Radial and bifurcating non-radial solutions for a singular perturbation problem in the case of exchange of stabilities

Georgia Karali

Department of Applied Mathematics, University of Crete, GR–714 09 Heraklion, Crete, Greece, and Institute of Applied and Computational Mathematics, IACM, FORTH, Greece.

Christos Sourdis

Department of Applied Mathematics, University of Crete, GR–714 09 Heraklion, Crete, Greece.

Abstract

We consider the singular perturbation problem $-\varepsilon^2\Delta u + (u - a(|x|))\,(u - b(|x|)) = 0$ in the unit ball of $\mathbb{R}^N$, $N \geq 1$, under Neumann boundary conditions. The assumption that $a(r) - b(r)$ changes sign in $(0, 1)$, known as the case of exchange of stabilities, is the main source of difficulty. More precisely, under the assumption that $a - b$ has one simple zero in $(0, 1)$, we prove the existence of two radial solutions u_+ and u_- that converge uniformly to $\max\{a, b\}$, as $\varepsilon \to 0$. The solution u_+ is asymptotically stable, whereas u_- has Morse index one, in the radial class. If $N \geq 2$, we prove that the Morse index of u_-, in the general class, is asymptotically given by $[c + o(1)]\varepsilon^{-\frac{2}{3}(N-1)}$ as $\varepsilon \to 0$, with $c > 0$ a certain positive constant. Furthermore, we prove the existence of a decreasing sequence of $\varepsilon_k > 0$, with $\varepsilon_k \to 0$ as $k \to +\infty$, such that non-radial solutions bifurcate from the unstable branch $\{(u_-(\varepsilon), \varepsilon)\,,\ \varepsilon > 0\}$ at $\varepsilon = \varepsilon_k$, $k = 1, 2, \cdots$. Our approach is perturbative, based on the existence and non-degeneracy of solutions of a "limit" problem. Moreover, our method of proof can be generalized to treat, in a unified manner, problems of the same nature where the singular limit is continuous but non-smooth.

Keywords: corner layer, exchange of stabilities, geometric singular perturbation theory, non-radial bifurcations
2000 MSC: 35J25, 35J20, 35B33, 35B40

1. Introduction

1.1. The problem

We consider the singularly perturbed elliptic problem

$$-\varepsilon^2\Delta u + (u - a(|x|))\,(u - b(|x|)) = 0 \ \text{ in } B_1, \quad \partial_\nu u = 0 \ \text{ on } \partial B_1, \tag{1}$$

in the unit ball of $\mathbb{R}^N$, $N \geq 1$, centered at the origin. The perturbation parameter ε is positive and small. The outward normal derivative of u on the boundary of B_1 is denoted by $\partial_\nu u$. The functions $a(r)$, $b(r)$ are in $C^3[0, 1]$, independent of ε, and there exists $r_0 \in (0, 1)$ such that

$$a(r) > b(r),\ r \in [0, r_0),\ \ a(r) < b(r),\ r \in (r_0, 1],\ \text{ and } \ a_r(r_0) < b_r(r_0) \quad \text{(see Figure 1.1)}. \tag{2}$$

This last assumption can be viewed as a *non-degeneracy* condition. Moreover, we assume that

$$a_r(0) = b_r(0) = 0, \quad \text{and } \ b_r(1) = 0. \tag{3}$$

(The case where $b_r(1) \neq 0$ can be treated by simply adding a boundary layer correction, see Remark 3.17).

The assumption that $a - b$ changes sign is related to the phenomenon of exchange of stabilities, and implies that, even in the case $N = 1$, the standard theory of singularly perturbed systems [21] cannot be applied.

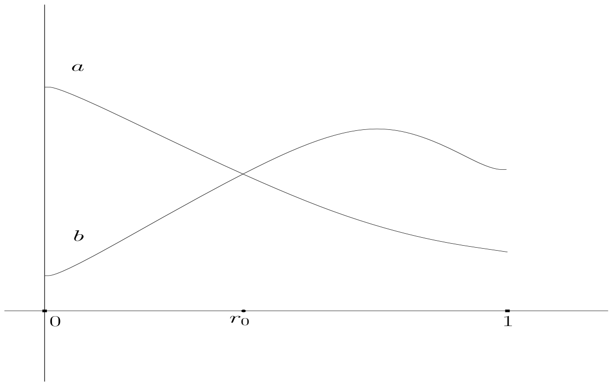

Figure 1: The graphs of a, b.

We are interested in solutions of (1), not necessarily radially symmetric, that converge *uniformly* to $\max\{a, b\}$ as $\varepsilon \to 0$. We say that such solutions have a *corner layer* at $|x| = r_0$. Furthermore, we are interested in estimating the convergence of such cornered layered solutions to $\max\{a, b\}$ as $\varepsilon \to 0$, and to study their stability properties.

Problem (1) is a characteristic case of the general problem

$$-\varepsilon^2 \Delta u + f(u, |x|) = 0 \text{ in } B_1, \quad \partial_\nu u = 0 \text{ on } \partial B_1,$$

where $f \in C^3 (\mathbb{R} \times [0, 1])$ is independent of $\varepsilon > 0$, and

$$f(a(r), r) = 0, \quad f(b(r), r) = 0, \quad r \in [0, 1],$$

$$f_u(a(r), r) > 0, \; r \in [0, r_0), \;\; f_u(b(r), r) > 0, \; r \in (r_0, 1], \;\; f_{uu}(a(r_0), r_0) > 0,$$

where $a, b \in C^3[0, 1]$ satisfy (2) and (3). However, in order to present the main ideas of the paper as clearly as possible, we have chosen to deal with the model problem (1). We remark that our approach can also be extended to cover the case where $a - b$ has finitely many simple zeroes in $(0, 1)$, as well as the case where f depends (suitably) on $\varepsilon > 0$.

1.2. Motivation for the current work

In the present paper, we deal with (1) via a technique widely used in the last past years: we look for solutions as

$$u = u_{ap} + \phi, \tag{4}$$

where u_{ap} is an approximate solution constructed by solutions of a limiting problem (see (14) below). The function ϕ will be found using the contraction mapping theorem. Although this approach has been used in many other papers in the context of spike or transition layer problems, some important differences occur with respect to the standard technique in the case of corner layer problems. Indeed, in other classes of equations, like Allen-Cahn or focusing Nonlinear Schrödinger, the solutions of the corresponding limiting problems give rise to a local approximate *inner solution*, typically having a spike or transition layer profile, that can be made global by a standard cut-off function argument (see [18], [45]). Actually, the one dimensional version of the previously mentioned equations fits in the framework of standard geometric singular perturbation theory [21], [37], [64]. In the present situation, and generally in problems involving corner layers, globalizing the inner solution, namely rigorously matching it with the outer, is not standard (see Subsection 1.5 for more details). Our motivation for the current work is to develop a matching procedure and a perturbation argument that have the flexibility to treat a class of corner layer problems in a unified manner, and the potential to deal with non-radial problems in general domains. We believe that the study of these problems, under the simplifying assumption of radial symmetry, is important in order to develop methods which may ultimately lead to the resolution of the general problems.

Singular perturbation problems of the same nature as (1) appear in population dynamics, when two or more species interact in a highly competitive way, and spatial segregation may occur. A wide literature is devoted to this topic, mainly for the case of competition models of Lotka-Volterra type (see for example [13], [16]). In [13] the behavior of the positive steady-states of a Lotka-Volterra model, in the case of two species, as the competition rate ε^{-2} tends to infinity, was reduced to the study of

$$-\varepsilon^2\Delta u + u(u - A(x)) = 0, \quad u > 0 \ \text{ in } \Omega, \quad u = 0 \ \text{ on } \partial\Omega, \tag{5}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, and A is the harmonic extension in Ω of a sign-changing $\mathcal{A} \in C(\partial\Omega)$. It was shown in [13], via the method of upper and lower solutions (using the corresponding limit problem (14)), that there exists a solution of (5) such that $u - \max\{A, 0\} = O\left(\varepsilon^{\frac{2}{3}}\right)$ as $\varepsilon \to 0$, uniformly in $\bar{\Omega}$. Note that, in this problem, the corresponding non-degeneracy condition (2) is ensured by Hopf's lemma [27] (see Proposition 3.16 in [9] for a result that allows more general A's and boundary conditions in (5)). A more complicated model was treated in [16], without making use of limit problem (14), and the convergence to the singular cornered layered solution was estimated in $L^2(\Omega)$.

Another problem that motivated our study of (1) is the semiclassical limit of the de-focusing nonlinear Schrödinger equation with a potential trap. In [35] the authors considered the harmonic trapping case, in $\mathbb{R}^2$, with a cubic nonlinearity. This leads to the study of the problem

$$-\varepsilon^2\Delta u + u(u^2 - B(x)) = 0, \quad u > 0 \ \text{ in } \mathbb{R}^N, \quad u \to 0 \ \text{ as } |x| \to +\infty, \tag{6}$$

where $B(x) = 1 - |x|_\Lambda^2$, with $|x|_\Lambda^2 = x_1^2 + \Lambda^2 x_2^2$, $0 < \Lambda \leq 1$. It was shown in [35], via variational methods and upper and lower solutions, that there exists a solution of (6) such that $u \to \sqrt{\max\{B, 0\}}$ as $\varepsilon \to 0$, uniformly in $\mathbb{R}^2$. Notice that the singular limit $\sqrt{\max\{B, 0\}}$ is continuous but non-smooth at the ellipse $|x|_\Lambda = 1$. If $N = 1$, a shooting argument approach, for a related problem, can be found in [23]. In the case where $\Lambda = 1$, the problem becomes radial, and an inner solution can be directly constructed as

$$\varepsilon^{\frac{1}{3}} U\left(\frac{r-1}{\varepsilon^{\frac{2}{3}}}\right), \quad \text{(see [25], [62])},$$

where U is the Hastings-McLeod solution of the Painlevé II equation, namely

$$-U_{\xi\xi} + U(U^2 - B_r(1)\xi) = 0, \ \ \xi \in \mathbb{R}, \ \ U - (B_r(1)\xi)^{\frac{1}{2}} \to 0 \ \text{ as } \xi \to -\infty, \ \ U \to 0 \ \text{ as } \xi \to +\infty, \ \ \text{(see [1], [33], [61])}.$$

Let us remark that the approach we develop in the present paper can be applied to the study of systems *without* variational structure.

The Lazer–McKenna conjecture, for a super-linear elliptic problem of Ambrosetti–Prodi type, is also related to our study of (1). The following problem was studied in [17]:

$$-\varepsilon^2\Delta u + |u|^p - \varphi_1(x) = 0 \ \text{ in } \Omega, \quad u = 0 \ \text{ on } \partial\Omega, \tag{7}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $p > 1$, and $\varphi_1 > 0$ is the principal Dirichlet eigenfunction of Ω. It was shown in [17], via the method of upper and lower solutions, that there exists a solution of (7) such that, for every compact subset $\mathcal{D}$ of Ω, $u - \varphi_1^{\frac{1}{p}} = O(\varepsilon^2)$ as $\varepsilon \to 0$, uniformly in $\mathcal{D}$. Note that, by Hopf's lemma [27], the function $\varphi_1^{\frac{1}{p}}$ is continuous but non-smooth at the boundary $\partial\Omega$, since $p > 1$. An inner solution, near the boundary $\partial\Omega$, can be constructed by the limiting problem

$$-U_{\xi\xi} + |U|^p - \xi = 0, \ \ \xi > 0, \ \ U(0) = 0, \ \ U - \xi^{\frac{1}{p}} \to 0 \ \text{ as } \ \xi \to +\infty, \ \ \text{(see [17])}.$$

Finally, let us mention that corner layer problems also arise in a class of nonlinear elliptic equations involving large or exponential nonlinearities, like the Brezis–Nirenberg problem (see [31]). After an appropriate rescaling, the corresponding limit problem is

$$U_{\xi\xi} + e^U = 0, \ \ \xi \in \mathbb{R}, \ \ \text{(see [31])}. \tag{8}$$

Note that (8) is invariant with respect to translations and dilations. Moreover, it is well known that all solutions of (8) diverge linearly as $\xi \to \pm\infty$, as is the case of the limit problem (14) in our situation. In the radial case, a perturbation argument has been developed in [31], based on the construction of approximate solutions from solutions of (8). However, there was no matching involved in that construction, thus making it hard to generalize the approach of [31] to deal with the non-radial scenario. Let us also mention that, in this class of problems, non-radial bifurcations from the radial corner layered solution branch have been studied in [28], [43] and [51]. (The one-dimensional profile U in (8) is unstable).

1.3. Known results

The known results for problem (1) concern the case $N = 1$, where (1) can be written as a geometric singular perturbation problem, and the general case $N \geq 1$, where stable solutions can be constructed by the method of upper and lower solutions.

1.3.1. Case N=1

If $N = 1$, problem (1) can be written as a geometric singular perturbation problem composed of two fast equations and a slow equation (see [37]). Let $u_1 = u,\ u_2 = \varepsilon \dot{u}_1$, where $\dot{} = \frac{d}{dt}$, then (1) is equivalent to the connection problem (see [64])

$$\begin{cases} \varepsilon \dot{u}_1 &= u_2, \\ \varepsilon \dot{u}_2 &= (u_1 - a(x))\,(u_1 - b(x)), \\ \dot{x} &= 1, \end{cases} \tag{9}$$

with boundary manifolds

$$\mathcal{B}_0 = \{u_1 \in \mathbb{R},\ u_2 = 0,\ x = 0\} \quad \text{and} \quad \mathcal{B}_1 = \{u_1 \in \mathbb{R},\ u_2 = 0,\ x = 1\}. \tag{10}$$

As $\varepsilon \to 0$, the limit of (9), which is only defined on the so-called slow manifold

$$\mathcal{S} = \{u_1 = a(x),\ u_2 = 0,\ x \in [0,1]\} \cup \{u_1 = b(x),\ u_2 = 0,\ x \in [0,1]\},$$

is plainly $\dot{x} = 1$. Hence, the one-dimensional *slow manifold* $\mathcal{S}$ undergoes a *transcritical bifurcation* at the point $\mathbf{c} = (a(r_0), 0, r_0)$ (recall (2)), as the slow variable x changes (and thus $\mathcal{S}$ is not actually a manifold, although we will refer to it as one). By transforming the *slow* system (9) to the fast variable $\tau := t/\varepsilon$, we obtain the equivalent *fast* system

$$\begin{cases} u_1' = u_2, \\ u_2' = (u_1 - a(x))\,(u_1 - b(x)), \\ x' = \varepsilon, \end{cases} \tag{11}$$

where $' = \frac{d}{d\tau}$. Letting $\varepsilon \to 0$ in (11), we obtain the fast limit system

$$\begin{cases} u_1' = u_2, \\ u_2' = (u_1 - a(x))\,(u_1 - b(x)), \\ x' = 0, \end{cases} \tag{12}$$

for which $\mathcal{S}$ is a manifold of equilibria. By virtue of (2), the branches of $\mathcal{S}$ defined by

$$\mathcal{S}^a = \{u_1 = a(x),\ u_2 = 0,\ x \in [0, r_0)\} \text{ and } \mathcal{S}^b = \{u_1 = b(x),\ u_2 = 0,\ x \in (r_0, 1]\},$$

consist of normally hyperbolic equilibria of (12) (see [37]), with one negative and one positive eigenvalue; whereas at the equilibrium $\mathbf{c} \in \mathcal{S}$ all eigenvalues are zero. Note that the *singular connecting orbit*

$$\Gamma_0 = \{u_1 = \max\{a(t), b(t)\},\ u_2 = 0,\ x = t,\ t \in [0,1]\}$$

parameterizes $\mathcal{S}^a \cup \{\mathbf{c}\} \cup \mathcal{S}^b$. Hence, the loss of normal hyperbolicity of the slow manifold $\mathcal{S}$ at the point $\mathbf{c}$ prohibits the use of standard geometric singular perturbation theory [21], [37], [39] in order to deduce the persistence of the singular orbit Γ_0, for small $\varepsilon > 0$. The fact that Γ_0 perturbs, for small $\varepsilon > 0$, to a connecting orbit Γ_ε of (9), (10) has been proven in [56], using the blow-up procedure for dealing with loss of normal hyperbolicity of the slow manifold [20], [42]. Actually, the problem treated in [56] was a Hamiltonian system in the whole real line, but the same proof applies thanks to [64]. One appends the equation $\varepsilon' = 0$ to (11), and performs a blow-up of the point $(\mathbf{c}, 0)$ of $u_1u_2x\varepsilon$-space to a 3-sphere by the transformation

$$u_1 = a(r_0) + R^2\bar{u}_1, \ \ u_2 = R^3\bar{u}_2, \ \ x = r_0 + R^2\bar{x}, \ \ \varepsilon = R^3\bar{\varepsilon}, \tag{13}$$

where $(\bar{u}_1, \bar{u}_2, \bar{x}, \bar{\varepsilon}) \in S^3$ and $R \geq 0$. Within the sphere, the de-singularized vector field has an equilibrium with a two-dimensional center-unstable manifold, and another with a two-dimensional center-stable manifold. It has been shown in [56] that these invariant manifolds intersect transversely along an *inner* solution, furnished by an asymptotically stable solution of the problem

$$U_{\xi\xi} = (U - a_r(r_0)\xi)(U - b_r(r_0)\xi), \ \ \xi \in \mathbb{R}, \quad U - a_r(r_0)\xi \to 0 \ \text{ as } \ \xi \to -\infty, \ U - b_r(r_0)\xi \to 0 \ \text{ as } \ \xi \to +\infty. \tag{14}$$

One then uses a shooting argument, together with the *Corner Lemma* of [55], to infer that, for small $\varepsilon > 0$, the unstable manifold $W^u(\mathcal{B}_0)$ intersects the stable manifold $W^s(\mathcal{B}_1)$ transversely along a solution Γ_ε of (9), (10). It follows that $\mathrm{dist}(\Gamma_\varepsilon, \Gamma_0) \to 0$ as $\varepsilon \to 0$ and, given any $D > 0$, we have

$$u_1(t) = a(r_0) + \varepsilon^{\frac{2}{3}} U\left(\frac{t - r_0}{\varepsilon^{\frac{2}{3}}}\right) + O\left(\varepsilon^{\frac{4}{3}}\right), \quad |t - r_0| \leq D\varepsilon^{\frac{2}{3}} \quad \text{as } \varepsilon \to 0. \tag{15}$$

We remark that no information about the stability properties of the obtained corner layered solution has been given in [56].

