Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

Website: http://AIMsciences.org

pp. **X–XX**

GLOBAL-IN-TIME BEHAVIOR OF THE SOLUTION TO A GIERER-MEINHARDT SYSTEM

Georgia Karali

Department of Applied Mathematics
University Crete, P.O. Box 2208
71409, Heraklion, Crete, Greece

Takashi Suzuki

Division of Mathematical Science
Graduate School of Engineering Science, Osaka University
Machikaneyamacho, Toyonakashi, 560-8531, Japan

Yoshio Yamada

Department of Applied Mathematics
Waseda University
3-4-1, Ohkubo, Shinjuku-ku, Tokyo, 169-8555, Japan

(Communicated by the associate editor name)

Abstract. Gierer-Meinhardt system is a mathematical model to describe biological pattern formation due to activator and inhibitor. Turing pattern is expected in the presense of local self-enhancement and long-range inhibition. The long-time behavior of the solution, however, has not yet been clarified mathematically. In this paper, we study the case when its ODE part takes periodic-in-time solutions, that is, $\tau = \frac{s+1}{p-1}$. Under some additional assumptions on parameters, we show that the solution exists global-in-time and absorbed into one of these ODE orbits. Thus spatial patterns eventually disappear if those parameters are in a region without local self-enhancement or long-range inhibition.

1. Introduction.

Several models in mathematical biology take the form of a reaction-diffusion system

$$\begin{aligned} u_t &= \varepsilon^2 \Delta u + f(u,v) \\ \tau v_t &= D\Delta v + g(u,v) \qquad \text{in } \Omega \times (0,T) \end{aligned} \tag{1}$$

with

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \times (0,T) \tag{2}$$

where ε, τ, and D are positive constants, Ω is a bounded domain in $\mathbf{R}^n$ with smooth boundary $\partial\Omega$, and ν is the outer unit normal vector. One of them is the Gierer-Meinhardt system in morphogenesis [2] which is the case of

$$f(u,v) = -u + \frac{u^p}{v^q}, \quad g(u,v) = -v + \frac{u^r}{v^s} \tag{3}$$

2000 *Mathematics Subject Classification.* Primary: 35K57; Secondary: 35Q92.

Key words and phrases. reaction-diffusion equation, Gierer-Meinhardt system, Turing pattern, Hamilton structure, asymptotic behavior of the solution.

with

$$p > 1, \quad q, r > 0, \quad s > -1. \tag{4}$$

It is concerned with pattern formations of spatial tissue structures of hydra, where $u = u(x,t) > 0$ and $v = v(x,t) > 0$ stand for the activator and inhibitor, respectively. Fundamental ideas of this model are from Turing [17], that is, instability of constant stationary solutions is driven by diffusion terms (see also Murray [13]).

Mathematical study on Gierer-Meinhardt system, the reaction - diffusion system (1)-(2) with the nonlinearity (3)-(4), has been done in details. "Turing pattern" is observed as spiky stable stationary solutions [15, 6] in the case of

$$0 < \varepsilon \ll 1, \quad D \gg 1, \quad 0 < \tau \ll 1, \tag{5}$$

and

$$\frac{p-1}{r} < \frac{q}{s+1}. \tag{6}$$

See also Wei [18] and the references therein for later studies.

Condition (6) takes the following roles in the ODE system

$$\frac{du}{dt} = -u + \frac{u^p}{v^q}, \quad \tau \frac{dv}{dt} = -v + \frac{u^r}{v^s}. \tag{7}$$

First, if (6) is the case, the ODE orbits near the equilibrium $(u,v) = (1,1)$ are cyclic. Next, in the case of

$$0 < \tau < \frac{s+1}{p-1} \tag{8}$$

the constant solution $(u,v) = (1,1)$ is stable as a steady state of (7) because the linearized equation takes the form

$$\frac{d}{dt}\begin{pmatrix} y \\ \tau z \end{pmatrix} = \begin{pmatrix} p-1 & -q \\ r & -(s+1) \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix}$$

and the real parts of all the eigenvalues of the matrix

$$A = \begin{pmatrix} p-1 & -q \\ r/\tau & -(s+1)/\tau \end{pmatrix}$$

are negative if and only if (8) is satisfied. Finally, if $\tau = 0$ the ODE system (7) is reduced to the single equation

$$\frac{du}{dt} = -u + u^\gamma, \quad \gamma = p - \frac{qr}{s+1}.$$

Then condition (6) implies $0 < \gamma < 1$ and hence global-in-time existence of the solution of this reduced system.

In spite of such a stable profile of the stationary solution $(u,v) = (1,1)$ in ODE, it becomes unstable as a steady state of

$$\begin{aligned} u_t &= \varepsilon^2 \Delta u - u + \frac{u^p}{v^q} \\ \tau v_t &= D\Delta v - v + \frac{u^r}{v^s} \qquad \text{in } \Omega \times (0,T) \end{aligned} \tag{9}$$

with

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \times (0,T) \tag{10}$$

if its linearized part

$$\frac{\partial}{\partial t}\begin{pmatrix} w \\ \tau z \end{pmatrix} = \begin{pmatrix} \varepsilon^2\Delta + p - 1 & -q \\ r & D\Delta - (s+1) \end{pmatrix}\begin{pmatrix} w \\ z \end{pmatrix} \quad \text{in } \Omega \times (0,T) \tag{11}$$

with

$$\frac{\partial w}{\partial \nu} = \frac{\partial z}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times (0,T) \tag{12}$$

is unstable. This property arises if

$$\lambda \begin{pmatrix} \alpha \\ \tau\beta \end{pmatrix} = \begin{pmatrix} -\mu\varepsilon^2 + p - 1 & -q \\ r & -\mu D - s - 1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{13}$$

has a positive eigenvalue λ, where μ is a positive eigenvalue of $-\Delta$ with $\frac{\partial}{\partial\nu}\cdot = 0$ on $\partial\Omega$. In fact, since

$$\lambda = -\mu\varepsilon^2 + (p-1) - \frac{qr}{\mu D + (s+1) + \tau\lambda}$$

the instability $\lambda = \lambda(\varepsilon, D, \tau) > 0$ occurs for $D \gg 1$ and $0 < \varepsilon \ll 1$. Condition (6), on the other hand, implies $\lambda(\varepsilon, D, \tau) < 0$ for $0 < D \ll 1$ and $0 < \tau \ll 1$. Thus, the instability of $(u, v) = (1, 1)$ as a stationary state of (9) with (10) arises if and only if $D \gg 1$, provided that (6) and $0 < \varepsilon, \tau \ll 1$ are the cases.

