

# RESONANCE PHENOMENA IN A SINGULAR PERTURBATION PROBLEM IN THE CASE OF EXCHANGE OF STABILITIES

GEORGIA KARALI AND CHRISTOS SOURDIS

ABSTRACT. We consider the following singularly perturbed elliptic problem:

$$\varepsilon^2 \Delta u = (u - a(\mathbf{y})) (u - b(\mathbf{y})) \quad \text{in } \Omega, \quad \frac{\partial u}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\Omega,$$

where  $\Omega$  is a bounded domain in  $\mathbb{R}^2$  with smooth boundary,  $\varepsilon > 0$  is a small parameter,  $\mathbf{n}$  denotes the outward normal of  $\partial\Omega$ , and  $a, b$  are smooth functions that do not depend on  $\varepsilon$ . We assume that the zero set of  $a - b$  is a simple closed curve  $\Gamma$ , contained in  $\Omega$ , and  $\nabla(a - b) \neq 0$  on  $\Gamma$ . We will construct solutions  $u_\varepsilon$  that converge in the Hölder sense to  $\max\{a, b\}$  in  $\Omega$ , and their Morse index tends to infinity, as  $\varepsilon \rightarrow 0$ , provided that  $\varepsilon$  stays away from certain *critical numbers*. Even in the case of stable solutions, whose existence is well established for *all* small  $\varepsilon > 0$ , our estimates improve previous results.

## 1. INTRODUCTION

**1.1. The problem and known results.** In this paper we consider the following elliptic problem:

$$\varepsilon^2 \Delta u - (u - a(\mathbf{y})) (u - b(\mathbf{y})) = 0 \quad \text{in } \Omega, \quad \frac{\partial u}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\Omega, \quad (1.1)$$

where  $\Omega$  is a smooth bounded domain in  $\mathbb{R}^2$ , containing the origin, the functions  $a, b$  are five times continuously differentiable in a tubular neighborhood of the curve  $\Gamma$ , defined below, and two times in the complementary points of  $\bar{\Omega}$ . By  $\mathbf{n}=\mathbf{n}(\mathbf{y})$  we denote the outward unit normal to  $\partial\Omega$ , and  $\varepsilon > 0$  is a small number. We assume that there exists a simple, smooth and closed curve  $\Gamma \subset \Omega$  which separates the domain into two disjoint components, namely

$$\Omega = \Omega_1 \cup \Gamma \cup \Omega_2, \quad (1.2)$$

such that

$$a(\mathbf{y}) > b(\mathbf{y}) \quad \text{in } \Omega_1, \quad a(\mathbf{y}) < b(\mathbf{y}) \quad \text{in } \Omega_2, \quad \frac{\partial a}{\partial \nu} < \frac{\partial b}{\partial \nu} \quad \text{on } \Gamma, \quad (1.3)$$

where  $\nu = \nu(\mathbf{y})$  denotes the outward unit normal to  $\partial\Omega_1$ . The last assumption in (1.3) be viewed as a non-degeneracy condition.

We are interested in solutions of (1.1) that converge to  $\max\{a, b\}$ , as  $\varepsilon \rightarrow 0$ , uniformly in  $\bar{\Omega}$ . Note that, by assumption (1.3), the singular limit function  $\max\{a, b\}$  is merely continuous and *not* differentiable across the curve  $\Gamma$ . In other words, the solutions we seek have a *corner layer* along the curve  $\Gamma$ . More precisely, our purpose in this paper is to show, via a perturbation argument, the existence of highly unstable solutions of this type. To the best of our knowledge, with the exception of one-dimensional or radially symmetric cases, elliptic or parabolic problems involving corner layered solutions have been treated thus far only by constructing upper

and lower solutions or by weak convergence arguments (see the discussion below). The case where  $a, b$  do not intersect is by now standard, see [16].

Let us mention that, even if posed in a one-dimensional domain, problem (1.1) (under (1.3)) is non-trivial. This is due to the fact that the solution set of the limit algebraic equation

$$(u - a(\mathbf{y}))(u - b(\mathbf{y})) = 0 \quad (1.4)$$

undergoes a transcritical bifurcation as  $\mathbf{y}$  crosses the point corresponding to  $\Gamma$ . At the bifurcation point, the two branches of solutions of (1.4), namely,

$$\mathcal{C}_a = \{(a(\mathbf{y}), \mathbf{y}), \mathbf{y} \in \Omega\}, \quad \mathcal{C}_b = \{(b(\mathbf{y}), \mathbf{y}), \mathbf{y} \in \Omega\},$$

*exchange stabilities* with respect to the dynamics of

$$\dot{u} = -(u - a(\mathbf{y}))(u - b(\mathbf{y})),$$

see [25, pg. 29]. Equation (1.4) is frequently referred to as the outer problem and, loosely speaking, determines the slow manifold of the slow-fast system corresponding to the one-dimensional version of equation (1.1) (see [30]). As predicted by the above discussion, the slow manifold loses its normal hyperbolicity, due to a transcritical bifurcation, which prevents the use of standard geometric singular perturbation theory [30] and one has to use a blow-up procedure (see [44]).

Motivated from reaction kinetics, it was shown in [9], by the method of upper and lower solutions, that problem (1.1) (under (1.3)) has a solution such that

$$\|u_\varepsilon - \max\{a, b\}\|_{L^\infty(\Omega)} = \mathcal{O}(\varepsilon^{\frac{2}{3}}), \quad \varepsilon \rightarrow 0, \quad \text{and } u_\varepsilon > a \text{ on } \Gamma. \quad (1.5)$$

In the special case where  $b$  is identically zero, problem (1.1) becomes the well known scalar logistic equation, see [12], and the existence of a positive solution which satisfies (1.5) was shown in [26], using a different construction of upper and lower solutions than [9]. In this context, relation (1.5) describes spatial segregation (see also [34]). Furthermore, it was also shown there that  $u_\varepsilon$  is asymptotically stable (with respect to the parabolic dynamics), and the principal eigenvalue of the associated linearized operator satisfies

$$\lambda_{1,\varepsilon} \geq c\varepsilon^{\frac{2}{3}} > 0,$$

for some constant  $c > 0$  and all small  $\varepsilon > 0$  (see also [10] for a different proof of this lower bound in the one-dimensional case). Let us point out here that the method of upper and lower solutions captures only stable solutions and, in general, is not applicable to the study of systems (see [45]). The possibility of formulating a theorem regarding the asymptotic behavior, as  $\varepsilon \rightarrow 0$ , of uniformly bounded stable solutions of (1.1), with  $a, b$  as in the present paper, has been speculated in pg. 79 of the review article [15].

On the other hand, in the radially symmetric case (in  $N \geq 2$  dimensions), with  $\Gamma = \{|\mathbf{y}| = r_0\}$ , it was shown in [32], by a perturbation argument, that problem (1.1) (under (1.3)) has an unstable radial solution  $u_\varepsilon^-$  such that, in particular,

$$\|u_\varepsilon^- - \max\{a, b\}\|_{L^\infty(\Omega)} = \mathcal{O}(\varepsilon^{\frac{2}{3}}), \quad \varepsilon \rightarrow 0, \quad \text{and } u_\varepsilon^- < a \text{ on } \Gamma. \quad (1.6)$$

Furthermore, given  $m \in \mathbb{N}$ , the first  $m$  eigenvalues of the linearized operator around  $u_\varepsilon^-$ , restricted to the radial class of functions, satisfy

$$\lambda_{i,\varepsilon} = \mu_i \varepsilon^{\frac{2}{3}} + \mathcal{O}(\varepsilon^{\frac{4}{3}}), \quad \varepsilon \rightarrow 0, \quad i = 1, \dots, m, \quad (1.7)$$

where  $\mu_i$  are the first  $m$  eigenvalues of a limit problem and satisfy

$$\mu_1 < 0, \quad \mu_2 > 0.$$

It was also shown, by expanding in polar coordinates, that the first  $K$  eigenvalues (not counting multiplicities) of the linearized operator, in the general class, satisfy

$$\Lambda_{i,\varepsilon} = \mu_1 \varepsilon^{\frac{2}{3}} + r_0^{-2} \tau_i \varepsilon^2 + \mathcal{O}(\varepsilon^{\frac{4}{3}}), \quad i = 1 \cdots, K, \quad \varepsilon \rightarrow 0, \quad (1.8)$$

where  $\tau_i = (i-1)(i+N-3)$ ,  $i = 1, \dots$ , are the eigenvalues of the Laplace–Beltrami operator of  $\mathbb{S}^{N-1}$ , provided

$$\mu_1 + r_0^{-2}(K-1)(K+N-3)\varepsilon^{\frac{4}{3}} \leq \frac{\mu_2}{4}.$$

As a consequence, it was shown that the Morse index  $M_\varepsilon$  of  $u_\varepsilon^-$ , namely the number of negative  $\Lambda_i$ 's (counting multiplicities), satisfies

$$\lim_{\varepsilon \rightarrow 0} M_\varepsilon / \varepsilon^{-\frac{2}{3}(N-1)} = \mu_*, \quad (1.9)$$

where  $\mu_*$  depends only on  $r_0$ ,  $\mu_1$ , and the dimension  $N$ . Moreover, it was proven that bifurcations of non-radial solutions take place along a certain infinite discrete set of values  $\varepsilon_j \rightarrow 0$ . The goal of the present paper is to show that the perturbation argument of [32], and the infinite–dimensional reduction of [17], can be adapted to capture analogous solutions, in two dimensions, without the simplifying assumption of radial symmetry.

Let us end this subsection by mentioning that, in the one–dimensional case, solutions of (1.1), oscillating close to  $a$ ,  $b$  or  $\min\{a, b\}$ , have been investigated in [24].

**1.2. Motivation for the current work.** Convergence, in the singular limit, to the maximum or minimum of a family of functions, as in (1.5) or (1.6), typically occurs in systems of nonlinear elliptic equations with competition, and describes phase separation, as the repulsive interaction tends to infinity. These include systems of Lotka–Volterra type [13], and systems arising in the Hartree–Fock theory of a mixture of Bose–Einstein condensates [47]. Actually, the original problem that led the authors of [9] to study (1.1) was a coupled system of this type. Elliptic systems where the singular limit has Hölder or Lipschitz regularity also arise in combustion theory [11]. Remarkably, unstable solutions with corner layer profile of the same nature as those considered in the present paper can be found in [11] (see Remark 3.1 below). In most cases, standard weak convergence arguments are quite sufficient to pass to the singular limit in the pointwise and strong  $L^2$  sense. An important question, treated in the previously mentioned references, is whether families of bounded solutions converge in spaces of Hölder continuous (or Lipschitz) functions and keeping track of the maximal global regularity available. For the problem at hand we answer this question, for the solutions we construct, and we hope that the perturbation approach of the current paper can be useful in the more general situations mentioned above. We note that, in contrast to weak convergence or upper and lower solution arguments, perturbation arguments can be applied to general systems without special monotonicity properties.

Elliptic singular perturbation problems, where the singular limit is Hölder continuous, but *not* Lipschitz, appear frequently in applications. In these situations, the limit algebraic equation (the analog of (1.4)) typically undergoes a pitchfork or saddle–node bifurcation as the parameter  $\mathbf{y}$  crosses a curve  $\Gamma$ . The case of pitchfork bifurcation has received a lot of attention recently, as it occurs when minimizing a Gross–Pitaevskii functional (see [1], [2], [3], [21], [27], and [40]). Due to the irregular nature of the singular limit (it does not belong in the Sobolev space  $H^1$ ),

standard weak convergence arguments are not applicable. Furthermore, it seems to be hard to construct a good pair of upper and lower solutions (especially). Actually, to the best of our knowledge, the behavior of solutions near  $\Gamma$  (estimates, monotonicity properties, etc), as  $\varepsilon \rightarrow 0$ , is well understood only in the case of radial symmetry<sup>1</sup>. Typically, the behavior of solutions can be satisfactory studied outside of an  $\varepsilon$ -dependent tubular neighborhood of the bifurcation curve  $\Gamma$ , by constructing suitable upper and lower solutions. Then, taking advantage of the radial symmetry, one shows that positive solutions are monotone in this tubular neighborhood and, therefore, is able to complete the picture (see [2]). A class of slow-fast Hamiltonian systems, in which the slow manifold loses normal hyperbolicity due to a pitchfork bifurcation, arises in the study of crystalline grain boundaries, see [4]. This problem has been treated successfully by a shooting argument in [20], geometric singular perturbation theory in [44], and a perturbation argument, in the spirit of the current paper, in [46]. A class of elliptic equations, where the corresponding limit algebraic equation admits a saddle-node bifurcation, appear in the proof of the Lazer–Mckenna conjecture for a superlinear elliptic problem of Ambrosetti-Prodi type, see [14] (in this case we have  $\Gamma \equiv \partial\Omega$ ). We believe that the perturbation approach we introduce in the current paper can be useful in these problems, since it provides very accurate estimates up to the bifurcation curve  $\Gamma$ , of the form (1.17) below. In turn, these are important for understanding interesting phenomena such as the appearance of vortices (see [2]), or the existence of small peak solutions (see [14]), close to  $\Gamma$ .

**1.3. Statement of the Main Result.** To state our main result, we feel that it is useful to first make some definitions available, and formally review some arguments, from the rest of the paper.

Let  $\Gamma$  be the closed smooth curve in (1.2), and let  $\ell = |\Gamma|$  denote its total length. We consider the natural parametrization  $\gamma = \gamma(\theta)$  of  $\Gamma$  with positive orientation, where  $\theta$  denotes an arc length parameter measured from a fixed point of  $\Gamma$ . Let  $\nu(\theta)$  denote the outer unit normal to  $\Gamma$ , as in (1.3). Points  $\mathbf{y}$  that are  $\delta_0$ -close to  $\Gamma$ , for sufficiently small  $\delta_0$ , can be represented in the form

$$\mathbf{y} = \gamma(\theta) + t\nu(\theta), \quad |t| < \delta_0, \quad \theta \in [0, \ell), \quad (1.10)$$

where the map  $\mathbf{y} \mapsto (t, \theta)$  is a local diffeomorphism. Note that  $t > 0$  in  $\Omega_2$ .

Blowing up (1.1) around the curve  $\Gamma$ , via the rescaling (3.2), (3.11) below, and keeping in mind that this inner blow up must match with the outer blow up of  $\max\{a, b\}$ , will lead us to the problem:

$$\begin{cases} v_{zz} + v_{xx} - v^2 + x^2 = 0, & (x, z) \in \mathbb{R}^2, \\ v - |x| \rightarrow 0, & x \rightarrow \pm\infty, \end{cases} \quad (1.11)$$

with  $v$  being  $\hat{\ell}/\varepsilon^{\frac{2}{3}}$ -periodic in the variable  $z$ , with  $\hat{\ell}$  as in (7.21) below. A stable one-dimensional solution  $V_+$ , of the above equation, can be constructed by the method of upper and lower solutions. We have that  $V_+$  is even,  $(V_+)_x > 0$ ,  $x > 0$ , and  $V_+ > x$ ,  $x \geq 0$ . We then seek other solutions of (1.11) in the form  $v = V_+ - W$ , and find that  $W$  has to satisfy:

---

<sup>1</sup>By adapting the approach of the current paper, we have recently removed this restrictive assumption in [33].

$$\begin{cases} W_{zz} + W_{xx} - 2V_+(x)W + W^2 = 0, & (x, z) \in \mathbb{R}^2, \\ W \rightarrow 0, & x \rightarrow \pm\infty, \end{cases} \quad (1.12)$$

with  $W$  being  $\hat{\ell}/\varepsilon^{\frac{2}{3}}$ -periodic in  $z$ . We will show, in Proposition 3.1, that (1.12) has a one-dimensional, positive, even solution  $w$  such that  $V(x) := V_+(x) - w(x) < |x|$ ,  $x \in \mathbb{R}$ , solves (1.11). Furthermore, the associated one-dimensional eigenvalue problem

$$\phi_{xx} - 2V\phi = \lambda\phi, \quad x \in \mathbb{R}, \quad \phi(\pm\infty) = 0, \quad (1.13)$$

has eigenvalues  $\lambda_0 > \lambda_1 > \dots$ , with  $\lambda_0 > 0$  and  $\lambda_1 < 0$ ; we denote by  $Z > 0$  the  $L^2$ -normalized eigenfunction corresponding to  $\lambda_0$ .

We define the number  $\lambda_*$  as

$$\lambda_* = \lambda_0 \frac{1}{4\pi^2} \left( \int_0^\ell \beta(\theta) d\theta \right)^2, \quad (1.14)$$

where  $\beta$  as in (3.10) below.

Now we can state our main result:

**Theorem 1.1.** *Given  $c > 0$ , there exists  $\varepsilon_0 > 0$  such that for all  $\varepsilon < \varepsilon_0$  satisfying the gap condition*

$$|\varepsilon^{\frac{4}{3}} i^2 - \lambda_*| \geq c\varepsilon^{\frac{2}{3}} \quad \forall i \in \mathbb{N}, \quad (1.15)$$

where  $\lambda_* > 0$  is the number in (1.14), problem (1.1) has a solution such that

$$\|u_\varepsilon - \max\{a, b\}\|_{L^\infty(\Omega)} = \mathcal{O}(\varepsilon^{\frac{2}{3}}) \quad \text{and} \quad u_\varepsilon < a \quad \text{on } \Gamma. \quad (1.16)$$

More precisely, there exist constants  $\delta > 0$  (small) and  $D > 0$  (large) such that the following estimates hold: Near the curve  $\Gamma$ , for  $\mathbf{y}$  given by (1.10),

$$u_\varepsilon(\mathbf{y}) = \begin{cases} a(0, \theta) + \varepsilon^{\frac{2}{3}} \beta^2(\theta) V\left(\beta(\theta) \frac{t}{\varepsilon^{\frac{2}{3}}}\right) + \frac{a_\varepsilon(0, \theta) + b_\varepsilon(0, \theta)}{2} t + \mathcal{O}\left(\varepsilon^{\frac{4}{3}} + t^2\right), & |t| \leq \delta, \\ \max\{a, b\}(\mathbf{y}) + \mathcal{O}(\varepsilon^{\frac{2}{3}}) G\left(\beta(\theta) \frac{t}{\varepsilon^{\frac{2}{3}}}\right) + \mathcal{O}(\varepsilon^2) t^{-1}, & D\varepsilon^{\frac{2}{3}} \leq |t| \leq \delta, \end{cases} \quad (1.17)$$

where

$$G(r) \equiv r^{-\frac{1}{4}} e^{-\frac{2\sqrt{2}}{3} r^{\frac{3}{2}}}, \quad r > 0. \quad (1.18)$$

Away from the curve  $\Gamma$ , we have

$$u_\varepsilon(\mathbf{y}) = \begin{cases} \max\{a, b\}(\mathbf{y}) + \mathcal{O}(\varepsilon^2), & \text{if } \text{dist}(\mathbf{y}, \Gamma) \geq \delta \text{ and } \text{dist}(\mathbf{y}, \partial\Omega) \geq \delta, \\ \max\{a, b\}(\mathbf{y}) + \mathcal{O}(\varepsilon), & \text{if } \text{dist}(\mathbf{y}, \partial\Omega) \leq \delta. \end{cases} \quad (1.19)$$

Moreover, there exists a constant  $C > 0$  such that

$$|\nabla u_\varepsilon(\mathbf{y})| \leq C, \quad \mathbf{y} \in \bar{\Omega}, \quad (1.20)$$

and  $u_\varepsilon \rightarrow \max\{a, b\}$  in  $C^{0, \alpha}(\bar{\Omega})$  as  $\varepsilon \rightarrow 0$ , satisfying (1.15), for every  $0 < \alpha < 1$  but not for  $\alpha = 1$ .

Let us briefly comment on our result, and in particular on the structure of the set in which the parameter  $\varepsilon$  can be chosen. As will be apparent in the proof, our construction does not hold for all values of the parameter  $\varepsilon$  close to zero. There is a resonance phenomenon which prevents the construction to hold for any small value of  $\varepsilon$  and which forces  $\varepsilon$  to be taken away from a set of critical numbers, as

described in (1.15). The latter condition is called resonance. Such a phenomenon is not new and, in the context of singularly perturbed semilinear elliptic equations, was originally found by A. Malchiodi and M. Montenegro in [38]. Since this seminal paper, this phenomenon has also been found in other instances, for example in the study of other semilinear partial differential equations [17, 18, 35, 37, 42] or in the study of constant mean curvature surfaces [36, 39]. Loosely speaking, it is caused by the presence of the tangential dimension  $\theta$  along the curve  $\Gamma$  and the fact that the profile in the normal  $t$  direction is unstable (see the discussion in the next paragraph). The significant difference of the problem at hand, with respect to those in the aforementioned references, occurs in the normal direction  $t$ , where we have a corner layer profile. This delicacy can already be seen from the fine estimate in the second relation of (1.17).

The fact that we are not able to construct the solutions for all values of  $\varepsilon$  close enough to zero is also reflected in another important feature of our solutions, namely that their Morse index (defined above (1.9)) tends to infinity as  $\varepsilon$  tends to zero.

**Proposition 1.1.** *As  $\varepsilon$  in (1.15) tends to zero, the Morse index of  $u_\varepsilon$  tends to infinity.*

We feel that it will be helpful to the reader to present at this point a formal argument that justifies part of (1.15): The linearization of the blown-up problem (1.11) on  $V$ , namely

$$\begin{cases} \Phi_{zz} + \Phi_{xx} - 2V(x)\Phi = \lambda\Phi, & (x, z) \in \mathbb{R}^2, \\ \Phi \rightarrow 0, & x \rightarrow \pm\infty, \end{cases} \quad (1.21)$$

with  $\Phi$  being  $2\pi\hat{\ell}/\varepsilon^{\frac{2}{3}}$ -periodic in the variable  $z$ , has functions of the form

$$\left[ a \sin\left(\frac{2\pi i}{\hat{\ell}}\varepsilon^{\frac{2}{3}}z\right) + b \cos\left(\frac{2\pi i}{\hat{\ell}}\varepsilon^{\frac{2}{3}}z\right) \right] Z(x)$$

as eigenvalues associated to eigenvalues

$$\lambda_0 - \frac{4\pi^2}{\hat{\ell}^2}i^2\varepsilon^{\frac{4}{3}}, \quad i \in \mathbb{N}.$$

Hence, when  $\varepsilon$  hits the critical numbers  $\varepsilon_i$ , with

$$\varepsilon_i^{\frac{4}{3}} = \lambda_0 \frac{1}{4\pi^2} \hat{\ell}^2 i^{-2} \stackrel{(7.21)}{=} \lambda_* i^{-2},$$

we have the presence of a kernel. Thus  $\varepsilon$  should stay away from these numbers, which motivates condition (1.15).

It is irresistible to relate our result to that of [17], where concentrating solutions along closed geodesics for the nonlinear Schrödinger equation were constructed in the semiclassical limit regime. There the blow up problem was (1.12) with constant positive potential  $V_+$  and power nonlinearity of exponent  $p > 1$ . (Recall that in our case the blow up problem is (1.11) and not the auxiliary problem (1.12)). In that reference a similar resonance phenomenon was observed and treated successfully by introducing an infinite dimensional Lyapunov-Schmidt reduction. The eigenvalues of the linearization, corresponding to (1.13), were  $\dots < \lambda_1 = 0 < \lambda_0$ , and the fact that  $\lambda_1 = 0$  caused a further difficulty that is not present in our case. An elliptic problem, involving resonance, in which the corresponding eigenvalues are as in the present situation, as described below (1.13), can be found in [42].

**Remark 1.1.** *The same result continuous to hold for the more general problem*

$$\varepsilon^2 \Delta u = f(u, \mathbf{y}) \text{ in } \Omega, \quad \frac{\partial u}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega,$$

*as treated originally in [9]. In order to bring out clearly the underline ideas we refrain from any such generalization.*

**Remark 1.2.** *Using the perturbation approach of the current paper, one can show that all the assertions of Theorem 1.1, with the exception of (1.15), (1.16), hold true for all small  $\varepsilon > 0$  if  $V$  is replaced by the one-dimensional stable solution  $V_+$  of (1.11). In this case, with the obvious notation, the same result continuous to hold if  $\Omega$  is a smooth domain in  $\mathbb{R}^N$ ,  $N \geq 1$ , and the zero set of  $a - b$  is an  $(N - 1)$ -dimensional submanifold  $\mathcal{M}$  of  $\Omega$  such that  $\nabla(a - b) \neq 0$  on  $\mathcal{M}$ . (One could also formulate a theorem for the case where  $\mathcal{M}$  is  $(N - m)$ -dimensional,  $1 \leq m \leq N - 1$ ). The corresponding estimates sharpen and are considerably more elaborate than those of [9] or [26] (recall (1.5)). Furthermore, we believe that they can be used in extending to arbitrary dimensions Theorem 1.5 in [34] which is proven in one space dimension.*

*Non-degeneracy conditions, as the last assumption in (1.3), are common in the study of transition layered solutions of elliptic equations with bistable nonlinearity (see [37] and the references therein). In that context, the curve  $\Gamma$  represents the interface of the layer. It turns out that, in some cases, the aforementioned conditions can be removed completely (see [16]). In particular, the interface may be non-smooth or intersect the boundary of the domain. Such generalizations may also be possible for problem (1.1), at least for the case of stable solutions. In this regard, we refer to [12, Prop. 3.16] for a related result.*

#### 1.4. Method of proof of the main theorem, and structure of the paper.

We briefly describe the reasons which cause the main difficulties in proving Theorem 1.1.