1.3.2. General $N \geq 1$

In [7], the authors considered the problem

$$-\varepsilon^2\Delta u + f(u, x) = 0 \ \text{ in } \Omega, \ \ \partial_\nu u = 0 \ \text{ on } \partial\Omega, \tag{16}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 1$, and $f \in C^2\left(\mathbb{R} \times \bar{\Omega}\right)$ (Ω, f independent of ε) satisfying the following hypothesis:
There exists a smooth $(N-1)$-dimensional sub-manifold $C \subset \Omega$ dividing Ω in two open connected components Ω_1, Ω_2, and $u_1, \ u_2 \in C^2\left(\bar{\Omega}\right)$ such that

$$u_1 > u_2 \ \text{ in } \Omega_1, \quad u_1 < u_2 \ \text{ in } \Omega_2, \tag{17}$$

$$f\left(u_i(x), x\right) = 0, \ \ x \in \Omega, \ f_u\left(u_i(x), x\right) > 0, \ \ x \in \Omega_i, \ f_u\left(u_i(x), x\right) < 0, \ \ x \in \Omega/\Omega_i, \ i = 1, 2, \tag{18}$$

$$f_u\left(u_0(x), x\right) > c|t|, \ \ |t| \leq d, \ \text{ where } u_0 = \max\{u_1, \ u_2\}, \tag{19}$$

and (θ, t) are the Fermi coordinates associated to the manifold C (see [22], [45]), and $c, d > 0$ are constants independent of $\varepsilon > 0$.

It was shown in [7], via the method of upper and lower solutions, that there exists a solution u_ε of (16) such that

$$|u_\varepsilon(x) - u_0(x)| \leq C\varepsilon^{\frac{2}{3}}, \quad x \in \bar{\Omega}, \tag{20}$$

where $C > 0$ is a constant independent of ε. Moreover, it has been shown in [6] that the principal eigenvalue of the linearization of (16) on u_ε satisfies

$$\Lambda_1 \geq c\varepsilon^{\frac{2}{3}}, \tag{21}$$

where $c > 0$ is a constant independent of ε. Hence, the solution u_ε is asymptotically stable (with respect to the parabolic dynamics). We remark that the method of upper and lower solutions renders only stable solutions, and, in general, is not applicable to the study of systems. Let us also point out that problem (16) was not linked to a limit problem (see (14)), as $\varepsilon \to 0$, in [6] or [7].

1.4. *Main results*

In Theorem 3.29 we establish the existence of two radially symmetric solutions $u_+,\ u_-$ of (1), with $u_-(r_0) < a(r_0) < u_+(r_0)$, converging uniformly to $\max\{a, b\}$ as $\varepsilon \to 0$, and, for any $D > 0$, we have

$$u_\pm(r) = a(r_0) + \varepsilon^{\frac{2}{3}} U_{1\pm}\left(\frac{r - r_0}{\varepsilon^{\frac{2}{3}}}\right) + \varepsilon^{\frac{4}{3}} U_{2\pm}\left(\frac{r - r_0}{\varepsilon^{\frac{2}{3}}}\right) + O(\varepsilon^2), \quad |r - r_0| \le D\varepsilon^{\frac{2}{3}} \quad \text{as } \varepsilon \to 0,$$

where $U_{1+} > U_{1-}$ are solutions of (14), whose existence and non-degeneracy are proven in Propositions 3.2 and 3.6 respectively, and $U_{2\pm}$ solve linear equations (36). We note that, besides establishing existence of two solutions, our estimate improves that of [56] (see (15) herein) if $N = 1$, as well as that of [7] (see (20) herein) in the case of radial symmetry. Moreover, we prove that the first m eigenvalues of the radial linearization of (1) on $u_\pm$ satisfy

$$\lambda_{i\pm} = \mu_{i\pm}\varepsilon^{\frac{2}{3}} + O\left(\varepsilon^{\frac{4}{3}}\right) \ \text{ as } \ \varepsilon \to 0, \ \ i = 1, \cdots, m, \tag{22}$$

where $\mu_{i\pm}$, $i = 1, \cdots, m$ are the first m eigenvalues of the limiting eigenvalue problem

$$-\psi_{\xi\xi} + (2U_{1\pm} - a_r(r_0)\xi - b_r(r_0)\xi)\psi = \mu\psi, \ \ \psi \in L^2(\mathbb{R}), \tag{23}$$

which is exactly the linearization of (14) on $U_{1\pm}$, in particular

$$\mu_{1+} > 0 \ \text{ and } \ \mu_{1-} < 0 < \mu_{2-}. \tag{24}$$

Hence the solution u_+ is asymptotically stable, whereas u_- is unstable with one negative (radial) eigenvalue.

Next we consider the linearization of (1) on u_+ and u_-, in the general class of functions, using a separation of variables. It is well known that the eigenfunction corresponding to the principal eigenvalue is radial and we may assume that it is positive. Hence, via $(22)_+$ and (24), we infer that the solution u_+ is asymptotically stable, in the general class, if $\varepsilon > 0$ is sufficiently small. Note that, in view of (24), estimate $(22)_+$, with $i = 1$, improves the corresponding estimate of [6] (see (21) herein). On the other hand, we will show that the linearized operator of (1) on u_- has asymptotically $\left[c\varepsilon^{-\frac{2}{3}}\right]$ negative non-radial eigenvalues, as $\varepsilon \to 0$, where $c > 0$ is a constant independent of $\varepsilon > 0$ (see Theorem 4.5). We give some accurate estimates for the small eigenvalues of the linearization of (1) at u_- (similar estimates can be shown for the linearization at u_+), and obtain a rather sharp asymptotic formula for the Morse index of u_-.

Finally, in Theorems 5.2, 5.4 and 5.6, we prove the existence of a plethora of non-radial solutions of (1) bifurcating from the unstable branch $(u_-(\varepsilon), \varepsilon)$, $\varepsilon > 0$ small.

1.5. *Strategy of the proof and structure of the paper*

In Section 3 we consider problem (1) in the class of radial solutions. One can calculate, via asymptotic analysis, a *formal* (non-standard) inner expansion

$$u_{in}(r) = a(r_0) + \varepsilon^{\frac{2}{3}} U_1\left(\frac{r - r_0}{\varepsilon^{\frac{2}{3}}}\right) + \varepsilon^{\frac{4}{3}} U_2\left(\frac{r - r_0}{\varepsilon^{\frac{2}{3}}}\right) + \cdots \tag{25}$$

near $r = r_0$ (note that this is compatible with the blow-up transformation (13)). The function U_1 in (25) has to satisfy the limit problem (14). Surprisingly, we obtain two solutions U_{1+} and U_{1-} of (14), and hence *two inner approximations*. Loosely speaking, the solution U_{1+} is a minimum, whereas U_{1-} is a mountain-pass [52]. Moreover, both U_{1+} and U_{1-} are non-degenerate solutions of (14). We remark that U_{1+} has appeared recently in a class of related singular perturbation problems in [13], [36], and [56]. The existence of U_{1-}, to the best of our knowledge, was not previously in the literature (see the appendix of [8] for a related result). The functions $U_{2\pm}$ in $(25)_\pm$ satisfy linear problems $(36)_\pm$, (41) below, which are solvable thanks to the non-degeneracy of $U_{1\pm}$. In this paper we have calculated the first two terms of the inner expansion (25). If these inner approximations $u_{in\pm}$ are substituted in (1), one finds that the remainder *grows* with respect to the distance from $r = r_0$ (see (34)); in contrast to the problems studied in [3], [54] where the remainder gets smaller (see also [18], [45]). Additionally, the second order term (that involves U_2) of the inner approximations $u_{in\pm}$ decays *slowly* to the corresponding term of the outer approximation $\max\{a, b\}$

(see Proposition 3.11). These facts pose important difficulties for matching the inner with the outer approximation, in order to construct a global approximation that is valid in the whole domain. We accomplish the desired matching by a *novel* procedure that glues u_{in} with a suitable perturbation of $\max\{a, b\}$ at $|r - r_0| = L\varepsilon^{\frac{2}{3}}$, $L > 0$ fixed, in a C^1 and piecewise C^2 manner. The obtained approximations $u_{ap\pm}$ satisfy

$$-\varepsilon^2\Delta u_{ap\pm} + (u_{ap\pm} - a(|x|))(u_{ap\pm} - b(|x|)) = O\left(\varepsilon^{\frac{8}{3}}\right) \text{ uniformly in } B_1 \cap \left\{||x| - r_0| \neq L\varepsilon^{\frac{2}{3}}\right\}, \tag{26}$$

and $u_{ap\pm} = \max\{a, b\} + O\left(\varepsilon^{\frac{2}{3}}\right)$ uniformly in $\bar{B}_1$, as $\varepsilon \to 0$. (Note that ϕ in (4) has to be radial, C^1 and piecewise C^2 with finite jumps at $|r - r_0| = L\varepsilon^{\frac{2}{3}}$). Our method, close in spirit to that of [62], provides *optimal* estimates, and the flexibility to deal with a variety of corner layer problems. Furthermore, it has the potential to treat *non-radial* problems in general domains. In a related corner layer problem in $\mathbb{R}$, of the same nature as (6), treated in [62], the gluing had to be performed at $|r - r_0| = |\ln\varepsilon|\varepsilon^{\frac{2}{3}}$. What allows us now to match at the optimal distance from $r = r_0$ is that we first suitably "prepare" the outer solution for the gluing, as described in (60). Next we study the linearization of (1) near $u_{ap\pm}$, in the radial class, namely

$$\mathbb{L}_\pm(\varphi) = -\varepsilon^2\Delta\varphi + \left(2u_{ap\pm} + e - a - b\right)\varphi, \;\; \partial_\nu\varphi = 0 \text{ on } \partial B_1, \; \varphi \text{ radial}, \tag{27}$$

where e is an arbitrary continuous radial function that is sufficiently small, say $\|e\|_{L^\infty(B_1)} = o(1)\varepsilon^{\frac{4}{3}}$. We prove that the following a-priori estimate holds,

$$\mathbb{L}_\pm(\varphi) = f \;\Rightarrow\; \|\varphi\|_{L^\infty(B_1)} \leq C\varepsilon^{-\frac{2}{3}}\|f\|_{L^\infty(B_1)}, \tag{28}$$

where f is radial, possibly discontinuous at $|r - r_0| = L\varepsilon^{\frac{2}{3}}$ (see Proposition 3.23). Furthermore, given any integer $m \geq 1$ independent of ε, we find that the first m eigenvalues of $\mathbb{L}_\pm$ satisfy (22), where μ_i, $i = 1, \cdots, m$ are the first m eigenvalues of the limiting problem (23) and satisfy (24) (see Proposition 3.25). We would like to mention that in many well known radial singular perturbation problems, such as the Allen-Cahn or focusing nonlinear Schrödinger equation, the corresponding linearization on the layered approximation typically has a small $O(\varepsilon^\alpha)$, $\alpha > 0$, nonzero eigenvalue and the rest of the spectrum is uniformly bounded away from zero, as $\varepsilon \to 0$ (see [3], [19]). By (26), (28), and the contraction mapping theorem, we can find $\phi_\pm = O\left(\varepsilon^2\right)$ as $\varepsilon \to 0$, uniformly in $\bar{B}_1$, such that $u_\pm$, defined by $(4)_\pm$, solve (1). Clearly,

$$u_\pm = u_{ap\pm} + O\left(\varepsilon^2\right) \text{ as } \varepsilon \to 0, \text{ uniformly in } \bar{B}_1,$$

and the radial linearizations of (1) on $u_\pm$ satisfy the eigenvalue distribution (22), see Theorem 3.29.

In Section 4, assuming $N \geq 2$, we linearize (1) on u_- and consider the following eigenvalue problem:

$$-\varepsilon^2\Delta\Psi + (2u_- - a - b)\Psi = \Lambda\Psi \text{ in } B_1, \quad \partial_\nu\Psi = 0 \text{ on } \partial B_1, \tag{29}$$

(here Ψ is not assumed to be radial). Following [19], we prove that, given $M > 0$ (independent of ε), the eigenvalues $\Lambda_1 < \Lambda_2 \leq \cdots$ of (29) behave qualitatively like

$$\Lambda_k = \mu_{1-}\varepsilon^{\frac{2}{3}} + \tau_k\varepsilon^2, \;\; 1 \leq k \leq \left[M\varepsilon^{-\frac{2}{3}}\right], \tag{30}$$

for small $\varepsilon > 0$, where $\tau_k = (k-1)(k+N-3)$ are the eigenvalues of the Laplace-Beltrami operator of S^{N-1} (see Theorem 4.5). We remark that our analysis is more delicate than that of [19] because the corresponding one dimensional profile (u_- for $N = 1$) has many small eigenvalues, as described by (22).

Relation (30) implies that the eigenvalues of (29) grow from a negative number (recall (24)) and eventually cross zero, as $\varepsilon \to 0$. This suggests the possibility of a great number of symmetry-breaking bifurcations from the radially symmetric branch $(u_-(\varepsilon), \varepsilon)$ (it is a smooth branch by the radial non-degeneracy of u_-). We show that this is indeed the case by making use of topological and equivariant bifurcation theory (see Section 5).

2. Notation

Throughout this paper, unless specified otherwise, we will denote by c/C positive small/ large generic constants, independent of ε, whose value will change from line to line. The values of ε will satisfy $0 < \varepsilon < \varepsilon_0$ with ε_0 getting smaller at each step (so that all previous relations still hold). Frequently we will suppress the obvious dependence of quantities on ε. Furthermore, Landau's symbols $O(1)$, $o(1)$ as $\varepsilon \to 0$ will be understood in the sense that $|O(1)| \leq C$ for small $\varepsilon > 0$ and $o(1) \to 0$ as $\varepsilon \to 0$. By $[d] \in \mathbb{N}$ we will denote the integer part of $d > 0$. Finally, if X is a linear space of functions defined in B_1, we will denote by $X_r \subset X$ the subspace of radial functions.