Henceforth we assume (6). Unique existence of the regular solution local-in-time to (9) with (10) is standard, for given smooth initial values

$$u|_{t=0} = u_0(x) \geq 0, \quad v|_{t=0} = v_0(x) > 0 \quad \text{on } \overline{\Omega}. \tag{14}$$

Its global-in-time existence was studied by Masuda-Takahashi [12], and recently, Jiang [7] has established this property for

$$\frac{p-1}{r} < 1. \tag{15}$$

Condition (15) is almost optimal, regarding the work of Ni-Suzuki-Takagi [14] concerning the ODE system (7). Namely, according to their classification of orbits, there is fintie time blowup in (7) for

$$\frac{p-1}{r} > 1.$$

One of other mathematical results on the Gierer-Meinhardt system (9) is the existence and non-existence of the global-in-time solution to the shadow system

$$\begin{aligned} u_t &= \varepsilon^2\Delta u - u + \frac{u^p}{\xi^q} && \text{in } \Omega \times (0,T) \\ \frac{\partial u}{\partial \nu} &= 0 && \text{on } \partial\Omega \times (0,T) \\ \tau\frac{d\xi}{dt} &= -\xi + \frac{1}{|\Omega|}\int_\Omega \frac{u^r}{\xi^s}dx && \text{in } (0,T) \end{aligned}$$

done by Li-Ni [11]. Yanagida [19, 20], on the other hand, formulated (9) as a skew-gradient system

$$\begin{aligned} ru_t &= r\varepsilon^2\Delta u + H_u(u,v) \\ \tau q v_t &= qD\Delta v - H_v(u,v) \quad \text{in } \Omega \times (0,T) \end{aligned}$$

with

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times (0,T)$$

in the case of

$$p+1=r, \quad q+1=s,$$

using

$$H(u,v) = -\frac{r}{2}u^2 + \frac{q}{2}v^2 + \frac{u^r}{v^q}.$$

Consequently, any non-degenerate mini-maximizer of

$$E(u,v) = \int_\Omega \frac{r\varepsilon^2}{2}|\nabla u|^2 - \frac{qD}{2}|\nabla v|^2 - H(u,v)\ dx$$

is linearly stable by general theory.

The asymptotic behavior of the solution as $t \uparrow +\infty$ of the Gierer-Meinhardt system, however, has been studied mostly for the system with supplementary terms

$$\begin{aligned} u_t &= \varepsilon^2 \Delta u - u + \frac{u^p}{v^q} + \sigma_1 \\ \tau v_t &= D\Delta v - v + \frac{u^r}{v^s} + \sigma_2 \qquad \text{in } \Omega \times (0,T) \end{aligned}$$

with

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \times (0,T)$$

where $\sigma_i = \sigma_i(x) \geq 0$, $i = 1, 2$, are smooth functions satisfying $\sigma_1 + \sigma_2 > 0$ (see [12, 7]). Technical difficulties to this problem lie on the uniform esitmate of $v > 0$ from below, and to our knowledge the present paper is the first challenge for the case without supplementary terms.

Our result, however, is restrited to the special case

$$\tau = \frac{s+1}{p-1} \tag{16}$$

where the ODE part takes the first integral (see [14])

$$H(u,v) = \frac{v^{s+1}}{u^{p-1}} + \frac{p-1}{r-p+1}u^{r-p+1} - \frac{s+1}{s+1-q}v^{s+1-q}. \tag{17}$$

Thus, any solution $(u,v) = (u(t), v(t))$ to (7) with $u(t), v(t) > 0$ satisfies

$$\frac{d}{dt}H(u(t), v(t)) = 0.$$

If

$$\frac{p-1}{r} \leq 1 \leq \frac{q}{s+1}, \tag{18}$$

furthermore, all the level curves of H are closed in uv plane with $u, v > 0$. Consequently, any solution $(u,v) = (u(t), v(t))$ to (7) is time-periodic with a period determined by the first integral (or energy) $H = H(u(t), v(t))$, which is constant in t.

Actually, periodic orbits to (7) arise if and only if (16) and (18) are the cases, according to the classification of the ODE orbits done by [14]. Our result is now stated as follows.

Theorem 1.1. *Let*

$$d_1 = \varepsilon^2, \quad d_2 = \tau^{-1}D, \tag{19}$$

and assume

$$\frac{2\sqrt{d_1 d_2}}{d_1 + d_2} \geq \sqrt{\frac{(s+1)(p-1)}{sp}}, \quad s > 0. \tag{20}$$

Assume, furthermore, (16), that is,

$$\tau = \frac{s+1}{p-1}$$

and

$$\frac{p}{r} < 1 < \frac{q}{s+1}. \tag{21}$$

Then, given a solution $(u, v) = (u(\cdot, t), v(\cdot, t))$ to the Gierer-Meinhardt system (9) with (10), we have an ODE orbit $\hat{O} \subset \mathbf{R}^2$ to (7) such that

$$\lim_{t \uparrow +\infty} dist_{C^2}((u(\cdot, t), v(\cdot, t)), \hat{O}) = 0. \tag{22}$$

Furthermore, if this $\hat{O}$ is not composed of a single point there is $\ell > 0$ such that

$$\lim_{t \uparrow +\infty} \|(u(\cdot, t+\ell), v(\cdot, t+\ell)) - (u(\cdot, t), v(\cdot, t))\|_{C^2} = 0. \tag{23}$$

The proof is based on the theory of dynamical systems. We introduce a Lyapunov function which has not been known so far. This Lyapunov function is valid only in the case of (20) and (21). In spite of these additional restrictions, parameters (p, q, r, s) satisfying all the requirements of Theorem 1.1 exist.

Actually, the set of the values of the left-hand of (20) is equal to $[0, 1)$ as d_i, $i = 1, 2$, varies. Hence inequality (20) requires

$$\frac{p-1}{s} \leq 1, \tag{24}$$

which, however, is consistent to (21). The extremal value 1 of the left-hand side of (20) is achieved if and only if $d_1 = d_2$. In other words the admissible parameter region of (p, q, r, s) assumed in Theorem 1.1 is wider as two diffusion coefficients d_i, $i = 1, 2$, are closer.

Since condition (21) is included by (15) and (18), in the parameter region treated in Theorem 1.1 any solution $(u, v) = (u(\cdot, t), v(\cdot, t))$ to (9) with (10) and (14) exists global-in-time, any solution to its ODE part (7) is time-periodic, and any PDE orbit $O = \{(u(\cdot, t), v(\cdot, t))\}_{t \geq 0}$ is absorbed into one of the periodic orbits of its ODE part, denoted by $\hat{O}$. In other words, any spatial patterns of the Gierer-Meinhardt system eventually disappear in the parameter region (21) of (p, q, r, s) under the assumptions of $\tau = (s+1)/(p-1)$ and $d_1 \approx d_2$. These assumptions are far from (5), the local self-enhancement $0 < \varepsilon \ll 1$ and the long-range inhibition $D \gg 1$, $0 < \tau \ll 1$. Thus Theorem 1.1 still supports the paradigm, Turing patterns expected under such environments [9].