The proof of Theorem 1.1 consists in showing that there exists a *genuine solution* near a suitably constructed *approximate solution*, provided the parameter  $\varepsilon$  is chosen small enough and away from a set where resonance occurs. We find it convenient to work exclusively with the equivalent formulation of (1.1) in stretched variables  $y = \varepsilon^{-\frac{2}{3}} \mathbf{y}$ , see (3.1) below. The approximate solution  $u_{ap}$  is carefully built in several steps throughout Sections 3-5. Using the one-dimensional solution  $V$  of (1.11), described above, and a lower order correction  $\phi_1$ , determined by the inhomogeneous linear problem (3.33)–(3.34) below, we construct an inner approximation  $u_{in}$ . This inner approximation is valid *only* near the (stretched) curve  $\Gamma_\varepsilon$ , in the sense that it leaves a remainder in the equation which grows with respect to the distance from the curve. The next step is to match the inner approximation  $u_{in}$  with the outer  $u_0 \equiv \max\{a(\varepsilon^{\frac{2}{3}} y), b(\varepsilon^{\frac{2}{3}} y)\}$ , which is valid for  $y \notin \Gamma_\varepsilon$ , in order to obtain a smooth approximation  $u_{ap}$  that is valid in the whole (stretched) domain  $\Omega_\varepsilon$ . The fact that the inner approximation  $u_{in}$  is valid only in a tubular neighborhood of  $\Gamma_\varepsilon$ , makes it already delicate to use a partition of unity type argument in order to smoothly interpolate between  $u_{in}$  and  $u_0$ . Worse than this, the lower order inner blow up profile  $\phi_1$  converges *algebraically slowly* to the blow-up of the corresponding outer (see Proposition 3.2). This difficulty is not present in other well known elliptic singular perturbation problems such as the nonlinear Schrödinger or Allen-Cahn equation, where the corresponding convergence is exponentially fast (see the references in the discussion following Theorem 1.1). We overcome these difficulties by adapting to

the general case a procedure we introduced for the radially symmetric case in [32] (see Sections 4, 5 below). Actually, by suitably incorporating a partition of unity type argument, we are able to considerably simplify some arguments of [32]. We emphasize that we are able to apply the standard partition of unity argument only after we carefully perturb, as in [32], the outer approximation  $u_0$  to an improved outer approximation  $\tilde{u}_{out}$ . We refer the interested reader to Remark 5.1 for this subtle point. Loosely speaking, the new outer approximation  $\tilde{u}_{out}$  is closer to  $u_{in}$ , and leaves a smaller remainder in the equation, than  $u_0$ . Smoothly interpolating between  $u_{in}$  and  $\tilde{u}_{out}$  leads us to an approximation  $\tilde{u}_{ap}$  which is valid in the entire domain but is more accurate near the curve  $\Gamma_\varepsilon$ . Finally, motivated from [32], we iterate once, using a modified Newton's method, to improve the accuracy of  $\tilde{u}_{ap}$  away from  $\Gamma_\varepsilon$ , and obtain the desired approximation  $u_{ap}$ .

Once we have constructed our approximate solution, we look for a genuine solution in the form

$$u = u_{ap} + \varphi, \tag{1.22}$$

hoping to find  $\varphi$  using the contraction mapping theorem. Therefore, in order to proceed, we need to study the associated linearized operator around  $u_{ap}$ . Here one faces a dramatically different situation compared to the radially symmetric case (recall (1.7)). This is clearly seen already by linearizing around the corresponding spherically symmetric approximate solution  $u_{ap}$ , since the eigenvalues closest to zero are given by formulas (1.8) divided by  $\varepsilon^{\frac{2}{3}}$  (due to the rescaling). These formulas predict that the linearized operator around  $u_{ap}$ , in the general case at hand, will also have an increasing number of negative eigenvalues, as  $\varepsilon \rightarrow 0$ , many of which accumulate to zero and sometimes, depending on the value of  $\varepsilon$ , we even have the presence of a kernel. This clearly causes difficulties in applying local inversion arguments to find  $\varphi$ , and  $\varepsilon$  must be taken away from these values. This kind of phenomena have been dealt with in various problems, see the references below Theorem 1.1. The scheme employed here follows the lines set in [17]. A difficulty we had to face was that, as we discussed in the previous subsection, the limit problem (1.12) in the present situation has a potential that grows, as  $x \rightarrow \pm\infty$ , in contrast to that in [17] which was a constant. The main steps of this scheme are the following: In Section 6 we adapt a very nice trick, already used in [17], in order to reduce the whole problem to a nonlocal problem in an infinite strip. The main advantage of working in the strip is that we can perform a separation of variables in order to study the associated linearized problem (see Section 7). Rather than solving the problem in the strip directly, we first solve a natural projected problem where the linear operator is uniformly invertible (see Section 8). Then, the resolution of the full problem becomes reduced to a nonlinear, nonlocal, second-order system of differential equations, which turns out to be directly solvable thanks to the assumptions made (see Sections 9-11). We end this paper by presenting some related open problems and conjectures in Section 12.

## 2. NOTATION

Throughout this paper, unless specified otherwise, we will denote by  $c/C$  positive small/ large generic constants, independent of  $\varepsilon$ , whose values will change from line to line. The value of  $\varepsilon$  will satisfy  $0 < \varepsilon < \varepsilon_0$  with  $\varepsilon_0$  getting smaller at each step (so that all previous relations still hold). Frequently we will suppress the obvious dependence of quantities on  $\varepsilon$ . Moreover, Landaus symbols  $\mathcal{O}(1), o(1)$ ,  $\varepsilon \rightarrow 0$ , will be understood in the sense that  $|\mathcal{O}(1)| \leq C$  for small  $\varepsilon > 0$ , and  $o(1) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ .



Abusing notation, frequently we will denote points  $\mathbf{y}$  simply by their image  $(t, \theta)$  under the mapping of (1.10). For some further notation, we will use in this paper, see Remark 3.2 below.

### 3. SETUP NEAR THE CURVE

In this section we will construct an approximation for (1.1) which is valid only near the curve  $\Gamma$ .

In the coordinates  $(t, \theta)$ , defined in (1.10), near  $\Gamma$  the metric can be parameterized as

$$g_{t,\theta} = dt^2 + (1 + kt)^2 d\theta^2,$$

and the Laplacian operator is

$$\Delta_{t,\theta} = \frac{\partial^2}{\partial t^2} + \frac{1}{(1 + kt)^2} \frac{\partial^2}{\partial \theta^2} + \frac{k}{1 + kt} \frac{\partial}{\partial t} - \frac{k't}{(1 + kt)^3} \frac{\partial}{\partial \theta},$$

where  $k(\theta)$  is the curvature of  $\Gamma$  (see for instance [37]).

In stretched variables  $y = \varepsilon^{-\frac{2}{3}} \mathbf{y}$ , recall (1.10), problem (1.1) becomes

$$\Delta u - \varepsilon^{-\frac{2}{3}} \left( u - a(\varepsilon^{\frac{2}{3}} y) \right) \left( u - b(\varepsilon^{\frac{2}{3}} y) \right) = 0 \quad \text{in } \Omega_\varepsilon, \quad \frac{\partial u}{\partial \eta} = 0 \quad \text{on } \partial\Omega_\varepsilon, \quad (3.1)$$

where  $\Omega_\varepsilon = \varepsilon^{-\frac{2}{3}} \Omega$ , and  $\eta = \eta(y)$  denotes the outward unit normal to  $\partial\Omega_\varepsilon$ . For future purposes, we denote  $\Gamma_\varepsilon = \varepsilon^{-\frac{2}{3}} \Gamma$  and  $\Omega_{i,\varepsilon} = \varepsilon^{-\frac{2}{3}} \Omega_i$ ,  $i = 1, 2$  (recall (1.2)).

Let

$$(s, z) = \varepsilon^{-\frac{2}{3}}(t, \theta) \quad (3.2)$$

be natural stretched coordinates associated to the curve  $\Gamma_\varepsilon$ , now defined for

$$s \in \left( -\delta_0 \varepsilon^{-\frac{2}{3}}, \delta_0 \varepsilon^{-\frac{2}{3}} \right), \quad z \in \left[ 0, \varepsilon^{-\frac{2}{3}} \ell \right]. \quad (3.3)$$

Near  $\Gamma_\varepsilon$ , the metric can be written as

$$g_{s,z} = ds^2 + (1 + \varepsilon^{\frac{2}{3}} k s)^2 dz^2,$$

and the Laplacian for  $u$  expressed in these coordinates becomes

$$\Delta u = u_{zz} + u_{ss} + B_1(u), \quad (3.4)$$

where

$$B_1(u) = -u_{zz} \left[ 1 - \frac{1}{\left( 1 + \varepsilon^{\frac{2}{3}} k(\varepsilon^{\frac{2}{3}} z) s \right)^2} \right] + \frac{\varepsilon^{\frac{2}{3}} k(\varepsilon^{\frac{2}{3}} z) u_s}{1 + \varepsilon^{\frac{2}{3}} k(\varepsilon^{\frac{2}{3}} z) s} - \frac{\varepsilon^{\frac{4}{3}} s k'(\varepsilon^{\frac{2}{3}} z) u_z}{\left( 1 + \varepsilon^{\frac{2}{3}} k(\varepsilon^{\frac{2}{3}} z) s \right)^3}. \quad (3.5)$$

Hence, equation (3.1) takes the form

$$u_{zz} + u_{ss} + B_1(u) - \varepsilon^{-\frac{2}{3}} \left( u - a(\varepsilon^{\frac{2}{3}} s, \varepsilon^{\frac{2}{3}} z) \right) \left( u - b(\varepsilon^{\frac{2}{3}} s, \varepsilon^{\frac{2}{3}} z) \right) = 0, \quad (3.6)$$

in the region (3.3). For further reference, it is convenient to expand the operator  $B_1$  as

$$B_1(u) = \left( \varepsilon^{\frac{2}{3}} k(\varepsilon^{\frac{2}{3}} z) - \varepsilon^{\frac{4}{3}} k^2(\varepsilon^{\frac{2}{3}} z) s \right) u_s + B_0(u), \quad (3.7)$$

where

$$B_0(u) = \varepsilon^{\frac{4}{3}} s a_1(\varepsilon^{\frac{2}{3}} s, \varepsilon^{\frac{2}{3}} z) u_z + \varepsilon^{\frac{2}{3}} s a_2(\varepsilon^{\frac{2}{3}} s, \varepsilon^{\frac{2}{3}} z) u_{zz} + \varepsilon^2 s^2 a_3(\varepsilon^{\frac{2}{3}} s, \varepsilon^{\frac{2}{3}} z) u_s, \quad (3.8)$$

for certain smooth functions  $a_j(t, \theta)$ ,  $j = 1, 2, 3$ . Observe that all terms in the operator  $B_1$  have  $\varepsilon^{\frac{2}{3}}$  as a common factor.

We now consider a further change of variables in equation (3.6). Note that, thanks to (1.3), we have

$$b_t(0, \theta) - a_t(0, \theta) \geq c, \quad \theta \in [0, \ell]. \quad (3.9)$$

Letting

$$\beta(\theta) = \left( \frac{b_t(0, \theta) - a_t(0, \theta)}{2} \right)^{\frac{1}{3}} > 0, \quad \theta \in [0, \ell], \quad (3.10)$$

we define  $v(x, z)$  by the relation

$$u(s, z) = a(0, \varepsilon^{\frac{2}{3}}z) + \varepsilon^{\frac{2}{3}}\beta^2(\varepsilon^{\frac{2}{3}}z)v(x, z) + \varepsilon^{\frac{2}{3}}\frac{a_t(0, \varepsilon^{\frac{2}{3}}z) + b_t(0, \varepsilon^{\frac{2}{3}}z)}{2}\beta^{-1}(\varepsilon^{\frac{2}{3}}z)x, \quad (3.11)$$

$$x = \beta(\varepsilon^{\frac{2}{3}}z)s.$$

Choosing a smaller  $\delta_0$ , if necessary, we may assume that the coordinates  $(x, z)$  are also defined for  $|x| \leq \delta_0\varepsilon^{-\frac{2}{3}}$ ,  $z \in [0, \varepsilon^{-\frac{2}{3}}\ell]$ . We want to express equation (3.6) in terms of these new coordinates. We compute:

$$\left\{ \begin{array}{l} u_s = \varepsilon^{\frac{2}{3}}\beta^3v_x + \varepsilon^{\frac{2}{3}}\frac{a_t(0, \varepsilon^{\frac{2}{3}}z) + b_t(0, \varepsilon^{\frac{2}{3}}z)}{2}, \\ u_{ss} = \varepsilon^{\frac{2}{3}}\beta^4v_{xx}, \\ u_z = \varepsilon^{\frac{2}{3}}a_\theta(0, \varepsilon^{\frac{2}{3}}z) + 2\varepsilon^{\frac{4}{3}}\beta'v + \varepsilon^{\frac{4}{3}}\beta'\beta xv_x + \varepsilon^{\frac{2}{3}}\beta^2v_z + \varepsilon^{\frac{4}{3}}\frac{a_{t\theta}(0, \varepsilon^{\frac{2}{3}}z) + b_{t\theta}(0, \varepsilon^{\frac{2}{3}}z)}{2}\beta^{-1}x, \\ u_{zz} = \varepsilon^{\frac{4}{3}}a_{\theta\theta}(0, \varepsilon^{\frac{2}{3}}z) + 2\varepsilon^2(\beta')^2v + 2\varepsilon^2\beta''v + 4\varepsilon^2(\beta')^2xv_x + 4\varepsilon^{\frac{4}{3}}\beta'\beta v_z + \varepsilon^2\beta''\beta xv_x \\ \quad + 2\varepsilon^{\frac{4}{3}}\beta'\beta xv_{xz} + \varepsilon^2(\beta')^2x^2v_{xx} + \varepsilon^{\frac{2}{3}}\beta^2v_{zz} + \varepsilon^2\frac{a_{t\theta\theta}(0, \varepsilon^{\frac{2}{3}}z) + b_{t\theta\theta}(0, \varepsilon^{\frac{2}{3}}z)}{2}\beta^{-1}x. \end{array} \right. \quad (3.12)$$

In order to write down the equation, it is also convenient to expand

$$a(\varepsilon^{\frac{2}{3}}s, \varepsilon^{\frac{2}{3}}z) = a(0, \varepsilon^{\frac{2}{3}}z) + a_t(0, \varepsilon^{\frac{2}{3}}z)\varepsilon^{\frac{2}{3}}s + \frac{1}{2}a_{tt}(0, \varepsilon^{\frac{2}{3}}z)\varepsilon^{\frac{4}{3}}s^2 + a_4(\varepsilon^{\frac{2}{3}}s, \varepsilon^{\frac{2}{3}}z)\varepsilon^2s^3, \quad (3.13)$$

$$b(\varepsilon^{\frac{2}{3}}s, \varepsilon^{\frac{2}{3}}z) = b(0, \varepsilon^{\frac{2}{3}}z) + b_t(0, \varepsilon^{\frac{2}{3}}z)\varepsilon^{\frac{2}{3}}s + \frac{1}{2}b_{tt}(0, \varepsilon^{\frac{2}{3}}z)\varepsilon^{\frac{4}{3}}s^2 + a_5(\varepsilon^{\frac{2}{3}}s, \varepsilon^{\frac{2}{3}}z)\varepsilon^2s^3, \quad (3.14)$$

for some smooth functions  $a_i(t, \theta)$ ,  $i = 4, 5$ , satisfying

$$a_4(\varepsilon^{\frac{2}{3}}s, \varepsilon^{\frac{2}{3}}z) = \frac{1}{6}a_{ttt}(0, \varepsilon^{\frac{2}{3}}z) + a_6(\varepsilon^{\frac{2}{3}}s, \varepsilon^{\frac{2}{3}}z)\varepsilon^{\frac{2}{3}}s, \quad (3.15)$$

$$a_5(\varepsilon^{\frac{2}{3}}s, \varepsilon^{\frac{2}{3}}z) = \frac{1}{6}b_{ttt}(0, \varepsilon^{\frac{2}{3}}z) + a_7(\varepsilon^{\frac{2}{3}}s, \varepsilon^{\frac{2}{3}}z)\varepsilon^{\frac{2}{3}}s, \quad (3.16)$$

for some smooth functions  $a_i(t, \theta)$ ,  $i = 6, 7$ . It turns out that  $u$  solves (3.6) if and only if  $v$ , defined by (3.11), solves

$$S(v) \equiv \beta^{-2}v_{zz} + v_{xx} - v^2 + x^2 + B_3(v) = 0, \quad (3.17)$$

where  $B_3(v)$  is a differential operator defined by

$$\begin{aligned}
 B_3(v) = & \beta^{-1}(\varepsilon^{\frac{2}{3}}k - \varepsilon^{\frac{4}{3}}k^2\beta^{-1}x)v_x + \beta^{-4}(\varepsilon^{\frac{2}{3}}k - \varepsilon^{\frac{4}{3}}k^2\beta^{-1}x)\frac{a_t(0,\varepsilon^{\frac{2}{3}}z)+b_t(0,\varepsilon^{\frac{2}{3}}z)}{2} \\
 & + \varepsilon^{\frac{2}{3}}\beta^{-4}a_{\theta\theta}(0,\varepsilon^{\frac{2}{3}}z) + 2\varepsilon^{\frac{4}{3}}(\beta')^2\beta^{-4}v + 2\varepsilon^{\frac{4}{3}}\beta''\beta^{-3}v + 4\varepsilon^{\frac{4}{3}}(\beta')^2\beta^{-4}xv_x \\
 & + 4\varepsilon^{\frac{2}{3}}\beta'\beta^{-3}v_z + \varepsilon^{\frac{4}{3}}\beta''\beta^{-3}xv_x + 2\varepsilon^{\frac{2}{3}}\beta'\beta^{-3}xv_{xz} + \varepsilon^{\frac{4}{3}}(\beta')^2\beta^{-4}x^2v_{xx} \\
 & + \varepsilon^{\frac{4}{3}}\frac{a_{t\theta\theta}(0,\varepsilon^{\frac{2}{3}}z)+b_{t\theta\theta}(0,\varepsilon^{\frac{2}{3}}z)}{2}\beta^{-5}x + \varepsilon^{\frac{2}{3}}\frac{b_{tt}(0,\varepsilon^{\frac{2}{3}}z)}{2}\beta^{-4}x^2v + \varepsilon^{\frac{2}{3}}\frac{b_{tt}(0,\varepsilon^{\frac{2}{3}}z)}{2}\beta^{-4}x^3 \\
 & + \varepsilon^{\frac{2}{3}}\frac{a_{tt}(0,\varepsilon^{\frac{2}{3}}z)}{2}\beta^{-4}x^2v - \varepsilon^{\frac{2}{3}}\frac{a_{tt}(0,\varepsilon^{\frac{2}{3}}z)}{2}\beta^{-4}x^3 + \varepsilon^{\frac{4}{3}}\frac{b_{ttt}(0,\varepsilon^{\frac{2}{3}}z)}{6}\beta^{-5}x^4 + \varepsilon^{\frac{4}{3}}\frac{b_{ttt}(0,\varepsilon^{\frac{2}{3}}z)}{6}\beta^{-5}x^3v \\
 & - \varepsilon^{\frac{4}{3}}\frac{a_{tt}(0,\varepsilon^{\frac{2}{3}}z)b_{tt}(0,\varepsilon^{\frac{2}{3}}z)}{4}\beta^{-8}x^4 + \varepsilon^{\frac{4}{3}}\frac{a_{ttt}(0,\varepsilon^{\frac{2}{3}}z)}{6}\beta^{-5}x^3v - \varepsilon^{\frac{4}{3}}\frac{a_{ttt}(0,\varepsilon^{\frac{2}{3}}z)}{6}\beta^{-5}x^4 + B_2(v),
 \end{aligned} \tag{3.18}$$

and

$$\begin{aligned}
 B_2(v) = & \varepsilon^2a_7\beta^{-6}x^5 + \varepsilon^2a_7\beta^{-6}x^4v + \varepsilon^2a_6\beta^{-6}x^4v - \varepsilon^2a_6\beta^{-6}x^5 - \varepsilon^2\frac{a_{tt}(0,\varepsilon^{\frac{2}{3}}z)}{2}a_5\beta^{-9}x^5 \\
 & - \varepsilon^2\frac{b_{tt}(0,\varepsilon^{\frac{2}{3}}z)}{2}a_4\beta^{-9}x^5 - \varepsilon^{\frac{8}{3}}a_4a_5\beta^{-10}x^6 + \varepsilon^{-\frac{2}{3}}\beta^{-4}B_0(u),
 \end{aligned} \tag{3.19}$$

where  $B_0(u)$  is the operator in (3.8) with derivatives expressed in terms of formulas (3.12) and  $s$  replaced by  $\beta^{-1}x$ ; the  $a_i$ 's,  $i = 4, \dots, 7$ , are given by (3.13) through (3.16) and are evaluated at  $(\varepsilon^{\frac{2}{3}}\beta^{-1}x, \varepsilon^{\frac{2}{3}}z)$ .

Let  $V(x)$  be as in the following proposition, proven in [32].

**Proposition 3.1.** *There exists a unique even solution  $V$  of*

$$\begin{cases} v_{xx} = v^2 - x^2, & x \in \mathbb{R}, \\ v(x) - |x| \rightarrow 0, & x \rightarrow \pm\infty, \end{cases} \tag{3.20}$$

satisfying  $V_x(x) > 0$ ,  $x > 0$ , and

$$V(x) < |x|, \quad x \in \mathbb{R}. \tag{3.21}$$

Furthermore, we have

$$|V(x) - |x|| \leq C(|x| + 1)^{-\frac{1}{4}}e^{-\frac{2\sqrt{2}}{3}|x|^{\frac{3}{2}}}, \quad x \in \mathbb{R}, \tag{3.22}$$

and  $|V_x - 1| \leq Ce^{-c|x|^{\frac{3}{2}}}$ ,  $x \geq 0$ . Moreover, the spectrum of the linearized operator, in  $L^2(\mathbb{R})$ ,

$$L_0(\psi) = \psi_{xx} - 2V\psi,$$

consists of simple eigenvalues  $\lambda_0 > \lambda_1 > \dots$  with  $\lambda_i \rightarrow -\infty$ ,  $i \rightarrow \infty$ , and

$$\lambda_0 > 0 > \lambda_1. \tag{3.23}$$

*Proof.* (Sketch) It is well known that problem (3.20) has an even solution  $V_+$  such that  $V_+(x) > |x|$ ,  $x \in \mathbb{R}$ , and  $(V_+)_x(x) > 0$ ,  $x > 0$ . This can be proven by the method of upper and lower solutions, see [13, 26, 28, 32, 44]. Actually, by a theorem of [8], this is the unique non-negative solution in  $\mathbb{R}^2$  of the elliptic equation of (1.11).

We then seek another solution in the form  $v = V_+ - w$ . In terms of  $w$ , problem (3.20) is equivalent to

$$\begin{cases} w_{xx} - 2V_+w + w^2 = 0, & x \in \mathbb{R}, \\ w(x) \rightarrow 0, & x \rightarrow \pm\infty, \end{cases} \quad (3.24)$$

which describes ground states of a nonlinear Schrödinger equation with potential  $2V_+$ , see [43]. Existence of a positive solution  $w$  of (3.24) follows from a variational, mountain pass type, argument which makes use of the positivity of  $V_+$  and the fact that  $V_+(x) \rightarrow \infty$  as  $x \rightarrow \pm\infty$ , see [43]. This last fact also implies that  $w$  decays to zero super-exponentially as  $x \rightarrow \pm\infty$ . Hence, since  $V_+$  is even, and increasing for  $x > 0$ , we can apply the method of moving planes [22] to show that  $w$  is even and strictly decreasing for  $x > 0$ . Letting  $V \equiv V_+ - w$ , it is clear that  $V$  solves (3.20), is even, and strictly increasing for  $x > 0$ . Furthermore, we can write

$$-(V - x)_{xx} + (V + x)(V - x) = 0, \quad x > 0.$$

Now, relation (3.22) follows at once thanks to the asymptotic theory of linear differential equations [6] and the evenness of  $V$ . Relation (3.21) follows from a standard maximum principle argument in the above equation. Again by the maximum principle, recalling that  $V_+ > 0$ , we deduce that any non-trivial solution of (3.24) is strictly positive. Hence, adapting the arguments of [31] (using crucially that  $V_+$  is even and  $(V_+)_x > 0$ ,  $x > 0$ ), we can show that  $w$  is the unique non-trivial solution of (3.24). In turn, this implies that the only solutions of (3.20) are  $V_+$  and  $V$ .