3. Radial corner layered solutions

In this section we will show that, for small $\varepsilon > 0$, problem (1) has two radial solutions which possess a corner layer at $r = r_0$.

In the class of radial solutions, problem (1) is equivalent to

$$-\varepsilon^2 u_{rr} - \varepsilon^2 \frac{N-1}{r} u_r + (u - a(r))(u - b(r)) = 0 \text{ in } (0,1), \quad u_r(0) = u_r(1) = 0. \tag{31}$$

3.1. The inner solution

We will begin by constructing an approximate solution for the equation of (31), valid only in a "small" neighborhood of $r = r_0$. We call such an approximation an inner solution.

Motivated from [56], we seek an inner solution near $r = r_0$ in the form

$$u_{in}(r) = a(r_0) + \varepsilon^{\frac{2}{3}} U_1\left(\frac{r - r_0}{\varepsilon^{\frac{2}{3}}}\right) + \varepsilon^{\frac{4}{3}} U_2\left(\frac{r - r_0}{\varepsilon^{\frac{2}{3}}}\right), \tag{32}$$

with $U_1,\ U_2$ to be determined.

Remark 3.1. *Another approach would be to seek an inner solution as*

$$u_{in}(r) = \varepsilon^{\beta} U_0\left(\frac{r - r_0}{\varepsilon^{\alpha}}\right) + \varepsilon^{\gamma} U_1\left(\frac{r - r_0}{\varepsilon^{\alpha}}\right) + \varepsilon^{\delta} U_2\left(\frac{r - r_0}{\varepsilon^{\alpha}}\right),$$

carry out the the calculation below, and find a-posteriori that $\alpha = \frac{2}{3},\ \beta = 0,\ U_0 = a(r_0),\ \gamma = \frac{2}{3},\ \delta = \frac{4}{3}$.

Let

$$\xi = \frac{r - r_0}{\varepsilon^{\frac{2}{3}}}. \tag{33}$$

Then, for $r - r_0 = o(1)$ or equivalently $\xi = o\left(\varepsilon^{-\frac{2}{3}}\right)$, we have

$$\begin{aligned}
&-\varepsilon^2 (u_{in})_{rr} - \varepsilon^2 \tfrac{N-1}{r}(u_{in})_r + (u_{in} - a(r))(u_{in} - b(r)) = \\
&-\varepsilon^{\frac{4}{3}}(U_1)_{\xi\xi} - \varepsilon^2 (U_2)_{\xi\xi} - \varepsilon^2 \tfrac{N-1}{r_0 + \varepsilon^{\frac{2}{3}}\xi}(U_1)_\xi - \varepsilon^{\frac{8}{3}} \tfrac{N-1}{r_0 + \varepsilon^{\frac{2}{3}}\xi}(U_2)_\xi \\
&+ \left(a(r_0) + \varepsilon^{\frac{2}{3}} U_1 + \varepsilon^{\frac{4}{3}} U_2 - a(r_0 + \varepsilon^{\frac{2}{3}}\xi)\right)\left(a(r_0) + \varepsilon^{\frac{2}{3}} U_1 + \varepsilon^{\frac{4}{3}} U_2 - b(r_0 + \varepsilon^{\frac{2}{3}}\xi)\right) = \\
&\left(-(U_1)_{\xi\xi} + a_r(r_0) b_r(r_0)\xi^2 - a_r(r_0)\xi U_1 - b_r(r_0)\xi U_1 + U_1^2\right)\varepsilon^{\frac{4}{3}} \\
&+ \Big(-(U_2)_{\xi\xi} - \tfrac{N-1}{r_0}(U_1)_\xi + \tfrac{1}{2} a_r(r_0) b_{rr}(r_0)\xi^3 - a_r(r_0)\xi U_2 + \tfrac{1}{2} a_{rr}(r_0) b_r(r_0)\xi^3 - \tfrac{1}{2} a_{rr}(r_0)\xi^2 U_1 \\
&-\tfrac{1}{2} b_{rr}(r_0)\xi^2 U_1 + 2U_1 U_2 - b_r(r_0)\xi U_2\Big)\varepsilon^2 \\
&+ O\left(\varepsilon^{\frac{8}{3}}\xi(U_1)_\xi + \varepsilon^{\frac{8}{3}}(U_2)_\xi + \varepsilon^{\frac{8}{3}}\xi^4 + \varepsilon^{\frac{10}{3}}\xi^5 + \varepsilon^{\frac{8}{3}}\xi^2 U_2 + \varepsilon^4 \xi^6 + \varepsilon^{\frac{8}{3}}\xi^3 U_1 + \varepsilon^{\frac{10}{3}}\xi^3 U_2 + \varepsilon^{\frac{8}{3}} U_2^2\right).
\end{aligned} \tag{34}$$

The above relation indicates that $U_1,\ U_2$ should satisfy

$$-(U_1)_{\xi\xi}+(U_1-a_r(r_0)\xi)(U_1-b_r(r_0)\xi)=0, \tag{35}$$

$$\begin{gathered}-(U_2)_{\xi\xi}+(2U_1-a_r(r_0)\xi-b_r(r_0)\xi)U_2=\\ \tfrac{N-1}{r_0}(U_1)_\xi-\tfrac{1}{2}a_r(r_0)b_{rr}(r_0)\xi^3-\tfrac{1}{2}a_{rr}(r_0)b_r(r_0)\xi^3+\tfrac{1}{2}a_{rr}(r_0)\xi^2U_1+\tfrac{1}{2}b_{rr}(r_0)\xi^2U_1,\end{gathered} \tag{36}$$

for $\xi\in\mathbb{R}$.

Let $K=K(\varepsilon)$ be any number satisfying

$$K(\varepsilon)\to+\infty,\ \ \varepsilon^{\frac{2}{3}}K(\varepsilon)\to 0\ \text{ as } \varepsilon\to 0. \tag{37}$$

Then

$$\begin{gathered}u_{in}(r_0-K\varepsilon^{\frac{2}{3}})-a(r_0-K\varepsilon^{\frac{2}{3}})=\\ a(r_0)+\varepsilon^{\frac{2}{3}}U_1(-K)+\varepsilon^{\frac{4}{3}}U_2(-K)-a(r_0)+a_r(r_0)K\varepsilon^{\frac{2}{3}}-\tfrac{1}{2}a_{rr}(r_0)K^2\varepsilon^{\frac{4}{3}}+O(K^3\varepsilon^2)=\\ (U_1(-K)+a_r(r_0)K)\,\varepsilon^{\frac{2}{3}}+\left(U_2(-K)-\tfrac{1}{2}a_{rr}(r_0)K^2\right)\varepsilon^{\frac{4}{3}}+O(K^3\varepsilon^2)\ \text{ as } \varepsilon\to 0.\end{gathered} \tag{38}$$

Similarly,

$$(u_{in}-b)\,(r_0+K\varepsilon^{\frac{2}{3}})=(U_1(K)-b_r(r_0)K)\,\varepsilon^{\frac{2}{3}}+\left(U_2(K)-\frac{1}{2}b_{rr}(r_0)K^2\right)\varepsilon^{\frac{4}{3}}+O(K^3\varepsilon^2)\ \text{ as } \varepsilon\to 0. \tag{39}$$

The inner approximate solution u_{in} should match with the outer approximation $\max\{a,\ b\}$ at the points $r_0\pm K\varepsilon^{\frac{2}{3}}$, as $\varepsilon\to 0$. Therefore, in view of (37), (38) and (39), the asymptotic behavior of $U_1,\ U_2$ should be

$$U_1(\xi)-a_r(r_0)\xi\to 0\ \text{ as }\ \xi\to-\infty,\quad U_1(\xi)-b_r(r_0)\xi\to 0\ \text{ as }\ \xi\to+\infty, \tag{40}$$

$$U_2(\xi)-\frac{1}{2}a_{rr}(r_0)\xi^2\to 0\ \text{ as }\ \xi\to-\infty,\quad U_2(\xi)-\frac{1}{2}b_{rr}(r_0)\xi^2\to 0\ \text{ as }\ \xi\to+\infty. \tag{41}$$

In the following proposition and remarks we will show, via the method of upper and lower solutions, the existence of an asymptotically stable solution U_{1+} of (35), (40).

Proposition 3.2. *There exists a solution U_{1+} of (35), (40) satisfying*

$$U_{1+}(\xi)>a_r(r_0)\xi,\ \ \xi\le 0\ \ \textit{and}\ \ U_{1+}(\xi)>b_r(r_0)\xi,\ \ \xi>0. \tag{42}$$

Moreover, there exists a constant $C>0$ such that

$$|U_{1+}(\xi)-\max\{a_r(r_0)\xi,\ b_r(r_0)\xi\}|\le C\,(|\xi|+1)^{-\frac{1}{4}}\,e^{-\frac{2}{3}(b_r(r_0)-a_r(r_0))^{\frac{1}{2}}|\xi|^{\frac{3}{2}}},\quad \xi\in\mathbb{R}. \tag{43}$$

Proof. This has been proven in [13] and [56] (see also [36]). For completeness, we present here a proof that is slightly simpler than the one of [56].

Let

$$\underline{u}=\max\{a_r(r_0)\xi,\ b_r(r_0)\xi\}=\begin{cases}a_r(r_0)\xi, & \xi\le 0,\\ b_r(r_0)\xi, & \xi>0.\end{cases} \tag{44}$$

Then $\underline{u}$ solves (35) for $\xi\neq 0$ and, recalling (2), we have $\underline{u}_\xi(0^-)<\underline{u}_\xi(0^+)$. Hence, it follows that $\underline{u}$ is a weak lower solution of (35), see [5], [49].

In view of (2), there exists a unique continuous $\phi\in L^2(\mathbb{R})$ satisfying

$$\begin{cases}-\phi_{\xi\xi}+(b_r(r_0)-a_r(r_0))\,|\xi|\phi=0, & \xi\neq 0,\\ \phi_\xi(0^-)-\phi_\xi(0^+)=b_r(r_0)-a_r(r_0).\end{cases} \tag{45}$$

Furthermore, the function ϕ is strictly positive, and bounded from above by the right hand side of (43) for some constant $C > 0$ (see [4, pg. 100]). Now let

$$\bar{u} = \underline{u} + \phi, \quad \xi \in \mathbb{R}.$$

Then, via (44), (45), we have that $\bar{u} \in C^2(\mathbb{R})$ (with $\bar{u}_{\xi\xi}(0) = 0$) and

$$-\bar{u}_{\xi\xi} + (\bar{u} - a_r(r_0)\xi)(\bar{u} - b_r(r_0)\xi) = -\phi_{\xi\xi} + \phi^2 + (b_r(r_0) - a_r(r_0))\,|\xi|\phi = \phi^2 > 0, \ \ \xi \in \mathbb{R}.$$

Hence, it follows that $\bar{u}$ is an upper solution of (35) such that $\underline{u}(\xi) < \bar{u}(\xi)$, $\xi \in \mathbb{R}$.

By a well known theorem [5], [49], we infer that there exists a stable solution U_{1+} of (35) such that $\underline{u}(\xi) < U_{1+}(\xi) < \bar{u}(\xi)$, $\xi \in \mathbb{R}$. The assertions of the proposition now follow at once.

The proof of the proposition is complete.

Remark 3.3. *From (35) and (43), it follows that* $(U_{1+} - \max\{a_r(r_0)\xi,\ b_r(r_0)\xi\})_\xi = O\left(e^{-c|\xi|^{\frac{3}{2}}}\right)$ *as* $\xi \to \pm\infty$.

Remark 3.4. *In view of (42), we have*

$$2U_{1+}(\xi) - a_r(r_0)\xi - b_r(r_0)\xi \geq c|\xi| + c, \ \ \xi \in \mathbb{R}, \tag{46}$$

and thus the spectrum of the linearized operator, in $L^2(\mathbb{R})$,

$$M_+(\psi) = -\psi_{\xi\xi} + (2U_{1+}(\xi) - a_r(r_0)\xi - b_r(r_0)\xi)\,\psi,$$

consists of simple positive eigenvalues $\mu_{1+} < \mu_{2+} < \cdots$ *with* $\mu_{i+} \to +\infty$ *as* $i \to +\infty$ *(see [34, Thm. 10.7]).*

In order to show the existence of an unstable solution of (35), (40), we will make use of the following lemma which is of independent interest.

Lemma 3.5. *If* $V \in C^1(\mathbb{R})$ *is even,* $V(0) > 0$, $V_\xi(\xi) > 0$, $\xi > 0$, *and* $\lim_{\xi\to+\infty} V(\xi) = +\infty$, *then there exists a positive solution of*

$$u_{\xi\xi} - 2V(\xi)u + u^2 = 0, \quad \xi \in \mathbb{R}, \tag{47}$$

such that u is even, $u_\xi(\xi) < 0$, $\xi > 0$, *and* $\lim_{\xi\to+\infty} u(\xi) = 0$.

Moreover, the spectrum of the linearized operator, in $L^2(\mathbb{R})$,

$$L(\varphi) = -\varphi_{\xi\xi} + 2\,(V(\xi) - u(\xi))\,\varphi,$$

consists of simple eigenvalues $\lambda_1 < \lambda_2 < \cdots$ *with* $\lambda_i \to +\infty$ *as* $i \to +\infty$, *and* $\lambda_1 < 0 < \lambda_2$.

Proof. Under the assumptions of the lemma, existence of a positive solution of (47) such that $\lim_{\xi\to\pm\infty} u(\xi) = 0$ has been shown by a "mountain pass" type argument in [52] (see Theorem 1.7 and Corollary 1.9 therein). Since V is even and $V_\xi(\xi) > 0$, $\xi > 0$, it follows from the moving plane method [26] that u is even and $u_\xi(\xi) < 0$, $\xi > 0$ (see also Lemma 2.3 in [38]).

Since $V(\xi) - u(\xi) \to +\infty$ as $\xi \to \pm\infty$, the spectrum of L consists of discrete eigenvalues $\lambda_1 < \lambda_2 < \cdots$ with $\lambda_i \to +\infty$ as $i \to +\infty$ (see [34, Thm. 10.7]). Each λ_i, $i \geq 1$, is simple and the corresponding eigenfunction φ_i has exactly $i - 1$ zeros in $(-\infty, +\infty)$ (obviously simple). This fact and the evenness of the potential $2\,(V(\xi) - u(\xi))$ imply that φ_i is even if i is odd, and φ_i is odd if i is even. Note also that $\varphi_i(\xi) \to 0$ super-exponentially as $\xi \to \pm\infty$, and the same holds for u as well. We may assume that $\varphi_i(\xi) > 0$ for sufficiently large $\xi > 0$ and $\|\varphi_i\|_{L^\infty(\mathbb{R})} = 1$, $i \geq 1$.

We have

$$-\,(\varphi_1)_{\xi\xi} + 2\,(V(\xi) - u(\xi))\,\varphi_1 = \lambda_1\varphi_1, \tag{48}$$

and

$$-\,u_{\xi\xi} + 2\,(V(\xi) - u(\xi))\,u = -u^2. \tag{49}$$

Multiplying (48) by u, (49) by φ_1, subtracting and integrating by parts over $(-\infty, +\infty)$, we arrive at

$$\lambda_1 \int_{-\infty}^{+\infty} \varphi_1 u d\xi = -\int_{-\infty}^{+\infty} u^2\varphi_1 d\xi.$$

Recalling that $\varphi_1(\xi),\ u(\xi) > 0,\ \xi \in \mathbb{R}$, we get $\lambda_1 < 0$.

We have

$$-(\varphi_2)_{\xi\xi} + 2\left(V(\xi) - u(\xi)\right)\varphi_2 = \lambda_2\varphi_2, \tag{50}$$

and

$$-w_{\xi\xi} + 2\left(V(\xi) - u(\xi)\right)w = -2V_\xi(\xi)u(\xi), \tag{51}$$

where $w = u_\xi$. Similarly as before, and making use of $w(0) = \varphi_2(0) = 0$, we obtain

$$\lambda_2 \int_0^{+\infty} \varphi_2 w d\xi = -2\int_0^{+\infty} V_\xi u\varphi_2 d\xi.$$

Recalling that $\varphi_2(\xi) > 0,\ w(\xi) < 0,\ V_\xi(\xi) > 0,\ u(\xi) > 0,\ \xi > 0$, we get $\lambda_2 > 0$.

The proof of the lemma is complete.