Assumption (16) may look restrictive. Here we emphasize again that this is the only case that the ODE part of (9)-(10) takes peiodic-in-time orbits. The proof of Theorem 1.1, furthermore, implies that the stationary state to (9)-(10) must be spatially homogeneous without this condition.

Theorem 1.2. *Let*

$$\frac{p}{r} < 1 < \frac{q}{s+1}, \quad \frac{p-1}{s} \leq 1 \tag{25}$$

and $d, D > 0$ be in the region

$$\frac{2\sqrt{dD}}{\sqrt{\frac{s+1}{p-1}}d + \sqrt{\frac{p-1}{s+1}}D} \geq \sqrt{\frac{(s+1)(p-1)}{sp}}. \tag{26}$$

Then any solution to

$$d\Delta u - u + \frac{u^p}{v^q} = 0, \quad D\Delta v - v + \frac{u^r}{v^s} = 0 \quad \text{in } \Omega \tag{27}$$

with

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \tag{28}$$

must be the constant $(1, 1)$.

Theorem 1.1 of Jiang-Ni [8] is also concerned with the uniqueness of the solution to (27)-(28). There, this property is established for

$$\max\{q, r\} < s + 1 \tag{29}$$

and $D/d \leq k$ with $k = k(p, q, r, s)$ calculated explicitly. We note that this case of [8] is a counter part of the one treated in Theorem 1.2, comparing (25) and (29). The other uniqueness result of [8] is Theorem 1.7, which is concerned with $n = 2$ and $\frac{p-1}{r} < 1$. Thus any $D_* > 0$ admits $d^* > 0$ such that there is no non-constant solution to (27)-(28) if $D \geq D_*$ and $d \geq d^*$.

The same properties as in Theorems 1.1 and 1.2 are observed in the classical prey-predator system

$$\begin{aligned} u_t &= \varepsilon^2 \Delta u + u(a - bv) \\ \tau v_t &= D\Delta v + v(-c + du) \qquad \text{in } \Omega \times (0, T) \end{aligned} \tag{30}$$

with

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \times (0, T), \tag{31}$$

where $a, b, c, d > 0$ are constants (see [1, 10]), that is, the ODE part takes always time-periodic orbits and the PDE solution is absorbed into one of them. We have, actually, common mathematical structures between these two models. In fact, first, any orbit of the ODE part of (30)

$$\frac{du}{dt} = u(a - bv), \quad \tau \frac{dv}{dt} = v(-c + du) \tag{32}$$

with $u = u(t) > 0$ and $v = v(t) > 0$ is time-periodic. This property follows from the fact that system (32) takes the first integral

$$H(u, v) = -a \log v + bv - \tau^{-1} c \log u + \tau^{-1} du \tag{33}$$

of which level curves are closed in uv plane, $u, v > 0$. Then,

$$\mathcal{H}(u(\cdot, t), v(\cdot, t)) = \int_\Omega H(u(x, t), v(x, t))dx \tag{34}$$

casts a Lyapunov function to (30) with (31).

Although these structures are common to (7) for (16), here we use additional technical ingredients. One is due to Masuda-Takahashi [12] estimating

$$\frac{d}{dt} \int_\Omega u^a v^{-b} dx, \quad a, b > 0.$$

The other is the use of a comparison principle to v^{-a} to derive the uniform estimate of $v = v(x, t) > 0$ from below.

In the following sections, first, we confirm that $H(u, v)$ defined by (17) is a Hamiltonian of an ODE system associated with (7) for (16) and then show that $H(u(\cdot, t), v(\cdot, t))$ defined by (34) and (17) acts as a Lyapunov function to (9) with

(10). This propery implies Theorem 1.2 immediately. The proof of Theorem 1.1 is given in the final section.

2. **Preliminaries.** The parabolic strong maximum principle to (9), (10), and (14) guarantees $u(\cdot,t)>0$ in $\overline{\Omega}\times(0,+\infty)$, provided that $u_0\not\equiv 0$ on $\overline{\Omega}$. Hence we shall treat positive solutions to (9) with (10) and to (7), mostly.

Writing (7) in the form of

$$u^{-p}(u_t+u)=v^{-q}, \quad v^s(v_t+\tau^{-1}v)=\tau^{-1}u^r,$$

we introduce new variables,

$$\xi=\frac{u^{-p+1}}{p-1}, \quad \eta=\frac{v^{s+1}}{s+1}. \tag{35}$$

Then it follows that

$$\xi_t=-u_tu^{-p}, \quad \eta_t=v^sv_t \tag{36}$$

and hence

$$\begin{aligned}\xi_t&=u^{-p+1}-v^{-q}=(p-1)\xi-\{(s+1)\eta\}^{-\frac{q}{s+1}}\\ \eta_t&=-\tau^{-1}v^{s+1}+\tau^{-1}u^r=-\tau^{-1}(s+1)\eta+\tau^{-1}\{(p-1)\xi\}^{-\frac{r}{p-1}}.\end{aligned}$$

It is not hard to formulate this system as a Hamilton system in the case of (16), that is, $p-1=\tau^{-1}(s+1)$. In fact, we have

$$\frac{d\xi}{dt}=H_\eta, \quad \frac{d\eta}{dt}=-H_\xi \tag{37}$$

using

$$H(\xi,\eta)=(p-1)\xi\eta+\left(\frac{r}{p-1}-1\right)^{-1}A(\xi)+\left(\frac{q}{s+1}-1\right)^{-1}B(\eta) \tag{38}$$

and

$$\begin{aligned}A(\xi)&=\tau^{-1}(p-1)^{-\frac{r}{p-1}}\xi^{1-\frac{r}{p-1}}\\ B(\eta)&=(s+1)^{-\frac{q}{s+1}}\eta^{1-\frac{q}{s+1}}.\end{aligned} \tag{39}$$

This Hamiltonian is actually equivalent to the first integral defined by (17).

Here we assume

$$\frac{p-1}{r}<1<\frac{q}{s+1}$$

and put

$$\alpha=\frac{r}{p-1}-1>0, \quad \beta=\frac{q}{s+1}-1>0. \tag{40}$$

Then it follows that

$$\begin{aligned}H(\xi,\eta)&=(p-1)\xi\eta+(s+1)^{-1}(p-1)^{-\alpha}\alpha^{-1}\xi^{-\alpha}\\ &\quad+(s+1)^{-\beta-1}\beta^{-1}\eta^{-\beta},\end{aligned} \tag{41}$$

recalling (16).