Since  $V \rightarrow \infty$  as  $x \rightarrow \pm\infty$ , the linearized operator, in  $L^2(\mathbb{R})$ ,

$$M(\psi) = \psi_{xx} - 2V\psi,$$

has spectrum consisting only of simple eigenvalues  $\lambda_0 > \lambda_1 > \dots$ . The eigenfunction associated to  $\lambda_i$  is even if  $i$  is even and vice versa. Testing the eigenvalue equation by  $w$  yields that  $\lambda_0 > 0$ ; whereas testing by  $w_x$  yields that  $\lambda_1 < 0$  (we make essential use of the evenness and strict monotonicity of  $V_+$  at this point).  $\square$

**Remark 3.1.** *The unstable solution  $V$  shares some similarities with the flame layer solution found in the appendix of [11].*

Then, taking  $V(x)$  as a first approximation in (3.17), the error produced is  $\varepsilon^{\frac{2}{3}}$  times a function with polynomial growth. Let us be more precise. We need to identify the terms of order  $\varepsilon^{\frac{2}{3}}$  and those of order  $\varepsilon^{\frac{4}{3}}$ :

$$S(V) = B_3(V) = \varepsilon^{\frac{2}{3}}S_1 + \varepsilon^{\frac{4}{3}}S_2 + B_2(V) + \varepsilon^{\frac{4}{3}}k'\beta^{-5}xa_\theta + 2\varepsilon^{\frac{4}{3}}k\beta^{-5}xa_{\theta\theta}, \quad (3.25)$$

where

$$S_1 = \beta^{-1}kV_x + \frac{a_t + b_t}{2}\beta^{-4}k + \beta^{-4}a_{\theta\theta} + \frac{b_{tt}}{2}\beta^{-4}x^2V + \frac{b_{tt}}{2}\beta^{-4}x^3 + \frac{a_{tt}}{2}\beta^{-4}x^2V - \frac{a_{tt}}{2}\beta^{-4}x^3, \quad (3.26)$$

$$\begin{aligned} S_2 = & -\beta^{-2}k^2xV_x - \frac{a_t + b_t}{2}\beta^{-5}k^2x + 2(\beta')^2\beta^{-4}V + 2\beta''\beta^{-3}V + 4(\beta')^2\beta^{-4}xV_x \\ & + \beta''\beta^{-3}xV_x + (\beta')^2\beta^{-4}x^2V_{xx} + \frac{a_{t\theta\theta} + b_{t\theta\theta}}{2}\beta^{-5}x + \frac{b_{ttt}}{6}\beta^{-5}x^4 + \frac{b_{ttt}}{6}\beta^{-5}x^3V \\ & - \frac{a_{tt}b_{tt}}{4}\beta^{-8}x^4 + \frac{a_{ttt}}{6}\beta^{-5}x^3V - \frac{a_{ttt}}{6}\beta^{-5}x^4 - k'\beta^{-5}xa_\theta - 2k\beta^{-5}xa_{\theta\theta}. \end{aligned} \quad (3.27)$$

Actually, it turns out that  $B_2(V) + \varepsilon^{\frac{4}{3}}k'\beta^{-5}xa_\theta + 2\varepsilon^{\frac{4}{3}}k\beta^{-5}xa_{\theta\theta}$  is of size  $\varepsilon^2$  (more precisely is bounded by  $C\varepsilon^2(1 + |x|^5)$ ).

**Remark 3.2. Notation.** *In the above relation and throughout this paper, unless specified otherwise, we will assume that  $\partial_t^i \partial_\theta^j u$ ,  $u = a, b$ ,  $i, j = 1, \dots$ , are evaluated at  $(0, \varepsilon^{\frac{2}{3}} z)$ , whereas the  $a_i$   $i = 1, \dots$ , at  $(\varepsilon^{\frac{2}{3}} \beta^{-1} x, \varepsilon^{\frac{2}{3}} z)$ .*

We now want to construct a further approximation to a solution that eliminates the terms of order  $\varepsilon^{\frac{2}{3}}$  in the error. We see that, for any smooth function  $\phi(x, z)$ ,

$$S(V + \phi) = S(V) + L_0(\phi) + B_4(\phi) + N_0(\phi), \quad (3.28)$$

where

$$L_0(\phi) = \beta^{-2} \phi_{zz} + \phi_{xx} - 2V(x)\phi, \quad (3.29)$$

$$B_4(\phi) = B_3(V + \phi) - B_3(V), \quad (3.30)$$

and

$$N_0(\phi) = -\phi^2. \quad (3.31)$$

We write

$$\begin{aligned} S(V + \phi) &= \left[ \varepsilon^{\frac{2}{3}} S_1 + \phi_{xx} - 2V\phi \right] + \varepsilon^{\frac{4}{3}} S_2 + B_2(V) + \varepsilon^{\frac{4}{3}} k' \beta^{-5} x a_\theta + 2\varepsilon^{\frac{4}{3}} k \beta^{-5} x a_{\theta\theta} \\ &\quad + \beta^{-2} \phi_{zz} + B_4(\phi) + N_0(\phi). \end{aligned} \quad (3.32)$$

We choose  $\phi = \phi_1$  in order to eliminate the term between brackets in the above expression. Namely, for fixed  $z$ , we need a solution of

$$-\phi_{xx} + 2V\phi = \varepsilon^{\frac{2}{3}} S_1, \quad -\infty < x < +\infty. \quad (3.33)$$

In order that the “inner” solution, described by (3.11) with  $v = V + \phi_1$ , matches with the “outer” solution  $\max \left\{ a(\varepsilon^{\frac{2}{3}} y), b(\varepsilon^{\frac{2}{3}} y) \right\}$  at  $\Omega_\varepsilon \cap \{x = \pm L\}$ , it is easy to check that the desired asymptotic behavior of  $\phi_1$  should be

$$\phi_{-\varepsilon^{\frac{2}{3}}} \frac{a_{tt}(0, \varepsilon^{\frac{2}{3}} z)}{2} \beta^{-4} x^2 \rightarrow 0 \text{ as } x \rightarrow -\infty, \quad \phi_{-\varepsilon^{\frac{2}{3}}} \frac{b_{tt}(0, \varepsilon^{\frac{2}{3}} z)}{2} \beta^{-4} x^2 \rightarrow 0 \text{ as } x \rightarrow \infty, \quad (3.34)$$

(see [32] for detailed computations in the radial case).

Based on the invertibility of the linear operator in the lefthand side of (3.33) (recall Proposition 3.1), we can show:

**Proposition 3.2.** *There exists a smooth solution  $\phi_1(x, z)$ ,  $(x, z) \in (-\infty, \infty) \times [0, \varepsilon^{-\frac{2}{3}} \ell]$  of (3.33)–(3.34). Moreover, we have*

$$\phi_1 = \varepsilon^{\frac{2}{3}} \frac{b_{tt}(0, \varepsilon^{\frac{2}{3}} z)}{2} \beta^{-4} x^2 + \varepsilon^{\frac{2}{3}} \mathcal{O}(x^{-1}), \quad x \rightarrow +\infty, \quad (3.35)$$

$$\phi_1 = \varepsilon^{\frac{2}{3}} \frac{a_{tt}(0, \varepsilon^{\frac{2}{3}} z)}{2} \beta^{-4} x^2 + \varepsilon^{\frac{2}{3}} \mathcal{O}(x^{-1}), \quad x \rightarrow -\infty,$$

$$\phi_{1,x} = \varepsilon^{\frac{2}{3}} a_{tt} \beta^{-4} x + \varepsilon^{\frac{2}{3}} \mathcal{O}(x^{-2}), \quad x \rightarrow -\infty, \quad \phi_{1,x} = \varepsilon^{\frac{2}{3}} b_{tt} \beta^{-4} x + \varepsilon^{\frac{2}{3}} \mathcal{O}(x^{-2}), \quad x \rightarrow \infty,$$

$$\phi_{1,xx} = \varepsilon^{\frac{2}{3}} a_{tt} \beta^{-4} + \varepsilon^{\frac{2}{3}} \mathcal{O}(x^{-3}), \quad x \rightarrow -\infty, \quad \phi_{1,xx} = \varepsilon^{\frac{2}{3}} b_{tt} \beta^{-4} + \varepsilon^{\frac{2}{3}} \mathcal{O}(x^{-3}), \quad x \rightarrow \infty,$$

uniformly in  $z \in [0, \varepsilon^{-\frac{2}{3}} \ell]$ , and

$$|\phi_{1,z}| \leq C\varepsilon^{\frac{4}{3}}(x^2 + 1), \quad |\phi_{1,xz}| \leq C\varepsilon^{\frac{4}{3}}(|x| + 1), \quad |\phi_{1,zz}| \leq C\varepsilon^2(x^2 + 1),$$

for  $(x, z) \in (-\infty, \infty) \times [0, \varepsilon^{-\frac{2}{3}} \ell]$ .

*Proof.* Let

$$\tilde{\phi}_1 = \begin{cases} \frac{1}{2}\varepsilon^{\frac{2}{3}}b_{tt}\beta^{-4}x^2, & x \geq 0, \\ \frac{1}{2}\varepsilon^{\frac{2}{3}}a_{tt}\beta^{-4}x^2, & x < 0. \end{cases}$$

We seek a solution of (3.33), (3.34) in the form  $\phi = \tilde{\phi}_1 + \varphi$  with  $\lim_{|x| \rightarrow \infty} \varphi(x) = 0$ . A direct calculation shows that  $\varphi$  should satisfy

$$-\varphi_{xx} + 2V\varphi = \varepsilon^{\frac{2}{3}}f, \quad (3.36)$$

where

$$f(x) = \begin{cases} b_{tt}\beta^{-4} + \beta^{-1}kV_x + \beta^{-4}k\frac{a_t+b_t}{2} + \beta^{-4}a_{\theta\theta} - \frac{b_{tt}-a_{tt}}{2}\beta^{-4}x^2(V-x), & x \geq 0, \\ a_{tt}\beta^{-4} + \beta^{-1}kV_x + \beta^{-4}k\frac{a_t+b_t}{2} + \beta^{-4}a_{\theta\theta} + \frac{b_{tt}-a_{tt}}{2}\beta^{-4}x^2(V+x), & x < 0. \end{cases}$$

In view of the estimates of Proposition 3.1, we search a solution of (3.36) in the form

$$\varphi = \frac{1}{2}n_1|x|^{-1}\varepsilon^{\frac{2}{3}}f_\infty + \psi,$$

where  $n_1$  is a smooth cutoff function (see (4.2) below), and

$$f_\infty(x) = \begin{cases} b_{tt}\beta^{-4} + \beta^{-1}k + \beta^{-4}k\frac{a_t+b_t}{2} + \beta^{-4}a_{\theta\theta}, & x \geq 0, \\ a_{tt}\beta^{-4} - \beta^{-1}k + \beta^{-4}k\frac{a_t+b_t}{2} + \beta^{-4}a_{\theta\theta}, & x < 0. \end{cases}$$

We find that  $\psi$  should satisfy

$$-\psi_{xx} + 2V\psi = g, \quad (3.37)$$

with  $g$  satisfying

$$|g| + \varepsilon^{-\frac{2}{3}}|g_z| + \varepsilon^{-\frac{4}{3}}|g_{zz}| \leq C(|x| + 1)^{-3}\varepsilon^{\frac{2}{3}}, \quad -\infty < x < \infty, \quad 0 \leq z \leq \varepsilon^{-\frac{2}{3}}\ell.$$

By Proposition 3.1, the linear operator in the lefthand side of (3.37) is invertible. Note that, for fixed  $z$ , we have  $g \in L^2(\mathbb{R})$ . Hence, for fixed  $z$ , there exists a unique solution of (3.37) such that  $\psi \in H^2(\mathbb{R})$  and  $V\psi \in L^2(\mathbb{R})$ . This solution clearly depends smoothly on  $z$ . This gives us the existence part of the proposition. The desired bounds follow from a barrier argument (see [32]) applied to (3.37) and its derivatives with respect to  $x, z$ .

The proof of the proposition is complete.  $\square$

**Remark 3.3.** *If  $b$  is a harmonic function, then  $f_\infty$ , defined above (3.37), becomes identically zero for  $x \geq 0$  (recall that  $a = b$  on  $\Gamma$ , (3.4), (3.5), and (3.10)). In turn, this implies that the rate of decay in (3.35) is super-exponential as  $x \rightarrow \infty$ . A similar comment applies in case  $a$  is harmonic.*

Substituting  $\phi = \phi_1$  into (3.32), we can compute the new error

$$S(V+\phi_1) = \varepsilon^{\frac{4}{3}}S_2 + B_2(V) + \varepsilon^{\frac{4}{3}}k'\beta^{-5}xa_\theta + 2\varepsilon^{\frac{4}{3}}k\beta^{-5}xa_{\theta\theta} + \beta^{-2}\phi_{1,zz} + B_4(\phi_1) + N_0(\phi_1). \quad (3.38)$$

Observe that since  $\phi_1$  is of size  $\mathcal{O}(\varepsilon^{\frac{2}{3}})$  (and has polynomial growth with respect to  $x$ ), all terms above carry  $\varepsilon^{\frac{4}{3}}$  in front. Observe also that all functions involved are expressed in  $(x, z)$ -variables, and the natural domain for those variables is the infinite strip

$$\mathcal{S} = \left\{ -\infty < x < \infty, \quad 0 \leq z \leq \varepsilon^{-\frac{2}{3}}\ell \right\}. \quad (3.39)$$

We now want to measure the size of the error in the  $L^2(\mathcal{S}_L)$ -norm, where

$$\mathcal{S}_L = \mathcal{S} \cap \{|x| \leq L\}, \quad (3.40)$$

and  $L > 0$  is a fixed number such that  $V(x) \geq 1$  for  $|x| \geq L$ . It is easy to verify, via (3.38) and the fact that  $|\mathcal{S}_L| \leq C\varepsilon^{-\frac{2}{3}}$ , that

$$\|S(V + \phi_1)\|_{L^2(\mathcal{S}_L)} \leq C\varepsilon. \quad (3.41)$$

At this point we can define the inner solution of (3.1), in  $\Omega_\varepsilon \cap \{|x| \leq \delta_0\varepsilon^{-\frac{2}{3}}\}$ , as

$$u_{in}(x, z) = a(0, \varepsilon^{\frac{2}{3}}z) + \varepsilon^{\frac{2}{3}}\beta^2(\varepsilon^{\frac{2}{3}}z)(V + \phi_1) + \varepsilon^{\frac{2}{3}} \frac{a_t(0, \varepsilon^{\frac{2}{3}}z) + b_t(0, \varepsilon^{\frac{2}{3}}z)}{2} \beta^{-1}(\varepsilon^{\frac{2}{3}}z)x, \quad (3.42)$$

(recall (3.11)).

The following proposition contains the main estimate regarding  $u_{in}$ .

**Proposition 3.3.** *The inner approximation  $u_{in}$ , defined in (3.42), satisfies*

$$\|\Delta u_{in} - \varepsilon^{-\frac{2}{3}} \left( u_{in} - a(\varepsilon^{\frac{2}{3}}y) \right) \left( u_{in} - b(\varepsilon^{\frac{2}{3}}y) \right)\|_{L^2(\Omega_\varepsilon \cap \{|x| \leq L\})} \leq C\varepsilon^{\frac{5}{3}}. \quad (3.43)$$

*Proof.* From (3.6) and (3.12), in  $\Omega_\varepsilon \cap \{|x| \leq L\}$ , we have

$$\Delta u_{in} - \varepsilon^{-\frac{2}{3}} \left( u_{in} - a(\varepsilon^{\frac{2}{3}}y) \right) \left( u_{in} - b(\varepsilon^{\frac{2}{3}}y) \right) = \varepsilon^{\frac{2}{3}}\beta^4 S(V + \varphi_1).$$

Now, the assertion of the proposition follows at once from (3.41).

The proof of the proposition is complete.  $\square$

#### 4. SET UP AWAY FROM THE CURVE

In this section we will suitably modify  $\max \left\{ a(\varepsilon^{\frac{2}{3}}y), b(\varepsilon^{\frac{2}{3}}y) \right\}$  in order to bring it closer to  $u_{in}$ , near the curve  $\Gamma_\varepsilon$ , while at the same time improving the remainder it leaves in (3.1) (away from the curve). We accomplish this, loosely speaking, by replacing the linear and quadratic order terms of the Taylor expansion of  $\max \left\{ a(\varepsilon^{\frac{2}{3}}y), b(\varepsilon^{\frac{2}{3}}y) \right\}$ , near  $\Gamma_\varepsilon$ , by the terms in  $u_{in}$  having linear and quadratic asymptotic behavior respectively.

For convenience purposes, we will additionally assume that

$$\frac{\partial a}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\Omega_1 \cap \partial\Omega \quad \text{and} \quad \frac{\partial b}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\Omega_2 \cap \partial\Omega. \quad (4.1)$$

The general case can be treated by simply adding a standard boundary layer correction, close to the boundary  $\partial\Omega$ , to the outer approximation we will construct in this section (see for instance [29]). (Under (4.1), the estimate in the first relation of (1.19) holds all the way up to  $\partial\Omega$ ).

Let  $\delta < \delta_0/100$  be a fixed number. We consider a smooth cutoff function

$$n_\delta(t) = \begin{cases} 1, & |t| \leq \delta, \\ 0, & |t| \geq 2\delta. \end{cases} \quad (4.2)$$

Motivated from the radial case [32], we define our first approximation in  $\Omega_\varepsilon/\{|x| < L\}$  to be

$$\begin{aligned} \tilde{u}_{out}(y) &= a(\varepsilon^{\frac{2}{3}}y) + n_\delta(\varepsilon^{\frac{2}{3}}x) \left( \varepsilon^{\frac{2}{3}}\beta^2V + \varepsilon^{\frac{2}{3}}\frac{b_t - a_t}{2}\beta^{-1}x + \varepsilon^{\frac{2}{3}}\beta^2\phi_1 - \frac{1}{2}\varepsilon^{\frac{4}{3}}a_{tt}\beta^{-2}x^2 \right) \\ &\stackrel{(3.10)}{=} a(\varepsilon^{\frac{2}{3}}y) + n_\delta(\varepsilon^{\frac{2}{3}}x)\varepsilon^{\frac{2}{3}}\beta^2 \left( V + x + \phi_1 - \frac{a_{tt}}{2}\varepsilon^{\frac{2}{3}}\beta^{-4}x^2 \right), \end{aligned} \quad (4.3)$$

for  $y \in \Omega_{1,\varepsilon}/\{-L < x < 0\}$ . Similarly we define  $\tilde{u}_{out}$  in  $\Omega_{2,\varepsilon}/\{0 < x < L\}$ .

The following lemma contains the main estimate regarding  $\tilde{u}_{out}$ .

**Lemma 4.1.** *Let*

$$\tilde{E}_{out}(y) \equiv \Delta\tilde{u}_{out} - \varepsilon^{-\frac{2}{3}} \left( \tilde{u}_{out} - a(\varepsilon^{\frac{2}{3}}y) \right) \left( \tilde{u}_{out} - b(\varepsilon^{\frac{2}{3}}y) \right),$$

then

$$\tilde{E}_{out}(y) = \begin{cases} \mathcal{O}(\varepsilon^2|x|), & y \in \Omega_\varepsilon \cap \{L \leq |x| \leq \delta\varepsilon^{-\frac{2}{3}}\}, \\ \mathcal{O}(\varepsilon^{\frac{4}{3}}), & y \in \Omega_\varepsilon/\{|x| \leq \delta\varepsilon^{-\frac{2}{3}}\}. \end{cases}$$

*Proof.* Making use of the definitions of  $\beta(\theta)$ ,  $V(x)$ ,  $\phi_1(x, z)$  (recall (3.10), and equations (3.20), (3.33)), one can calculate that, in  $\Omega_\varepsilon \cap \{-\delta\varepsilon^{-\frac{2}{3}} \leq x \leq -L\}$ , we have

$$\begin{aligned} &\Delta\tilde{u}_{out} - \varepsilon^{-\frac{2}{3}} \left( \tilde{u}_{out} - a(\varepsilon^{\frac{2}{3}}y) \right) \left( \tilde{u}_{out} - b(\varepsilon^{\frac{2}{3}}y) \right) = \\ &\varepsilon^{\frac{4}{3}} \left( a_{tt}(\varepsilon^{\frac{2}{3}}\beta^{-1}x, \varepsilon^{\frac{2}{3}}z) - a_{tt} + a_{\theta\theta}(\varepsilon^{\frac{2}{3}}\beta^{-1}x, \varepsilon^{\frac{2}{3}}z) - a_{\theta\theta} + ka_t(\varepsilon^{\frac{2}{3}}\beta^{-1}x, \varepsilon^{\frac{2}{3}}z) - ka_t \right) \\ &- \varepsilon^{\frac{2}{3}}\beta^4\phi_1 \left( \phi_1 - \frac{a_{tt}}{2}\varepsilon^{\frac{2}{3}}\beta^{-4}x^2 \right) + \varepsilon^{\frac{4}{3}}\frac{b_{tt}}{2}x^2 \left( \phi_1 - \frac{a_{tt}}{2}\varepsilon^{\frac{2}{3}}\beta^{-4}x^2 \right) - (a_4 - a_5)\varepsilon^2\beta^{-1}x^3(V + x) \\ &- (a_4 - a_5)\varepsilon^2\beta^{-1}x^3 \left( \phi_1 - \frac{a_{tt}}{2}\varepsilon^{\frac{2}{3}}\beta^{-4}x^2 \right) + \varepsilon^2 [2(\beta')^2V + 2\beta''\beta V + 4(\beta')^2xV_x + \beta''\beta xV_x \\ &+ (\beta')^2x^2V_{xx} + \frac{b_{t\theta\theta} - a_{t\theta\theta}}{2}\beta^{-1}x + \varepsilon^{-\frac{2}{3}}\beta^3k\phi_{1,x} - ka_{tt}\beta^{-1}x - \beta^{-1}xk^2a_t(\varepsilon^{\frac{2}{3}}\beta^{-1}x, \varepsilon^{\frac{2}{3}}z) \\ &- \beta^2k^2xV_x - k^2\frac{b_t - a_t}{2}\beta^{-1}x + a_1a_\theta(\varepsilon^{\frac{2}{3}}\beta^{-1}x, \varepsilon^{\frac{2}{3}}z)\beta^{-1}x + a_2a_{\theta\theta}(\varepsilon^{\frac{2}{3}}\beta^{-1}x, \varepsilon^{\frac{2}{3}}z)\beta^{-1}x] \\ &+ \varepsilon^{\frac{8}{3}} \left[ 2\varepsilon^{-\frac{2}{3}}(\beta')^2\phi_1 + 2\varepsilon^{-\frac{2}{3}}\beta''\beta\phi_1 + 4\varepsilon^{-\frac{2}{3}}(\beta')^2x\phi_{1,x} + \varepsilon^{-\frac{2}{3}}\beta''\beta x\phi_{1,x} + \varepsilon^{-\frac{2}{3}}(\beta')^2x^2\phi_{1,xx} \right. \\ &\quad \left. - \frac{a_{t\theta\theta}}{2}\beta^{-2}x^2 - \varepsilon^{-\frac{2}{3}}\beta^2k^2x\phi_{1,x} + a_{tt}\beta^{-2}kx^2 + a_3a_t(\varepsilon^{\frac{2}{3}}\beta^{-1}x, \varepsilon^{\frac{2}{3}}z)\beta^{-2}x^2 \right. \\ &\quad \left. + a_1x(2\beta'V + \beta'xV_x) + a_2\beta^{-1}x(2(\beta')^2V + 2\beta''\beta V + 4(\beta')^2xV_x + \beta''\beta xV_x + (\beta')^2x^2V_{xx}) \right. \\ &\quad \left. + a_3\beta x^2V_x + \frac{b_{t\theta} - a_{t\theta}}{2}a_1\beta^{-2}x^2 + \frac{b_{t\theta\theta} - a_{t\theta\theta}}{2}a_2\beta^{-2}x^2 + \frac{b_t - a_t}{2}a_3\beta^{-2}x^2 \right. \\ &\quad \left. + a_1x(2\beta'\phi_1 + \beta'x\phi_{1,x}) + a_2\beta^{-1}x(2(\beta')^2\phi_1 + 2\beta''\beta\phi_1 + 4(\beta')^2x\phi_{1,x} + \beta''\beta x\phi_{1,x} + (\beta')^2x^2\phi_{1,xx}) \right. \\ &\quad \left. + a_3\beta x^2\phi_{1,x} \right] - \varepsilon^{\frac{10}{3}} \left[ a_1\frac{a_{t\theta}}{2}\beta^{-3}x^3 + a_2\frac{a_{t\theta\theta}}{2}\beta^{-3}x^3 + a_3a_{tt}\beta^{-3}x^3 \right]. \end{aligned}$$