We can now establish the existence of an unstable solution U_{1-} of (35), (40).

Proposition 3.6. *There exists a solution U_{1-} of (35), (40) satisfying*

$$U_{1-}(\xi) < a_r(r_0)\xi,\ \ \xi \le 0\ \ \textit{and}\ \ U_{1-}(\xi) < b_r(r_0)\xi,\ \ \xi > 0. \tag{52}$$

Moreover, the spectrum of the linearized operator, in $L^2(\mathbb{R})$,

$$M_-(\psi) = -\psi_{\xi\xi} + \left(2U_{1-}(\xi) - a_r(r_0)\xi - b_r(r_0)\xi\right)\psi,$$

consists of simple eigenvalues $\mu_{1-} < \mu_{2-} < \cdots$ with $\mu_{i-} \to +\infty$ as $i \to +\infty$, and

$$\mu_{1-} < 0 < \mu_{2-}. \tag{53}$$

Proof. We make the substitution

$$U(\xi) = \left(\frac{b_r(r_0) - a_r(r_0)}{2}\right)^{\frac{2}{3}} v\left(\left(\frac{b_r(r_0) - a_r(r_0)}{2}\right)^{\frac{1}{3}}\xi\right) + \frac{a_r(r_0) + b_r(r_0)}{2}\xi. \tag{54}$$

In terms of v, problem (35), (40) is equivalent to

$$v_{\xi\xi}(\xi) = v^2(\xi) - \xi^2,\ \ \xi \in \mathbb{R}, \tag{55}$$

and

$$v(\xi) - |\xi| \to 0 \quad \text{as} \ \ \xi \to \pm\infty. \tag{56}$$

We can apply Proposition 3.2 (with $a_r(r_0) = -1,\ b_r(r_0) = 1$) to obtain a solution V_+ of (55), (56) such that $V_+(\xi) > |\xi|,\ \xi \in \mathbb{R}$. It is easy to see that V_+ is even and $(V_+)_\xi(\xi) > 0,\ \xi > 0$. (Note that if $\tilde{V}$ solves (55), (56), and $\tilde{V}(\xi) \ge -|\xi|,\ \xi \in \mathbb{R}$, then $\tilde{V} \equiv V_+$).

We search for another solution of (55), (56) in the form

$$V_- = V_+ - u, \quad \text{with}\ \ u(\xi) \to 0\ \ \text{as}\ \ \xi \to \pm\infty,$$

and see that u has to solve (47) with $V = V_+$. We therefore choose u to be the solution given in Lemma 3.5, and find that V_- is even, increasing for $\xi > 0$, and solves (55), (56). Since $V_-(0) < 0$, there exists a unique $\xi_0 > 0$ such that $\xi_0 + V_-(\xi_0) = 0$. In $[0, \xi_0)$, we have $\xi + V_-(\xi) < 0$, and thus $V_-(\xi) < \xi$. The same inequality also holds true in $[\xi_0, +\infty)$. To see this, let $w = \xi - V_-,\ \xi \ge \xi_0$, then $w(\xi_0) = 2\xi_0 > 0$, $\lim_{\xi\to+\infty} w(\xi) = 0$, and $-w_{\xi\xi} + (V_- + \xi)w = 0,\ \xi > \xi_0$. Recalling that $V_-(\xi) + \xi > 0$ in $(\xi_0, +\infty)$, by the maximum principle, we deduce that $w > 0,\ \xi \ge \xi_0$. Hence, by the evenness of V_-, we infer that $V_-(\xi) < |\xi|,\ \ \xi \in \mathbb{R}$.

It is straightforward to verify that U_{1-} given by (54) with $v = V_-$ satisfies the assertions of the proposition, and the proof is complete.

Remark 3.7. *Note that U_{1-} enjoys the same asymptotic behavior as U_{1+} (see Proposition 3.2 and Remark 3.3).*

Remark 3.8. *For notational simplicity, we will sometimes drop the subscripts $+,\ -$.*

Remark 3.9. *Note that the function $2U_1(\xi) - a_r(r_0)\xi - b_r(r_0)\xi$ is even.*

In the sequel we will make use of the following lemma which is a consequence of the maximum principle.

Lemma 3.10. *Suppose that $\psi \in C^2$ satisfies*

$$-\psi_{\xi\xi} + p(\xi)\psi = f(\xi), \quad \psi(\xi) \to 0 \text{ as } \xi \to +\infty, \tag{57}$$

where $p,\ f$ are continuous, and

$$p(\xi) \geq c\xi, \quad |f(\xi)| \leq C\xi^{-\alpha},$$

for large $\xi > 0$, and some positive constants $c,\ C,\ \alpha$.

Then

$$\psi(\xi) = O\left(\xi^{-\alpha-1}\right) \text{ as } \xi \to +\infty.$$

Proof. Let $\bar{\psi} = D\xi^{-\alpha-1}$, $\xi > 0$, with $D > 0$ large to be determined. Then

$$\begin{aligned} -\bar{\psi}_{\xi\xi} + p\bar{\psi} - f &\geq -(\alpha+1)(\alpha+2)D\xi^{-\alpha-3} + cD\xi^{-\alpha} - C\xi^{-\alpha} \\ &= D\xi^{-\alpha-3}\left(-(\alpha+1)(\alpha+2) + (c - CD^{-1})\xi^3\right) > 0, \end{aligned}$$

provided $D > 2c^{-1}C$ and $\xi \geq \xi_1 = \left(2c^{-1}(\alpha+1)(\alpha+2)\right)^{\frac{1}{3}}$. We chose $D > 2c^{-1}C$ such that $|\psi(\xi_1)| < \bar{\psi}(\xi_1)$. The assertion of the lemma now follows readily from the maximum principle, since p is positive (recall also that $\psi \to 0$ as $\xi \to +\infty$ and $-\bar{\psi}$ is a lower solution of (57)). □

In the following proposition, based on the non-degeneracy of $U_{1\pm}$ and Lemma 3.10, we will solve for $U_{2\pm}$ the problems $(36)_\pm$, (41).

Proposition 3.11. *Given $U_1 = U_{1+}$ or $U_1 = U_{1-}$, there exists a unique solution $U_{2+},\ U_{2-}$ of $(36)_\pm$, (41) respectively.*

Moreover, for every $m \in \mathbb{N}$, we have

$$U_{2\pm}(\xi) = \frac{1}{2}a_{rr}(r_0)\xi^2 + \sum_{i=1}^{m} \frac{(3i-3)!}{3^{i-1}(i-1)!} \frac{a_{rr}(r_0) + \frac{N-1}{r_0}a_r(r_0)}{(a_r(r_0) - b_r(r_0))^i}\xi^{2-3i} + O\left(\xi^{-3m-1}\right) \text{ as } \xi \to -\infty,$$

$$U_{2\pm}(\xi) = \frac{1}{2}b_{rr}(r_0)\xi^2 + \sum_{i=1}^{m} \frac{(3i-3)!}{3^{i-1}(i-1)!} \frac{b_{rr}(r_0) + \frac{N-1}{r_0}b_r(r_0)}{(b_r(r_0) - a_r(r_0))^i}\xi^{2-3i} + O\left(\xi^{-3m-1}\right) \text{ as } \xi \to +\infty.$$

Proof. Given $m \in \mathbb{N}$, we define a $\tilde{U}_2 \in C^3(\mathbb{R})$ such that

$$\tilde{U}_2(\xi) = \begin{cases} \frac{1}{2}a_{rr}(r_0)\xi^2 + \sum_{i=1}^{m} \frac{(3i-3)!}{3^{i-1}(i-1)!} \frac{a_{rr}(r_0) + \frac{N-1}{r_0}a_r(r_0)}{(a_r(r_0)-b_r(r_0))^i}\xi^{2-3i}, & \xi \leq -1 \\ \\ \frac{1}{2}b_{rr}(r_0)\xi^2 + \sum_{i=1}^{m} \frac{(3i-3)!}{3^{i-1}(i-1)!} \frac{b_{rr}(r_0) + \frac{N-1}{r_0}b_r(r_0)}{(b_r(r_0)-a_r(r_0))^i}\xi^{2-3i}, & \xi \geq 1. \end{cases}$$

We search for solutions of $(36)_\pm$, (41) in the form

$$U_2 = \tilde{U}_2 + \psi, \quad \psi \in L^2(\mathbb{R}).$$

Recalling the asymptotic behavior of $U_1,\ (U_1)_\xi$, from Proposition 3.2 and Remarks 3.3, 3.7, it is straightforward to see that equation (36) becomes

$$-\psi_{\xi\xi}+(2U_1-a_r(r_0)\xi-b_r(r_0)\xi)\,\psi=f_1(\xi)+f_2(\xi),\quad \xi\in\mathbb{R}, \tag{58}$$

with $f_1,\ f_2\in C^1(\mathbb{R})$ satisfying

$$f_1(\xi)=\begin{cases}\frac{(3m-3)!(2-3m)(1-3m)}{3^{m-1}(m-1)!}\frac{a_{rr}(r_0)+\frac{N-1}{r_0}a_r(r_0)}{(a_r(r_0)-b_r(r_0))^m}\xi^{-3m}, & \xi\le -1,\\ \\ \frac{(3m-3)!(2-3m)(1-3m)}{3^{m-1}(m-1)!}\frac{b_{rr}(r_0)+\frac{N-1}{r_0}b_r(r_0)}{(b_r(r_0)-a_r(r_0))^m}\xi^{-3m}, & \xi\ge 1,\end{cases}$$

and $f_2,\ (f_2)_\xi=O\left(e^{-c|\xi|^{\frac{3}{2}}}\right)$ as $\xi\to\pm\infty$. In view of Remark 3.4 and Proposition 3.6, we know that the linear operators $M_\pm$ appearing in the left hand side of (58) are invertible. Hence, we obtain unique $\psi_+,\ \psi_-\in W^{1,2}(\mathbb{R})$ satisfying $(58)_\pm$ respectively (note that $f_1+f_2\in L^2(\mathbb{R})$). Then, clearly $U_{2\pm}=\tilde{U}_2+\psi_\pm$ solve $(36)_\pm$ and (41) respectively. Finally, using Lemma 3.10, we obtain that $\psi_\pm=O\left(\xi^{-3m-1}\right)$ as $\xi\to\pm\infty$.

The proof of the proposition is complete.

Remark 3.12. *By differentiating (58), and using Lemma 3.10, we find that* $\left(U_{2\pm}-\tilde{U}_2\right)_\xi=O\left(\xi^{-3m-2}\right)$ *as* $\xi\to\pm\infty$.

The properties of the inner solution we have constructed are summarized in

Proposition 3.13. *The inner approximation u_{in}, defined in (32), satisfies*

$$-\varepsilon^2(u_{in})_{rr}-\varepsilon^2\frac{N-1}{r}(u_{in})_r+(u_{in}-a(r))(u_{in}-b(r))=O\left(\varepsilon^{\frac{8}{3}}\right),\quad |r-r_0|\le L\varepsilon^{\frac{2}{3}}, \tag{59}$$

as $\varepsilon\to 0$, where $L>0$ is any fixed constant.

Proof. Relation (59) follows immediately from (34), by recalling (35) and (36).

3.2. The outer solution

Now we will suitably modify $\max\{a,b\}$ and construct outer approximations $u_{out\pm}$, valid for $|r-r_0|\ge L\varepsilon^{\frac{2}{3}}$, that glue continuously with the inner approximations $u_{in\pm}$ at $|r-r_0|=L\varepsilon^{\frac{2}{3}}$, where $L>0$ is a constant independent of $\varepsilon>0$.

3.2.1. The first outer approximation $\tilde{u}_{out}$

Let $L>0$ be a constant to be chosen large, but independent of ε. First we define the outer solution of (31), in $[0,r_0-L\varepsilon^{\frac{2}{3}}]$, as

$$\tilde{u}_{out}=a(r)+\left(\varepsilon^{\frac{2}{3}}U_1\left(\frac{r-r_0}{\varepsilon^{\frac{2}{3}}}\right)-a_r(r_0)(r-r_0)+\varepsilon^{\frac{4}{3}}U_2\left(\frac{r-r_0}{\varepsilon^{\frac{2}{3}}}\right)-\frac{1}{2}a_{rr}(r_0)(r-r_0)^2\right)\zeta(r), \tag{60}$$

$0\le r\le r_0-L\varepsilon^{\frac{2}{3}}$, where $0\le\zeta\le 1$ is a smooth cut-off function such that

$$\zeta(r)=\begin{cases}1, & |r-r_0|\le\delta,\\ \\ 0, & |r-r_0|\ge 2\delta,\end{cases} \tag{61}$$

for some small fixed $\delta>0$ such that $(r_0-10\delta,r_0+10\delta)\subset(0,1)$. Similarly we define $\tilde{u}_{out}$ in $[r_0+L\varepsilon^{\frac{2}{3}},1]$. Notice that, from (3), we have

$$(\tilde{u}_{out})_r(0)=(\tilde{u}_{out})_r(1)=0. \tag{62}$$

The following lemma contains the fundamental estimate regarding $\tilde{u}_{out}$.

Lemma 3.14. *Let*

$$\tilde{E}_{out}(r) = -\varepsilon^2(\tilde{u}_{out})_{rr} - \varepsilon^2\frac{N-1}{r}(\tilde{u}_{out})_r + (\tilde{u}_{out} - a(r))(\tilde{u}_{out} - b(r)), \quad r \in (0, r_0 - L\varepsilon^{\frac{2}{3}}) \cup (r_0 + L\varepsilon^{\frac{2}{3}}, 1).$$

Then

$$\tilde{E}_{out}(r) = \begin{cases} O\left(\varepsilon^{\frac{8}{3}} + \varepsilon^2|r - r_0|\right), & L\varepsilon^{\frac{2}{3}} < |r - r_0| \leq \delta, \\ O\left(\varepsilon^2\right), & |r - r_0| > \delta, \end{cases}$$

as $\varepsilon \to 0$.

Proof. In $(r_0 - \delta, r_0 - L\varepsilon^{\frac{2}{3}})$, by (60), (61), we have

$$-\varepsilon^2(\tilde{u}_{out})_{rr} - \varepsilon^2\tfrac{N-1}{r}(\tilde{u}_{out})_r + (\tilde{u}_{out} - a(r))(\tilde{u}_{out} - b(r)) =$$

$$-\varepsilon^2 a_{rr}(r) - \varepsilon^{\frac{4}{3}}(U_1)_{\xi\xi} - \varepsilon^2(U_2)_{\xi\xi} + \varepsilon^2 a_{rr}(r_0)$$

$$-\varepsilon^2\tfrac{N-1}{r}a_r(r) - \varepsilon^2\tfrac{N-1}{r}(U_1)_\xi + \varepsilon^2\tfrac{N-1}{r}a_r(r_0) - \varepsilon^{\frac{8}{3}}\tfrac{N-1}{r}(U_2)_\xi + \varepsilon^2\tfrac{N-1}{r}a_{rr}(r_0)(r - r_0)$$

$$+\left(\varepsilon^{\frac{2}{3}}U_1 - a_r(r_0)(r - r_0) + \varepsilon^{\frac{4}{3}}U_2 - \tfrac{1}{2}a_{rr}(r_0)(r - r_0)^2\right)\cdot$$
$$\cdot\left(\varepsilon^{\frac{2}{3}}U_1 + \varepsilon^{\frac{4}{3}}U_2 - b_r(r_0)(r - r_0) - \tfrac{1}{2}b_{rr}(r_0)(r - r_0)^2 + O\left((r - r_0)^3\right)\right) =$$

where $U_i,\ (U_i)_\xi,\ (U_i)_{\xi\xi},\ i = 1, 2,$ are evaluated at $\xi = \frac{r - r_0}{\varepsilon^{\frac{2}{3}}}$, and in view of (35), (36),

$$-\varepsilon^2\left(a_{rr}(r) - a_{rr}(r_0)\right) - \varepsilon^2\tfrac{N-1}{r}\left(a_r(r) - a_r(r_0) - \tfrac{(r-r_0)}{r_0}(U_1)_\xi + \varepsilon^{\frac{2}{3}}(U_2)_\xi - a_{rr}(r_0)(r - r_0)\right)$$

$$+\varepsilon^{\frac{8}{3}}\left(U_2^2 - \tfrac{1}{2}a_{rr}(r_0)\xi^2U_2 - \tfrac{1}{2}b_{rr}(r_0)\xi^2U_2 + \tfrac{1}{4}a_{rr}(r_0)b_{rr}(r_0)\xi^4\right)$$

$$+\varepsilon^{\frac{8}{3}}\left(U_1 - a_r(r_0)\xi\right)O\left(\xi^3\right) + \varepsilon^{\frac{10}{3}}\left(U_2 - \tfrac{1}{2}a_{rr}(r_0)\xi^2\right)O\left(\xi^3\right) =$$

$$= O\left(\varepsilon^{\frac{8}{3}} + |r - r_0|\varepsilon^2\right),$$

where we used the estimates of Propositions 3.2, 3.11 and Remarks 3.3, 3.7, 3.12. Hence, the assertion of the lemma holds in $(r_0 - \delta, r_0 - L\varepsilon^{\frac{2}{3}})$. In $(r_0 - 2\delta, r_0 - \delta)$, the previously mentioned estimates imply that

$$|\tilde{u}_{out} - a(r)| + |(\tilde{u}_{out} - a(r))_r| + |(\tilde{u}_{out} - a(r))_{rr}| = O\left(\varepsilon^2\right),$$

and in $(0, r_0 - 2\delta)$ we have $\tilde{u}_{out} = a$. Thus, the assertion of the lemma holds in $(0, r_0 - \delta)$ as well (recall (3)). Identical calculations also apply in $(r_0 + L\varepsilon^{\frac{2}{3}}, 1)$.