Now we use (19) and (35). First, (36) implies

$$\begin{aligned}\xi_t&=-d_1u^{-p}\Delta u+u^{-p+1}-v^{-q}\\ \eta_t&=d_2v^s\Delta v-\tau^{-1}v^{s+1}+\tau^{-1}u^r.\end{aligned}$$

Then, (9) and (10) read as

$$\begin{aligned} &\xi_t = -d_1(p-1)\xi^{\frac{p}{p-1}}\Delta\xi^{-\frac{1}{p-1}} + H_\eta, \quad \xi > 0 \\ &\eta_t = d_2(s+1)\eta^{\frac{s}{s+1}}\Delta\eta^{\frac{1}{s+1}} - H_\xi, \qquad \eta > 0 \qquad \text{in } \Omega\times(0,T) \end{aligned}$$

and

$$\frac{\partial\xi}{\partial\nu} = \frac{\partial\eta}{\partial\nu} = 0 \qquad \text{on } \partial\Omega\times(0,T),$$

respectively. This formulation implies

$$\begin{aligned} \frac{d}{dt}\int_\Omega H(\xi,\eta)dx &= \int_\Omega H_\xi\xi_t + H_\eta\eta_t \ dx \\ &= \int_\Omega -H_\xi d_1(p-1)\xi^{\frac{p}{p-1}}\Delta\xi^{-\frac{1}{p-1}} + H_\eta d_2(s+1)\eta^{\frac{s}{s+1}}\Delta\eta^{\frac{1}{s+1}} \ dx, \end{aligned}$$

while

$$\begin{aligned} H_\xi &= (p-1)\eta - (s+1)^{-1}(p-1)^{-\alpha}\xi^{-\alpha-1} \\ H_\eta &= (p-1)\xi - (s+1)^{-\beta-1}\eta^{-\beta-1} \end{aligned}$$

holds by (41). Then it follows that

$$\begin{aligned} \frac{d}{dt}\int_\Omega H(\xi,\eta)dx = (p-1)&\int_\Omega -d_1(p-1)\eta\xi^{\frac{p}{p-1}}\Delta\xi^{-\frac{1}{p-1}} \\ &+d_2(s+1)\xi\eta^{\frac{s}{s+1}}\Delta\eta^{\frac{1}{s+1}} \ dx \\ +&\int_\Omega d_1(s+1)^{-1}(p-1)^{-\alpha+1}\xi^{-\alpha+\frac{1}{p-1}}\Delta\xi^{-\frac{1}{p-1}} \\ &-d_2(s+1)^{-\beta}\eta^{-\beta-\frac{1}{s+1}}\Delta\eta^{\frac{1}{s+1}} \ dx. \end{aligned} \tag{42}$$

The last two terms of the right-hand side of (42) are treated by

$$\begin{aligned} \int_\Omega \xi^{-\alpha+\frac{1}{p-1}}\Delta\xi^{-\frac{1}{p-1}}dx &= \left(-\alpha+\frac{1}{p-1}\right)\frac{1}{p-1}\int_\Omega \xi^{-\alpha-2}|\nabla\xi|^2dx \\ \int_\Omega \eta^{-\beta-\frac{1}{s+1}}\Delta\eta^{\frac{1}{s+1}}dx &= \left(\beta+\frac{1}{s+1}\right)\frac{1}{s+1}\int_\Omega \eta^{-\beta-2}|\nabla\eta|^2dx \end{aligned}$$

using (10), while for the first two terms we note

$$\begin{aligned} \int_\Omega \eta\xi^{\frac{p}{p-1}}\Delta\xi^{-\frac{1}{p-1}}dx &= \frac{1}{p-1}\int_\Omega \nabla(\eta\xi^{\frac{p}{p-1}})\cdot\xi^{-\frac{p}{p-1}}\nabla\xi \ dx \\ &= \frac{1}{p-1}\int_\Omega \nabla\xi\cdot\nabla\eta + \frac{p}{p-1}\eta\xi^{-1}|\nabla\xi|^2 \ dx \end{aligned}$$

and

$$\begin{aligned} \int_\Omega \xi\eta^{\frac{s}{s+1}}\Delta\eta^{\frac{1}{s+1}}dx &= -\frac{1}{s+1}\int_\Omega \nabla(\xi\eta^{\frac{s}{s+1}})\cdot\eta^{-\frac{s}{s+1}}\nabla\eta \ dx \\ &= -\frac{1}{s+1}\int_\Omega \nabla\xi\cdot\nabla\eta + \frac{s}{s+1}\xi\eta^{-1}|\nabla\eta|^2dx. \end{aligned}$$

Therefore, it follows that

$$\begin{aligned}\frac{1}{p-1}\frac{d}{dt}\int_\Omega H(\xi,\eta)dx &= -\int_\Omega d_1\left(\nabla\xi\cdot\nabla\eta+\frac{p}{p-1}\eta\xi^{-1}|\nabla\xi|^2\right)\\ &\quad +d_2\left(\nabla\xi\cdot\nabla\eta+\frac{s}{s+1}\xi\eta^{-1}|\nabla\eta|^2\right)\\ &\quad +d_1(s+1)^{-1}(p-1)^{-\alpha-1}\left(\alpha-\frac{1}{p-1}\right)\xi^{-\alpha-2}|\nabla\xi|^2\\ &\quad +d_2(s+1)^{-\beta-1}(p-1)^{-1}\left(\beta+\frac{1}{s+1}\right)\eta^{-\beta-2}|\nabla\eta|^2\ dx\\ &= -\int_\Omega (d_1+d_2)\nabla\xi\cdot\nabla\eta+d_1\cdot\frac{p}{p-1}\xi^{-1}\eta|\nabla\xi|^2\\ &\quad +d_2\cdot\frac{s}{s+1}\xi\eta^{-1}|\nabla\eta|^2\\ &\quad +d_1(s+1)^{-1}(p-1)^{-\alpha-1}\left(\alpha-\frac{1}{p-1}\right)\xi^{-\alpha-2}|\nabla\xi|^2\\ &\quad +d_2(s+1)^{-\beta-1}(p-1)^{-1}\left(\beta+\frac{1}{s+1}\right)\eta^{-\beta-2}|\nabla\eta|^2\ dx.\end{aligned}$$

Here, the inequality $\alpha-\frac{1}{p-1}>0$ is equivalent to $\frac{p}{r}<1$ and the quadratic form

$$Q(X,Y)=d_1\cdot\frac{p}{p-1}X^2+d_2\cdot\frac{s}{s+1}Y^2+(d_1+d_2)XY$$

is non-negative definite if and only if (20). We thus end up with the following lemma.