Hence, by the estimates of Propositions 3.1, 3.2, we infer that the assertion of the lemma holds true in  $\Omega_\varepsilon \cap \{-\delta\varepsilon^{-\frac{2}{3}} \leq x \leq -L\}$ . The previously mentioned estimates,



and (4.3), also imply that

$$\tilde{u}_{out} - a(\varepsilon^{\frac{2}{3}}y) = n_\delta(\varepsilon^{\frac{2}{3}}x)\varepsilon^{\frac{2}{3}}\beta^2 \left( V + x + \phi_1 - \frac{a_{tt}}{2}\varepsilon^{\frac{2}{3}}\beta^{-4}x^2 \right) = \mathcal{O}(\varepsilon^2), \quad (4.4)$$

uniformly in  $\Omega_\varepsilon \cap \{-2\delta\varepsilon^{-\frac{2}{3}} \leq x \leq -\delta\varepsilon^{-\frac{2}{3}}\}$ . Similarly, one can check that

$$u(y) = u(s, z) = v(x, z) = \tilde{u}_{out} - a(\varepsilon^{\frac{2}{3}}y)$$

satisfies

$$|v_x| \leq C\varepsilon^{\frac{8}{3}}, \quad |v_z| \leq C\varepsilon^{\frac{2}{3}}, \quad |v_{xx}| \leq C\varepsilon^{\frac{10}{3}}, \quad |v_{xz}| \leq C\varepsilon^{\frac{4}{3}}, \quad |v_{zz}| \leq C\varepsilon^{\frac{4}{3}},$$

for  $x \in [-2\delta\varepsilon^{-\frac{2}{3}}, -\delta\varepsilon^{-\frac{2}{3}}]$ ,  $z \in [0, \varepsilon^{-\frac{2}{3}}\ell]$ . In turn, via the relations

$$\begin{cases} u_s = \beta v_x, \\ u_{ss} = \beta^2 v_{xx}, \\ u_z = \varepsilon^{\frac{2}{3}}\beta'\beta^{-1}xv_x + v_z, \\ u_{zz} = \varepsilon^{\frac{4}{3}}\beta''\beta^{-1}xv_x + \varepsilon^{\frac{4}{3}}(\beta')^2\beta^{-2}x^2v_{xx} + 2\varepsilon^{\frac{2}{3}}\beta'\beta^{-1}xv_{xz} + v_{zz}, \end{cases} \quad (4.5)$$

these estimates imply that

$$|u_s| \leq C\varepsilon^{\frac{8}{3}}, \quad |u_z| \leq C\varepsilon^{\frac{2}{3}}, \quad |u_{ss}| \leq C\varepsilon^{\frac{10}{3}}, \quad |u_{zz}| \leq C\varepsilon^{\frac{4}{3}},$$

for  $s \in [-2\delta\varepsilon^{-\frac{2}{3}}\beta^{-1}, -\delta\varepsilon^{-\frac{2}{3}}\beta^{-1}]$ ,  $z \in [0, \varepsilon^{-\frac{2}{3}}\ell]$ . These last estimates, by virtue of (3.4), yield that

$$\Delta u = \mathcal{O}(\varepsilon^{\frac{4}{3}}) \quad \text{uniformly in } \Omega_\varepsilon \cap \{-2\delta\varepsilon^{-\frac{2}{3}} \leq x \leq -\delta\varepsilon^{-\frac{2}{3}}\}.$$

So,

$$\Delta \tilde{u}_{out} = \Delta \left( a(\varepsilon^{\frac{2}{3}}y) \right) + \mathcal{O}(\varepsilon^{\frac{4}{3}}) = \mathcal{O}(\varepsilon^{\frac{4}{3}}) \quad \text{uniformly in } \Omega_\varepsilon \cap \{-2\delta\varepsilon^{-\frac{2}{3}} \leq x \leq -\delta\varepsilon^{-\frac{2}{3}}\}.$$

In view of (4.4), and the above relation, we find that the assertion of the lemma holds true in  $\Omega_\varepsilon \cap \{-2\delta\varepsilon^{-\frac{2}{3}} \leq x \leq -\delta\varepsilon^{-\frac{2}{3}}\}$  as well. In the remaining region  $\Omega_{1,\varepsilon}/\{-2\delta\varepsilon^{-\frac{2}{3}} \leq x < 0\}$ , we have  $\tilde{u}_{out} = a(\varepsilon^{\frac{2}{3}}y)$  and the assertion of the lemma clearly holds. Identical calculations also apply in  $\Omega_{2,\varepsilon}/\{0 < x < L\}$ .

The proof of the lemma is complete.  $\square$

The following estimates will be useful in the sequel:

**Lemma 4.2.** *We have*

$$\tilde{u}_{out} - u_{in} = \mathcal{O}(\varepsilon^2x^3) \quad \text{in } \Omega_\varepsilon \cap \{L \leq |x| \leq \delta\varepsilon^{-\frac{2}{3}}\}, \quad (4.6)$$

and

$$|\nabla(\tilde{u}_{out} - u_{in})| + |\Delta(\tilde{u}_{out} - u_{in})| \leq C\varepsilon^2 \quad \text{in } \Omega_\varepsilon \cap \{L \leq |x| \leq 3L\}.$$

*Proof.* From (3.42), (4.3), we find that

$$(\tilde{u}_{out} - u_{in})(s, z) = a(\varepsilon^{\frac{2}{3}}s, \varepsilon^{\frac{2}{3}}z) - a - a_t\varepsilon^{\frac{2}{3}}s - \frac{1}{2}a_{tt}\varepsilon^{\frac{4}{3}}s^2, \quad (4.7)$$

for  $y = (s, z) = (\beta^{-1}x, z) \in \Omega_\varepsilon \cap \{-\delta\varepsilon^{-\frac{2}{3}} \leq x \leq -L\}$ . An analogous relation, with  $a$  replaced by  $b$ , holds true in the corresponding region with  $x > 0$ . The first assertion of the lemma now follows at once.

Note that, from (4.7), we also get

$$(\tilde{u}_{out} - u_{in})_s = \mathcal{O}(\varepsilon^2 s^2), \quad (\tilde{u}_{out} - u_{in})_{ss} = \mathcal{O}(\varepsilon^2 s),$$

$$(\tilde{u}_{out} - u_{in})_z = \mathcal{O}(\varepsilon^{\frac{8}{3}} s^3), \quad (\tilde{u}_{out} - u_{in})_{zz} = \mathcal{O}(\varepsilon^{\frac{10}{3}} s^3),$$

for  $y = (s, z) = (\beta^{-1}x, z) \in \Omega_\varepsilon \cap \{-\delta\varepsilon^{-\frac{2}{3}} \leq x \leq -L\}$ . Analogous relations hold true in the corresponding region with  $x > 0$ . The second assertion of the lemma now follows readily, via (3.4), and the fact that there exist constants  $c, C > 0$  such that

$$c(|u_s| + |u_z|) \leq |u_{y_1}| + |u_{y_2}| \leq C(|u_s| + |u_z|), \quad (4.8)$$

for  $y = (y_1, y_2) \in \{(s, z) \mid |s| \leq \delta_0\varepsilon^{-\frac{2}{3}}, z \in [0, \varepsilon^{-\frac{2}{3}}\ell]\}$ , and any smooth function  $u$  defined in this region (recall that the mapping  $\mathbf{y} \rightarrow (t(\mathbf{y}), \theta(\mathbf{y}))$ , defined below (1.10), is a diffeomorphism close to the curve  $\Gamma$ , and  $(t(\mathbf{y}), \theta(\mathbf{y})) = \varepsilon^{\frac{2}{3}}(s(\varepsilon^{-\frac{2}{3}}\mathbf{y}), z(\varepsilon^{-\frac{2}{3}}\mathbf{y}))$ ).

The proof of the lemma is complete.  $\square$

## 5. THE MATCHING PROCEDURE

In this section we will carefully interpolate between  $u_{in}$  and  $\tilde{u}_{out}$  in order to get a smooth approximation  $\tilde{u}_{ap}$  in  $\Omega_\varepsilon$ , without affecting the order of remainder that  $u_{in}, \tilde{u}_{out}$  left in (3.1) separately. Then we will appropriately iterate once in (3.1) in order to obtain an even better approximation  $u_{ap}$ .

**5.1. The first global approximation  $\tilde{u}_{ap}$ .** Let

$$\tilde{u}_{ap} = u_{in} + (1 - n_L(x))(\tilde{u}_{out} - u_{in}), \quad (5.1)$$

where the cutoff function  $n_L$  is as in (4.2).

**Lemma 5.1.** *If  $\varepsilon > 0$  is sufficiently small, we have*

$$\Delta\tilde{u}_{ap} - \varepsilon^{-\frac{2}{3}}(\tilde{u}_{ap} - a(\varepsilon^{\frac{2}{3}}y))(\tilde{u}_{ap} - b(\varepsilon^{\frac{2}{3}}y)) = \begin{cases} \mathcal{O}(\varepsilon^2|x| + \varepsilon^2), & y \in \Omega_\varepsilon \cap \{|x| \leq \delta\varepsilon^{-\frac{2}{3}}\}, \\ \mathcal{O}(\varepsilon^{\frac{4}{3}}), & y \in \Omega_\varepsilon / \{|x| \leq \delta\varepsilon^{-\frac{2}{3}}\}, \end{cases} \quad (5.2)$$

and

$$\varepsilon^{-\frac{2}{3}}(2\tilde{u}_{ap}(y) - a(\varepsilon^{\frac{2}{3}}y) - b(\varepsilon^{\frac{2}{3}}y)) \geq \begin{cases} c|x|, & \Omega_\varepsilon \cap \{L \leq |x| \leq \delta\varepsilon^{-\frac{2}{3}}\}, \\ c\varepsilon^{-\frac{2}{3}}, & \Omega_\varepsilon / \{|x| < \delta\varepsilon^{-\frac{2}{3}}\}. \end{cases} \quad (5.3)$$

*Proof.* In view of the proof of Proposition 3.3 and Lemmas 4.1, it remains to show estimate (5.2) in the intermediate region  $\Omega_\varepsilon \cap \{L \leq |x| \leq 2L\}$ . There, we have

$$\begin{aligned} \Delta\tilde{u}_{ap} - \varepsilon^{-\frac{2}{3}}(\tilde{u}_{ap} - a(\varepsilon^{\frac{2}{3}}y))(\tilde{u}_{ap} - b(\varepsilon^{\frac{2}{3}}y)) &= \Delta u_{in} - \varepsilon^{-\frac{2}{3}}(u_{in} - a(\varepsilon^{\frac{2}{3}}y))(u_{in} - b(\varepsilon^{\frac{2}{3}}y)) \\ &\quad - (\Delta n_L)(\tilde{u}_{out} - u_{in}) - 2\nabla n_L \nabla(\tilde{u}_{out} - u_{in}) \\ &\quad + (1 - n_L)\Delta(\tilde{u}_{out} - u_{in}) - \varepsilon^{-\frac{2}{3}}(1 - n_L)^2(\tilde{u}_{out} - u_{in})^2 \\ &\quad - \varepsilon^{-\frac{2}{3}}(1 - n_L)(\tilde{u}_{out} - u_{in})(2u_{in} - a(\varepsilon^{\frac{2}{3}}y) - b(\varepsilon^{\frac{2}{3}}y)). \end{aligned}$$

The first line of the righthand side of the above relation can be estimated directly, as before, from the proof of Proposition 3.3; the second and third by Lemma 4.2;

the fourth by Lemma 4.2 and (3.42) (recall that  $a = b$  on  $\Gamma$ ). The desired estimate (5.2), in the region  $\Omega_\varepsilon \cap \{L \leq |x| \leq 2L\}$ , follows at once.

The proof of the lower bound (5.3) proceeds as follows: In  $\Omega_\varepsilon \cap \{-2L \leq x \leq -L\}$ , recalling (3.42) and the definition of  $L$  (from (3.40)), we have

$$2u_{in}(y) - a(\varepsilon^{\frac{2}{3}}y) - b(\varepsilon^{\frac{2}{3}}y) = 2\varepsilon^{\frac{2}{3}}\beta^2V + \mathcal{O}(\varepsilon^{\frac{4}{3}}) \geq c\varepsilon^{\frac{2}{3}} \geq \frac{c}{2L}\varepsilon^{\frac{2}{3}}|x|.$$

In  $\Omega_\varepsilon \cap \{-\delta\varepsilon^{-\frac{2}{3}} \leq x \leq -2L\}$ , by the estimates of Propositions 3.1, 3.2, and (4.3), we obtain that

$$\begin{aligned} 2\tilde{u}_{out}(y) - a(\varepsilon^{\frac{2}{3}}y) - b(\varepsilon^{\frac{2}{3}}y) &= a(\varepsilon^{\frac{2}{3}}y) - b(\varepsilon^{\frac{2}{3}}y) + 2\varepsilon^{\frac{2}{3}}\beta^2(V+x) + 2\varepsilon^{\frac{2}{3}}\beta^2\left(\phi_1 - \frac{\alpha_{tt}}{2}\varepsilon^{\frac{2}{3}}\beta^{-4}x^2\right) \\ &= (a_t - b_t)\varepsilon^{\frac{2}{3}}\beta^{-1}x + \mathcal{O}(\varepsilon^{\frac{4}{3}}x^2) + \mathcal{O}(\varepsilon^{\frac{2}{3}}e^{-cL^{\frac{3}{2}}}) + \mathcal{O}(\varepsilon^{\frac{4}{3}}L^{-1}) \\ &\stackrel{(3.9)}{\geq} c\varepsilon^{\frac{2}{3}}|x| - C\delta\varepsilon^{\frac{2}{3}}|x| - C\varepsilon^{\frac{2}{3}}L^{-1} \\ &\geq c\varepsilon^{\frac{2}{3}}|x| - C\delta\varepsilon^{\frac{2}{3}}|x| - C\varepsilon^{\frac{2}{3}}L^{-2}|x| \\ &\geq c\varepsilon^{\frac{2}{3}}|x|, \end{aligned} \tag{5.4}$$

where we have decreased  $\delta_0 > 0$  and increased  $L > 0$  if necessary (independently of  $\varepsilon$ ). Whereas, in  $\Omega_{1,\varepsilon}/\{-\delta\varepsilon^{-\frac{2}{3}} < x < 0\}$ , thanks to (1.3) and (4.4), we have

$$2\tilde{u}_{out}(y) - a(\varepsilon^{\frac{2}{3}}y) - b(\varepsilon^{\frac{2}{3}}y) = a(\varepsilon^{\frac{2}{3}}y) - b(\varepsilon^{\frac{2}{3}}y) + \mathcal{O}(\varepsilon^2) \geq c.$$

The above estimates also hold true in  $\Omega_{2,\varepsilon}/\{0 < x < L\}$ . Hence, recalling Lemma 4.2 and (5.1), we conclude that relation (5.3) holds.

The proof of the lemma is complete.  $\square$

**Remark 5.1.** *It is tempting to simply define a global approximation in the form*

$$\tilde{v}_{ap} = u_{in} + (1 - n_{M_\varepsilon}(x))(u_0 - u_{in}),$$

where  $u_0 \equiv \max\{a(\varepsilon^{\frac{2}{3}}y), b(\varepsilon^{\frac{2}{3}}y)\}$ , and  $L \leq M_\varepsilon \leq \delta\varepsilon^{-\frac{2}{3}}$  to be chosen. However, as one can readily verify, we would get

$$\Delta\tilde{v}_{ap} - \varepsilon^{-\frac{2}{3}}\left(\tilde{v}_{ap} - a(\varepsilon^{\frac{2}{3}}y)\right)\left(\tilde{v}_{ap} - b(\varepsilon^{\frac{2}{3}}y)\right) = \varepsilon^{\frac{2}{3}}\tilde{G}(M_\varepsilon) + M_\varepsilon^{-q_1}\varepsilon^{\frac{4}{3}} + M_\varepsilon^{q_2}\varepsilon^2,$$

in the region  $M_\varepsilon \leq |x| \leq 2M_\varepsilon$ , where  $\tilde{G}$  is a positive super-exponentially decaying function of the form (1.18), and  $q_1, q_2$  are positive constants. The natural choice  $M_\varepsilon = |\ln \varepsilon|$  leads to an estimate of order  $|\ln \varepsilon|^{-q_1}\varepsilon^{\frac{4}{3}}$  for the righthand side of the above relation. This is much weaker than that provided by (5.2), for  $x$  in the same region, which is of order  $|\ln \varepsilon|\varepsilon^2$ .

**5.2. The improved global approximation  $u_{ap}$ .** Starting from  $\tilde{u}_{ap}$ , we will apply one step of a modified Newton's method in (3.1) in order to improve the accuracy of  $\tilde{u}_{ap}$  away from the curve  $\Gamma_\varepsilon$ . (As one can check, iterating more than once does not lead to further improvements).

5.2.1. *An approximate linearization.* As we have discussed in the introduction, we expect the linearized operator

$$-\Delta + \varepsilon^{-\frac{2}{3}} \left( 2\tilde{u}_{ap} - a(\varepsilon^{\frac{2}{3}}y) - b(\varepsilon^{\frac{2}{3}}y) \right) \quad (5.5)$$

to be near non-invertible. We will improve the approximate solution  $\tilde{u}_{ap}$  using one step of a modified Newton's method where, instead of the linear operator (5.5), we will use an invertible approximation which is obtained by suitably modifying the potential in (5.5). This technique was already used in [32], for the radial case, but for the purpose of matching a perturbation of  $\tilde{u}_{out}$  with  $u_{in}$ . The issue in [32] was that the potential in (5.5) becomes negative close to  $\Gamma_\varepsilon$  (the radial operator (5.5) is invertible, recall (1.7)).

Let

$$p(y) = \begin{cases} \frac{p(\beta^{-1}L, z) - p(-\beta^{-1}L, z)}{2L}(x + L) + p(-\beta^{-1}L, z), & y = (\beta^{-1}x, z) \in \Omega_\varepsilon \cap \{|x| \leq L\}, \\ \varepsilon^{-\frac{2}{3}} \left( 2u_{ap}(y) - a(\varepsilon^{\frac{2}{3}}y) - b(\varepsilon^{\frac{2}{3}}y) \right), & y \in \bar{\Omega}_\varepsilon / \{|x| < L\}. \end{cases} \quad (5.6)$$

Note that  $p \in C(\bar{\Omega}_\varepsilon)$ . Furthermore, in view of (5.3), we have

$$p(y) \geq \begin{cases} c, & y = (\beta^{-1}x, z) \in \Omega_\varepsilon \cap \{|x| \leq L\}, \\ c|x|, & y = (\beta^{-1}x, z) \in \Omega_\varepsilon \cap \{L \leq |x| \leq \delta\varepsilon^{-\frac{2}{3}}\}, \\ c\varepsilon^{-\frac{2}{3}}, & y \in \Omega_\varepsilon / \{|x| < \delta\varepsilon^{-\frac{2}{3}}\}, \end{cases} \quad (5.7)$$

and, recalling (3.42) and the first part of (5.4),

$$\left| p(y) - \varepsilon^{-\frac{2}{3}} \left( 2\tilde{u}_{ap}(y) - a(\varepsilon^{\frac{2}{3}}y) - b(\varepsilon^{\frac{2}{3}}y) \right) \right| \leq \begin{cases} C, & \text{in } \Omega_\varepsilon \cap \{|x| \leq L\}, \\ 0, & \text{in } \bar{\Omega}_\varepsilon / \{|x| < L\}. \end{cases} \quad (5.8)$$

**Remark 5.2.** *This type of modification was originally used in the matching procedure performed in [46]. However, in light of Lemma 2.2 of the recent paper [21], it turns out that the potential of the linear operator in [46] is positive. Therefore, there is no need for using this modification in [46].*

5.2.2. *A modified Newton's method.* We define a new approximate solution for (3.1) as

$$u_{ap} = \tilde{u}_{ap} + \sigma, \quad (5.9)$$

where  $\sigma$  is determined from the following modified Newton's method:

$$\begin{cases} -\Delta\sigma + p\sigma = \Delta\tilde{u}_{ap} - \varepsilon^{-\frac{2}{3}} \left( \tilde{u}_{ap} - a(\varepsilon^{\frac{2}{3}}y) \right) \left( \tilde{u}_{ap} - b(\varepsilon^{\frac{2}{3}}y) \right) & \text{in } \Omega_\varepsilon, \\ \frac{\partial\sigma}{\partial\eta} = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases} \quad (5.10)$$

**Lemma 5.2.** *If  $\varepsilon > 0$  is sufficiently small, there exists a unique solution of (5.10), and satisfies*

$$\|\sigma\|_{L^\infty(\Omega_\varepsilon)} \leq C\varepsilon^2. \quad (5.11)$$

*Proof.* By virtue of (5.7), the linear operator in the lefthand side of (5.10) is invertible, and existence and uniqueness of  $\sigma$  follow immediately.

Let  $y_\varepsilon \in \bar{\Omega}_{1,\varepsilon}/\{-L < x \leq 0\}$  be such that

$$\sigma(y_\varepsilon) = \max_{\bar{\Omega}_\varepsilon} \sigma.$$

Without loss of generality we may assume that  $\sigma(y_\varepsilon) \geq 0$ . Three possibilities can occur:

- (1) If  $y_\varepsilon \in \Omega_\varepsilon \cap \{|x| \leq L\}$ , then  $\Delta\sigma(y_\varepsilon) \leq 0$ . Thus, by (5.2), (5.7), and (5.10), we deduce that

$$c\sigma(y_\varepsilon) \leq C\varepsilon^2,$$

$$\text{i.e., } \sigma(y_\varepsilon) = \mathcal{O}(\varepsilon^2).$$

- (2) If  $y_\varepsilon = \left(\beta^{-1}(\varepsilon^{\frac{2}{3}}z_\varepsilon)x_\varepsilon, z_\varepsilon\right) \in \Omega_\varepsilon \cap \{L < |x| \leq \delta\varepsilon^{-\frac{2}{3}}\}$ , we have  $\Delta\sigma(y_\varepsilon) \leq 0$ , and we obtain as before that

$$c|x_\varepsilon|\sigma(y_\varepsilon) \leq C\varepsilon^2|x_\varepsilon|,$$

$$\text{i.e., } \sigma(y_\varepsilon) = \mathcal{O}(\varepsilon^2).$$

- (3) If  $y_\varepsilon \in \bar{\Omega}_\varepsilon/\{|x| \leq \delta\varepsilon^{-\frac{2}{3}}\}$ , we have  $\Delta\sigma(y_\varepsilon) \leq 0$  (if  $y_\varepsilon \in \partial\Omega_\varepsilon$  we have to use the strong maximum principle), and as before we get

$$c\varepsilon^{-\frac{2}{3}}\sigma(y_\varepsilon) \leq C\varepsilon^{\frac{4}{3}},$$

$$\text{i.e., } \sigma(y_\varepsilon) = \mathcal{O}(\varepsilon^2).$$

Hence, we have  $\max_{\bar{\Omega}_\varepsilon} \sigma = \mathcal{O}(\varepsilon^2)$ . Similarly we can show that  $\min_{\bar{\Omega}_\varepsilon} \sigma = \mathcal{O}(\varepsilon^2)$ .

The proof of the lemma is complete.  $\square$

The following proposition contains the fundamental estimates regarding the approximation  $u_{ap}$ , and will be used in an essential way in the next sections.

