h(v)dv\right\|_{C^{1+\alpha}(S^2)}$$
$$=\epsilon^3 O\left(\frac{\rho_h\|r_h\|^2_{C^{1+\alpha}(S^2)}}{|\xi_i-\xi_h|}+\frac{\rho_h(\rho_i+\rho_h)\|r_h\|_{C^{1+\alpha}(S^2)}}{|\xi_i-\xi_h|^2}+\frac{\rho_h(\rho_i^2+\rho_h^2)}{|\xi_i-\xi_h|^3}\right.$$
$$+\frac{\rho_h\rho_i\|r_i\|_{C^{1+\alpha}(S^2)}+\rho_h^2\|r_h\|_{C^{1+\alpha}(S^2)}}{|\xi_i-\xi_h|^2}+\frac{\rho_h\|r_h\|^2_{C^{1+\alpha}(S^2)}}{l}$$
$$+\rho_h(\rho_i^2+\rho_h^2)\frac{1}{l^3}+\rho_h(\rho_i\|r_i\|_{C^{1+\alpha}(S^2)}+\rho_h\|r_h\|_{C^{1+\alpha}(S^2)}$$
$$\left.+\rho_i\|r_h\|_{C^{1+\alpha}(S^2)})\frac{1}{l^2}\right)\|V_h\|_{C^\alpha(S^2)}. \tag{17}$$

The proof of Proposition 1 is complete. □

Consider the conservation condition (9.2) in [1]

$$\int_\Gamma V = \sum_h \int_{\Gamma_h} V = 0, \tag{18}$$

and recall that

$$W_h = H_h - E. \tag{19}$$

Proposition 2. *The conservation condition (18) determines uniquely the constant E. Moreover we have the following expression:*

$$\begin{aligned} E = & \frac{1}{\epsilon\bar\rho} - \frac{1}{\epsilon N\bar\rho} \sum_h 4\pi\epsilon\rho_h\gamma(\xi_i,\xi_h)\left(1-\frac{\rho_h}{\bar\rho}\right) \\ & -\frac{1}{\epsilon N\bar\rho} \sum_h{}' \frac{\epsilon\rho_i}{|\xi_i-\xi_h|}\left(1-\frac{\rho_h}{\bar\rho}\right) + \frac{1}{\epsilon N\bar\rho} O(\epsilon\|r_i\|_{C^{1+\alpha}(S^2)}) \\ & +\frac{1}{\epsilon N\bar\rho} O(\epsilon^2\|r_i\|^2_{C^{1+\alpha}(S^2)}) + \frac{1}{\epsilon N\bar\rho} O(\epsilon\|r_i\|_{C^{1+\alpha}(S^2)}\epsilon) + \frac{1}{\epsilon N\bar\rho} O(\epsilon^2), \end{aligned} \tag{20}$$

where $\rho = (\rho_1, \ldots, \rho_N)$, $r = (r_1, \ \ldots, r_N)$, $\bar\rho = \frac{1}{N}\sum_i \rho_i.$