Lemma 2.1. *Under the assumptions of Theorem 1.1, it holds that*

$$\frac{d}{dt}\int_\Omega H(\xi,\eta)dx\le-\int_\Omega c_1|\nabla\xi^{-\alpha/2}|^2+c_2|\nabla\eta^{-\beta/2}|^2\ dx \tag{43}$$

where $c_i>0$, $i=1,2$, *are constants.*

Concluding this section, we note the following. First, by (35) and (40) it holds that

$$\int_\Omega H(\xi,\eta)dx\approx\int_\Omega u^{-p+1}v^{s+1}+u^{r-p+1}+v^{-q+s+1}\ dx \tag{44}$$

and

$$\int_\Omega|\nabla\xi^{-\alpha/2}|^2+|\nabla\eta^{-\beta/2}|^2\ dx\approx\int_\Omega|\nabla u^{\frac{r-p+1}{2}}|^2+|\nabla v^{\frac{-q+s+1}{2}}|^2\ dx,$$

where

$$r-p+1>0>-q+s+1.$$

Next, Lemma 2.1 implies the following proof.

Proof of Theorem 1.2. Any solution (u,v) to (27)-(28) is regarded as a stationary solution (u,v) to (9)-(10) for $\varepsilon^2=d$. This stationary system of (9)-(10) is independent of τ, so that (u,v) may be regarded as a stationary solution to (9)-(10) for $\tau=\frac{s+1}{p-1}$. Then, the left-hand side of (43) vanishes because this (u,v) is independent of t. Therefore, it follows that that (ξ,η) and hence (u,v) are spatially homogeneous under the assumptions of Theorem 1.1. Here, condition (20) for $d_1=\varepsilon^2=d$ and $d_2=\tau^{-1}D=\frac{p-1}{s+1}D$ means (26). Thus $(u,v)=(1,1)$ follows if (25)-(26) are the cases. □

3. **Proof of Theorem 1.1.** Henceforth, C_i, $i = 1, 2, \cdots, 13$, denote positive constants independent of t. To clarify their dependnce on parameters, say, $a, b, \cdots$, we sometimes write them as $C_i(a, b, \cdots)$. Furthermore, we shall use standard semi-group estimates (see, for instance, [4, 16] and the references in [5]).

First, we show the following lemma.

Lemma 3.1. *Under the assumptions of Theorem 1.1, it holds that*

$$\|v(\cdot, t)^{-1}\|_\infty \leq C_1 \tag{45}$$

for any $t \geq 0$.

Proof. Given $\ell > \max\{n/2, 1\}$, we put $a = \frac{\ell}{q-s-1} > 0$ and $v = w^{-a} > 0$. Then we obtain

$$\|w(\cdot, t)\|_\ell \leq C_2 \tag{46}$$

by Lemma 2.1. It also holds that

$$\begin{aligned} w_t &= d_2\Delta w - d_2(a+1)w^{-1}|\nabla w|^2 + a^{-1}\tau^{-1}(w - u^r w^{a(s+1)+1}) \\ &\leq d_2\Delta w + a^{-1}\tau^{-1}w \end{aligned}$$

and hence

$$\begin{aligned} & w_t \leq (d_2\Delta - \mu)w + (\mu + \tau^{-1}a^{-1})w && \text{in } \Omega \times (0, +\infty) \\ & \frac{\partial w}{\partial \nu} = 0 && \text{on } \partial\Omega \times (0, +\infty), \end{aligned}$$

where $\mu > 0$. Then, the standard maximum principle guarantees

$$0 < w \leq \overline{w} \qquad \text{on } \overline{\Omega} \times [0, +\infty) \tag{47}$$

using the solution $\overline{w} = \overline{w}(x, t)$ to

$$\begin{aligned} & \overline{w}_t = (d_2\Delta - \mu)\overline{w} + (\mu + \tau^{-1}a^{-1})w && \text{in } \Omega \times (0, +\infty) \\ & \frac{\partial \overline{w}}{\partial \nu} = 0 && \text{on } \partial\Omega \times (0, +\infty) \\ & \overline{w}|_{t=0} = v_0(x)^{-a} && \text{in } \Omega. \end{aligned}$$

It holds that

$$\overline{w}(\cdot, t) = e^{t(d_2\Delta - \mu)}v_0^{-a} + (\mu + a^{-1}\tau^{-1}) \cdot \int_0^t e^{(t-s)(d_2\Delta - \mu)}w(\cdot, s)ds$$

with

$$\|e^{t\Delta}\|_{L^\ell(\Omega) \to L^\ell(\Omega)} \leq C_3(\ell).$$

Therefore, we obtain

$$\|\Delta^\gamma \overline{w}(\cdot, t)\|_\ell \leq C_4(\gamma, \ell)$$

by (46), where $0 < \gamma < 1$. Then it follows that

$$\|\overline{w}(\cdot, t)\|_{W^{2\gamma, \ell}} \leq C_5(\gamma, \ell)$$

which implies

$$\|\overline{w}(\cdot, t)\|_\infty \leq C_6$$

by $\ell > \max\{n/2, 1\}$ and Morrey's theorem because $0 < \gamma < 1$ is arbitrary. Hence we obtain

$$\|v(\cdot, t)^{-a}\|_\infty = \|w(\cdot, t)\|_\infty \leq C_6$$

by (47). Since $a > 0$, the result follows with $C_1 = C_6^{1/a}$. □

Following [12], now we estimate

$$\frac{d}{dt}\int_\Omega u^a v^{-b}dx$$

from above, where $a, b > 0$. First, we have

$$\begin{aligned}
\frac{d}{dt}\int_\Omega u^a v^{-b}dx &= \int_\Omega a u^{a-1}v^{-b}u_t - b u^a v^{-b-1}v_t\ dx \\
&= \int_\Omega -d_1 a\nabla u\cdot\nabla(u^{a-1}v^{-b}) - a u^a v^{-b} + a u^{a-1+p}v^{-b-q} \\
&\quad + d_2 b\nabla v\cdot\nabla(u^a v^{-b-1}) + \tau^{-1}b(u^a v^{-b} - u^{a+r}v^{-b-1-s})dx \\
&= \int_\Omega -a(a-1)d_1 u^{a-2}v^{-b}|\nabla u|^2 \\
&\qquad + ab(d_1+d_2)u^{a-1}v^{-b-1}\nabla u\cdot\nabla v - b(b+1)d_2 u^a v^{-b-2}|\nabla v|^2 \\
&\qquad + (\tau^{-1}b - a)u^a v^{-b} + a u^{a+p-1}v^{-b-q} \\
&\qquad - \tau^{-1}b u^{a+r}v^{-b-s-1}\ dx.
\end{aligned}$$

The following lemma is essentially obtained in the proof of Lemma 2 of [7].