The proof of the lemma is complete.

3.2.2. The refined outer approximation u_{out}

Motivated from [62], we now define the outer solution of (31), in $[0, r_0 - L\varepsilon^{\frac{2}{3}}]$, as

$$u_{out} = \tilde{u}_{out} + \sigma, \tag{63}$$

where σ solves

$$\begin{cases} -\varepsilon^2\sigma_{rr} - \varepsilon^2\tfrac{N-1}{r}\sigma_r + (2\tilde{u}_{out} - a(r) - b(r))\,\sigma = -\tilde{E}_{out}(r), & r \in (0, r_0 - L\varepsilon^{\frac{2}{3}}), \\ \sigma_r(0) = 0, \quad \sigma(r_0 - L\varepsilon^{\frac{2}{3}}) = u_{in}(r_0 - L\varepsilon^{\frac{2}{3}}) - \tilde{u}_{out}(r_0 - L\varepsilon^{\frac{2}{3}}), \end{cases} \tag{64}$$

($\tilde{E}_{out}$ is as in Lemma 3.14). Similarly we define u_{out} in $[r_0 + L\varepsilon^{\frac{2}{3}}, 1]$. It is useful to note at this point that u_{out} is determined from $\tilde{u}_{out}$ by one step of Newton's iteration applied to (31).

Existence and estimates for σ are provided by the following lemma.

Lemma 3.15. *If $\varepsilon > 0$ is sufficiently small, there exists a unique solution of (64). Moreover,*

$$\sigma(r) = O\left(\varepsilon^2\right), \quad 0 \le r \le r_0 - L\varepsilon^{\frac{2}{3}}, \tag{65}$$

and

$$\sigma_r(r_0 - L\varepsilon^{\frac{2}{3}}) = O\left(\varepsilon^{\frac{4}{3}}\right) \quad \text{as } \varepsilon \to 0. \tag{66}$$

Analogous estimates also hold for σ in $[r_0 + L\varepsilon^{\frac{2}{3}}, 1]$.

Proof. Note that, thanks to (2) and (40), we can choose an $L > 0$ such that

$$U_1(\xi) \ge \max\{a_r(r_0)\xi,\ b_r(r_0)\xi\} - \frac{b_r(r_0) - a_r(r_0)}{4}|\xi|, \quad |\xi| \ge L.$$

Then

$$2U_1(\xi) - a_r(r_0)\xi - b_r(r_0)\xi \ge \frac{b_r(r_0) - a_r(r_0)}{2}|\xi|, \quad |\xi| \ge L, \tag{67}$$

(in the case where $U_1 = U_{1+}$, by (46), we have a stronger estimate).

If $0 \le r \le r_0 - L\varepsilon^{\frac{2}{3}}$, we have

$$\begin{aligned} 2\tilde{u}_{out}(r) - a(r) - b(r) \quad &= a(r) - b(r) + 2\varepsilon^{\frac{2}{3}}\left(U_1\left(\tfrac{r-r_0}{\varepsilon^{\frac{2}{3}}}\right) - a_r(r_0)\tfrac{r-r_0}{\varepsilon^{\frac{2}{3}}}\right)\zeta(r) \\ &+2\varepsilon^{\frac{4}{3}}\left(U_2\left(\tfrac{r-r_0}{\varepsilon^{\frac{2}{3}}}\right) - \tfrac{1}{2}a_{rr}(r_0)\left(\tfrac{r-r_0}{\varepsilon^{\frac{2}{3}}}\right)^2\right)\zeta(r) \\ &\ge c|r-r_0| - C\varepsilon^{\frac{2}{3}}e^{-cL^{\frac{3}{2}}} - C\varepsilon^{\frac{4}{3}} \ge \tfrac{c}{2}\left(|r-r_0| + \varepsilon^{\frac{2}{3}}\right), \end{aligned} \tag{68}$$

provided $\varepsilon > 0$ is sufficiently small, where we used (2), (41), (43), Remark 3.7, and possibly increased L (independently of ε). From now on we fix such an $L > 0$.

Hence, the linear elliptic boundary value problem

$$\begin{cases} -\varepsilon^2\Delta\sigma + (2\tilde{u}_{out} - a - b)\,\sigma = -\tilde{E}_{out}, & |x| < r_0 - L\varepsilon^{\frac{2}{3}}, \\ \sigma = u_{in}(r_0 - L\varepsilon^{\frac{2}{3}}) - \tilde{u}_{out}(r_0 - L\varepsilon^{\frac{2}{3}}), & |x| = r_0 - L\varepsilon^{\frac{2}{3}}, \end{cases} \tag{69}$$

where $\tilde{u}_{out} = \tilde{u}_{out}(|x|)$, $a = a(|x|)$, $b = b(|x|)$, $\tilde{E}_{out} = \tilde{E}_{out}(|x|)$, has a unique solution $\sigma = \sigma(x)$. This solution is radially symmetric, i.e., $\sigma = \sigma(|x|)$ (otherwise (69) would have infinitely many different solutions through rotations around the origin). Furthermore, equation (69) implies that $\Delta(\tilde{u}_{out} + \sigma) \in C^\alpha(|x| \le r_0 - L\varepsilon^{\frac{2}{3}})$, for some $0 < \alpha < 1$, and thus $\tilde{u}_{out} + \sigma \in C^{2+\alpha}(|x| \le r_0 - L\varepsilon^{\frac{2}{3}})$ (see [27]). Then, identifying $\sigma(r)$ with $\sigma(|x|)$, it is easy to see that $\sigma \in C^2[0, r_0 - L\varepsilon^{\frac{2}{3}}]$ and solves (64).

Let x_ε, with $|x_\varepsilon| = r_\varepsilon \le r_0 - L\varepsilon^{\frac{2}{3}}$, be such that

$$\sigma(x_\varepsilon) = \max_{|x| \le r_0 - L\varepsilon^{\frac{2}{3}}} \sigma.$$

Without loss of generality, we may assume that $\sigma(x_\varepsilon) > 0$. Three possibilities can occur:

1. If $|x_\varepsilon| = r_0 - L\varepsilon^{\frac{2}{3}}$, from (32), (60), (61) and (64), we have

$$\sigma(x_\varepsilon) = -\left(a(r_0 - L\varepsilon^{\frac{2}{3}}) - a(r_0) + a_r(r_0)L\varepsilon^{\frac{2}{3}} - \frac{1}{2}a_{rr}(r_0)L^2\varepsilon^{\frac{4}{3}}\right) = O\left(\varepsilon^2\right).$$

2. If $r_0 - \delta \le |x_\varepsilon| < r_0 - L\varepsilon^{\frac{2}{3}}$, we have $\Delta\sigma(x_\varepsilon) \le 0$ and, from Lemma 3.14 together with (68), (69), we obtain that

$$c\left(|r_\varepsilon - r_0| + \varepsilon^{\frac{2}{3}}\right)\sigma(x_\varepsilon) \le C\varepsilon^2\left(\varepsilon^{\frac{2}{3}} + |r_\varepsilon - r_0|\right),$$

i.e., $\sigma(x_\varepsilon) = O\left(\varepsilon^2\right)$.

3. If $|x_\varepsilon| \le r_0 - \delta$, we have $\Delta\sigma(x_\varepsilon) \le 0$ and, via Lemma 3.14 together with (68), (69), we arrive at

$$c\sigma(x_\varepsilon) \le C\varepsilon^2.$$

Similarly we can show that $\min_{|x|\le r_0 - L\varepsilon^{\frac{2}{3}}} \sigma = O\left(\varepsilon^2\right)$, and (65) follows.

In $\left(r_0 - 2L\varepsilon^{\frac{2}{3}},\ r_0 - L\varepsilon^{\frac{2}{3}}\right)$, we have $2\tilde{u}_{out} - a - b = O\left(\varepsilon^{\frac{2}{3}}\right)$ (see (68)) and, via Lemma 3.14 and (65), equation (64) can be written as

$$-\varepsilon^2\left(r^{N-1}\sigma_r\right)_r + O\left(\varepsilon^{\frac{2}{3}}\varepsilon^2\right) = O\left(\varepsilon^{\frac{8}{3}}\right),\ \text{i.e.,}\ \left(r^{N-1}\sigma_r\right)_r = O\left(\varepsilon^{\frac{2}{3}}\right).$$

So,

$$\int_{r_0-2L\varepsilon^{\frac{2}{3}}}^{r_0-L\varepsilon^{\frac{2}{3}}} \left(r - r_0 + 2L\varepsilon^{\frac{2}{3}}\right)\left(r^{N-1}\sigma_r\right)_r dr = O\left(\varepsilon^2\right),$$

and an integration by parts yields

$$-\int_{r_0-2L\varepsilon^{\frac{2}{3}}}^{r_0-L\varepsilon^{\frac{2}{3}}} r^{N-1}\sigma_r dr + L\varepsilon^{\frac{2}{3}}(r_0 - L\varepsilon^{\frac{2}{3}})^{N-1}\sigma_r(r_0 - L\varepsilon^{\frac{2}{3}}) = O\left(\varepsilon^2\right).$$

Integrating by parts one more time, we find that

$$\int_{r_0-2L\varepsilon^{\frac{2}{3}}}^{r_0-L\varepsilon^{\frac{2}{3}}} (N-1)r^{N-2}\sigma dr + O\left(\|\sigma\|_{L^\infty}\right) + L\varepsilon^{\frac{2}{3}}(r_0 - L\varepsilon^{\frac{2}{3}})^{N-1}\sigma_r(r_0 - L\varepsilon^{\frac{2}{3}}) = O\left(\varepsilon^2\right),$$

and by using again (65), we obtain relation (66).

Identical calculations also hold for σ in $[r_0 + L\varepsilon^{\frac{2}{3}}, 1]$, and the proof of the lemma is complete.

The refined outer solution we have constructed satisfies the following proposition.

Proposition 3.16. *The outer approximation u_{out}, defined in (63), satisfies*

$$-\varepsilon^2(u_{out})_{rr} - \varepsilon^2\frac{N-1}{r}(u_{out})_r + (u_{out} - a(r))(u_{out} - b(r)) = O\left(\varepsilon^4\right), \quad r \in (0, r_0 - L\varepsilon^{\frac{2}{3}}) \cup (r_0 + L\varepsilon^{\frac{2}{3}}, 1), \tag{70}$$

$$(u_{out})_r(0) = (u_{out})_r(1) = 0, \tag{71}$$

$$(u_{out} - u_{in})(r_0 \pm L\varepsilon^{\frac{2}{3}}) = 0, \ \ (u_{out} - u_{in})_r(r_0 \pm L\varepsilon^{\frac{2}{3}}) = O\left(\varepsilon^{\frac{4}{3}}\right), \tag{72}$$

and

$$u_{out}(r) - u_{in}(r) = O\left((r - r_0)^3 + \varepsilon^2\right) \ \ in \ \ (r_0 - \delta, r_0 - L\varepsilon^{\frac{2}{3}}) \cup (r_0 + L\varepsilon^{\frac{2}{3}}, r_0 + \delta), \tag{73}$$

as $\varepsilon \to 0$.

Proof. By (63), (64), and their analogs in $(r_0 + L\varepsilon^{\frac{2}{3}}, 1)$, we derive that

$$-\varepsilon^2(u_{out})_{rr} - \varepsilon^2\frac{N-1}{r}(u_{out})_r + (u_{out} - a(r))(u_{out} - b(r)) = \sigma^2,$$

$r \in (0, r_0 - L\varepsilon^{\frac{2}{3}}) \cup (r_0 + L\varepsilon^{\frac{2}{3}}, 1)$, and (70) follows from the first assertion of Lemma 3.15. Relation (71) is a direct consequence of (62) and the definition of σ (recall (64)). In $[r_0 - \delta, r_0 - L\varepsilon^{\frac{2}{3}}]$, by (32), (60), (61), we have

$$(u_{out} - u_{in})(r) = a(r) - a(r_0) - a_r(r_0)(r - r_0) - \frac{1}{2}a_{rr}(r_0)(r - r_0)^2 + \sigma(r),$$

(a similar relation also holds in $[r_0 + L\varepsilon^{\frac{2}{3}}, r_0 + \delta]$), and now (72), (73) follow readily from the definition of σ (recall (64)) and Lemma 3.15.

The proof of the proposition is complete.

Remark 3.17. *If we had not assumed that $b_r(1) = 0$, in (3), then we simply replace b, in the analog of (60), by $b_1 = b + \zeta(1 - r + r_0)\beta$, where β solves*

$$\begin{cases} -\varepsilon^2\beta_{rr} - \varepsilon^2\frac{N-1}{r}\beta_r + (b(r) - a(r))\beta = 0, & r \in (1-3\delta, 1), \\ \beta(1-3\delta) = 0, \quad \beta_r(1) = -b_r(1), & \end{cases}$$

and ζ is as in (61). Since $b(r) - a(r) \geq c$, $r \in [1-3\delta, 1]$, it follows that $|\beta(r)| \leq C\varepsilon \exp\left(c\frac{r-1}{\varepsilon}\right)$, $r \in [1-3\delta, 1]$. The addition of this boundary layer correction to b does not affect our proofs at all, but note that the bound in (121), below, would become $O(\varepsilon)$ if $1 - \frac{2}{c}|\ln\varepsilon|\varepsilon \leq r \leq 1$ as $\varepsilon \to 0$.

3.3. The gluing procedure

Up to this point, we have constructed inner and outer approximations for (31) that glue continuously at $|r - r_0| = L\varepsilon^{\frac{2}{3}}$. Now, with the addition of a suitable correction, we will glue them in a C^1, piecewise C^2 manner, and construct global approximate solutions $u_{ap\pm}$ that are valid in the whole domain.

3.3.1. The continuous approximation $\tilde{u}_{ap}$

First we define the approximate solution of (31) as

$$\tilde{u}_{ap} = \begin{cases} u_{out}, & r \in [0, r_0 - L\varepsilon^{\frac{2}{3}}] \cup [r_0 + L\varepsilon^{\frac{2}{3}}, 1], \\ u_{in}, & r \in (r_0 - L\varepsilon^{\frac{2}{3}}, r_0 + L\varepsilon^{\frac{2}{3}}). \end{cases} \tag{74}$$

In view of (72), we know that $\tilde{u}_{ap} \in C([0,1]) \cap C^2\left([0,1] - \{r_0 \pm L\varepsilon^{\frac{2}{3}}\}\right)$, and the jump discontinuities of $\left(\tilde{u}_{ap}\right)_r$ at $r_0 \pm L\varepsilon^{\frac{2}{3}}$ satisfy

$$(\tilde{u}_{ap})_r\left((r_0 \pm L\varepsilon^{\frac{2}{3}})^-\right) - (\tilde{u}_{ap})_r\left((r_0 \pm L\varepsilon^{\frac{2}{3}})^+\right) = O\left(\varepsilon^{\frac{4}{3}}\right). \tag{75}$$

3.3.2. Balancing the jumps of $(\tilde{u}_{ap})_r$ at $|r - r_0| = L\varepsilon^{\frac{2}{3}}$

Our next task is to construct a small function with the property that, when added to $\tilde{u}_{ap}$, it balances the jump discontinuities of $(\tilde{u}_{ap})_r$ at $r = r_0 \pm L\varepsilon^{\frac{2}{3}}$ while preserving the remainder that $\tilde{u}_{ap}$ leaves in (31) for $r \neq r_0 \pm L\varepsilon^{\frac{2}{3}}$ (recall (59) and (70)).