**Proposition 5.1.** *The approximate solution  $u_{ap}$ , defined in (5.9), satisfies*

$$\|\Delta u_{ap} - \varepsilon^{-\frac{2}{3}} \left(u_{ap} - a(\varepsilon^{\frac{2}{3}}y)\right) \left(u_{ap} - b(\varepsilon^{\frac{2}{3}}y)\right)\|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon^{\frac{5}{3}}, \quad (5.12)$$

$$\frac{\partial u_{ap}}{\partial \eta} = 0 \quad \text{on } \partial\Omega_\varepsilon, \quad (5.13)$$

and

$$(u_{ap} - u_{in})(y) = \mathcal{O}(\varepsilon^2x^3 + \varepsilon^2) \quad \text{if } y = (\beta^{-1}x, z) \in \Omega_\varepsilon \cap \{|x| \leq \delta\varepsilon^{-\frac{2}{3}}\}, \quad (5.14)$$

as  $\varepsilon \rightarrow 0$ , where  $u_{in}$  is the inner solution as defined in (3.42).

*Proof.* By (5.10), we find that

$$\begin{aligned} \Delta u_{ap} - \varepsilon^{-\frac{2}{3}} \left(u_{ap} - a(\varepsilon^{\frac{2}{3}}y)\right) \left(u_{ap} - b(\varepsilon^{\frac{2}{3}}y)\right) &= \left[ p - \varepsilon^{-\frac{2}{3}} \left(2\tilde{u}_{ap} - a(\varepsilon^{\frac{2}{3}}y) - b(\varepsilon^{\frac{2}{3}}y)\right) \right] \sigma \\ &\quad - \varepsilon^{-\frac{2}{3}}\sigma^2. \end{aligned}$$

Thus, by (5.8), (5.11), we infer that

$$\Delta u_{ap} - \varepsilon^{-\frac{2}{3}} \left(u_{ap} - a(\varepsilon^{\frac{2}{3}}y)\right) \left(u_{ap} - b(\varepsilon^{\frac{2}{3}}y)\right) = \begin{cases} \mathcal{O}(\varepsilon^2), & y \in \Omega_\varepsilon \cap \{|x| \leq L\}, \\ \mathcal{O}(\varepsilon^{\frac{10}{3}}), & y \in \Omega_\varepsilon/\{|x| < L\}. \end{cases} \quad (5.15)$$

Now, estimate (5.12) follows by simply noting that  $|\Omega_\varepsilon \cap \{|x| \leq L\}| = \mathcal{O}(\varepsilon^{-\frac{2}{3}})$  and  $|\Omega_\varepsilon/\{|x| < L\}| \leq C\varepsilon^{-\frac{4}{3}}$ . Relation (5.13) follows at once from (4.1), (4.3) and (5.10). Finally, relation (5.14) follows readily from Lemma 4.2, (5.1), and (5.11).

The proof of the proposition is complete.  $\square$

**5.3. Further improved approximation close to the curve  $\Gamma_\varepsilon$ .** To further improve the approximation for a solution, we need to introduce a parameter  $e$ .

We let  $(\lambda_0, Z(x))$  be the principal eigenvalue- eigenfunction pair of the problem

$$\phi_{xx} - 2V\phi = \lambda\phi, \quad \phi(\pm\infty) = 0. \quad (5.16)$$

Then, from Proposition 3.1, we know that  $\lambda_0 > 0$  and  $Z(x)$  is one signed and even in  $x$ . Furthermore, the eigenfunction  $Z$  decays super-exponentially to zero, as  $x \rightarrow \pm\infty$ , with the same rate as the righthand side of (3.22). Moreover, without loss of generality, we may assume that  $\|Z\|_{L^2(\mathbb{R})} = 1$ .

Let  $e(\theta)$  be a twice differentiable,  $\ell$ -periodic, function which will be determined later. We define our basic approximation to a solution to the problem, near the curve  $\Gamma_\varepsilon$ , to be

$$w = u_{ap} + \varepsilon^{\frac{2}{3}} e(\varepsilon^{\frac{2}{3}} z) Z, \quad |x| \leq \delta_0 \varepsilon^{-\frac{2}{3}}, \quad z \in [0, \ell \varepsilon^{-\frac{2}{3}}]. \quad (5.17)$$

In all that follows, we will assume the validity of the following constraint:

$$\|e\|_b := \varepsilon^{\frac{4}{3}} \|e''\|_{L^2(0,\ell)} + \varepsilon^{\frac{2}{3}} \|e'\|_{L^2(0,\ell)} + \|e\|_{L^\infty(0,\ell)} \leq \varepsilon^{\frac{2}{3}}. \quad (5.18)$$

In reality, a posteriori, this parameter will turn out to be smaller than stated here.

The new error of approximation is

$$\begin{aligned} E_1 &= \Delta w - \varepsilon^{-\frac{2}{3}} \left( w - a(\varepsilon^{\frac{2}{3}} y) \right) \left( w - b(\varepsilon^{\frac{2}{3}} y) \right) \\ &= E_0 + \varepsilon^{\frac{2}{3}} \Delta(eZ) - \left( 2u_{ap} - a(\varepsilon^{\frac{2}{3}} y) - b(\varepsilon^{\frac{2}{3}} y) \right) eZ - \varepsilon^{\frac{2}{3}} e^2 Z^2, \end{aligned} \quad (5.19)$$

where

$$E_0 = \Delta u_{ap} - \varepsilon^{-\frac{2}{3}} \left( u_{ap} - a(\varepsilon^{\frac{2}{3}} y) \right) \left( u_{ap} - b(\varepsilon^{\frac{2}{3}} y) \right). \quad (5.20)$$

Let us recall that, thanks to (3.4), (4.5), the Laplacian of  $u(y) = u(s, z) = v(x, z)$ , in coordinated  $(x, z)$ , becomes

$$\Delta u = v_{zz} + \beta^2 v_{xx} + \tilde{B}_1(v), \quad (5.21)$$

where

$$\tilde{B}_1(v) = \varepsilon^{\frac{4}{3}} \beta'' \beta^{-1} x v_x + \varepsilon^{\frac{4}{3}} (\beta')^2 \beta^{-2} x^2 v_{xx} + 2\varepsilon^{\frac{2}{3}} \beta' \beta^{-1} x v_{xz} + B_1(v), \quad (5.22)$$

and  $B_1$  is the differential operator in (3.5) with derivatives expressed by (4.5) and  $s$  replaced by  $\beta^{-1}x$ . Note also that from (3.42), (4.6), and (5.11), we obtain that

$$2u_{ap} - a(\varepsilon^{\frac{2}{3}} y) - b(\varepsilon^{\frac{2}{3}} y) = 2\varepsilon^{\frac{2}{3}} \beta^2 V + \varepsilon^{\frac{4}{3}} \mathcal{O}(x^2 + 1), \quad |x| \leq \delta_0 \varepsilon^{-\frac{2}{3}}. \quad (5.23)$$

A short calculation, using (5.21), (5.22), and the above relation, shows that

$$\varepsilon^{\frac{2}{3}} \Delta(eZ) - \left( 2u_{ap} - a(\varepsilon^{\frac{2}{3}} y) - b(\varepsilon^{\frac{2}{3}} y) \right) eZ = \varepsilon^2 e'' Z + \lambda_0 \varepsilon^{\frac{2}{3}} \beta^2 eZ + \varepsilon^{\frac{2}{3}} \tilde{B}_1(eZ) + \varepsilon^{\frac{4}{3}} \mathcal{O}(x^2 + 1) eZ. \quad (5.24)$$

We write

$$E_{11} = \varepsilon^2 e'' Z + \lambda_0 \varepsilon^{\frac{2}{3}} \beta^2 eZ \quad \text{and} \quad E_{12} = E_1 - E_{11} \quad (\text{recall (5.19)}). \quad (5.25)$$

We set up the full problem, close to the curve, in the form

$$\Delta(w + \phi) - \varepsilon^{-\frac{2}{3}} \left( w + \phi - a(\varepsilon^{\frac{2}{3}} y) \right) \left( w + \phi - b(\varepsilon^{\frac{2}{3}} y) \right) = 0,$$

which can be expanded in the following way:

$$\Delta \phi - \varepsilon^{-\frac{2}{3}} \left( 2w - a(\varepsilon^{\frac{2}{3}} y) - b(\varepsilon^{\frac{2}{3}} y) \right) \phi - \varepsilon^{-\frac{2}{3}} \phi^2 + \Delta w - \varepsilon^{-\frac{2}{3}} \left( w - a(\varepsilon^{\frac{2}{3}} y) \right) \left( w - b(\varepsilon^{\frac{2}{3}} y) \right) = 0.$$

In summary, near the curve, the problem takes the form:

$$\beta^2 L_1(\phi) + \tilde{B}_1(\phi) + N_1(\phi) + E_1 = 0,$$

where  $E_1$ ,  $\tilde{B}_1$  are described in (5.19), (5.22) respectively, and

$$L_1(\phi) = \beta^{-2} \phi_{zz} + \phi_{xx} - \varepsilon^{-\frac{2}{3}} \beta^{-2} \left( 2w - a(\varepsilon^{\frac{2}{3}} y) - b(\varepsilon^{\frac{2}{3}} y) \right) \phi, \quad N_1(\phi) = -\varepsilon^{-\frac{2}{3}} \phi^2. \quad (5.26)$$

We recall that the description made here is only local (for  $|x| \leq \delta_0 \varepsilon^{-\frac{2}{3}}$ ). We will be able however to reduce the problem to one qualitatively similar to that of the above form in the infinite strip  $\mathcal{S}$  (recall (3.39)).

## 6. THE GLUING PROCEDURE

Let us first define some useful cutoff functions

$$\chi_\delta^\varepsilon(x) := n_\delta(\varepsilon^{\frac{1}{3}} x), \quad \delta > 0, \quad (6.1)$$

where  $n_\delta$  is the smooth cutoff function defined in (4.2). The choice of the power 1/3 will become clear in the proof of Proposition 7.1 below (in particular, see relation (7.28)).

We define our new global approximation to be simply

$$\mathbf{w} = u_{ap} + \varepsilon^{\frac{2}{3}} \chi_{30\gamma}^\varepsilon(x) eZ, \quad (6.2)$$

recall (5.17), where  $\gamma > 0$  is a small constant, independent of  $\varepsilon$ , to be chosen (until then, unless specified otherwise, all the following constants will implicitly depend on  $\gamma$ ). Then, the function  $\mathbf{w} + \tilde{\phi}$  solves (3.1) if and only if

$$\tilde{L}(\tilde{\phi}) = \tilde{N}(\tilde{\phi}) + \tilde{E} \quad \text{in } \Omega_\varepsilon, \quad \frac{\partial \tilde{\phi}}{\partial \eta} = 0 \quad \text{on } \partial\Omega_\varepsilon,$$

where

$$\tilde{L}(\tilde{\phi}) = \Delta \tilde{\phi} - \varepsilon^{-\frac{2}{3}} \left( 2\mathbf{w} - a(\varepsilon^{\frac{2}{3}} y) - b(\varepsilon^{\frac{2}{3}} y) \right) \tilde{\phi}, \quad (6.3)$$

$$\tilde{N}(\tilde{\phi}) = \varepsilon^{-\frac{2}{3}} \tilde{\phi}^2 \quad \text{and} \quad \tilde{E} = -\Delta \mathbf{w} + \varepsilon^{-\frac{2}{3}} \left( \mathbf{w} - a(\varepsilon^{\frac{2}{3}} y) \right) \left( \mathbf{w} - b(\varepsilon^{\frac{2}{3}} y) \right).$$

We now use a very nice trick which was already used in [17]. This trick amounts to decompose the function  $\tilde{\phi}$  into two functions, one of which is supported in a tubular neighborhood of  $\Gamma_\varepsilon$  and the other one being globally defined in  $\Omega_\varepsilon$ . Therefore, we decompose  $\tilde{\phi}$  in the following form:

$$\tilde{\phi} = \chi_{3\gamma}^\varepsilon \phi + \psi,$$

where, in coordinates  $(x, z)$ , we assume that  $\phi$  is defined in the whole strip  $\mathcal{S}$ . We want

$$\tilde{L}(\chi_{3\gamma}^\varepsilon \phi) + \tilde{L}(\psi) = \tilde{E} + \tilde{N}(\chi_{3\gamma}^\varepsilon \phi + \psi) \quad \text{in } \Omega_\varepsilon, \quad \frac{\partial \psi}{\partial \eta} = 0 \quad \text{on } \partial\Omega_\varepsilon.$$

We achieve this if the pair  $(\phi, \psi)$  satisfies the following nonlinear coupled system:

$$\chi_{3\gamma}^\varepsilon \tilde{L}(\phi) = \chi_\gamma^\varepsilon \tilde{E} + \chi_\gamma^\varepsilon \tilde{N}(\phi + \psi) + 2\chi_\gamma^\varepsilon \varepsilon^{-\frac{2}{3}} (\mathbf{w} - \tilde{u}) \psi \quad \text{in } \mathcal{S}, \quad (6.4)$$

$$\Delta \psi - Q\psi - 2(1 - \chi_\gamma^\varepsilon) \varepsilon^{-\frac{2}{3}} (\mathbf{w} - \tilde{u}) \psi = (1 - \chi_\gamma^\varepsilon) \tilde{E} + (1 - \chi_\gamma^\varepsilon) \tilde{N}(\chi_{3\gamma}^\varepsilon \phi + \psi) - 2\nabla(\chi_{3\gamma}^\varepsilon) \nabla \phi - \Delta(\chi_{3\gamma}^\varepsilon) \phi \quad (6.5)$$

in  $\Omega_\varepsilon$ , and  $\frac{\partial \psi}{\partial \eta} = 0$  on  $\partial\Omega_\varepsilon$ , where

$$\check{u} = \max \left\{ a(\varepsilon^{\frac{2}{3}}y), \frac{a(\varepsilon^{\frac{2}{3}}y) + b(\varepsilon^{\frac{2}{3}}y)}{2} + \varepsilon^{\frac{2}{3}}, b(\varepsilon^{\frac{2}{3}}y) \right\} \text{ and } Q = \varepsilon^{-\frac{2}{3}} \left( 2\check{u} - a(\varepsilon^{\frac{2}{3}}y) - b(\varepsilon^{\frac{2}{3}}y) \right). \quad (6.6)$$

Notice that the operator  $\tilde{L}$  in the strip  $\mathcal{S}$  may be taken as any compatible extension outside the  $\{|x| \leq 6\gamma\varepsilon^{-\frac{1}{3}}\}$ -neighborhood of the curve. For future reference, it is useful to note at this point some properties of  $\check{u}$  and  $Q$ : Recalling (1.3), it is easy to see that there exists an  $\mathcal{O}(1)$  neighborhood  $U_\varepsilon$  of the curve  $\Gamma_\varepsilon$  such that

$$\check{u}(y) = \begin{cases} a(\varepsilon^{\frac{2}{3}}y), & y \in \Omega_1 \setminus U_\varepsilon, \\ \frac{a(\varepsilon^{\frac{2}{3}}y) + b(\varepsilon^{\frac{2}{3}}y)}{2} + \varepsilon^{\frac{2}{3}}, & y \in U_\varepsilon, \\ b(\varepsilon^{\frac{2}{3}}y), & y \in \Omega_2 \setminus U_\varepsilon. \end{cases} \quad (6.7)$$

Furthermore, observe that relation (3.9) implies that

$$Q(y) \geq \begin{cases} c, & y = (\beta^{-1}x, z) \in \Omega_\varepsilon \cap \{|x| \leq \gamma\varepsilon^{-\frac{1}{3}}\}, \\ c\varepsilon^{-\frac{1}{3}}, & y \in \Omega_\varepsilon \setminus \{|x| \leq \gamma\varepsilon^{-\frac{1}{3}}\}. \end{cases} \quad (6.8)$$

Moreover, recalling (4.4), (5.1), (5.9), (5.11), (5.18), and the super-exponential decay of  $Z$ , we find that

$$\left| (1 - \chi_\gamma^\varepsilon)\varepsilon^{-\frac{2}{3}}(\mathbf{w} - \check{u}) \right| \leq C\varepsilon \text{ in } \bar{\Omega}_\varepsilon. \quad (6.9)$$

**Remark 6.1.** *Everything in this paper, except from relation (6.8), still holds with the choice  $\check{u}(y) = \max\{a(\varepsilon^{\frac{2}{3}}y), b(\varepsilon^{\frac{2}{3}}y)\}$ .*

What we want to do next is to reduce the problem to a problem in the strip. To do this, we solve, given a small  $\phi$ , problem (6.5) for  $\psi$ . This can be done in a straightforward manner. Assume that  $\phi$  satisfies the following conditions:

$$|\phi(x, z)| \leq \mathcal{M}\varepsilon^{\frac{5}{3}}, \quad |x| \geq \gamma\varepsilon^{-\frac{1}{3}} \text{ and } |\nabla\phi(x, z)| + |\phi(x, z)| \leq \exp\{-\varepsilon^{-\frac{1}{3}}\}, \quad |x| \geq 3\gamma\varepsilon^{-\frac{1}{3}}, \quad (6.10)$$

for a certain constant  $\mathcal{M} > 0$ . Firstly, we can use (6.8), (6.9), and a maximum principle argument to bound the inverse of the operator in the lefthand side of (6.5) (one obtains better estimates if the righthand side has support in  $|x| \leq 2\gamma\varepsilon^{-\frac{1}{3}}$ ). Then, a direct application of the contraction mapping principle, recalling (5.15), (5.18), and the super-exponential decay of  $Z$ , yields that problem (6.5) has a unique (small) solution  $\psi = \psi(\phi)$  with

$$\|\psi(\phi)\|_{L^\infty(\Omega_\varepsilon)} \leq C\mathcal{M}^2\varepsilon^3, \quad (6.11)$$

if  $\varepsilon < \varepsilon(\mathcal{M})$ , with  $C$  independent of  $\varepsilon, \mathcal{M}$ . Furthermore, the nonlinear operator  $\psi$  satisfies a Lipschitz condition of the form

$$\|\psi(\phi_1) - \psi(\phi_2)\|_{L^\infty(\Omega_\varepsilon)} \leq C\varepsilon \|\phi_1 - \phi_2\|_{L^\infty(|x| \geq \gamma\varepsilon^{-\frac{1}{3}})} + C\varepsilon^{\frac{2}{3}} \|\nabla(\phi_1 - \phi_2)\|_{L^\infty(|x| \geq 3\gamma\varepsilon^{-\frac{1}{3}})}, \quad (6.12)$$

( $C$  independent of  $\varepsilon, \mathcal{M}$ ) where, with some abuse of notation, by  $\{|x| \geq r\varepsilon^{-\frac{1}{3}}\}$ ,  $r > 0$ , we denote the complement of the  $\{|x| < r\varepsilon^{-\frac{1}{3}}\}$ -neighborhood of  $\Gamma_\varepsilon$ . For



future reference, note that from (6.5), thanks to (6.8) and (6.9), we have

$$\|\psi(\phi_1) - \psi(\phi_2)\|_{H^1(\Omega_\varepsilon)} \leq C\varepsilon^{\frac{1}{3}}\|\phi_1 - \phi_2\|_{H^1(\mathcal{S})}, \quad \varepsilon < \varepsilon(\mathcal{M}), \quad (6.13)$$

with  $C$  independent of  $\varepsilon, \mathcal{M}$ .

The full problem has thus been reduced to solving the (nonlocal) problem in the infinite strip  $\mathcal{S}$ :

$$L_2(\phi) = \chi_\gamma^\varepsilon \tilde{E} + \chi_\gamma^\varepsilon \tilde{N}(\phi + \psi(\phi)) + 2\chi_\gamma^\varepsilon \varepsilon^{-\frac{2}{3}}(\mathbf{w} - \tilde{u})\psi(\phi) \quad (6.14)$$

for a  $\phi \in H^2(\mathcal{S})$  satisfying condition (6.10). Here  $L_2$  denotes a linear operator that coincides with  $\tilde{L}$  on the region  $\{|x| \leq 10\gamma\varepsilon^{-\frac{1}{3}}\}$  (then we can multiply both sides of (6.14) by  $\chi_{3\gamma}^\varepsilon$  and get (6.4)). We shall define this operator next. The operator  $\tilde{L}$  for  $|x| \leq 10\gamma\varepsilon^{-\frac{1}{3}}$ , in coordinates  $(x, z)$ , is given by  $\beta^2 L_1 + \tilde{B}_1$  (recall (5.17), (5.21), (5.26), (6.2), and (6.3)). We extend it for functions  $\phi$  defined in the entire strip  $\mathcal{S}$ , in terms of coordinates  $(x, z)$ , as follows:

$$L_2(\phi) := \beta^2 L_0(\phi) + \chi_{10\gamma}^\varepsilon \tilde{B}_1(\phi) - \chi_{10\gamma}^\varepsilon \left[ \varepsilon^{-\frac{2}{3}} \left( 2w - a(\varepsilon^{\frac{2}{3}}y) - b(\varepsilon^{\frac{2}{3}}y) \right) - 2\beta^2 V \right] \phi, \quad (6.15)$$

where, we recall,

$$L_0(\phi) = \beta^{-2} \phi_{zz} + \phi_{xx} - 2V\phi. \quad (6.16)$$

Rather than solving problem (6.14) directly, we shall do it in steps. Firstly, we consider the following projected problem in  $H^2(\mathcal{S})$ : given  $e$  satisfying bound (5.18), find functions  $\phi \in H^2(\mathcal{S})$ ,  $d \in L^2(0, \ell)$ ,  $\ell$ -periodic, such that

$$L_2(\phi) = -\chi_\gamma^\varepsilon E_1 + N_2(\phi) + d(\varepsilon^{\frac{2}{3}}z)\chi_\gamma^\varepsilon Z \quad \text{in } \mathcal{S}, \quad (6.17)$$

$$\phi(x, 0) = \phi(x, \ell/\varepsilon^{\frac{2}{3}}), \quad \phi_z(x, 0) = \phi_z(x, \ell/\varepsilon^{\frac{2}{3}}), \quad -\infty < x < \infty, \quad (6.18)$$

$$\int_{-\infty}^{\infty} \phi(x, z)Z(x)dx = 0, \quad 0 < z < \frac{\ell}{\varepsilon^{\frac{2}{3}}}. \quad (6.19)$$

Here

$$N_2(\phi) = \chi_\gamma^\varepsilon \tilde{N}(\phi + \psi(\phi)) + 2\chi_\gamma^\varepsilon \varepsilon^{-\frac{2}{3}}(w - \tilde{u})\psi(\phi), \quad (6.20)$$

(recall that  $\mathbf{w} = w$  and  $\tilde{E} = -E_1$  for  $|x| \leq 30\gamma\varepsilon^{-\frac{1}{3}}$ ). We will prove that this problem has a unique solution whose norm is controlled by the  $L^2$ -norm of  $E_{12}$ , and *not* by that of the whole  $E_1$  (recall (5.25)). After this has been done, our task is to adjust the parameter  $e$  in such a way that  $d$  is identically zero. As we will see, this turns out to be equivalent to solving a nonlocal, nonlinear second-order differential equation for  $e$  under periodic boundary conditions. We will deal with this next. We will carry out this program in the following sections. To solve (6.17)-(6.19), we need to investigate the invertibility of  $L_2$  in an  $L^2 - H^2$  setting under periodic boundary and orthogonality conditions. Let us mention that such infinite dimensional Lyapunov-Schmidt reduction arguments were first introduced by Pacard and Ritoré [41] in the context of the Allen-Cahn equation.

## 7. INVERTIBILITY OF $L_2$

Let  $L_2$  be the operator defined, in  $H^2(\mathcal{S})$ , by (6.15). In this section we study the linear problem

$$L_2(\phi) = h + d(\varepsilon^{\frac{2}{3}}z)\chi_\gamma^\varepsilon Z \quad \text{in } \mathcal{S}, \quad (7.1)$$

$$\phi(x, 0) = \phi(x, \ell/\varepsilon^{\frac{2}{3}}), \quad \phi_z(x, 0) = \phi_z(x, \ell/\varepsilon^{\frac{2}{3}}), \quad -\infty < x < \infty, \quad (7.2)$$

$$\int_{-\infty}^{\infty} \phi(x, z)Z(x)dx = 0, \quad 0 < z < \frac{\ell}{\varepsilon^{\frac{2}{3}}}, \quad (7.3)$$

for given  $h \in L^2(\mathcal{S})$ .