Lemma 3.2. *Let (6) and (15) be satisfied. Then, given $a > 1$ and $b > 0$ such that*

$$\frac{2\sqrt{d_1 d_2}}{d_1+d_2} \geq \sqrt{\frac{ab}{(a-1)(b+1)}}, \tag{48}$$

it holds that

$$\begin{aligned}
\frac{d}{dt}\int_\Omega u^a v^{-b}dx &\leq (-a+\tau^{-1}b)\int_\Omega u^a v^{-b}dx \\
&+ C_7(a,b)\left(\int_\Omega v^{-\theta/\varepsilon}dx\right)^{\varepsilon}\left(\int_\Omega u^a v^{-b}dx\right)^{1-\varepsilon}
\end{aligned} \tag{49}$$

with ε and θ defined by

$$\begin{aligned}
\theta &= \frac{r}{r-p+1-\delta}\left[q - \frac{(p-1)(s+1)}{r} - \left(\frac{s+1}{r} - \frac{b}{a}\right)\delta\right] \\
\varepsilon &= \frac{\delta}{a}\left(\frac{r}{r-p+1-\delta}\right).
\end{aligned} \tag{50}$$

Here we take $0 < \delta \ll 1$ so that $0 < \varepsilon < 1$ and $\theta > 0$ are achieved, recalling (6).

Proof. By (48) the quadratic form

$$Q(X,Y) = a(a-1)d_1X^2 + b(b+1)d_2Y^2 - ab(d_1+d_2)XY$$

is non-negative definite. Hence it holds that

$$\begin{aligned}
\frac{d}{dt}\int_\Omega u^a v^{-b}dx &\leq (-a+\tau^{-1}b)\int_\Omega u^a v^{-b}dx \\
&+ \int_\Omega a u^{a+p-1}v^{-b-q} - \tau^{-1}b u^{a+r}v^{-b-s-1}\ dx.
\end{aligned}$$

First, we use

$$u^{a+p-1}v^{-b-q} = \left\{v^{-\theta}(u^a v^{-b})^{1-\varepsilon}\right\}^{1-\frac{p-1+\delta}{r}}\cdot\left(u^{r+a}v^{-s-b-1}\right)^{\frac{p-1+\delta}{r}} \tag{51}$$

derived from (50). In fact, we have

$$
\begin{aligned}
& a(1-\varepsilon)\cdot\{1-\frac{p-1+\delta}{r}\}+(r+a)\cdot\frac{p-1+\delta}{r} \\
&\quad = a(1-\varepsilon)+(r+a\varepsilon)\cdot\frac{p-1+\delta}{r} \\
&\quad = a+p-1+\delta+a\varepsilon\cdot\frac{p-1+\delta-r}{r} \\
&\quad = a+p-1
\end{aligned}
$$

and

$$
\begin{aligned}
& \{\theta+b(1-\varepsilon)\}\{1-\frac{p-1+\delta}{r}\}+(s+b+1)\cdot\frac{p-1+\delta}{r} \\
&\quad = \theta+b(1-\varepsilon)+\{-\theta+b\varepsilon+s+1\}\cdot\frac{p-1+\delta}{r} \\
&\quad = \theta\cdot\frac{r-p+1-\delta}{r}+b\varepsilon\cdot\frac{-r+p-1+\delta}{r}+b \\
&\qquad +(s+1)\cdot\frac{p-1+\delta}{r} \\
&\quad = \left\{q-\frac{(p-1)(s+1)}{r}-(\frac{s+1}{r}-\frac{b}{a})\delta\right\} \\
&\qquad -\frac{b}{a}\delta+b+(s+1)\cdot\frac{p-1+\delta}{r}=q+b.
\end{aligned}
$$

Hence (51) follows.

Next, we use Young's inequality as

$$
\begin{aligned}
& au^{a+p-1}v^{-b-q} \\
&\quad = a\left\{v^{-\theta}(u^a v^{-b})^{1-\varepsilon}\right\}^{1-\frac{p-1+\delta}{r}}\cdot\left(u^{r+a}v^{-s-b-1}\right)^{\frac{p-1+\delta}{r}} \\
&\quad = \left\{\frac{p-1+\delta}{r}\cdot\tau^{-1}b\cdot u^{r+a}v^{-s-b-1}\right\}^{\frac{p-1+\delta}{r}} \\
&\qquad \cdot\left\{\left[a\left(\tau^{-1}b\cdot\frac{p-1+\delta}{r}\right)^{-\frac{p-1-\delta}{r}}\right]^{(1-\frac{p-1+\delta}{r})^{-1}}\right. \\
&\qquad \left.\cdot v^{-\theta}(u^a v^{-b})^{1-\varepsilon}\right\}^{1-\frac{p-1+\delta}{r}} \\
&\quad \le \tau^{-1}b\cdot u^{r+a}v^{-s-b-1}+C_8(a,b)\cdot v^{-\theta}\cdot(u^a v^{-b})^{1-\varepsilon},
\end{aligned}
$$

where

$$
C_8(a,b)=(1-\frac{p-1+\delta}{r})\cdot\left[a\left(\tau^{-1}b\cdot\frac{p-1+\delta}{r}\right)^{-\frac{p-1-\delta}{r}}\right]^{(1-\frac{p-1+\delta}{r})^{-1}}.
$$

We thus end up with

$$
\begin{aligned}
& \frac{d}{dt}\int_\Omega u^a v^{-b}dx \le (-a+\tau^{-1}b)\int_\Omega u^a v^{-b}dx \\
&\qquad +C_8(a,b)\int_\Omega v^{-\theta}(u^a v^{-b})^{1-\varepsilon}dx.
\end{aligned}
$$

Since

$$\int_\Omega v^{-\theta}(u^a v^{-b})^{1-\varepsilon} dx \le \left\{ \int_\Omega u^a v^{-b} dx \right\}^{1-\varepsilon} \cdot \left\{ \int_\Omega v^{-\theta/\varepsilon} dx \right\}^{\varepsilon},$$

we obtain (49) with $C_7(a,b) = C_8(a,b)$. □

Now we show the following lemma.

Lemma 3.3. *Under the assumptions of Theorem 1.1, any $a > 1$ admits $0 < b \ll 1$ such that*

$$\int_\Omega u^a v^{-b} dx \le C_9, \quad t \ge 0. \tag{52}$$

Proof. If $a > 1$ and $b > 0$ satisfy (48), we have

$$\frac{d}{dt}\int_\Omega u^a v^{-b} dx \le (-a + \tau^{-1} b) \int_\Omega u^a v^{-b} dx + C_{10} \left\{ \int_\Omega u^a v^{-b} dx \right\}^{1-\varepsilon}$$

by Lemmas 3.1 and 3.2. Given $a > 1$, on the other hand, we can take $0 < b \ll 1$ satisfying (48) and $a > \tau^{-1} b$. Then inequality (52) follows from Lemma 2.2 of [12] or Lemma 3 of [7]. □

We proceed to the following lemma.