From (2), (32), (65), and (68), it follows that

$$\begin{cases} 2\tilde{u}_{ap} - a - b \geq c\varepsilon^{\frac{2}{3}}, & r \in [0, r_0 - L\varepsilon^{\frac{2}{3}}] \cup [r_0 + L\varepsilon^{\frac{2}{3}}, 1], \\ 2\tilde{u}_{ap} - a - b = O\left(\varepsilon^{\frac{2}{3}}\right), & r \in (r_0 - L\varepsilon^{\frac{2}{3}}, r_0 + L\varepsilon^{\frac{2}{3}}). \end{cases} \tag{76}$$

Remark 3.18. *In the case where $U_1 = U_{1+}$, the first relation of (76) holds for every $r \in [0,1]$ (recall (46)). On the other hand, in the case where $U_1 = U_{1-}$, we have $2\tilde{u}_{ap} - a - b \leq -c\varepsilon^{\frac{2}{3}}$ for some $r \in (r_0 - L\varepsilon^{\frac{2}{3}}, r_0 + L\varepsilon^{\frac{2}{3}})$.*

Let

$$q = \begin{cases} 2\tilde{u}_{ap} - a - b, & r \in [0, r_0 - L\varepsilon^{\frac{2}{3}}] \cup [r_0 + L\varepsilon^{\frac{2}{3}}, 1], \\ \frac{q(r_0 + L\varepsilon^{\frac{2}{3}}) - q(r_0 - L\varepsilon^{\frac{2}{3}})}{2L\varepsilon^{\frac{2}{3}}}(r - r_0 + L\varepsilon^{\frac{2}{3}}) + q(r_0 - L\varepsilon^{\frac{2}{3}}), & r \in (r_0 - L\varepsilon^{\frac{2}{3}}, r_0 + L\varepsilon^{\frac{2}{3}}). \end{cases} \tag{77}$$

Then $q \in C([0,1])$ and, by (76),

$$q(r) \geq c\varepsilon^{\frac{2}{3}}, \quad r \in [0,1]. \tag{78}$$

Relations (77) and (78) suggest the following lemma.

Lemma 3.19. *If $\varepsilon > 0$ is sufficiently small, there exists a unique $\rho \in C([r_0 - 3\delta, r_0 + 3\delta]) \cap C^1\left((r_0 - 3\delta, r_0 + 3\delta) - \{r_0 - L\varepsilon^{\frac{2}{3}}\}\right)$ such that*

$$-\varepsilon^2 \rho_{rr} - \varepsilon^2 \frac{N-1}{r}\rho_r + q\rho = 0 \quad in \ (r_0 - 3\delta, r_0 + 3\delta) - \{r_0 - L\varepsilon^{\frac{2}{3}}\}, \tag{79}$$

$$\rho(r_0 - 3\delta) = 0, \ \ \rho(r_0 - L\varepsilon^{\frac{2}{3}}) = \varepsilon^2, \ \ \rho(r_0 + 3\delta) = 0. \tag{80}$$

Moreover, for some numbers $c, C > 0$,

$$0 < \rho(r) \leq C\varepsilon^2 \exp\left(-c\frac{|r - r_0 + L\varepsilon^{\frac{2}{3}}|}{\varepsilon^{\frac{2}{3}}}\right), \ \ r \in (r_0 - 3\delta, r_0 + 3\delta), \tag{81}$$

and the jump discontinuity of ρ_r at $r_0 - L\varepsilon^{\frac{2}{3}}$ satisfies

$$c\varepsilon^{\frac{4}{3}} \leq \rho_r\left((r_0 - L\varepsilon^{\frac{2}{3}})^-\right) - \rho_r\left((r_0 - L\varepsilon^{\frac{2}{3}})^+\right) \leq C\varepsilon^{\frac{4}{3}}. \tag{82}$$

Proof. Existence and uniqueness follow readily from (78). The fact that $\rho > 0$ in $(r_0 - 3\delta, r_0 + 3\delta)$ is a consequence of the maximum principle. The upper bound in (81) follows from Lemma 3.3 in [50], see also [22, p. 230].

To show (82) we will use a re-scaling argument. Let

$$\tilde{\rho}(s) = \varepsilon^{-2}\rho(r_0 - L\varepsilon^{\frac{2}{3}} + \varepsilon^{\frac{2}{3}} s), \ \ L - \frac{3\delta}{\varepsilon^{\frac{2}{3}}} \leq s \leq 0.$$

Then

$$\begin{cases} -\tilde{\rho}_{ss} - \varepsilon^{\frac{2}{3}} \frac{N-1}{r_0 - L\varepsilon^{\frac{2}{3}} + \varepsilon^{\frac{2}{3}} s}\tilde{\rho}_s + \varepsilon^{-\frac{2}{3}} q(r_0 - L\varepsilon^{\frac{2}{3}} + \varepsilon^{\frac{2}{3}} s)\tilde{\rho} = 0, \quad L - \frac{3\delta}{\varepsilon^{\frac{2}{3}}} < s < 0, \\ \tilde{\rho}(L - \frac{3\delta}{\varepsilon^{\frac{2}{3}}}) = 0, \ \ \tilde{\rho}(0) = 1, \ \ \text{and} \ \ 0 < \tilde{\rho}(s) \leq Ce^{cs}, \ \ L - \frac{3\delta}{\varepsilon^{\frac{2}{3}}} < s \leq 0. \end{cases} \tag{83}$$

In view of (60) and (65), it is straightforward to verify that

$$\varepsilon^{-\frac{2}{3}} q(r_0 - L\varepsilon^{\frac{2}{3}} + \varepsilon^{\frac{2}{3}} s) \to 2U_1(s - L) - a_r(r_0)(s - L) - b_r(r_0)(s - L) \ \text{ as } \varepsilon \to 0, \tag{84}$$

uniformly in compact subsets of $(-\infty, 0]$. Therefore, applying standard interior and boundary elliptic estimates (see [27]) to (83), we can extract a subsequence $\varepsilon_n \to 0, \ n \to +\infty$, such that $\tilde{\rho}_{\varepsilon_n} \to \tilde{\rho}_0$ as $n \to +\infty$ in $C^1_{loc}((-\infty, 0])$. From (83) and (84), we find that $\tilde{\rho}_0$ satisfies

$$\begin{cases} -(\tilde{\rho}_0)_{ss} + (2U_1(s - L) - a_r(r_0)(s - L) - b_r(r_0)(s - L))\tilde{\rho}_0 = 0, \ \ s < 0, \\ \tilde{\rho}_0(0) = 1, \ \ \text{and} \ \ 0 < \tilde{\rho}_0(s) \leq Ce^{cs}, \ \ s \leq 0. \end{cases} \tag{85}$$

By the uniqueness of the limiting function (recall (67)), we deduce that

$$\tilde{\rho}_\varepsilon \to \tilde{\rho}_0 \ \text{ in } \ C^1_{loc}((-\infty, 0]) \ \text{ as } \ \varepsilon \to 0.$$

In particular, we have that $(\tilde{\rho}_\varepsilon)_s(0^-) - (\tilde{\rho}_0)_s(0^-) = o(1)$ as $\varepsilon \to 0$, i.e.,

$$\rho_r\left((r_0 - L\varepsilon^{\frac{2}{3}})^-\right) = (\tilde{\rho}_0)_s(0^-)\varepsilon^{\frac{4}{3}} + o\left(\varepsilon^{\frac{4}{3}}\right) = c\varepsilon^{\frac{4}{3}} + o\left(\varepsilon^{\frac{4}{3}}\right) \ \text{ as } \varepsilon \to 0, \tag{86}$$

with $c = (\tilde{\rho}_0)_s(0^-) > 0$. (From (67), (85), we see that $(\tilde{\rho}_0)_{ss} > 0, \ s < 0, (\tilde{\rho}_0)_s \to 0$ as $s \to -\infty$, and it follows that $(\tilde{\rho}_0)_s > 0, \ s \leq 0$). Similarly we can show that

$$\rho_r\left((r_0 - L\varepsilon^{\frac{2}{3}})^+\right) = -c\varepsilon^{\frac{4}{3}} + o\left(\varepsilon^{\frac{4}{3}}\right) \ \text{ as } \varepsilon \to 0, \tag{87}$$

for some $c > 0$. (We have only to note that

$$\varepsilon^{-\frac{2}{3}} q(r_0 - L\varepsilon^{\frac{2}{3}} + \varepsilon^{\frac{2}{3}} s) \to q_0(s) \ \text{ in } C_{loc}([0, +\infty)) \ \text{ as } \ \varepsilon \to 0,$$

with $q_0(s) > 0, \ s \geq 0$). Relation (82) now follows immediately from (86) and (87).

The proof of the lemma is complete.

Similarly we have

Lemma 3.20. *If $\varepsilon > 0$ is sufficiently small, there exists a unique $\varrho \in C([r_0-3\delta, r_0+3\delta]) \cap C^1\left((r_0-3\delta, r_0+3\delta) - \{r_0+L\varepsilon^{\frac{2}{3}}\}\right)$ such that*

$$-\varepsilon^2\varrho_{rr} - \varepsilon^2\frac{N-1}{r}\varrho_r + q\varrho = 0 \quad in \ (r_0-3\delta, r_0+3\delta) - \{r_0+L\varepsilon^{\frac{2}{3}}\}, \tag{88}$$

$$\varrho(r_0-3\delta) = 0, \ \ \varrho(r_0+L\varepsilon^{\frac{2}{3}}) = \varepsilon^2, \ \ \varrho(r_0+3\delta) = 0. \tag{89}$$

Moreover, for some numbers $c, C > 0$,

$$0 < \varrho(r) \le C\varepsilon^2 \exp\left(-c\frac{|r-r_0-L\varepsilon^{\frac{2}{3}}|}{\varepsilon^{\frac{2}{3}}}\right), \ \ r \in (r_0-3\delta, r_0+3\delta), \tag{90}$$

and the jump discontinuity of ϱ_r at $r_0+L\varepsilon^{\frac{2}{3}}$ satisfies

$$c\varepsilon^{\frac{4}{3}} \le \varrho_r\left((r_0+L\varepsilon^{\frac{2}{3}})^-\right) - \varrho_r\left((r_0+L\varepsilon^{\frac{2}{3}})^+\right) \le C\varepsilon^{\frac{4}{3}}. \tag{91}$$

Remark 3.21. *Ideally we would like ρ to solve the distributional equation*

$$-\varepsilon^2\Delta\rho + \left(2\tilde{u}_{ap} - a - b\right)\rho = \varepsilon^{\frac{10}{3}}\delta_{\{|x|=r_0-L\varepsilon^{\frac{2}{3}}\}} \ \ in \ \mathbb{R}^N,$$

(similarly for ϱ). However, in the case where $U_1 = U_{1-}$, it is not obvious to us how to establish existence and estimates for the above equation, as the potential of the Schrödinger operator in the left-hand side takes some negative values (recall Remark 3.18). A possible approach could make use of the non-degeneracy of the linear operator M_-, defined in Proposition 3.6, and re-scaling arguments as in Subsection 3.4. Estimates for the fundamental solution of a class of one-dimensional Schrödinger operators with nonnegative potentials, vanishing at some points, have been obtained recently in [24].

Let

$$\omega = (A\rho + B\varrho)\,\zeta, \quad r \in [0,1], \tag{92}$$

where

$$A = \frac{(u_{in})_r(r_0-L\varepsilon^{\frac{2}{3}}) - (u_{out})_r(r_0-L\varepsilon^{\frac{2}{3}})}{\rho_r\left((r_0-L\varepsilon^{\frac{2}{3}})^-\right) - \rho_r\left((r_0-L\varepsilon^{\frac{2}{3}})^+\right)}, \ \ B = \frac{(u_{out})_r(r_0+L\varepsilon^{\frac{2}{3}}) - (u_{in})_r(r_0+L\varepsilon^{\frac{2}{3}})}{\varrho_r\left((r_0+L\varepsilon^{\frac{2}{3}})^-\right) - \varrho_r\left((r_0+L\varepsilon^{\frac{2}{3}})^+\right)}, \tag{93}$$

and ζ was defined (61). Note that $\omega \in C([0,1]) \cap C^1\left([0,1] - \{r_0 \pm L\varepsilon^{\frac{2}{3}}\}\right)$.

3.3.3. The C^1, piecewise C^2, approximation u_{ap}

Let

$$u_{ap} = \tilde{u}_{ap} + \omega \in C^1([0,1]) \cap C^2\left([0,1] - \{r_0 \pm L\varepsilon^{\frac{2}{3}}\}\right), \tag{94}$$

with $(u_{ap})_{rr}$ having finite jump discontinuities at $|r-r_0| = L\varepsilon^{\frac{2}{3}}$ (recall (74), (92), (93)).

From (72), (82) and (91), we see that

$$|A| + |B| \le C.$$

Hence, by (61), (81), (90), we easily deduce that

$$\omega = O\left(\varepsilon^2\right), \ r \in [0,1]; \quad \omega = 0, \ r \in (0, r_0-2\delta) \cup (r_0+2\delta, 1), \tag{95}$$

and, via equations (79), (88),

$$|\omega| + |\omega_r| + |\omega_{rr}| \le C\exp\left(-\frac{c}{\varepsilon^{\frac{2}{3}}}\right), \ r \in (r_0-2\delta, r_0-\delta) \cup (r_0+\delta, r_0+2\delta). \tag{96}$$

Note also that, by equations (79) and (88), we have

$$-\varepsilon^2\omega_{rr} - \varepsilon^2\frac{N-1}{r}\omega_r + q\omega = 0 \quad \text{in} \ \ (r_0-\delta, r_0+\delta) - \{r_0 \pm L\varepsilon^{\frac{2}{3}}\}. \tag{97}$$

Everything we have done so far has led us to the following proposition.

Proposition 3.22. *The approximate solution u_{ap}, defined in (94), satisfies*

$$-\varepsilon^2(u_{ap})_{rr} - \varepsilon^2\frac{N-1}{r}(u_{ap})_r + (u_{ap} - a(r))(u_{ap} - b(r)) = O\left(\varepsilon^{\frac{8}{3}}\right), \quad r \in (0,1) - \{r_0 \pm L\varepsilon^{\frac{2}{3}}\}, \tag{98}$$

$$(u_{ap})_r(0) = (u_{ap})_r(1) = 0, \tag{99}$$

and

$$u_{ap}(r) - u_{in}(r) = O\left((r - r_0)^3 + \varepsilon^2\right), \quad r \in (r_0 - \delta, r_0 + \delta), \tag{100}$$

as $\varepsilon \to 0$.

Proof. In $(r_0 - \delta, r_0 + \delta) - \{r_0 \pm L\varepsilon^{\frac{2}{3}}\}$,

$$-\varepsilon^2(u_{ap})_{rr} - \varepsilon^2\tfrac{N-1}{r}(u_{ap})_r + (u_{ap} - a(r))(u_{ap} - b(r)) =$$

$$-\varepsilon^2(\tilde{u}_{ap})_{rr} - \varepsilon^2\tfrac{N-1}{r}(\tilde{u}_{ap})_r + (\tilde{u}_{ap} - a(r))(\tilde{u}_{ap} - b(r))$$

$$-\varepsilon^2\omega_{rr} - \varepsilon^2\tfrac{N-1}{r}\omega_r + \left(2\tilde{u}_{ap} - a(r) - b(r)\right)\omega + \omega^2$$

and by (59), (70), (74), (95), (97),

$$= O\left(\varepsilon^{\frac{8}{3}}\right) + \left(2\tilde{u}_{ap} - a(r) - b(r) - q(r)\right)\omega = O\left(\varepsilon^{\frac{8}{3}}\right) + O\left(\varepsilon^{\frac{2}{3}}\varepsilon^2\right),$$

as $\varepsilon \to 0$, where we used (76), (77) and (95). Thus, relation (98) is valid in $(r_0 - \delta, r_0 + \delta) - \{r_0 \pm L\varepsilon^{\frac{2}{3}}\}$. In $(0, r_0 - \delta) \cup (r_0 + \delta, 1)$, relation (98) follows readily from (70), (95), and (96). In view of (71) and (95), we find that (99) holds. Relation (100) is a direct consequence of (73) and (95).