Our main result in this section is the following:

**Proposition 7.1.** *There exist constants  $\gamma_*$ ,  $\varepsilon_0$ ,  $C > 0$  such that, for all  $\varepsilon \in (0, \varepsilon_0)$  and  $h \in L^2(\mathcal{S})$ , problem (7.1)-(7.3), with  $\gamma = \gamma_*$ , has a unique solution  $\phi = T(h)$ . Furthermore, we have the estimate*

$$\|T(h)\|_{H^2(\mathcal{S})} \leq C\|h\|_{L^2(\mathcal{S})}.$$

For the proof of this result, we need to show the validity of the corresponding assertion for a simpler operator that does not depend on  $\varepsilon$ . Let us first consider the problem:

$$L(\phi) = -\Delta\phi + 2V\phi = \chi_\gamma^\varepsilon h \quad \text{in } \mathcal{S}, \quad (7.4)$$

$$\phi(x, 0) = \phi(x, \ell/\varepsilon^{\frac{2}{3}}), \quad \phi_z(x, 0) = \phi_z(x, \ell/\varepsilon^{\frac{2}{3}}), \quad -\infty < x < \infty, \quad (7.5)$$

$$\int_{-\infty}^{\infty} \phi(x, z)Z(x)dx = 0, \quad 0 < z < \frac{\ell}{\varepsilon^{\frac{2}{3}}}. \quad (7.6)$$

The following a-priori estimate holds:

**Lemma 7.1.** *There exists a constant  $C > 0$ , independent of  $\varepsilon$ ,  $\gamma$ ,  $h$ , such that solutions of (7.4)-(7.6), with  $h \in L^2(\mathcal{S})$  and  $\gamma > 0$ , satisfy the a-priori estimate*

$$\|\phi\|_{H^2(\mathcal{S})} \leq C\|h\|_{L^2(\mathcal{S})}.$$

*Proof.* Let us consider Fourier series decompositions for  $\phi$  and  $h$  of the form

$$\phi(x, z) = \sum_{k=0}^{\infty} \left[ \phi_{1k}(x) \cos\left(\frac{2\pi k}{\ell} \varepsilon^{\frac{2}{3}} z\right) + \phi_{2k}(x) \sin\left(\frac{2\pi k}{\ell} \varepsilon^{\frac{2}{3}} z\right) \right],$$

$$h(x, z) = \sum_{k=0}^{\infty} \left[ h_{1k}(x) \cos\left(\frac{2\pi k}{\ell} \varepsilon^{\frac{2}{3}} z\right) + h_{2k}(x) \sin\left(\frac{2\pi k}{\ell} \varepsilon^{\frac{2}{3}} z\right) \right].$$

Then we have the validity of the equations

$$\frac{4\pi^2 k^2}{\ell^2} \varepsilon^{\frac{4}{3}} \phi_{lk} - L_0(\phi_{lk}) = \chi_\gamma^\varepsilon h_{lk}, \quad x \in \mathbb{R}, \quad (7.7)$$

with the orthogonality condition

$$\int_{-\infty}^{\infty} \phi_{lk} Z dx = 0, \quad (7.8)$$

for  $k = 0, 1, \dots, l = 1, 2$ , where  $L_0$  is the linear operator described in Proposition 3.1. Since the righthand side of (7.7) has compact support, and  $V(x) \rightarrow \infty$  as  $x \rightarrow \pm\infty$ , a rather standard barrier argument can be used to show that each  $\phi_{lk}$  decays super-exponentially to zero as  $x \rightarrow \pm\infty$ . The same property also holds true for  $\phi_{lk,x}$ . Let us consider the bilinear form associated to the operator  $-L_0$ , namely

$$B(\psi, \psi) = \int_{-\infty}^{\infty} (\psi_x^2 + 2V\psi^2) dx, \quad \psi \in H^1(\mathbb{R}) \quad \text{with} \quad \int_{-\infty}^{\infty} V\psi^2 dx < \infty.$$

Since (7.8) holds, recalling the spectral properties of  $L_0$  from Proposition 3.1, we conclude that

$$c \left( \|\phi_{lk}\|_{L^2(\mathbb{R})}^2 + \|\phi_{lk,x}\|_{L^2(\mathbb{R})}^2 \right) \leq B(\phi_{lk}, \phi_{lk}). \quad (7.9)$$

Here, and throughout this proof, by  $c/C$  we denote positive generic constants, independent of  $\varepsilon, \gamma, k$ , whose value will decrease/increase from line to line. Using this fact, and testing equation (7.7) by  $\phi_{lk}$ , we find that

$$\|\phi_{lk}\|_{L^2(\mathbb{R})}^2 + \|\phi_{lk,x}\|_{L^2(\mathbb{R})}^2 \leq C\|\chi_\gamma^\varepsilon h_{lk}\|_{L^2(\mathbb{R})}^2. \quad (7.10)$$

Testing (7.7) once again, this time by  $\phi_{lk,xx}$ , we arrive at

$$c \int_{-\infty}^{\infty} (\phi_{lk,xx})^2 dx \leq 2 \int_{-\infty}^{\infty} V \phi_{lk} \phi_{lk,xx} dx + C \int_{-\infty}^{\infty} (\chi_\gamma^\varepsilon h_{lk})^2 dx. \quad (7.11)$$

An integration by parts, which is possible by the super-exponential decay of  $\phi_{lk}$ ,  $\phi_{lk,x}$ , yields

$$\int_{-\infty}^{\infty} V \phi_{lk} \phi_{lk,xx} dx = - \int_{-\infty}^{\infty} V (\phi_{lk,x})^2 dx - \int_{-\infty}^{\infty} V_x \phi_{lk} \phi_{lk,x} dx. \quad (7.12)$$

By Proposition 3.1,

$$-V \leq C, \quad |V_x| \leq 1, \quad x \in \mathbb{R},$$

and, taking into account (7.10), (7.11), (7.12), we arrive at

$$\|\phi_{lk,xx}\|_{L^2(\mathbb{R})}^2 \leq C\|\chi_\gamma^\varepsilon h_{lk}\|_{L^2(\mathbb{R})}^2. \quad (7.13)$$

Adding up estimates (7.10) and (7.13) in  $k$  and  $l$ , we conclude that

$$\|D^2\phi\|_{L^2(\mathcal{S})}^2 + \|D\phi\|_{L^2(\mathcal{S})}^2 + \|\phi\|_{L^2(\mathcal{S})}^2 \leq C\|\chi_\gamma^\varepsilon h\|_{L^2(\mathcal{S})}^2 \leq C\|h\|_{L^2(\mathcal{S})}^2,$$

which ends the proof.  $\square$

Next, we consider the following problem: given  $h \in L^2(\mathcal{S})$ , find functions  $\phi \in H^2(\mathcal{S})$ ,  $d \in L^2(0, \ell)$ ,  $\ell$ -periodic, such that

$$L(\phi) = \chi_m^\varepsilon h + d(\varepsilon^{\frac{2}{3}} z) \chi_\gamma^\varepsilon Z \quad \text{in } \mathcal{S}, \quad (7.14)$$

$$\phi(x, 0) = \phi(x, \ell/\varepsilon^{\frac{2}{3}}), \quad \phi_z(x, 0) = \phi_z(x, \ell/\varepsilon^{\frac{2}{3}}), \quad -\infty < x < \infty, \quad (7.15)$$

$$\int_{-\infty}^{\infty} \phi(x, z) Z(x) dx = 0, \quad 0 < z < \frac{\ell}{\varepsilon^{\frac{2}{3}}}, \quad (7.16)$$

where  $m \in \mathbb{N}$ . Note that, for large  $m > 0$ , the righthand side of (7.14) approximates, in some sense, that of (7.1).

The following lemma provides us with existence of solutions as well as estimates that will be used in the sequel for passing to the limit  $m \rightarrow \infty$ .

**Lemma 7.2.** *Problem (7.14)-(7.16) possesses a unique solution  $\phi = \tilde{T}_m(h)$ . Moreover, there exists a constant  $C > 0$ , and an  $\varepsilon_0(\gamma) > 0$ , such that*

$$\|\tilde{T}_m(h)\|_{H^2(\mathcal{S})} \leq C\|h\|_{L^2(\mathcal{S})},$$

for all  $\varepsilon \in (0, \varepsilon_0(\gamma))$ ,  $\gamma > 0$ ,  $h \in L^2(\mathcal{S})$ ,  $m \in \mathbb{N}$ .

*Proof.* To establish existence, we assume that

$$h(x, z) = \sum_{k=0}^{\infty} \left[ h_{1k}(x) \cos\left(\frac{2\pi k}{\ell} \varepsilon^{\frac{2}{3}} z\right) + h_{2k}(x) \sin\left(\frac{2\pi k}{\ell} \varepsilon^{\frac{2}{3}} z\right) \right],$$

and consider the problem of finding  $\phi_{lk} \in H^2(\mathbb{R})$  and constants  $d_{lk}$  such that

$$\frac{4\pi^2 k^2}{\ell^2} \varepsilon^{\frac{4}{3}} \phi_{lk} - L_0(\phi_{lk}) = \chi_m^\varepsilon h_{lk} + d_{lk} \chi_\gamma^\varepsilon Z, \quad x \in \mathbb{R},$$

and

$$\int_{-\infty}^{\infty} \phi_{lk} Z dx = 0, \quad k = 0, 1, \dots, l = 1, 2.$$

This problem is solvable provided

$$d_{lk} = -\frac{\int_{-\infty}^{\infty} \chi_m^\varepsilon h_{lk} Z dx}{\int_{-\infty}^{\infty} \chi_\gamma^\varepsilon Z^2 dx}.$$

Observe in particular that

$$\sum_{k=0}^{\infty} |d_{lk}|^2 \leq \frac{C}{\left(\int_{-\infty}^{\infty} \chi_\gamma^\varepsilon Z^2 dx\right)^2 \varepsilon^{\frac{2}{3}} \|\chi_m^\varepsilon h\|_{L^2(\mathcal{S})}^2} \leq \frac{4C}{\left(\int_{-\infty}^{\infty} Z^2 dx\right)^2 \varepsilon^{\frac{2}{3}} \|h\|_{L^2(\mathcal{S})}^2}, \quad (7.17)$$

if  $\varepsilon \in (0, \varepsilon_0(\gamma))$ , for some  $\varepsilon_0(\gamma) \rightarrow 0$  as  $\gamma \rightarrow 0$ , and constant  $C > 0$  independent of  $\varepsilon, \gamma, m, h$ .

Finally, define

$$\phi(x, z) = \sum_{k=0}^{\infty} \left[ \phi_{1k}(x) \cos\left(\frac{2\pi k}{\ell} \varepsilon^{\frac{2}{3}} z\right) + \phi_{2k}(x) \sin\left(\frac{2\pi k}{\ell} \varepsilon^{\frac{2}{3}} z\right) \right], \quad (x, z) \in \mathcal{S},$$

and correspondingly

$$d(\theta) = \sum_{k=0}^{\infty} \left[ d_{1k} \cos\left(\frac{2\pi k}{\ell} \theta\right) + d_{2k} \sin\left(\frac{2\pi k}{\ell} \theta\right) \right], \quad \theta \in [0, \ell].$$

Estimate (7.17) implies that  $d(\varepsilon^{\frac{2}{3}} z)Z$  has its  $L^2(\mathcal{S})$  norm controlled by that of  $h$ . The a-priori estimates of the previous lemma tell us that the series for  $\phi$  is convergent in  $H^2(\mathcal{S})$ , and defines a unique solution for the problem that satisfies the desired bound (with constant *independent* of  $\varepsilon, \gamma, m, h$ ).

The proof of the lemma is complete.  $\square$

We consider now the following problem: given  $h \in L^2(\mathcal{S})$ , find functions  $\phi \in H^2(\mathcal{S})$ ,  $d \in L^2(0, \ell)$ ,  $\ell$ -periodic, such that

$$L(\phi) = h + d(\varepsilon^{\frac{2}{3}} z) \chi_\gamma^\varepsilon Z \quad \text{in } \mathcal{S}, \quad (7.18)$$

$$\phi(x, 0) = \phi(x, \ell/\varepsilon^{\frac{2}{3}}), \quad \phi_z(x, 0) = \phi_z(x, \ell/\varepsilon^{\frac{2}{3}}), \quad -\infty < x < \infty, \quad (7.19)$$

$$\int_{-\infty}^{\infty} \phi(x, z) Z(x) dx = 0, \quad 0 < z < \frac{\ell}{\varepsilon^{\frac{2}{3}}}. \quad (7.20)$$

Letting  $m \rightarrow \infty$  in (7.14)-(7.16), we can obtain the following:

**Lemma 7.3.** *Problem (7.18)-(7.20) possesses a unique solution  $\phi = \tilde{T}(h)$ . Moreover, there exists a constant  $C > 0$ , and an  $\varepsilon_0(\gamma) > 0$ , such that*

$$\|\tilde{T}(h)\|_{H^2(\mathcal{S})} \leq C \|h\|_{L^2(\mathcal{S})},$$

for all  $\varepsilon \in (0, \varepsilon_0(\gamma))$ ,  $\gamma > 0$ ,  $h \in L^2(\mathcal{S})$ .

*Proof.* From Lemma 7.2, given  $m \in \mathbb{N}$ , problem (7.14)-(7.16) possesses a unique solution  $\phi_m = \tilde{T}_m(h)$ , if  $0 < \varepsilon < \varepsilon_0(\gamma)$ . Furthermore, the sequence  $\{\phi_m\}$  is bounded in  $H^2(\mathcal{S})$ . Hence, passing to a subsequence, we may assume that

$$\phi_m \rightharpoonup \phi \quad \text{weakly in } H^2(\mathcal{S}).$$

Keeping everything fixed and letting  $m \rightarrow \infty$  in the weak form of (7.14)-(7.16), we find that  $\tilde{T}(h) := \phi$  solves (7.18)-(7.20), and satisfies the desired bound (by the weak lower semi-continuity of the  $H^2$ -norm).

The proof of the lemma is complete.  $\square$

We can now give the

**PROOF OF PROPOSITION 7.1:** We will reduce problem (7.1)-(7.3) to a small perturbation of a problem of the form (7.18)-(7.20) in which Lemma 7.3 is applicable. We will achieve this by introducing a change of variables that eliminates the weight  $\beta^{-2}$  in front of  $\phi_{zz}$  (recall (6.15), (6.16)).

We let

$$\phi(x, z) = \varphi(x, \alpha(z)), \quad \alpha(z) = \varepsilon^{-\frac{2}{3}} \int_0^{\varepsilon^{\frac{2}{3}} z} \beta(r) dr.$$

The map  $\alpha : [0, \ell/\varepsilon^{\frac{2}{3}}] \rightarrow [0, \hat{\ell}/\varepsilon^{\frac{2}{3}}]$  is a diffeomorphism, where

$$\hat{\ell} = \int_0^\ell \beta(r) dr. \quad (7.21)$$

We denote then

$$\phi_z = \beta \varphi_{z'}, \quad \phi_{zz} = \beta^2 (\varepsilon^{\frac{2}{3}} z) \varphi_{z'z'} + \varepsilon^{\frac{2}{3}} \beta' (\varepsilon^{\frac{2}{3}} z) \varphi_{z'}, \quad (7.22)$$

while differentiation in  $x$  does not change. Recalling the definition of  $L_2$  from (6.15), the equation in terms of  $\varphi$  now reads

$$\begin{aligned} -\Delta \varphi + 2V\varphi &= \beta^{-2} \chi_{10\gamma}^\varepsilon \hat{B}_1(\varphi) - \chi_{10\gamma}^\varepsilon \left[ \varepsilon^{-\frac{2}{3}} \beta^{-2} \left( 2w - a(\varepsilon^{\frac{2}{3}} y) - b(\varepsilon^{\frac{2}{3}} y) \right) - 2V \right] \varphi \\ &\quad + \varepsilon^{\frac{2}{3}} \beta' \beta^{-2} \varphi_{z'} - \beta^{-2} \hat{h} - \hat{d}(\varepsilon^{\frac{2}{3}} z') \beta^{-2} \chi_{10\gamma}^\varepsilon Z, \end{aligned} \quad (7.23)$$

for  $(x, z') \in \hat{\mathcal{S}} := \{x \in \mathbb{R}, z' \in [0, \hat{\ell}/\varepsilon^{\frac{2}{3}}]\}$ , together with the conditions

$$\varphi(x, 0) = \varphi(x, \hat{\ell}/\varepsilon^{\frac{2}{3}}), \quad \varphi_{z'}(x, 0) = \varphi_{z'}(x, \hat{\ell}/\varepsilon^{\frac{2}{3}}), \quad -\infty < x < \infty, \quad (7.24)$$

$$\int_{-\infty}^{\infty} \varphi(x, z') Z(x) dx = 0, \quad 0 < z' < \frac{\hat{\ell}}{\varepsilon^{\frac{2}{3}}}. \quad (7.25)$$

Here  $\hat{h}(x, z') = h(x, \alpha^{-1}(z'))$ ,  $\hat{d}(\varepsilon^{\frac{2}{3}} z') = d(\varepsilon^{\frac{2}{3}} \alpha^{-1}(z'))$ , and the operator  $\hat{B}_1$  is defined by using formulas (7.22) to replace the  $z$ -derivatives by  $z'$ -derivatives, and the variable  $z$  by  $\alpha^{-1}(z')$ , in the operator  $\tilde{B}_1$ . We set

$$\hat{B}_2(\varphi) = \beta^{-2} \chi_{10\gamma}^\varepsilon \hat{B}_1(\varphi) \quad \text{and} \quad \hat{B}_3(\varphi) = -\chi_{10\gamma}^\varepsilon \left[ \varepsilon^{-\frac{2}{3}} \beta^{-2} \left( 2w - a(\varepsilon^{\frac{2}{3}} y) - b(\varepsilon^{\frac{2}{3}} y) \right) - 2V \right] \varphi.$$

From Lemma 7.3, we know that equations (7.23)–(7.25) are equivalent to a fixed point problem

$$\varphi = \tilde{T} \left( \hat{B}_2(\varphi) + \hat{B}_3(\varphi) + \varepsilon^{\frac{2}{3}} \beta' \beta^{-2} \varphi_{z'} - \beta^{-2} \hat{h} \right). \quad (7.26)$$

Notice that the operator  $\hat{B}_2$  is small in the sense that

$$\|\hat{B}_2(\varphi)\|_{L^2(\hat{\mathcal{S}})} \leq C(\gamma \varepsilon^{\frac{1}{3}} + \varepsilon^{\frac{2}{3}}) \|\varphi\|_{H^2(\hat{\mathcal{S}})}, \quad (7.27)$$

for some constant  $C$  independent of  $\varphi$  and small  $\varepsilon, \gamma > 0$ . This last estimate is a rather straightforward consequence of the fact that  $\varepsilon^{\frac{2}{3}} |x| \leq 20\gamma \varepsilon^{\frac{1}{3}}$  whenever the operator  $\hat{B}_2$  is supported, and relations (5.22), (7.22). Furthermore, from (5.18), (5.23), we have

$$\left| \chi_{10\gamma}^\varepsilon \left[ \varepsilon^{-\frac{2}{3}} \beta^{-2} \left( 2w - a(\varepsilon^{\frac{2}{3}} y) - b(\varepsilon^{\frac{2}{3}} y) \right) - 2V \right] \right| \leq C \varepsilon^{\frac{2}{3}} \chi_{10\gamma}^\varepsilon (x^2 + 1) \leq C(\gamma^2 + \varepsilon^{\frac{2}{3}}), \quad (x, z) \in \mathcal{S}, \quad (7.28)$$

where  $C$  is independent of small  $\varepsilon, \gamma > 0$ . Hence, it follows that

$$\|\hat{B}_3(\varphi)\|_{L^2(\hat{\mathcal{S}})} \leq C(\gamma^2 + \varepsilon^{\frac{2}{3}})\|\varphi\|_{L^2(\hat{\mathcal{S}})},$$

with  $C$  independent of  $\varphi$  and small  $\varepsilon, \gamma > 0$ . Let's recall at this point that the constant in the estimate of Lemma 7.3 does *not* depend on  $\gamma$  (nor on  $\varepsilon, h$ ). This means that we can fix a small  $\gamma_* > 0$ , and find an  $\varepsilon_0 > 0$ , so that we can apply the contraction mapping principle in (7.26), with  $\gamma = \gamma_*$  and  $\varepsilon \in (0, \varepsilon_0)$ . Thus, if  $\varepsilon \in (0, \varepsilon_0)$ , equations (7.23)-(7.25), with  $\gamma = \gamma_*$ , have a unique solution  $\varphi(\hat{h})$ , satisfying the estimate  $\|\varphi(\hat{h})\|_{H^2(\hat{\mathcal{S}})} \leq C\|\hat{h}\|_{L^2(\hat{\mathcal{S}})}$  for some constant  $C$  independent of  $\varepsilon, \hat{h}$ .

The result now follows by transforming the estimate for  $\varphi$  into a similar one for  $\phi$  via a change of variables. This concludes the proof.  $\square$

**Remark 7.1.** *From now on  $\gamma$  will be fixed equal to  $\gamma_*$ .*

## 8. SOLVING THE NONLINEAR INTERMEDIATE PROBLEM

In this section we will solve the nonlinear problem (6.17)–(6.19), i.e.,

$$L_2(\phi) = -\chi_\gamma^\varepsilon E_1 + N_2(\phi) + d(\varepsilon^{\frac{2}{3}}z)\chi_\gamma^\varepsilon Z, \quad (8.1)$$

under periodic boundary and orthogonality conditions in  $\mathcal{S}$ . Here  $N_2$  is as in (6.20), whenever this operator is well defined, namely, for  $\phi$  satisfying (6.10). A first elementary but crucial observation is that the term

$$E_{11} = (\varepsilon^2 e'' + \lambda_0 \varepsilon^{\frac{2}{3}} \beta^2 e)Z,$$

in the decomposition of  $E_1$  (recall (5.25)), has precisely the form  $d(\varepsilon^{\frac{2}{3}}z)Z$  and can therefore be absorbed for now in that term. Thus, the equivalent problem we will look at is

$$L_2(\phi) = -\chi_\gamma^\varepsilon E_{12} + N_2(\phi) + d(\varepsilon^{\frac{2}{3}}z)\chi_\gamma^\varepsilon Z, \quad (8.2)$$

under periodic boundary and orthogonality conditions in  $\mathcal{S}$ . The big difference between the terms  $E_{11}$  and  $E_{12}$  is their sizes. Notice that, by Proposition 5.1, and (5.18),

$$\|\chi_\gamma^\varepsilon E_{12}\|_{L^2(\mathcal{S})} \leq C\varepsilon^{\frac{5}{3}}, \quad (8.3)$$

while  $\chi_\gamma^\varepsilon E_{11}$  is a-priori only of size  $\mathcal{O}(\varepsilon)$  in  $L^2(\mathcal{S})$ . We call

$$E_2 = \chi_\gamma^\varepsilon E_{12}. \quad (8.4)$$

For future reference, it is useful to point out the Lipschitz dependence of the term of error  $E_2$  on the parameter  $e$  for the norm defined in (5.18). One can readily check that we have the validity of the estimate

$$\|E_2(e_1) - E_2(e_2)\|_{L^2(\mathcal{S})} \leq C\varepsilon\|e_1 - e_2\|_b. \quad (8.5)$$

Let  $T$  be the operator defined in Proposition 7.1. Then, the equation (8.2) is equivalent to the fixed point problem

$$\phi = T(-E_2 + N_2(\phi)) \equiv \mathcal{A}(\phi). \quad (8.6)$$

The operator  $T$  has a useful property: Assume that  $h$  has support on  $|x| \leq 2\gamma\varepsilon^{-\frac{1}{3}}$ , then  $\phi = T(h)$  satisfies the estimate

$$|\phi(x, z)| + |\nabla\phi(x, z)| \leq \|\phi\|_{L^\infty(\mathcal{S})} e^{-2\mu\varepsilon^{-\frac{1}{2}}} \quad \text{for } |x| \geq 3\gamma\varepsilon^{-\frac{1}{3}}, \quad (8.7)$$

for some constant  $\mu > 0$  independent of  $\varepsilon, h$ . Indeed, since the term involving  $d$  is supported on  $|x| \leq 2\gamma\varepsilon^{-\frac{1}{3}}$ , recalling the comments below (7.27), and (7.28), it follows that  $\phi$  satisfies, for  $|x| \geq 2\gamma\varepsilon^{-\frac{1}{3}}$ , an equation of the form

$$\beta^{-2}\phi_{zz} + \phi_{xx} - 2V\phi + \mathcal{O}(1)\phi + \chi_{10\gamma}^\varepsilon \mathcal{O}\left(\varepsilon^{\frac{1}{3}}(|\phi_x| + |\phi_z| + |\phi_{xx}| + |\phi_{xz}| + |\phi_{zz}|)\right) = 0, \quad \varepsilon \rightarrow 0,$$

uniformly in  $x, z$ . Then, for  $x \geq 2\gamma\varepsilon^{-\frac{1}{3}}$ , we can use a barrier of the form

$$\bar{\phi}(x, z) = \|\phi\|_{L^\infty(\mathcal{S})} \exp\left\{-D(x - 2\gamma\varepsilon^{-\frac{1}{3}})^{\frac{3}{2}}\right\}, \quad D > 0,$$

(recall that  $V - |x| \rightarrow 0$  as  $x \rightarrow \pm\infty$ ) to conclude that

$$\phi(x, z) \leq \|\phi\|_{L^\infty(\mathcal{S})} \exp\left\{-2\mu\varepsilon^{-\frac{1}{2}}\right\}, \quad x \geq 3\gamma\varepsilon^{-\frac{1}{3}},$$

for some constant  $\mu > 0$  independent of  $\varepsilon, h$ . The remaining inequalities for  $\phi$  are found in the same way. The bound for  $\nabla\phi$  follows simply by local elliptic estimates.