Lemma 3.4. *Under the assumptions of Theorem 1.1, it holds that*

$$\lim_{t \uparrow +\infty} \|(u(\cdot,t), v(\cdot,t)) - (\overline{u}(t), \overline{v}(t))\|_{C^2} = 0, \tag{53}$$

where

$$\overline{u}(t) = \int_\Omega u(x,t)\ dx, \quad \overline{v}(t) = \int_\Omega v(x,t)\ dx. \tag{54}$$

Proof. Lemmas 3.1 and 3.3 guarantee

$$\left\| \frac{u^p}{v^q}(\cdot,t) \right\|_\ell + \left\| \frac{u^r}{v^s}(\cdot,t) \right\|_\ell \le C_{11}$$

for $\ell > \max\{n/2, 1\}$. Then we obtain

$$\|u(\cdot,t)\|_\infty + \|v(\cdot,t)\|_\infty \le C_{12} \tag{55}$$

similarly to the proof of Lemma 3.1. The orbit

$$O = \{(u(\cdot,t), v(\cdot,t)\}_{t \ge 0}$$

is thus compact in $C^2(\overline{\Omega}) \times C^2(\overline{\Omega})$ by the parabolic regularity using inequalities (45) and (55).

From the classical theory of dynamical systems (see [3], for example), ω-limit set of the above O is defined by

$$\begin{aligned} \omega(u_0, v_0) = \{(u_*, v_*) \mid \exists t_k \uparrow +\infty \\ \text{s.t. } \|(u(\cdot,t_k), v(\cdot,t_k)) - (u_*, v_*)\|_{C^2} = 0\}. \end{aligned}$$

It is compact and connected in $C^2(\overline{\Omega}) \times C^2(\overline{\Omega})$. On the other hand, we have

$$v_* > 0 \quad \text{on } \overline{\Omega} \tag{56}$$

by (45) and therefore, a semi-flow is well-defined on $\omega(u_0, v_0)$ using (9) and (10).

The set $\omega(u_0, v_0)$, furthermore, is invariant under this flow, and the solution $(\tilde{u}, \tilde{v}) = (\tilde{u}(\cdot, t), \tilde{v}(\cdot, t))$ to (9), (10), and

$$\tilde{u}|_{t=0} = u_* \geq 0, \quad \tilde{v}|_{t=0} = v_* > 0 \qquad \text{on } \overline{\Omega}$$

satisfies

$$\frac{d}{dt}\int_\Omega H(\tilde{\xi}(\cdot, t), \tilde{\eta}(\cdot, t))\ dx = 0, \quad t > 0, \tag{57}$$

where

$$\tilde{\xi} = \frac{\tilde{u}^{-p+1}}{p-1}, \quad \tilde{\eta} = \frac{\tilde{v}^{s+1}}{s+1}.$$

In fact, Fatou's lemma guarantees

$$\int_\Omega u_*^{-p+1} v_*^{s+1} + u_*^{r-p+1} + v_*^{-q+s+1}\ dx < +\infty,$$

recalling (44). Since $v_* > 0$ on $\overline{\Omega}$, it holds that

$$\int_\Omega u_*^{-p+1} dx < +\infty$$

and hence $u_* \not\equiv 0$. Then we obtain $\tilde{u}(\cdot, t) > 0$ on $\overline{\Omega}$ for $t > 0$ by the parabolic strong maximum principle to (9) with (10).

Consequently, the value $H(\tilde{\xi}(\cdot, t), \tilde{\eta}(\cdot, t))$ is well-defined for $t > 0$, which is invariant from the LaSalle principle. This property implies (57). Then, it holds that

$$\int_\Omega c_1 |\nabla \xi^{-\alpha/2}|^2 + c_2 |\nabla \eta^{-\beta/2}|^2\ dx \leq 0$$

by (43). Hence this $(\tilde{u}, \tilde{v}) = (\tilde{u}(\cdot, t), \tilde{v}(\cdot, t))$, $t > 0$, is a pair of spatially homogeneous functions. Namely, the above (u_*, v_*) must be a pair of positive constant functions.

We have proven that $\omega(u_0, v_0)$ is contained in the set of pairs of positive constants. Then it holds that

$$\lim_{t \uparrow +\infty} \{\|\nabla u(\cdot, t)\|_{C^1} + \|\nabla v(\cdot, t)\|_{C^1}\} = 0 \tag{58}$$

and hence (53) with (54). □

We show the first part of Theorem 1.1.

Lemma 3.5. *Under the assumptions of Theorem 1.1 there is an ODE orbit $\hat{O} \subset \mathbf{R}^2$ satisfying (22).*

Proof. Given the solution $(u, v) = (u(\cdot, t), v(\cdot, t))$ to (9), (10), and (14) with $u_0 \not\equiv 0$, the orbit $O = \{(u(\cdot,), v(\cdot, t))\}_{t \geq 0}$ exists global-in-time and is compact in $C^2(\overline{\Omega}) \times C^2(\overline{\Omega})$. By (43) the value

$$H_\infty = \lim_{t \uparrow +\infty} \int_\Omega H(\xi(x, t), \eta(x, t)) dx \tag{59}$$

exists, where $(\xi, \eta) = (\xi(\cdot, t), \eta(\cdot, t))$ is defined by (35). Furthermore, any $(u_*, v_*) \in \omega(u_0, v_0)$ is a pair of positive constants.

Since the set $\omega(u_0, v_0)$ is invariant under the flow defined by (9) and (10), this (u_*, v_*) lies on one of the ODE orbit of (7) which is always time-periodic in the case of $\tau = (p-1)/(s+1)$. Since this ODE system takes the Hamilton formalism (37), the above orbit is detemined by the first integral, that is, H_∞ defined by (59). Hence $\omega(u_0, v_0)$ is contained in a definite ODE orbit denoted by $\hat{O}$, and then it holds that (22). □

The following lemma is used for the proof of the second part of Theorem 1.1.

Lemma 3.6. *Under the assumptions of Theorem 1.1 each $t_k \uparrow +\infty$ admits $\{t'_k\} \subset \{t_k\}$ and an ODE solution $(\hat{u}(t), \hat{v}(t))$ such that $\hat{O} = \{(\hat{u}(t), \hat{v}(t))\}_{t\in\mathbf{R}}$ and*

$$\lim_{k\to\infty} \sup_{t\in[-T,T]} \|(u(\cdot, t+t'_k), v(\cdot, t+t'_k)) - (\hat{u}(t), \hat{v}(t))\|_{C^2} = 0 \tag{60}$$

for any $T > 0$.