Proof. Analogous to that of Proposition 9.1 in [1] and is omitted. □

Proposition 3. *We have that*

$$\begin{aligned} V_i = & \frac{1}{\epsilon\rho_i}\left(\frac{1}{\epsilon\rho_i} - \frac{1}{\epsilon\bar\rho}\right) \\ & -\frac{\epsilon}{\epsilon^2\rho_i^2} T_0 L r_i - \frac{1}{\epsilon\rho_i}\frac{1}{\epsilon N\bar\rho}\sum_h 4\pi\epsilon\rho_i\gamma(\xi_i,\xi_h)\left(\frac{\rho_h}{\bar\rho}-1\right) \\ & -\frac{1}{\epsilon\rho_i}\frac{1}{\epsilon N\bar\rho}\sum_{i,h}{}' \frac{\epsilon\rho_i}{|\xi_i-\xi_h|}\left(\frac{\rho_h}{\bar\rho}-1\right) \\ & -\frac{1}{\epsilon\rho_i}\sum_h{}' \frac{1}{|\xi_i-\xi_h|}\left(\frac{\rho_h}{\bar\rho}-1\right) \\ & -\frac{1}{\epsilon\rho_i}\sum_h \gamma(\xi_i,\xi_h)4\pi\left(\frac{\rho_h}{\bar\rho}-1\right) + \frac{1}{\epsilon\rho_i}\frac{1}{\epsilon N\bar\rho} O(\epsilon\|r_i\|_{C^{1+\alpha}(S^2)}) \\ & +\frac{1}{\epsilon\rho_i}\frac{1}{\epsilon N\bar\rho} O\left(\epsilon^2\|r_i\|^2_{C^{1+\alpha}(S^2)}\right) + \frac{1}{\epsilon\rho_i}\left(K_i'' - K_i\right) \\ & -\frac{\epsilon}{\epsilon\rho_i}\left(\frac{1}{\epsilon\rho_i}-\frac{1}{\epsilon\bar\rho}\right)\left(\frac{3}{2}r_i - \frac{1}{2}T_0 r_i\right) + \frac{1}{\epsilon^2\rho_i^2} O\left(\epsilon^2\|r_i\|^2_{C^{1+\alpha}(S^2)}\right) \\ & -\frac{1}{\epsilon\rho_i}\frac{1}{\epsilon N\bar\rho} O\left(\epsilon^2\|r_i\|_{C^{1+\alpha}(S^2)}\right) \\ & +\frac{\epsilon}{\epsilon\rho_i}\frac{1}{\epsilon N\bar\rho}\sum 4\pi\epsilon\rho_h\gamma(\xi_i,\xi_h)\left(\frac{\rho_h}{\bar\rho}-1\right)\left(\frac{3}{2}r_i - \frac{1}{2}T_0 r_i\right) \\ & +\frac{\epsilon}{\epsilon\rho_i}\frac{1}{\epsilon N\bar\rho}\sum_h{}' \frac{\epsilon\rho_i}{|\xi_i-\xi_h|}\left(\frac{\rho_h}{\bar\rho}-1\right)\left(\frac{3}{2}r_i - \frac{1}{2}T_0 r_i\right) \end{aligned}$$

$$+\frac{1}{\epsilon\rho_i}\sum_h{}' \frac{\epsilon^2\rho_h\rho_i}{|\xi_i-\xi_h|^2}\,3\left\langle u,\frac{\xi_i-\xi_h}{|\xi_i-\xi_h|}\right\rangle\frac{1}{\epsilon}\left(\frac{1}{\rho_h}-\frac{1}{\bar\rho}\right)$$
$$-\frac{1}{\epsilon\rho_i}\sum_h{}' \epsilon^2\rho_h^2 3\left\langle u,\frac{\partial\gamma(\xi_i,\xi_h)}{\partial x}\right\rangle\frac{1}{\epsilon}\left(\frac{1}{\rho_h}-\frac{1}{\bar\rho}\right)$$
$$+\frac{1}{\epsilon\rho_i}\epsilon^2\,T_0\int_{S^2}\frac{r_i(\cdot)+r_i(v)}{4\pi|\cdot-v|}\,r_i(v)(T_0W_i)(v)dv$$
$$+\left(\frac{\epsilon|\rho|}{\text{dist}}+\epsilon\|r_i\|_{C^{1+\alpha}(S^2)}\right)^2\sum_h\frac{\epsilon\rho_h}{\text{dist}}\,\|T_0W_h\|_{C^\alpha(S^2)}$$
$$+O\left(\epsilon^3\|r_i\|^3_{C^{1+\alpha}(S^2)}\right)\|T_0W_i\|, \tag{21}$$

where K_i'' satisfies the estimate

$$K_i''=\frac{1}{\epsilon N\bar\rho}\,O(\epsilon^2\|r_i\|)+\frac{1}{\epsilon N\bar\rho}\,O(\epsilon^2). \tag{22}$$

Proof. Analogous to that of Proposition 9.2, and it is omitted. □

From (21), (22) we obtain (1) in the same way as (1.2) in [1] is obtained from (9.9).

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