The proof of the proposition is complete.

3.4. Linear theory for the radial problem

Now we will study the linearization of (31) near the approximate solutions $u_{ap\pm}$.

3.4.1. The linear operator $\mathbb{L}$

Throughout this subsection we will consider the linear operator

$$\mathbb{L}(\varphi) = -\varepsilon^2\Delta\varphi + Q(|x|)\varphi, \quad D(\mathbb{L}) = \{\varphi \in W_r^{2,2}(B_1) \ : \ \partial_\nu\varphi = 0 \text{ on } \partial B_1\}, \tag{101}$$

where

$$Q = 2u_{ap} - a - b + e \ \text{ with } \|e\|_{C_r(\bar{B}_1)} = o(1)\varepsilon^{\frac{4}{3}} \ \text{ as } \ \varepsilon \to 0, \quad (e \text{ otherwise arbitrary}). \tag{102}$$

The linear operator $\mathbb{L}$ is self-adjoint in $L_r^2(B_1)$. It is easy to see, from (65), (68), (76), (95) and (102), that

$$Q(r) \geq c\left(|r - r_0| + \varepsilon^{\frac{2}{3}}\right) \ \text{ in } [0, r_0 - L\varepsilon^{\frac{2}{3}}] \cup [r_0 + L\varepsilon^{\frac{2}{3}}, 1], \tag{103}$$

$$Q(r) \geq -C\varepsilon^{\frac{2}{3}} \ \text{ in } [0,1], \tag{104}$$

if $\varepsilon > 0$ is sufficiently small. Moreover, letting

$$\tilde{Q}(\xi) = \varepsilon^{-\frac{2}{3}}Q\left(r_0 + \varepsilon^{\frac{2}{3}}\xi\right), \quad \xi \in \left(-\frac{r_0}{\varepsilon^{\frac{2}{3}}}, \frac{1 - r_0}{\varepsilon^{\frac{2}{3}}}\right), \tag{105}$$

we find, via (100), that

$$\tilde{Q}(\xi) = 2U_1(\xi) - a_r(r_0)\xi - b_r(r_0)\xi + O\left((\xi^2 + 1)\varepsilon^{\frac{2}{3}}\right), \quad \xi \in \left(-\frac{\delta}{\varepsilon^{\frac{2}{3}}}, \frac{\delta}{\varepsilon^{\frac{2}{3}}}\right) \ \text{ as } \ \varepsilon \to 0. \tag{106}$$

3.4.2. A-priori estimates for the equation $\mathbb{L}(\varphi) = f$

The a-priori estimates, in the uniform norm, of the following proposition will be crucially used later on for showing the existence of solutions of (31), uniformly close to the approximations $u_{ap\pm}$, for $\varepsilon > 0$ small.

Proposition 3.23. *Suppose that $\varphi,\ f$ are radial, $\varphi \in C^1([0,1]) \cap C^2\left((0,1] - \{r_0 \pm L\varepsilon^{\frac{2}{3}}\}\right)$ with φ_{rr} possibly having finite jump discontinuities at $r_0 \pm L\varepsilon^{\frac{2}{3}}$, and $f \in C\left([0,1] - \{r_0 \pm L\varepsilon^{\frac{2}{3}}\}\right)$ possibly having finite jump discontinuities at $r_0 \pm L\varepsilon^{\frac{2}{3}}$.*

If

$$\mathbb{L}(\varphi) = f \text{ in } B_1, \ \ \partial_\nu \varphi = 0 \text{ on } \partial B_1,$$

then

$$\|\varphi\|_{L^\infty_r(B_1)} \le C\varepsilon^{-\frac{2}{3}}\|f\|_{L^\infty_r(B_1)},$$

provided $\varepsilon \in (0, \varepsilon_0)$, where $\varepsilon_0,\ C > 0$ are independent of $f,\ \varepsilon$.

If

$$\mathbb{L}(\varphi) = \big||x| - r_0\big| f \text{ in } B_1, \ \ \partial_\nu \varphi = 0 \text{ on } \partial B_1,$$

then

$$\|\varphi\|_{L^\infty_r(B_1)} \le C\|f\|_{L^\infty_r(B_1)},$$

provided $\varepsilon \in (0, \varepsilon_0)$, where $\varepsilon_0,\ C > 0$ are independent of $f,\ \varepsilon$.

Proof. We will prove the first assertion of the proposition, and leave the other one to the interested reader. We will argue by contradiction. Let us assume the existence of sequences $\varepsilon_n > 0$, $\varphi_n \in C^1([0,1]) \cap C^2\left((0,1] - \{r_0 \pm L\varepsilon_n^{\frac{2}{3}}\}\right)$ with $(\varphi_n)_{rr}$ possibly having finite jump discontinuities at $r_0 \pm L\varepsilon_n^{\frac{2}{3}}$, $f_n \in C\left([0,1] - \{r_0 \pm L\varepsilon_n^{\frac{2}{3}}\}\right)$ possibly having finite jump discontinuities at $r_0 \pm L\varepsilon_n^{\frac{2}{3}}$ such that

$$\varepsilon_n \to 0, \ \ \varepsilon_n^{-\frac{2}{3}}\|f_n\|_{L^\infty_r(B_1)} \to 0 \ \text{ as } n \to +\infty, \ \ \|\varphi_n\|_{L^\infty_r(B_1)} = 1, \tag{107}$$

and

$$-\varepsilon_n^2 \Delta\varphi_n + Q_n(|x|)\varphi_n = f_n \ \text{ in } B_1, \ \ \partial_\nu\varphi_n = 0 \ \text{ on } \partial B_1, \ \ n \ge 1. \tag{108}$$

Without loss of generality we may assume that $\|\varphi_n\|_{L^\infty_r(B_1)} = \varphi_n(x_n) = 1$ with $|x_n| = r_n,\ 0 \le r_n \le 1$. Note that

$$r_n \in [r_0 - 2L\varepsilon_n^{\frac{2}{3}}, r_0 + 2L\varepsilon_n^{\frac{2}{3}}] \ \text{ for all large } n \ge 1. \tag{109}$$

Indeed, if for a subsequence $|x_n| < r_0 - 2L\varepsilon_n^{\frac{2}{3}}$ or $r_0 + 2L\varepsilon_n^{\frac{2}{3}} < |x_n| \le 1$, then $\Delta\varphi_n(x_n) \le 0$. To see this, first of all note that $\Delta\varphi_n$ is continuous at x_n (from (108)). Supposing that $\Delta\varphi_n(x_n) > 0$, then there exists a ball B_n, contained in $\bar{B}_1 - \{0\}$, such that $\Delta\varphi_n(x) > 0$ in B_n, $x_n \in \partial B_n$, and $\varphi_n(x) < \varphi_n(x_n) = 1$ in B_n (note that $\varphi_n \in C^1\left(\bar{B}_n\right) \cap C^2(B_n)$). Therefore, by the Hopf boundary lemma [27], we have $\partial_\nu\varphi_n > 0$ at x_n, where ν is any outward normal vector with respect to B_n. This is a contradiction because if $x_n \in B_1$ then $\nabla\varphi_n(x_n) = 0$, and if $x_n \in \partial B_1$ then $\partial_\nu\varphi_n = 0$ at x_n. Hence, via (103) and (108), we get $c\varepsilon_n^{\frac{2}{3}} \le f_n(x_n)$ which is not possible, if n is sufficiently large, by (107).

On the other hand, $\tilde{\varphi}_n(\xi) = \varphi_n\left(r_0 + \varepsilon_n^{\frac{2}{3}}\xi\right)$ clearly satisfies

$$\begin{cases} -(\tilde{\varphi}_n)_{\xi\xi} - \varepsilon_n^{\frac{2}{3}} \frac{N-1}{r_0 + \varepsilon_n^{\frac{2}{3}}\xi}(\tilde{\varphi}_n)_\xi + \tilde{Q}_n\tilde{\varphi}_n = \varepsilon_n^{-\frac{2}{3}} f_n(r_0 + \varepsilon_n^{\frac{2}{3}}\xi), \ \ |\tilde{\varphi}_n| \le 1, \ \ -\frac{r_0}{\varepsilon_n^{\frac{2}{3}}} < \xi < \frac{1-r_0}{\varepsilon_n^{\frac{2}{3}}}, \\ \tilde{\varphi}_n(\xi_n) = 1, \ \ \xi_n = \frac{r_n - r_0}{\varepsilon_n^{\frac{2}{3}}}, \ \ n \ge 1, \end{cases} \tag{110}$$

where $\tilde{Q}$ was defined in (105). Using (106), (107), and a standard compactness argument, as in the proof of Lemma 3.19, we find that, after passing to a suitable subsequence,

$$\tilde{\varphi}_n \to \tilde{\varphi}_0 \ \text{ in } C^1_{loc}(\mathbb{R}) \ \text{ and } \ \xi_n \to \xi_0 \in [-2L, 2L] \ \text{ as } \ n \to +\infty,$$

where for the second relation we used (109). Passing to the limit, along this subsequence, in (110) yields

$$-(\tilde{\varphi}_0)_{\xi\xi} + (2U_1(\xi) - a_r(r_0)\xi - b_r(r_0)\xi)\,\tilde{\varphi}_0 = 0, \quad |\tilde{\varphi}_0| \le 1, \;\; \xi \in \mathbb{R}, \;\; \text{and } \tilde{\varphi}_0(\xi_0) = 1.$$

Since $2U_1(\xi) - a_r(r_0)\xi - b_r(r_0)\xi \to +\infty$ linearly as $\xi \to \pm\infty$, by a standard barrier argument, we get $\tilde{\varphi}_0 = O\left(e^{-c|\xi|^{\frac{3}{2}}}\right)$ as $\xi \to \pm\infty$, in particular $\tilde{\varphi}_0 \in L^2(\mathbb{R})$ which implies that Kernel$\{M\} \neq \emptyset$ (M as in Remark 3.4 and Proposition 3.6). However, in view of Remark 3.4 and Proposition 3.6, this is not possible. We have thus reached a contradiction, and the proof is complete.

Remark 3.24. *In the case where $U_1 = U_{1+}$, the assertion of Proposition 3.23 can be derived directly from a maximum principle argument (recall Remark 3.18).*

3.4.3. Spectral analysis of $\mathbb{L}$

We will show that the spectrum of $\mathbb{L}_\pm$ is linked, as $\varepsilon \to 0$, to that of the limit operators $M_\pm$, defined in Remark 3.4 and Proposition 3.6. Let us recall that the spectrum of the linear operator, in $L^2(\mathbb{R})$,

$$M(\psi) = -\psi_{\xi\xi} + (2U_1(\xi) - a_r(r_0)\xi - b_r(r_0)\xi)\,\psi,$$

consists of simple eigenvalues $\mu_1 < \mu_2 < \cdots$ with $\mu_i \to +\infty$ as $i \to +\infty$, see Remark 3.4 and Proposition 3.6. Furthermore, the corresponding L^∞-normalized eigenfunctions ψ_i satisfy

$$|(\psi_i)_{\xi\xi}| + |(\psi_i)_\xi| + |\psi_i| \le C_i \exp\left(-c_i|\xi|^{\frac{3}{2}}\right), \quad \xi \in \mathbb{R}, \tag{111}$$

and ψ_i has $i-1$ zeros in $\mathbb{R}$, $i = 1, 2, \cdots$.

The following proposition will be the basis for studying the stability properties, in the radial class, of the radial solutions that we will construct close to $u_{ap\pm}$.

Proposition 3.25. *Given $m \in \mathbb{N}$ (independent of ε), the first m eigenvalues $\lambda_1 < \cdots < \lambda_m$ of $\mathbb{L}$, in the radial class, and the corresponding L^∞_r-normalized eigenfunctions φ_i satisfy*

$$\lambda_i = \mu_i\varepsilon^{\frac{2}{3}} + O\left(\varepsilon^{\frac{4}{3}}\right) \text{ and } \varphi_i(r_0 + \varepsilon^{\frac{2}{3}}\xi) \to \psi_i \text{ in } C^1_{loc}(\mathbb{R}) \text{ as } \varepsilon \to 0, \; i = 1, \cdots, m. \tag{112}$$

Proof. Let us consider

$$\mu_i\varepsilon^{\frac{2}{3}}, \quad \Phi_i = \psi_i\left(\frac{|x| - r_0}{\varepsilon^{\frac{2}{3}}}\right)\zeta(|x|), \quad x \in B_1, \quad i = 1, \cdots, m, \tag{113}$$

(ζ was defined in (61)) as approximate eigenvalue-eigenfunction pairs for $\mathbb{L}$. Note that, from (111),

$$\|\Phi_i\|^2_{L^2_r(B_1)} = \varepsilon^{\frac{2}{3}} \int_{-\frac{\delta}{\varepsilon^{\frac{2}{3}}}}^{\frac{\delta}{\varepsilon^{\frac{2}{3}}}} (r_0 + \varepsilon^{\frac{2}{3}}\xi)^{N-1}\psi_i^2(\xi)d\xi + O\left(e^{-\frac{c}{\varepsilon}}\right) = \varepsilon^{\frac{2}{3}} r_0^{N-1} \int_{-\infty}^{+\infty} \psi_i^2 d\xi + o(\varepsilon^{\frac{2}{3}}) \text{ as } \varepsilon \to 0, \tag{114}$$

where we used Lebesgue's dominated convergence theorem. For $r \in (r_0 - \delta, r_0 + \delta)$ or equivalently $\xi \in \left(-\frac{\delta}{\varepsilon^{\frac{2}{3}}}, \frac{\delta}{\varepsilon^{\frac{2}{3}}}\right)$, via (106) and (111), we have

$$\varepsilon^{-\frac{2}{3}}\left(\mathbb{L}(\Phi_i) - \mu_i\varepsilon^{\frac{2}{3}}\Phi_i\right)(r_0 + \varepsilon^{\frac{2}{3}}\xi) = -(\psi_i)_{\xi\xi} - \frac{N-1}{r_0 + \varepsilon^{\frac{2}{3}}\xi}\varepsilon^{\frac{2}{3}}(\psi_i)_\xi + \tilde{Q}\psi_i - \mu_i\psi_i = O\left(\varepsilon^{\frac{2}{3}}\right)\exp\left(-c|\xi|^{\frac{3}{2}}\right).$$

For $r \in (0, r_0 - \delta) \cup (r_0 + \delta, 1)$, we have $\left(\mathbb{L}(\Phi_i) - \mu_i\varepsilon^{\frac{2}{3}}\Phi_i\right)(r) = O\left(e^{-\frac{c}{\varepsilon}}\right)$. Similarly as in (114), we find that

$$\|\mathbb{L}(\Phi_i) - \mu_i\varepsilon^{\frac{2}{3}}\Phi_i\|_{L^2_r(B_1)} = O\left(\varepsilon^{\frac{5}{3}}\right) = O\left(\varepsilon^{\frac{4}{3}}\right)\|\Phi_i\|_{L^2_r(B_1)}. \tag{115}$$

Since $\mathbb{L}$ is self-adjoint in $L^2_r(B_1)$ with domain $D(\mathbb{L})$ as in (101), by employing regular perturbation theory for self-adjoint operators (see [34, pg. 53–54]), we deduce from (115) that

$$\sigma(\mathbb{L}) \cap \left(\mu_i\varepsilon^{\frac{2}{3}} - O(\varepsilon^{\frac{4}{3}}), \mu_i\varepsilon^{\frac{2}{3}} + O(\varepsilon^{\frac{4}{3}})\right) \neq \emptyset \text{ as } \varepsilon \to 0, \quad i = 1, \cdots, m. \tag{116}$$

We denote by $\lambda_i,\ i = 1, \cdots, m$, the first m eigenvalues of $\mathbb{L}$. In view of (104) and (116), we infer that

$$-C\varepsilon^{\frac{2}{3}} \le \lambda_i \le \mu_m \varepsilon^{\frac{2}{3}} + O\left(\varepsilon^{\frac{4}{3}}\right) \ \text{ as } \varepsilon \to 0, \ \ i = 1, \cdots, m. \tag{117}$$

Since $\mathbb{L}$ is a radial operator, it follows that to each λ_i there corresponds a unique L_r^∞-normalized eigenfunction φ_i. Moreover, it is well known (see [66, Ch. VI]) that $\varphi_i(r)$ has $i-1$ zeros in $(0,1)$ (all of them simple). Note that, from (103) and the Neumann boundary conditions, the zeros of $\varphi_i,\ i = 1, \cdots, m$ are contained in $(r_0 - C_m\varepsilon^{\frac{2}{3}}, r_0 + C_m\varepsilon^{\frac{2}{3}})$ for some large constant $C_m > L$.