Now we recall that, for every  $\phi$  fulfilling (6.10), the operator  $\psi(\phi)$  satisfies relations (6.11), (6.12), and (6.13). These facts will allow us to construct a region where the contraction mapping principle applies to (8.6). We consider the following closed, bounded, subset of  $H^2(\mathcal{S})$ :

$$\mathcal{B} = \left\{ \phi \in H^2(\mathcal{S}) : \|\phi\|_{H^2(\mathcal{S})} \leq M\varepsilon^{\frac{5}{3}}, \quad \|\nabla\phi + |\phi|\|_{L^\infty(|x| \geq 3\gamma\varepsilon^{-\frac{1}{3}})} \leq \|\phi\|_{H^2(\mathcal{S})} e^{-\mu\varepsilon^{-\frac{1}{2}}} \right\},$$

where  $\mu > 0$  is as in (8.7), and  $M$  a constant to be determined (independently of  $\varepsilon$ ). Note that, thanks to Sobolev's imbedding

$$\|\phi\|_{L^\infty(\mathcal{S})} \leq C\|\phi\|_{H^2(\mathcal{S})} \quad (C \text{ independent of } \varepsilon, \phi), \quad (8.8)$$

functions  $\phi$  in  $\mathcal{B}$  fulfill condition (6.10) provided  $\varepsilon$  is sufficiently small.

We claim that if the constant  $M$  is fixed sufficiently large, then the map  $\mathcal{A}$ , defined in (8.6), is a contraction from  $\mathcal{B}$  into itself for all small  $\varepsilon > 0$ . For functions  $\phi$  in  $\mathcal{B}$ , we have the following estimate for  $N_2(\phi) = \varepsilon^{-\frac{2}{3}}\chi_\gamma^\varepsilon(\phi + \psi(\phi))^2 + 2\chi_\gamma^\varepsilon\varepsilon^{-\frac{2}{3}}(w - \tilde{u})\psi(\phi)$ :

$$\|N_2(\phi)\|_{L^2(\mathcal{S})} \leq CM^2\varepsilon^{\frac{5}{2}}, \quad \varepsilon < \varepsilon(M). \quad (8.9)$$

(Here, and throughout this section, by  $C > 0$  we denote a generic constant that is independent of  $M$  and all small  $\varepsilon > 0$ ; by  $\varepsilon(M)$  we denote a generic small constant that depends only on  $M$  such that  $\varepsilon(M) \rightarrow 0$  as  $M \rightarrow \infty$ ). To see the above estimate, let  $\mathcal{S}_\gamma = \mathcal{S} \cap \{|x| \leq 2\gamma\varepsilon^{-\frac{1}{3}}\}$ . Then, for  $\phi \in \mathcal{B}$ , we have that

$$\|N_2(\phi)\|_{L^2(\mathcal{S})} \leq C\varepsilon^{-\frac{2}{3}} \left[ \|\phi\|_{L^4(\mathcal{S})}^2 + \|\psi(\phi)\|_{L^4(\mathcal{S}_\gamma)}^2 \right] + C\|\psi(\phi)\|_{L^2(\mathcal{S}_\gamma)},$$

where we also used the easily derived estimate  $w - \tilde{u} = \mathcal{O}(\varepsilon^{\frac{2}{3}})$ ,  $\varepsilon \rightarrow 0$ , uniformly in  $\Omega_\varepsilon$ . Using Sobolev's imbedding, we get

$$\|\phi\|_{L^4(\mathcal{S})}^2 \leq C\|\phi\|_{H^1(\mathcal{S})}^2 \leq C\|\phi\|_{H^2(\mathcal{S})}^2 \leq CM^2\varepsilon^{\frac{10}{3}}.$$

From (6.11), and the fact that the area of  $\mathcal{S}_\gamma$  is  $\mathcal{O}(\varepsilon^{-1})$ , we obtain that

$$\varepsilon^{-\frac{2}{3}}\|\psi(\phi)\|_{L^4(\mathcal{S}_\gamma)}^2 + \|\psi(\phi)\|_{L^2(\mathcal{S}_\gamma)} \leq CM^2\varepsilon^{\frac{5}{2}}, \quad \varepsilon < \varepsilon(M).$$

Hence, estimate (8.9) holds true.

We claim that a Lipschitz property holds for  $N_2$ . More precisely, there exists a constant  $C$  such that

$$\|N_2(\phi_1) - N_2(\phi_2)\|_{L^2(\mathcal{S})} \leq C\varepsilon^{\frac{1}{3}}\|\phi_1 - \phi_2\|_{H^2(\mathcal{S})} \quad \forall \phi_1, \phi_2 \in \mathcal{B}, \quad (8.10)$$

provided  $\varepsilon < \varepsilon(M)$ . Indeed, by direct computations, we obtain that

$$\|\phi_1^2 - \phi_2^2\|_{L^2(\mathcal{S}_\gamma)} \leq (\|\phi_1\|_{L^4(\mathcal{S}_\gamma)} + \|\phi_2\|_{L^4(\mathcal{S}_\gamma)}) \|\phi_1 - \phi_2\|_{L^4(\mathcal{S}_\gamma)} \quad \forall \phi_1, \phi_2 \in H^2(\mathcal{S}_\gamma).$$

Therefore, we have

$$\begin{aligned} \|N_2(\phi_1) - N_2(\phi_2)\|_{L^2(\mathcal{S})} &\leq \varepsilon^{-\frac{2}{3}} \|(\phi_1 + \psi(\phi_1))^2 - (\phi_2 + \psi(\phi_2))^2\|_{L^2(\mathcal{S}_\gamma)} \\ &\quad + C \|\psi(\phi_1) - \psi(\phi_2)\|_{L^2(\mathcal{S}_\gamma)} \\ &\leq (A_1 + A_2) (\|\phi_1 - \phi_2\|_{L^4(\mathcal{S}_\gamma)} + \|\psi(\phi_1) - \psi(\phi_2)\|_{L^4(\mathcal{S}_\gamma)}) \\ &\quad + C \|\psi(\phi_1) - \psi(\phi_2)\|_{L^2(\mathcal{S}_\gamma)}, \end{aligned}$$

where  $A_i = \varepsilon^{-\frac{2}{3}} (\|\phi_i\|_{L^4(\mathcal{S}_\gamma)} + \|\psi(\phi_i)\|_{L^4(\mathcal{S}_\gamma)})$ ,  $i = 1, 2$ . Arguing as before, and recalling relation (6.13), we deduce that (8.10) holds.

Now, for  $\phi \in \mathcal{B}$ , it follows from Proposition 7.1, (8.3), (8.4), (8.6), and (8.9), that

$$\|\mathcal{A}(\phi)\|_{H^2(\mathcal{S})} \leq C_* \varepsilon^{\frac{5}{3}} + CM^2 \varepsilon^{\frac{5}{2}},$$

for some constants  $C_*$ ,  $C$  independent of  $M$  and all small  $\varepsilon$ . Thus, choosing any number  $M > C_*$ , we have

$$\|\mathcal{A}(\phi)\|_{H^2(\mathcal{S})} \leq M \varepsilon^{\frac{5}{3}},$$

provided  $\varepsilon$  is sufficiently small. On the other hand, the function  $\varphi = \mathcal{A}(\phi)$  satisfies an equation of the form  $L_2(\varphi) = h$  with  $h$  compactly supported on  $|x| \leq 2\gamma\varepsilon^{-\frac{1}{3}}$ . Hence, by the discussion leading to (8.7), and Sobolev's imbedding (8.8), we infer that  $\varphi$  belongs to  $\mathcal{B}$ , provided  $\varepsilon$  is sufficiently small. Furthermore, thanks to Proposition 7.1, and (8.10), the map  $\mathcal{A}$  is a contraction. We conclude that (8.6) has a unique fixed point in  $\mathcal{B}$ . In turn, this fixed point supplies us with the desired solution of (8.2).

We recall that the operator  $\mathcal{A}$  carries the function  $e$  as a parameter. Hence, the fixed point  $\phi$  of  $\mathcal{A}$  depends on  $e$ , and we can write  $\phi = \phi(e)$ . A tedious but straightforward analysis of the terms involved in the differential operator  $L_2$ , the nonlinear operator  $N_2$ , and in the error  $E_2$ , yields that this dependence is indeed Lipschitz with respect to the  $H^2$ -norm (for each fixed  $\varepsilon$ ). Indeed, from (5.19), (5.25), (6.5), (6.11), (8.4), and the super-exponential decay of  $Z$ , in the same way as (6.13), we get that, for  $\phi \in \mathcal{B}$ ,

$$\|\psi_{e_1}(\phi) - \psi_{e_2}(\phi)\|_{H^1(\Omega_\varepsilon)} \leq e^{-c\varepsilon^{-\frac{1}{3}}} \|e_1 - e_2\|_b,$$

where we have emphasized the dependence of the operator  $\psi$  on functions  $e_1, e_2$  satisfying (5.18). Now, arguing as before, we can show that, for  $\phi \in \mathcal{B}$ ,

$$\|N_{2,e_1}(\phi) - N_{2,e_2}(\phi)\|_{L^2(\mathcal{S})} \leq C\varepsilon^{\frac{8}{3}} \|e_1 - e_2\|_b. \quad (8.11)$$

Notice that in view of  $\phi(e_i) = \mathcal{A}(\phi(e_i))$ ,  $i = 1, 2$ , we can write

$$L_{2,0}(\phi(e_i)) = -E_2(e_i) + N_{2,e_i}(\phi(e_i)) + 2\chi_{10\gamma}^\varepsilon Z e_i \phi(e_i) + d_{e_i}(\varepsilon^{\frac{2}{3}} z) \chi_\gamma^\varepsilon Z, \quad i = 1, 2,$$

where  $L_{2,0}$  is the linear operator described in (6.15) with  $e = 0$ . So,

$$\begin{aligned} L_{2,0}(\phi(e_1) - \phi(e_2)) &= -E_2(e_1) + E_2(e_2) + N_{2,e_1}(\phi(e_1)) - N_{2,e_2}(\phi(e_2)) \\ &\quad + 2\chi_{10\gamma}^\varepsilon Z (e_1 \phi(e_1) - e_2 \phi(e_2)) + \tilde{d}(\varepsilon^{\frac{2}{3}} z) \chi_\gamma^\varepsilon Z, \end{aligned}$$



where  $\tilde{d}(\theta) = d_{e_1}(\theta) - d_{e_2}(\theta)$ . Finally, by (5.18), Proposition 7.1, (8.5), (8.10), (8.11), and the above relation, we infer that

$$\|\phi(e_1) - \phi(e_2)\|_{H^2(\mathcal{S})} \leq C\varepsilon\|e_1 - e_2\|_b. \quad (8.12)$$

We summarize the results we have obtained in this section in the following:

**Proposition 8.1.** *There are numbers  $\mu, M > 0$  such that for all sufficiently small  $\varepsilon > 0$  and all  $e$  satisfying (5.18), problem (6.17)–(6.19) has a unique solution  $\phi = \phi(e)$  that satisfies*

$$\|\phi\|_{H^2(\mathcal{S})} \leq M\varepsilon^{\frac{5}{3}},$$

and

$$\| |\nabla\phi| + |\phi| \|_{L^\infty(|x| \geq 3\gamma\varepsilon^{-\frac{1}{3}})} \leq \|\phi\|_{H^2(\mathcal{S})} e^{-\mu\varepsilon^{-\frac{1}{2}}}.$$

Besides,  $\phi$  depends Lipschitz-continuously on  $e$  in the sense of estimate (8.12).

Next, we carry out the second part of the program, which is to set up an equation for  $e$  that is equivalent to making  $d$  identically zero. This equation will be obtained by simply integrating (only in  $x$ ) equation (8.1) against  $Z$ . It is therefore of crucial importance to carry out computations of the term  $\int_{-\infty}^{\infty} \chi_\gamma^\varepsilon E_1 Z dx$ . We do that in the next section.

## 9. ESTIMATES FOR PROJECTIONS OF THE ERROR

In this section we carry out estimates for the term  $\int_{-\infty}^{\infty} \chi_\gamma^\varepsilon E_1 Z dx$ , where  $E_1$ , we recall, was defined in (5.19).

The main component in this expression is given by

$$(\varepsilon^2 e'' + \lambda_0 \varepsilon^{\frac{2}{3}} \beta^2 e) \int_{-\infty}^{\infty} \chi_\gamma^\varepsilon Z^2 dx = (\varepsilon^2 e'' + \lambda_0 \varepsilon^{\frac{2}{3}} \beta^2 e) \left( 1 + \mathcal{O}(e^{-c\varepsilon^{-\frac{1}{3}}}) \right),$$

where we used that  $\int_{-\infty}^{\infty} Z^2 dx = 1$ , and the (super) exponential decay of  $Z$ .

By relations (5.15), (5.20), we infer that

$$\int_{-\infty}^{\infty} \chi_\gamma^\varepsilon E_0 Z dx = \varepsilon^2 a_1(\varepsilon^{\frac{2}{3}} z),$$

for some smooth, uniformly bounded (in  $\varepsilon$ ), function  $a_1$  of  $\theta = \varepsilon^{\frac{2}{3}} z$ .

We will estimate the remaining terms coming from (5.19), (5.24). In view of (5.22), the evenness and decay of  $Z$ , and recalling the normalization  $\|Z\|_{L^2(\mathbb{R})} = 1$ , we can show that

$$\varepsilon^{\frac{2}{3}} \int_{-\infty}^{\infty} \chi_\gamma^\varepsilon \tilde{B}_1(eZ) Z dx = \varepsilon^{\frac{10}{3}} a_2 e'' - \varepsilon^2 \left( \beta' \beta^{-1} + \varepsilon^{\frac{4}{3}} a_3 \right) e' + \varepsilon^2 a_4 e,$$

for some smooth, uniformly bounded (in  $\varepsilon$ ), functions  $a_i$ ,  $i = 2, 3, 4$ , of  $\theta = \varepsilon^{\frac{2}{3}} z$ .

Combining (5.19), (5.24), and the above relations, we conclude that

$$\begin{aligned} \int_{-\infty}^{\infty} \chi_\gamma^\varepsilon E_1 Z dx &= \varepsilon^2 (1 + \varepsilon^{\frac{4}{3}} b_1) e'' - \varepsilon^2 (\beta' \beta^{-1} + \varepsilon^{\frac{4}{3}} b_2) e' \\ &\quad + \varepsilon^{\frac{2}{3}} (\lambda_0 \beta^2 + \varepsilon^{\frac{2}{3}} b_3) e + \varepsilon^{\frac{2}{3}} b_4 e^2 + b_5 \varepsilon^2, \end{aligned} \quad (9.1)$$

for some smooth, uniformly bounded (in  $\varepsilon$ ), functions  $b_i$ ,  $i = 1, \dots, 5$ , of  $\theta = \varepsilon^{\frac{2}{3}} z$ . In particular, we have that  $b_i$  are independent of  $e$ , and actually  $b_4 = -\int_{-\infty}^{\infty} \chi_\gamma^\varepsilon Z^3 dx$ .

10. PROJECTIONS OF TERMS INVOLVING  $\phi$ 

We will estimate next the terms that involve  $\phi$  in (8.1) tested against  $Z$ . We call the sum of them  $\varphi(\phi)$  (recall (6.15)), which can be decomposed as  $\varphi = \sum_{i=1}^3 \varphi_i$  below.

Firstly, note that

$$\int_{-\infty}^{\infty} L_0(\phi)Zdx = \beta^{-2} \frac{d^2}{dz^2} \left( \int_{-\infty}^{\infty} \phi Zdx \right) + \lambda_0 \int_{-\infty}^{\infty} \phi Zdx = 0,$$

thanks to (6.19).

Let

$$\varphi_1(\varepsilon^{\frac{2}{3}}z) = \int_{-\infty}^{\infty} \chi_{10\gamma}^{\varepsilon} \tilde{B}_1(\phi)Zdx.$$

We make the following observation: all terms in  $\tilde{B}_1(\phi)$  carry  $\varepsilon^{\frac{2}{3}}$  and involve powers of  $x$  times derivatives of zero, one, or two orders of  $\phi$ . Since  $Z$  has super-exponential decay, the conclusion is that

$$\int_0^{\ell} |\varphi_1(\theta)|^2 d\theta \leq C\varepsilon^2 \|\phi\|_{H^2(\mathcal{S})}^2.$$

Hence, by Proposition 8.1, we get

$$\|\varphi_1\|_{L^2(0,\ell)} \leq C\varepsilon^{\frac{8}{3}}. \quad (10.1)$$

In  $\varphi_1$  we single out a less regular term, arising from a second order derivative in  $z$  for  $\phi$ , namely

$$\varphi_{1*} := - \int_{-\infty}^{\infty} \chi_{10\gamma}^{\varepsilon} \left[ 1 - (1 + \varepsilon^{\frac{2}{3}}k\beta^{-1}x)^{-2} \right] \phi_{zz}Zdx.$$

Emphasizing the dependence on  $e$ , we readily see that

$$|\varphi_{1*}(e_1) - \varphi_{1*}(e_2)|^2 \leq C\varepsilon^{\frac{4}{3}} \int_{-\infty}^{\infty} |\phi_{zz}(e_1) - \phi_{zz}(e_2)|^2 dx.$$

So, we get

$$\|\varphi_{1*}(e_1) - \varphi_{1*}(e_2)\|_{L^2(0,\ell)}^2 \leq C\varepsilon^2 \|\phi(e_1) - \phi(e_2)\|_{H^2(\mathcal{S})}^2 \stackrel{(8.12)}{\leq} C\varepsilon^4 \|e_1 - e_2\|_b^2,$$

i.e.,

$$\|\varphi_{1*}(e_1) - \varphi_{1*}(e_2)\|_{L^2(0,\ell)} \leq C\varepsilon^2 \|e_1 - e_2\|_b. \quad (10.2)$$

The remainder  $\varphi_1 - \varphi_{1*}$  actually defines, for fixed  $\varepsilon$ , a compact operator of  $e$  into  $L^2(0,\ell)$ . This is a consequence of the fact that weak convergence in  $H^2(\mathcal{S})$  implies local strong convergence in  $H^1(\mathcal{S})$ , and the same is the case for  $H^2(0,\ell)$  and  $C^1[0,\ell]$ . If  $e_j$  is a weakly convergent sequence in  $H^2(0,\ell)$ , then clearly the functions  $\phi(e_j)$  constitute a bounded sequence in  $H^2(\mathcal{S})$ . In the above remainder, one can integrate by parts if necessary once in  $x$ . Averaging against  $Z$ , which decays super-exponentially, localizes the situation and the desired fact follows.

Let us now consider the term

$$\varphi_2(\varepsilon^{\frac{2}{3}}z) := - \int_{-\infty}^{\infty} \chi_{10\gamma}^{\varepsilon} \left[ \varepsilon^{-\frac{2}{3}} \left( 2w - a(\varepsilon^{\frac{2}{3}}y) - b(\varepsilon^{\frac{2}{3}}y) \right) - 2\beta^2 V \right] \phi Zdx.$$

By the first inequality in (7.28), the decay of  $Z$ , and Proposition 8.1, we obtain that

$$\|\varphi_2\|_{L^2(0,\ell)} \leq C\varepsilon \|\phi\|_{L^2(\mathcal{S})} \leq C\varepsilon^{\frac{8}{3}}. \quad (10.3)$$

Finally, let

$$\varphi_3(\varepsilon^{\frac{2}{3}}z) = - \int_{-\infty}^{\infty} N_2(\phi)Zdx.$$

By (8.9), we deduce that

$$\|\varphi_3\|_{L^2(0,\ell)} \leq C\varepsilon^{\frac{1}{3}}\|N_2(\phi)\|_{L^2(\mathcal{S})} \leq C\varepsilon^{\frac{17}{6}}. \quad (10.4)$$

The term  $\varphi_2 + \varphi_3$  defines a compact operator for  $e$  similarly as the remainder  $\varphi_1 - \varphi_{1*}$ .

## 11. SOLVING THE REDUCED PROBLEM FOR $e$

In this section we set up an equation for  $e$  such that, for the solution  $\phi$  of (6.17)-(6.19), predicted by Proposition 8.1, one has that the coefficient  $d(\varepsilon^{\frac{2}{3}}z)$  is identically zero. To achieve this, we multiply the equation against  $Z$  and integrate only in  $x$ . The equation  $d = 0$  is then equivalent to the relation

$$\varphi(\phi) = - \int_{-\infty}^{\infty} \chi_\gamma^\varepsilon E_1 Z dx.$$

Using the estimates of the previous sections, we then find that the above relation is equivalent to the following nonlinear, nonlocal differential equation for  $e$ :

$$\mathcal{L}(e) \equiv \varepsilon^{\frac{4}{3}}(\beta^{-2}e'' - \beta'\beta^{-3}e') + \lambda_0 e = \varepsilon^{\frac{8}{3}}\tilde{b}_1 e'' + \varepsilon^{\frac{8}{3}}\tilde{b}_2 e' + \varepsilon^{\frac{2}{3}}\tilde{b}_3 e + \tilde{b}_4 e^2 + \varepsilon^{\frac{4}{3}}\tilde{b}_5 - \varepsilon^2 \beta^{-2} M_\varepsilon. \quad (11.1)$$

The operator  $M_\varepsilon = M_\varepsilon(e)$  can be decomposed in the following form:

$$M_\varepsilon(e) = A_\varepsilon(e) + K_\varepsilon(e),$$

where  $K_\varepsilon = \varepsilon^{-\frac{8}{3}}(\varphi_1 - \varphi_{1*} + \varphi_2 + \varphi_3)$  is uniformly bounded in  $L^2(0, \ell)$  for  $e$  satisfying constraint (5.18), recall (10.1), (10.3), (10.4), and is also compact. The operator  $A_\varepsilon = \varepsilon^{-\frac{8}{3}}\varphi_{1*}$  is also uniformly bounded in  $L^2(0, \ell)$ , and, due to (10.2), satisfies the Lipschitz condition:

$$\|A_\varepsilon(e_1) - A_\varepsilon(e_2)\|_{L^2(0,\ell)} \leq C\varepsilon^{-\frac{2}{3}}\|e_1 - e_2\|_b. \quad (11.2)$$

The functions  $\tilde{b}_i = -\beta^{-2}b_i$ ,  $i = 1, \dots, 5$ , are smooth, uniformly bounded (in  $\varepsilon$ ), and independent of  $e$  (recall (9.1)).

We now use assumption (1.15) to deal with the invertibility of  $\mathcal{L}$ . We have the following:

**Lemma 11.1.** *Assume that condition (1.15) holds. If  $f \in L^2(0, \ell)$ , then there is a unique solution  $e \in H^2(0, \ell)$  of  $\mathcal{L}(e) = f$  that is  $\ell$ -periodic and satisfies*

$$\varepsilon^{\frac{4}{3}}\|e''\|_{L^2(0,\ell)} + \varepsilon^{\frac{2}{3}}\|e'\|_{L^2(0,\ell)} + \|e\|_{L^\infty(0,\ell)} \leq C\varepsilon^{-\frac{2}{3}}\|f\|_{L^2(0,\ell)},$$

with  $C$  independent of  $\varepsilon, f$ . Moreover, if  $f$  is in  $H^2(0, \ell)$ , then

$$\varepsilon^{\frac{4}{3}}\|e''\|_{L^2(0,\ell)} + \|e'\|_{L^2(0,\ell)} + \|e\|_{L^\infty(0,\ell)} \leq C\|f\|_{H^2(0,\ell)},$$

with  $C$  independent of  $\varepsilon, f$ .