Proof. Inequalities (45) and (55) imply

$$\|u_t(\cdot, t)\|_{C^2} + \|v_t(\cdot, t)\|_{C^2} \leq C_{13}, \quad t \geq 1$$

by the parabolic regularity. Hence by the Ascoli-Arzelá theorem $t_k \uparrow +\infty$ admits $\{t'_k\} \subset \{t_k\}$ and a solution $(\hat{u}, \hat{v}) = (\hat{u}(\cdot, t), \hat{v}(\cdot, t))$ to (9)-(10) such that

$$\lim_{k\to\infty} \sup_{t\in[-T,T]} \|(u(\cdot, t+t'_k), v(\cdot, t+t'_k)) - (\hat{u}(\cdot, t), \hat{v}(\cdot, t))\|_{C^2} = 0 \tag{61}$$

for any $T > 0$. Since (58) implies

$$\nabla \hat{u}(\cdot, t) = \nabla \hat{v}(\cdot, t) = 0, \quad t \in [-T, T]$$

this $(\hat{u}, \hat{v}) = (\hat{u}(\cdot, t), \hat{v}(\cdot, t))$ must be spatially homogeneous, denoted by $(\hat{u}, \hat{v}) = (\hat{u}(t), \hat{v}(t))$. Consequently, it is a solution to (7), and then (60) follows from (61). □

We are ready to complete the following proof.

Proof of Theorem 1.1. It remains to show (23). Let $\ell \geq 0$ be the time period of the solution to (7) on $\hat{O}$ in Lemma 3.6. Unless $\hat{O}$ is composed of a single point, it holds that $\ell > 0$. Then we take $T > 2\ell$.

By Lemma 3.6, any $t_k \uparrow +\infty$ admits $\{t'_k\} \subset \{t_k\}$ and a solution $(\hat{u}(t), \hat{v}(t))$ to (7) such that $\hat{O} = \{(\hat{u}(t), \hat{v}(t))\}_{t\in\mathbf{R}}$ and (60). Let $t \in [-\ell, \ell]$ be fixed. Then it holds that

$$(\hat{u}(t+\ell), \hat{v}(t+\ell)) = (\hat{u}(t), \hat{v}(t)) \tag{62}$$

and therefore,

$$\begin{aligned}
&\limsup_{k\to\infty} \|(u(\cdot, t+\ell+t'_k), v(\cdot, t+\ell+t'_k)) \\
&\quad -(u(\cdot, t+t'_k), v(\cdot, t+t'_k))\|_{C^2} \\
&\leq \lim_{k\to\infty} \|(u(\cdot, t+\ell+t'_k), v(\cdot, t+\ell+t'_k)) - (\hat{u}(t+\ell), \hat{v}(t+\ell))\|_{C^2} \\
&\quad + \lim_{k\to\infty} \|(u(\cdot, t+t'_k), v(\cdot, t+t'_k)) - (\hat{u}(t), \hat{v}(t))\|_{C^2} = 0.
\end{aligned}$$

This property means

$$\lim_{s\uparrow+\infty} \|(u(\cdot, t+\ell+s), v(\cdot, t+\ell+s)) - (u(\cdot, t+s), v(\cdot, t+s))\|_{C^2} = 0$$

and in particular, (23) follows. □

Acknowledgments. The first author is supported by a Marie Curie International Research Staff Exchange Scheme (IRSES) with the 7th European Community Framework Programme, under the grant PIRSESGA-2009-247486. The first author is also partially supported by the FP7-REGPOT-2009-1 project "Archimedes Center for Modeling, Analysis and Computation". The second author was supported by a JST project, CREST (Alliance for Breakthrough Between Mathematics and Sciences). Part of this work was done while he was visiting Department of Applied Mathematics in Crete University, April, 2011. He thanks the warm hospitality.

REFERENCES

[1] N.D. Alikakos, *An application of the invariance principle to reaction-diffusion equations*, J. Differential Equations **33** (1979) 201-225. MR0542670

[2] A. Gierer and H. Meinhardt, *A theory of biological pattern formation*, Kybernetik **12** (1972) 30-39.

[3] J.K. Hale, *Asymptotic Behavior of Dissipative Systems*, Amer. Math. Soc., Providence, 1988. MR0941371

[4] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Math. **840** Sprinver-Verlag, Berlin, 1981. MR0610244

[5] H. Hoshino and Y. Yamada, *Sovability and smoothing effect for semilinear parabolic equations*, Funkcialaj Ekvacioj **34** (1991) 475-494. MR1150875

[6] D. Iron, M.J. Ward, and J. Wei, *The stability of spike solutions to the one-dimensional Gierer-Meinhardt model*, Physica D **150** (2001) 25-62. MR1818735

[7] H. Jiang, *Global existence of solutions of an activator-inhibitor system*, Discrete and Continuous Dynamical Systems **14** (2006) 737-751. MR2177095

[8] H. Jiang and W.-M. Ni, *A priori estimates of stationary solutions of an activator-inhibitor system*, Indiana Univ. Math. J. **56** (2007) 681-732. MR2317543

[9] A.J. Koch and H. Meinhardt, *Biological pattern formation: from basic mechanism to complex structure*, Rev. Modern Physiscs **66** (1994) 1481-1510.

[10] E. Latos, T. Suzuki, and Y. Yamada, *Transient and asymptotic dynamics of a prey-predator system with diffusion*, to appear in; Math. Meth. Appl. Sci.

[11] F. Li and W.-M. Ni, *On the global existence and finite time blow-up of shadow systems*, J. Differential Equations **247** (2009) 1762-1776. MR2553858

[12] K. Masuda and T. Takahashi, *Reaction-diffusion systems in Gierer-Meinhardt theory in biological pattern formation*, Japan J. Appl. Math. **4** (1987) 47-58. MR0899203

[13] J.D. Murray, *Mathematical Biology II: Spatial Models and Biomedical Applications*, third edition, Springer, New York, 2003. MR1952568

[14] W.-M. Ni, K. Suzuki, and I. Takagi, *The dynamics of a kinetic activator-inhibitor system*, J. Differential Equations **229** (2006) 426-465. MR2263562

[15] W.-M. Ni and I. Takagi, *On the Neumann problem for some semilinear elliptic equations and systems of activator-inhibitor type*, Trans. Amer. Math. Soc. **297** (1986) 351-368. MR0849484

[16] F. Rothe, *Global Solutions of Reaction-Diffusion Equations*, Lecture Notes in Math. **1072** Springer-Verlag, 1984. MR0755878

[17] A. Turing, *The chemical basis of morphogenesis*, Philos. Transl. Roy. Soc. London, **B237** (1952) 37-72.

[18] J. Wei, *Existence and stability of spikes for the Gierer-Meinhardt sytem*, Handbook of Differential Equations, Stationary Partial Differential Equations **5** (Edited by M. Chipot), Elsevier, Amsterdam, 2008. MR2497911

[19] E. Yanagida, *Reaction-diffusion systems with skew-gradient structure*, Meth. Appl. Anal. **8** (2001) 209-226. MR1904526

[20] E. Yanagida, *Mini-maximizers for reaction-diffusion systems with skew-gradient structure*, J. Differential Equations **179** (2002) 311-335. MR1883746

Received xxxx 20xx; revised xxxx 20xx.

E-mail address: gkarali@tem.uoc.gr
E-mail address: suzuki@sigmath.es.osaka-u.ac.jp
E-mail address: yamadsa@waseda.jp