*Proof.* By setting  $\epsilon = \varepsilon^{\frac{2}{3}}$ , this is exactly Lemma 8.1 in [17].  $\square$

We are now ready for the

**PROOF OF THEOREM 1.1:** We first solve  $\mathcal{L}(e_0) = \varepsilon^{\frac{4}{3}}\tilde{b}_5$ , and replace  $e = e_0 + \tilde{e}$ . By the second assertion of Lemma 11.1, we have

$$\varepsilon^{\frac{4}{3}}\|e_0''\|_{L^2(0,\ell)} + \|e_0'\|_{L^2(0,\ell)} + \|e_0\|_{L^\infty(0,\ell)} \leq C\varepsilon^{\frac{4}{3}}.$$

The resulting equation for  $\tilde{e}$  has the same form as (11.1) except that now the term  $\varepsilon^{\frac{4}{3}}\tilde{b}_5$  disappears. Let us observe that, by the first assertion of Lemma 11.1, the linear operator  $\mathcal{L}$  is invertible with bounds for  $\mathcal{L}(e) = f$  given by

$$\|e\|_b \leq C\varepsilon^{-\frac{2}{3}}\|f\|_{L^2(0,\ell)}.$$

It then follows from the contraction mapping principle, and (11.2), that the problem

$$[\mathcal{L} + \varepsilon^2\beta^{-2}A_\varepsilon](e) = f$$

is uniquely solvable for  $e$  satisfying (5.18) if  $\|f\|_{L^2(0,\ell)} \leq C\varepsilon^{\frac{4}{3}+\rho}$  for some  $\rho > 0$ . The desired result for the full problem (11.1) then follows directly from Schauder's fixed point theorem. In fact, refining the fixed-point region, we can actually get

$$\|e\|_b = \mathcal{O}(\varepsilon^{\frac{4}{3}}) \quad (11.3)$$

for the solution.

The corresponding estimates (1.17), (1.19) for the solution  $u_\varepsilon$  we constructed for the stretched problem follow readily by combining (5.1), (5.11), the asymptotic behavior of  $V, \phi_1$  from Propositions 3.1, 3.2 respectively, Proposition 8.1, and (6.11). By elliptic regularity [23], or the interpolation-type inequality of Lemma A.1 in [7], it follows that

$$|\nabla\sigma| + |\nabla\phi(e)| + |\nabla\psi(\phi)| \leq C\varepsilon^{\frac{4}{3}}, \quad (11.4)$$

and the corresponding estimate (1.20) follows by noting that  $|\nabla u_{in}| \leq C\varepsilon^{\frac{2}{3}}$  in  $\Omega_\varepsilon \cap \{|x| \leq \delta\varepsilon^{-\frac{2}{3}}\}$  whereas  $|\nabla \tilde{u}_{out}| \leq C\varepsilon^{\frac{2}{3}}$  in the remaining domain, Lemma 4.2, and (5.18). Finally, scaling back to the original variables yields (1.17), (1.19), (1.20). The first assertion below (1.20) follows by the compactness of the imbedding  $C^1(\bar{\Omega}) \subset C^{0,\alpha}(\bar{\Omega})$ ,  $0 < \alpha < 1$  and (1.16); the second follows by noting that the error term in (1.17) is of order  $\varepsilon^{\frac{2}{3}}$  in the  $C^1$  sense (by Lemma 4.2, (11.4), and (11.3)). $\square$

Next we present the

**PROOF OF PROPOSITION 1.1:** For convenience, we will continue working in stretched variables. Let

$$\psi(y) = Z(x)\Theta(\varepsilon^{\frac{2}{3}}z),$$

be a function defined in the tubular neighborhood of  $\Gamma_\varepsilon$  described by

$$U_\varepsilon = \Omega_\varepsilon \cap \{|x| \leq \delta\varepsilon^{-\frac{2}{3}}\},$$

where  $Z$  was defined in (5.16), and  $\Theta$  any smooth,  $\ell$ -periodic function. Making use of (5.16), (5.21), a direct calculation shows that

$$\Delta\psi - \varepsilon^{-\frac{2}{3}}\left(2u_\varepsilon - a(\varepsilon^{\frac{2}{3}}y) - b(\varepsilon^{\frac{2}{3}}y)\right)\psi = (\varepsilon^{\frac{4}{3}}\Theta_{\theta\theta} + \lambda_0\beta^2\Theta)Z + \left(\varepsilon^{\frac{2}{3}}\Theta + \varepsilon^{\frac{4}{3}}\Theta_\theta + \varepsilon^2\Theta_{\theta\theta}\right)F(x, z),$$

in  $U_\varepsilon$ , with  $|F(x, z)| \leq Ce^{-c|x|}$ . Testing the above equality by  $\psi$ , recalling the super-exponential decay of  $Z$ , the fact that  $\|Z\|_{L^2(\mathbb{R})} = 1$ , and using the elementary inequalities

$$2|\Theta||\Theta_\theta| \leq \left(\varepsilon^{-\frac{2}{3}}\Theta^2 + \varepsilon^{\frac{2}{3}}\Theta_\theta^2\right), \quad 2|\Theta||\Theta_{\theta\theta}| \leq \left(\varepsilon^{-\frac{4}{3}}\Theta^2 + \varepsilon^{\frac{4}{3}}\Theta_{\theta\theta}^2\right),$$

we infer that

$$\int_{U_\varepsilon} \left[-\Delta\psi + \varepsilon^{-\frac{2}{3}}\left(2u_\varepsilon - a(\varepsilon^{\frac{2}{3}}y) - b(\varepsilon^{\frac{2}{3}}y)\right)\psi\right]\psi dy =$$

$$\varepsilon^{-\frac{2}{3}} \int_0^\ell (\varepsilon^{\frac{4}{3}} \Theta_\theta^2 - \lambda_0 \beta^2 \Theta^2) d\theta + \mathcal{O}(1) \int_0^\ell \left( \Theta^2 + \varepsilon^{\frac{4}{3}} \Theta_\theta^2 + \varepsilon^{\frac{8}{3}} \Theta_{\theta\theta}^2 \right) d\theta.$$

(To be more accurate, the righthand side is multiplied by  $1 + \mathcal{O}(\delta)$  arising mainly from the Jacobian of the transformation  $y \rightarrow (x, z)$ ).

The above relation motivates us to consider the following *geometric eigenvalue problem*:

$$\begin{cases} -\varepsilon^{\frac{4}{3}} \Theta_{\theta\theta} - \lambda_0 \beta^2(\theta) \Theta = \Lambda \Theta & \text{in } (0, \ell), \\ \Theta(0) = \Theta(\ell), \quad \Theta_\theta(0) = \Theta_\theta(\ell). \end{cases} \quad (11.5)$$

It's well known that the above problem has a sequence of different eigenvalues  $\Lambda_1 < \Lambda_2 < \dots$  with corresponding eigenfunctions  $\Theta_1, \Theta_2, \dots$ . It is obvious that  $\Lambda_1 < 0$  because of the positivity of  $\lambda_0 \beta^2$  and  $\Theta_1$  can be chosen positive. Moreover, by adapting the proof of Lemma 2.4 in [48], all critical eigenvalues of (11.5) have good estimates: If  $\varepsilon$  is small then we have, as  $i \rightarrow \infty$ ,

$$\Lambda_i = \frac{4\pi^2}{\lambda_0 \hat{\ell}^2} (i^2 \varepsilon^{\frac{4}{3}} - \lambda_*) + \mathcal{O}\left(\frac{\varepsilon^{\frac{4}{3}}}{i^2}\right), \quad (11.6)$$

where  $\lambda_*$ ,  $\hat{\ell}$  as defined in (1.14), (7.21) respectively.

Recalling the definition of the cutoff function  $n_\delta$  from (4.2), let us define the function  $\psi_i \in H^1(\Omega_\varepsilon)$  as

$$\psi_i(y) = \Theta_i(\varepsilon^{\frac{2}{3}} z) Z(x) n_\delta(\varepsilon^{\frac{2}{3}} x).$$

In view of the calculation above (11.5), and the super-exponential decay of  $Z$ , given  $C > 0$ , if  $1 \leq i \leq [C\varepsilon^{-\frac{2}{3}}]$ , we have

$$\varepsilon^{\frac{2}{3}} \int_{\Omega_\varepsilon} \left[ |\nabla \psi_i|^2 - \varepsilon^{-\frac{2}{3}} \left( 2u_\varepsilon - a(\varepsilon^{\frac{2}{3}} y) - b(\varepsilon^{\frac{2}{3}} y) \right) \psi_i^2 \right] dy = \frac{4\pi^2}{\lambda_0 \hat{\ell}^2} (i^2 \varepsilon^{\frac{4}{3}} - \lambda_*) + \mathcal{O}\left(\varepsilon^{\frac{2}{3}}\right),$$

provided  $\varepsilon$  is sufficiently small and (1.15) holds. Finally, noticing that

$$\int_{\Omega_\varepsilon} \psi_i \psi_j = \mathcal{O}(e^{-\frac{\varepsilon}{\varepsilon}}), \quad i \neq j,$$

we conclude that the Morse index of the solution  $u_\varepsilon$  is greater than  $\left\lfloor \frac{\sqrt{\lambda_*}}{2} \varepsilon^{-\frac{2}{3}} \right\rfloor$  as  $\varepsilon \rightarrow 0$  and (1.15) holds. (This is in agreement with the asymptotic formula (1.9) in the radial case).

The proof of the proposition is complete.  $\square$

## 12. OPEN PROBLEMS

An interesting question is whether one can remove the resonance condition (1.15) in Theorem 1.1. In problems of constructing solutions concentrating on curves for the nonlinear Schrödinger equation, this has been achieved (in some cases) very recently in [5], [19] by constructing approximate solutions from two dimensional solutions of the corresponding blown up problem, which roughly was (1.12) with constant positive potential. Let us point out that, in the current situation, solutions  $W \in H^1(\mathbb{R}^2)$  of the equation (1.12) are not radial and, to the best of our knowledge, their non-degeneracy is not known. Here, by non-degeneracy, we mean that the kernel in  $H^1(\mathbb{R}^2)$  of the linearized operator  $-\Delta + 2(V_+ - W)$  is spanned by  $\frac{\partial W}{\partial z}$ . (Existence of such  $W$  can be established by the same arguments used for the solution

$\mathbf{w} \in H^1(\mathbb{R})$  in Proposition 3.1 and noting that (1.12) is translation invariant in the direction  $z$ ).

Naturally, an interesting open problem is to prove an analog of Theorem 1.1 for problem (1.1) posed in any dimension  $N \geq 2$ , where the zero set of  $a - b$  would be an  $(N - 1)$ -dimensional submanifold  $\mathcal{M}$  of  $\Omega$  and  $\nabla(a - b) \neq 0$  on  $\mathcal{M}$ . (One could also formulate a theorem for the case where  $\mathcal{M}$  is  $(N - m)$ -dimensional,  $1 \leq m \leq N - 1$ ). The same question has been the subject of a lot of recent research in other semilinear elliptic problems involving resonance. If  $N$  is arbitrary, instead of the infinite dimensional reduction of [17], which we use in the present paper, the preferred approach seems to be that of [38] (see [35, 37, 42]). Let us briefly discuss this point and what are the difficulties in employing the scheme of [38] in the present situation. Our construction of the approximate solution can easily be adapted to problem (1.1) posed in  $N$  dimensions. Based on the radially symmetric case, we expect that the eigenvalues of the linearization of (1.1) around this approximation should behave qualitatively as  $\Lambda_i$  described in (1.8) where  $\tau_i$  now are the eigenvalues of the Laplace-Beltrami operator of the manifold  $\mathcal{M}$ . (Instead of expanding in polar coordinates, one can use (naively) a Fourier decomposition with respect to the eigenfunctions of the Laplace-Beltrami operator and the normal Laplacian on the manifold  $\mathcal{M}$ , see [35] for a related result). Assuming that this expectation holds, by Weyl's asymptotic formula, the eigenvalues that are closest to zero should behave qualitatively like

$$\Lambda_i = -\varepsilon^{\frac{2}{3}} + i^{\frac{2}{N-1}} \varepsilon^2. \quad (12.1)$$

Notice that in this way the average distance between two consecutive  $\Lambda_i$ 's (when they are close to zero) is of order  $\varepsilon^{N-1}$  so, even if we have invertibility, the distance of the spectrum to zero is (in the best cases) of order  $\varepsilon^{N-1}$ . Therefore, the inverse operator is always large in norm. By this reason, to apply a nonlinear scheme to perturb the approximate solution into a genuine solution of (1.1), we need first to find very good approximate solutions, with a precision depending on  $N$ , and then prove that the linearized operator is invertible for suitable values of  $\varepsilon$ . This is indeed a rather delicate issue: for reasons of brevity we do not discuss it here but we refer directly to [37]. Improving inductively the accuracy of the approximation by adding lower order terms, determined from linear inhomogeneous problems of the form (3.33)-(3.34), seems to be difficult and complicated because each inhomogeneous term and boundary condition grows polynomially instead of decaying exponentially to zero as was the case in [37] or [38], where arbitrarily high-order approximations for other equations were constructed in this manner. Unlike the previous references, the problem at hand has in common with [42] the fact that the linearization of the profile normal to  $\Gamma$  is invertible (recall Proposition 3.1), and another possibility for improving the approximate solution could be to adapt the iteration scheme of that paper.

Problem (1.1) posed in  $N$  dimensions, as in the previous paragraph, has variational structure, and the stable solution satisfying (1.5) is a local minimizer of the corresponding functional. So, it is easy to see that (1.1) has a *mountain-pass* solution. Motivated from [14], we conjecture that, for all small  $\varepsilon > 0$ , a mountain-pass solution satisfies the first inequality of (1.5), and the second outside of an  $\mathcal{O}(\varepsilon^{\frac{2}{3}})$ -neighborhood of a point  $P$  on  $\Gamma$  where the function  $\frac{\partial b}{\partial \nu} - \frac{\partial a}{\partial \nu}$  attains its global minimum value (recall (1.3)). The profile of a mountain-pass solution should be

that of a stable solution with the addition of a small downward peak near  $P$  (the point  $P$  depends on the family of solutions).

In view of hypothesis (1.3), the function  $a - b$  attains its maximum value at a point  $Q$  in  $\Omega_1$ . Motivated from [48], we conjecture that one can add a sharp downward spike near  $Q$  to the solution of Theorem 1.1 (at least in two dimensions) or to the stable solution found in [9] and construct highly or not too unstable solutions respectively with *both corner layer and spike*.

Based on the two conjectures we formulated above, and motivated from Remark 3.10 in [14], and [15], we further conjecture that, if  $N = 2$ , solutions of uniformly bounded Morse index, for small  $\varepsilon > 0$ , are a superposition of a stable solution, as described in (1.5), and a finite number of sharp downward spikes in  $\bar{\Omega} \setminus \Gamma$  and small downward peaks close to  $\Gamma$ .

**Acknowledgments.** We would like to thank M. Kowalczyk for stimulating discussions on [17], [19], and M. Ward for bringing reference [24] to our attention. We would also like to thank M. del Pino for drawing our attention to [16] and [31]. The research leading to these results has received funding from the European Union's Seventh Framework Programme (FP7-REGPOT-2009-1) under grant agreement n° 245749.

#### REFERENCES

- [1] A. AFTALION, S. ALAMA, and L. BRONSARD, *Giant vortex and the breakdown of strong pinning in a rotating Bose–Einstein condensate*, Arch. Ration. Mech. Anal. **178** (2005), 247–286.
- [2] A. AFTALION, R. L. JERRARD, and J. ROYO-LETELIER, *Non-existence of vortices in the small density region of a condensate*, Journal of Functional Analysis **260** (2011), 2387–2406.
- [3] S. ALAMA, L. BRONSARD, and J. A. MONTERO, *Vortices for a rotating toroidal Bose–Einstein condensate*, Arch. Rational Mech. Anal. **187** (2008), 481–522.
- [4] N. D. ALIKAKOS, P. W. BATES, J. W. CAHN, P. C. FIFE, G. FUSCO, and G. B. TANOGLU, *Analysis of the corner layer problem in anisotropy*, Discrete Contin. Dyn. Syst. **6** (2006), 237–255.
- [5] W. AO, M. MUSSO, J. WEI, *On spikes concentrating on line-segments to a semilinear Neumann problem*, J. Differential Equations **251** (2011), 881–901.
- [6] C. M. BENDER, and S. A. ORSZAG, *Advanced Mathematical Methods for Scientists and Engineers: Asymptotic Methods and Perturbation Theory*, Springer-Verlag, New York, 1999.
- [7] F. BETHUEL, H. BREZIS, and F. HÉLEIN, *Asymptotics for the minimization of a Ginzburg–Landau functional*, Calc. Var. Partial Differential Equations **1** (1993), 123–148.
- [8] H. BREZIS, *Semilinear equations in  $\mathbb{R}^n$  without conditions at infinity*, Appl. Math. Optimization **12** (1984), 271–282.
- [9] V. F. BUTUZOV, N. N. NEFEDOV, and K. R. SCHNEIDER, *Singularly perturbed elliptic problems in the case of exchange of stabilities*, J. Differential Equations **169** (2001), 373–395.
- [10] V. F. BUTUZOV, *On the stability and domain of attraction of asymptotically non-smooth stationary solutions to a singularly perturbed parabolic equation*, Computational Mathematics and Mathematical Physics **46** (2006), 413–424.
- [11] L. A. CAFFARELLI, and J.-M. ROQUEJOFFRE, *Uniform Hölder estimates in a class of elliptic systems and applications to singular limits in models for diffusion flames*, Arch. Ration. Mech. Anal. **183** (2007), 457–487.
- [12] R. S. CANTRELL, and C. COSNER, *Spatial ecology via reaction–diffusion equations*, Wiley Series in Mathematical and Computational Biology, 2003.
- [13] M. CONTI, S. TERRACINI, and G. VERZINI, *Asymptotic estimates for the spatial segregation of competitive systems*, Adv. Math. **195** (2005), 524–560.
- [14] E. N. DANCER, and S. YAN, *On the superlinear Lazer–McKenna conjecture*, J. Differential Equations **210** (2005), 317–351.
- [15] E. N. DANCER, *Stable and not too unstable solutions on  $\mathbb{R}^n$  for small diffusion*, in Non-linear Dynamics and Evolution Equations, Fields Institute Communications **48**, American Mathematical Society, (2006), 67–93.

- [16] M. DEL PINO, *Layers with nonsmooth interface in a semilinear elliptic problem*, Comm. Partial Differential Equations **17** (1992), 1695–1708.
- [17] M. DEL PINO, M. KOWALCZYK, and J. WEI, *Concentration on curves for nonlinear Schrödinger equations*, Comm. Pure Appl. Math. **60** (2007), 113–146.
- [18] M. DEL PINO, M. KOWALCZYK, and J. WEI, *Resonance and interior layers in an inhomogeneous phase transition model*, SIAM J. Math. Anal. **38** (2007), 1542–1564.
- [19] M. DEL PINO, M. KOWALCZYK, F. PACARD, and J. WEI, *The Toda system and multiple-end solutions of autonomous planar elliptic problems*, Adv. Math. **224** (2010), 1462–1516.
- [20] P. C. FIFE, *A phase plane analysis of a corner layer problem arising in the study of crystalline grain boundaries*, unpublished preprint (2004).
- [21] C. GALLO, and D. PELINOVSKY, *On the Thomas–Fermi ground state in a harmonic potential*, Asymptotic Analysis **73** (2011), 53–96.
- [22] B. GIDAS, W. M. NI, and L. NIRENBERG, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. **68** (1979), 209–243.
- [23] D. GILBARG, and N. S. TRUDINGER, *Elliptic Partial Differential Equations of Second Order*, second ed., Springer-Verlag, New York, 1983.
- [24] R. HABERMAN *Slowly varying jump and transition phenomena associated with algebraic bifurcation problems*, Siam J. Appl. Math. **37** (1979), 69–106.
- [25] J. HALE, and H. KOÇAK, *Dynamics and Bifurcations*, Texts in Applied Mathematics **3**, Springer-Verlag, New York, 1991.
- [26] V. HUTSON, Y. LOU, and K. MISCHAIKOW, *Spatial heterogeneity of resources versus Lotka–Volterra dynamics*, J. Differential Equations **185** (2002), 97–136.
- [27] R. IGNAT, and V. MILLOT, *The critical velocity for vortex existence in a twodimensional rotating BoseEinstein condensate*, J. Funct. Anal. **233** (2006), 260–306.
- [28] M. IIDA, K. NAKASHIMA, and E. YANAGIDA, *On certain one-dimensional elliptic systems under different growth conditions at respective infinities*, in Asymptotic Analysis and Singularities, Advanced Studies in Pure Mathematics **47-2** (2007), 565–572.
- [29] H. IKEDA, *On the asymptotic solutions for a weakly coupled elliptic boundary value problem with a small parameter*, Hiroshima Math. J. **16** (1986), 227–250.
- [30] C. K. R. T. JONES, *Geometric singular perturbation theory*, in Dynamical Systems (Montecatini Terme, 1994), Lecture Notes in Math. **1609**, Springer, Berlin, 1995, 44–118.
- [31] Y. KABEYA, and K. TANAKA, *Uniqueness of positive radial solutions of semilinear elliptic equations in  $\mathbb{R}^N$  and Sérés non-degeneracy condition*, Comm. Partial Differential Equations **24** (1999), 563–598.
- [32] G. KARALI, and C. SOURDIS, *Radial and bifurcating non-radial solutions for a singular perturbation problem in the case of exchange of stabilities*, Ann. I. H. Poincaré - AN **29** (2012), 131–170.
- [33] G. KARALI, and C. SOURDIS, *Study of the ground state of a Gross–Pitaevskii energy with general potential in the Thomas–Fermi limit*, preprint (2012).
- [34] Y. LOU, *On the effects of migration and spatial heterogeneity on single and multiple species*, J. Differential Equations **223** (2006), 400–426.
- [35] F. MAHMOUDI, and A. MALCHIODI, *Concentration on minimal submanifolds for a singularly perturbed Neumann problem*, Adv. in Math. **209** (2007), 460–525.
- [36] F. MAHMOUDI, R. MAZZEO, and F. PACARD, *Constant mean curvature hypersurfaces condensing along a submanifold*, Geom. Funct. Anal. **16** (2006), 924–958.
- [37] F. MAHMOUDI, A. MALCHIODI, and J. WEI, *Transition layer for the heterogeneous Allen–Cahn equation*, Ann. Inst. H. Poincaré Anal. Non Linéaire **25** (2008), 609–631.
- [38] A. MALCHIODI, and M. MONTENEGRO, *Boundary concentration phenomena for a singularly perturbed elliptic problem*. Comm. Pure and Applied Math. **55** (2002), 1507–1568.
- [39] R. MAZZEO, and F. PACARD, *Foliations by constant mean curvature tubes*, Communications in Analysis and Geometry **13** (2005), 633–670.
- [40] A. MONTERO, *Hodge decomposition with degenerate weights and the Gross–Pitaevskii energy*, Journal of Functional Analysis **254** (2008), 1926–1973.
- [41] F. PACARD, and M. RITORÉ, *From constant mean curvature hypersurfaces to the gradient theory of phase transitions*, Journal of Differential Geometry **64** (2003), 359–423.
- [42] F. PACARD, F. PACELLA, and B. SCIUNZI, *Solutions of semilinear elliptic equations in tubes*, preprint (2011).
- [43] P. H. RABINOWITZ, *On a class of nonlinear Schrödinger equations*, Z. Angew. Math. Phys. **43** (1992), 270–291.



- [44] S. SCHECTER, and C. SOURDIS, *Heteroclinic orbits in slow-fast Hamiltonian systems with slow manifold bifurcations*, J. Dyn. Diff. Equat. **22** (2010), 629-655.
- [45] J. SMOLLER, *Shock Waves and Reaction–Diffusion Equations*, Springer-Verlag, New York, 1980.
- [46] C. SOURDIS, and P. C. FIFE, *Existence of heteroclinic orbits for a corner layer problem in anisotropic interfaces*, Adv. Differential Equations **12** (2007), 623-668.
- [47] J. WEI, and T. WETH, *Asymptotic behavior of solutions of planar elliptic systems with strong competition*, Nonlinearity **21** (2008), 305–317.
- [48] J. WEI, and J. YANG, *Solutions with transition layer and spike in an inhomogeneous phase transition model*, J. Differential Equations **246** (2009), 3642–3667.

DEPARTMENT OF APPLIED MATHEMATICS, UNIVERSITY OF CRETE, GR-714 09 HERAKLION, CRETE, GREECE, AND INSTITUTE OF APPLIED AND COMPUTATIONAL MATHEMATICS, IACM, FORTH, GREECE.

*E-mail address:* `gkarali@tem.uoc.gr`

DEPARTMENT OF APPLIED MATHEMATICS, UNIVERSITY OF CRETE, GR-714 09 HERAKLION, CRETE, GREECE.

*E-mail address:* `csourdis@tem.uoc.gr`