Clearly $\tilde{\varphi}_i(\xi) = \varphi_i(r_0 + \varepsilon^{\frac{2}{3}}\xi)$ satisfies

$$-(\tilde{\varphi}_i)_{\xi\xi} - \varepsilon^{\frac{2}{3}} \frac{N-1}{r_0 + \varepsilon^{\frac{2}{3}}\xi} (\tilde{\varphi}_i)_\xi + \tilde{Q}\tilde{\varphi}_i = \lambda_i \varepsilon^{-\frac{2}{3}} \tilde{\varphi}_i, \quad \xi \in \left(-\frac{r_0}{\varepsilon^{\frac{2}{3}}}, \frac{1-r_0}{\varepsilon^{\frac{2}{3}}}\right), \tag{118}$$

and $\|\tilde{\varphi}_i\|_{L^\infty} = 1,\ i = 1, \cdots, m$. Using (106), (117), and passing to a subsequence $\varepsilon_n \to 0$, $n \to +\infty$, as in (110), we find that

$$\tilde{\varphi}_{i,n} \to \tilde{\varphi}_{i,0} \ \text{ in } C^1_{loc}(\mathbb{R}), \ \ \lambda_{i,n}\varepsilon_n^{-\frac{2}{3}} \to \tilde{\lambda}_{i,0} \ \text{ as } n \to +\infty,$$

and

$$-(\tilde{\varphi}_{i,0})_{\xi\xi} + (2U_1(\xi) - a_r(r_0)\xi - b_r(r_0)\xi)\tilde{\varphi}_{i,0} = \tilde{\lambda}_{i,0}\tilde{\varphi}_{i,0}, \quad \xi \in \mathbb{R}, \ \ \|\tilde{\varphi}_{i,0}\|_{L^\infty(\mathbb{R})} = 1, \ \ i = 1, \cdots, m. \tag{119}$$

As in the proof of Proposition 3.23, we see that $\tilde{\varphi}_{i,0} \to 0$ super-exponentially as $\xi \to \pm\infty$ and, in particular, that $\tilde{\varphi}_{i,0} \in L^2(\mathbb{R})$. Since each $\tilde{\varphi}_{i,n}$, $n \ge 1$, has $i-1$ zeros, all of them simple and contained in $(-C_m, C_m)$, it follows that $\tilde{\varphi}_{i,0}$ has $i-1$ simple zeros in $(-2C_m, 2C_m)$ (we also made use of the uniqueness theorem of initial value problems for equation (119) at this point). On the other hand, since $C_m > L$, we see from (67) that $\tilde{\varphi}_{i,0}$ does not have any zeros outside of $(-C_m, C_m)$. Hence $\tilde{\varphi}_{i,0}$ has $i-1$ zeros in $(-\infty, +\infty)$. Consequently, we obtain that $\tilde{\lambda}_{i,0} = \mu_i$ and $\tilde{\varphi}_{i,0} = \psi_i,\ i = 1, \cdots, m$. By the uniqueness of the limit, and (116), we deduce that (112) holds.

The proof of the proposition is complete.

Remark 3.26. *By using (120), it is possible to obtain higher order approximations of the eigenvalues $\lambda_i,\ i \ge 1$, in Proposition 3.25. Although, this is of interest in its own right, we do not exhibit the details in this paper.*

Remark 3.27. *It is not obvious to us, how to conclude the validity of the L_r^∞-bounds of Proposition 3.23 directly from Proposition 3.25.*

Remark 3.28. *Since*

$$2U_1(\xi) - a_r(r_0)\xi - b_r(r_0)\xi = (b_r(r_0) - a_r(r_0))\,|\xi| + O\left(e^{-c|\xi|^{\frac{3}{2}}}\right) \ \ as \ \ \xi \to \pm\infty,$$

it follows from the WKB eigenvalue condition [4, pg. 521] that

$$\mu_i = ci^{\frac{2}{3}} + o(i^{\frac{2}{3}}) \ \ \ as \ \ i \to +\infty, \ \ for\ some\ constant\ c > 0.$$

Hence, by examining the proof of Proposition 3.25, we expect that there exists a constant $d > 0$ such that the first $\left[\frac{d}{\varepsilon}\right]$ (radial) eigenvalues of $\mathbb{L}$ behave qualitatively like $\mu_i\varepsilon^{\frac{2}{3}}$, $i = 1, \cdots, \left[\frac{d}{\varepsilon}\right]$ as $\varepsilon \to 0$.

3.5. Existence and stability of radial corner layered solutions

We are now in position to show, via the contraction mapping theorem, the existence of solutions $u_\pm$ of (31) near the approximations $u_{ap\pm}$, for small $\varepsilon > 0$, and study their stability properties.

Theorem 3.29. *Problem (31) admits two distinct solutions u_+, u_- such that*

$$u_\pm(r) = a(r_0) + \varepsilon^{\frac{2}{3}} U_{1\pm}\left(\frac{r-r_0}{\varepsilon^{\frac{2}{3}}}\right) + \varepsilon^{\frac{4}{3}} U_{2\pm}\left(\frac{r-r_0}{\varepsilon^{\frac{2}{3}}}\right) + O\left(\varepsilon^2 + (r-r_0)^3\right), \ \ r \in (r_0 - \delta, r_0 + \delta), \tag{120}$$

and

$$u_\pm - \max\{a,\ b\} = O(\varepsilon^2),\quad r \in [0, r_0 - \delta] \cup [r_0 + \delta, 1] \quad \textit{as } \varepsilon \to 0, \tag{121}$$

where $U_1,\ U_2$ are as is Propositions 3.2, 3.6, 3.11.

Moreover, given $m \in \mathbb{N}$, the first m eigenvalues of the radial linearized operators

$$\mathbb{L}_\pm(\varphi) = -\varepsilon^2 \Delta\varphi + (2u_\pm - a - b)\varphi,\quad \partial_\nu \varphi = 0 \ \textit{ on } \partial B_1,$$

satisfy

$$\lambda_{i\pm} = \mu_{i\pm}\varepsilon^{\frac{2}{3}} + O\left(\varepsilon^{\frac{4}{3}}\right) \ \textit{ as } \varepsilon \to 0,\ \ i = 1, \cdots, m,$$

where

$$0 < \mu_{1+} < \mu_{2+} < \cdots \quad \textit{and} \quad \mu_{1-} < 0 < \mu_{2-} < \cdots,$$

were defined in Remark 3.4 and Proposition 3.6.

Proof. We search for a solution of (31) as

$$u = u_{ap} + \phi,$$

with $\phi \in C^1([0,1]) \cap C^2\left((0,1) - \{r_0 \pm L\varepsilon^{\frac{2}{3}}\}\right)$. We find that ϕ satisfies

$$\mathbb{L}(\phi) = N(\phi) + E, \tag{122}$$

where $\mathbb{L}$ is as in (101), (102), with $e = 0$,

$$N(\phi) = -\phi^2 \quad \text{and} \quad E = \varepsilon^2 (u_{ap})_{rr} + \varepsilon^2 \frac{N-1}{r}(u_{ap})_r - (u_{ap} - a(r))(u_{ap} - b(r)),$$

(note that the equality in (122) holds in the $L^2_r(B_1)$ sense). Given $\phi \in C^\gamma_r\left(\bar{B}_1\right)$, for some $0 < \gamma < 1$, the right hand side of (122) is in $L^p_r(B_1)$ for every $p > 1$. Hence, by Proposition 3.23 and elliptic regularity theory [27], there exists a unique $T(\phi) \in W^{2,p}_r(B_1) \cap D(\mathbb{L})$ such that

$$\mathbb{L}\left(T(\phi)\right) = N(\phi) + E. \tag{123}$$

By choosing $p > N$ large, we find that $T(\phi) \in C^{1+\gamma}_r\left(\bar{B}_1\right)$ (see [27]). Now, via (123) and elliptic regularity theory, we obtain that

$$u_{ap} + T(\phi) \in C^{2+\gamma}_r\left(\bar{B}_1\right). \tag{124}$$

Let

$$X_M = \{\phi \in C^\gamma_r\left(\bar{B}_1\right) \ :\ \|\phi\|_{L^\infty_r(B_1)} \le M\varepsilon^2\},$$

where $M > 0$ is a large constant, independent of ε, to be determined so that $T(X_M) \subseteq X_M$ and T is a contraction in X_M with respect to the L^∞_r-norm, if $\varepsilon > 0$ is sufficiently small. If $\phi \in X_M$, then from (98), Proposition 3.23 (which can be applied thanks to (124)) and (123), we obtain that

$$\|T(\phi)\|_{L^\infty_r(B_1)} \le C\varepsilon^{-\frac{2}{3}}\|\phi\|^2_{L^\infty_r(B_1)} + C\varepsilon^2 \le CM^2\varepsilon^{\frac{10}{3}} + C\varepsilon^2 \le M\varepsilon^2,$$

for small $\varepsilon > 0$, provided M is fixed sufficiently large. Hence, we have that $T(X_M) \subseteq X_M$ for small $\varepsilon > 0$. Similarly, if $\phi_1,\ \phi_2 \in X_M$, we derive that

$$\|T(\phi_1) - T(\phi_2)\|_{L^\infty_r(B_1)} \le CM\varepsilon^{\frac{4}{3}}\|\phi_1 - \phi_2\|_{L^\infty_r(B_1)}.$$

Thus, if $\varepsilon > 0$ is sufficiently small, the mapping T is a contraction in X_M. Therefore, by the contraction mapping principle, we deduce that T has a unique fixed point $\phi_* \in X_M$, if $\varepsilon > 0$ is sufficiently small. Recalling (124), we see that $u_\pm = u_{ap\pm} + \phi_{*\pm} = u_{ap\pm} + T(\phi_{*\pm}) \in C^{2+\gamma}_r\left(\bar{B}_1\right)$, and solve (31) (recall also (99)). Note that

$$u_\pm = u_{ap\pm} + O(\varepsilon^2), \quad \text{uniformly in } \bar{B}_1, \text{ as } \varepsilon \to 0,$$

and (120) now follows from (100). Relation (121) follows readily by recalling (60), (65), and (95). The asymptotic estimates on the first m eigenvalues are a direct consequence of Proposition 3.25, with $Q = 2u_\pm - a - b = 2u_{ap\pm} - a - b + 2\phi_{*\pm}$, see (102).

The proof of the theorem is complete.

3.5.1. Smoothness of the radial corner layered solutions $u_\pm$ with respect to $\varepsilon > 0$

The bifurcation problems, we will consider in Section 5, require smoothness of the solution u_-, with respect to $\varepsilon > 0$, and information on the behavior of $\frac{\partial}{\partial\varepsilon}u_-$ as $\varepsilon \to 0$. A formal calculation, starting from (120), predicts the following

Lemma 3.30. *There exists $\varepsilon_0 > 0$ such that the mappings $u_\pm : (0, \varepsilon_0) \to C^{2+\gamma}\left(\bar{B}_1\right)$ are C^2, where $0 < \gamma < 1$. Moreover,*

$$\varepsilon^{\frac{1}{3}}\left(\frac{\partial}{\partial\varepsilon}u_\pm\right)(r_0 + \varepsilon^{\frac{2}{3}}\xi) \to \frac{2}{3}\left(U_{1\pm} - \xi(U_{1\pm})_\xi\right) \text{ in } C^1_{loc}(\mathbb{R}) \text{ as } \varepsilon \to 0.$$

Proof. Let $Z = C^\gamma\left(\bar{B}_1\right)$, where $0 < \gamma < 1$, endowed with the usual L^2 inner product,

$$X = \{u \in C^{2+\gamma}\left(\bar{B}_1\right) \,:\, \partial_\nu u = 0 \text{ on } \partial B_1\}, \text{ and } I = (0, \varepsilon_0).$$

We associate to (1) the map $F : X \times I \to Z$ defined by

$$F(u, \varepsilon) = -\varepsilon^2\Delta u + (u - a(|x|))\,(u - b(|x|))\,.$$

Clearly $F \in C^2\,(X \times I,\ Z)$, i.e., $F \in C^2\,(X_r \times I,\ Z_r)$, and one has

$$F_u(u, \varepsilon)v = -\varepsilon^2\Delta v + (2u(x) - a(|x|) - b(|x|))\,v, \quad u,\ v \in X, \quad \varepsilon \in I.$$

In view of Theorem 3.29, the linear operators $(F_u(u_\pm, \varepsilon))^{-1} : Z_r \to X_r$ exist, and, by the closed graph theorem, they are bounded. The implicit function theorem then implies that, for each $\varepsilon \in (0, \varepsilon_0)$, $u_\pm$ are isolated solutions of (1) in X_r, and $u_\pm : (0, \varepsilon_0) \to X$ are C^2.

For convenience, let us drop the subscripts $\pm$ and write $\frac{\partial}{\partial\varepsilon}u(x) = \dot{u}(x),\ x \in B_1$. By differentiating (1) (at $u = u_\pm(\varepsilon)$) with respect to ε, we obtain that

$$\mathbb{L}(\dot{u}) = 2\varepsilon\Delta u = 2\varepsilon^{-1}\,(u - a(|x|))\,(u - b(|x|)) \text{ in } B_1,\ \partial_\nu\dot{u} = 0 \text{ on } \partial B_1,$$

where $\mathbb{L}$ is as in Theorem 3.29. From (60), (65), (95), and Theorem 3.29, we infer that $u - \max\{a,\ b\} = O\left(\varepsilon^{\frac{2}{3}}\right)$, uniformly in $\bar{B}_1$, as $\varepsilon \to 0$. Furthermore, from (2), we have $|a(|x|) - b(|x|)| \le C\,||x| - r_0|\,,\ x \in \bar{B}_1$. So,

$$\mathbb{L}(\dot{u}) = O\left(\varepsilon^{\frac{1}{3}} + \varepsilon^{-\frac{1}{3}}\,||x| - r_0|\right) \text{ in } B_1,\ \partial_\nu\dot{u} = 0 \text{ on } \partial B_1.$$

Hence, via Proposition 3.23 and a standard comparison argument, we derive that

$$||\dot{u}||_{L^\infty_r(B_1)} \le C\varepsilon^{-\frac{1}{3}}. \tag{125}$$

Let $w(\xi) = \varepsilon^{\frac{1}{3}}(\dot{u})(r_0 + \varepsilon^{\frac{2}{3}}\xi)$, then

$$-w_{\xi\xi} - \varepsilon^{\frac{2}{3}}\frac{N-1}{r_0 + \varepsilon^{\frac{2}{3}}\xi}w_\xi + \tilde{Q}w = 2\varepsilon^{-\frac{4}{3}}\left(u(r_0 + \varepsilon^{\frac{2}{3}}\xi) - a(r_0 + \varepsilon^{\frac{2}{3}}\xi)\right)\left(u(r_0 + \varepsilon^{\frac{2}{3}}\xi) - b(r_0 + \varepsilon^{\frac{2}{3}}\xi)\right),$$

$|w(\xi)| \le C,\ \xi \in \left(-\frac{r_0}{\varepsilon^{\frac{2}{3}}}, \frac{1-r_0}{\varepsilon^{\frac{2}{3}}}\right)$ (recall (125)), where $\tilde{Q}$ is as in (105) with $e = 2(u - u_{ap})$. In view of (106), (120), and the standard compactness argument, we can pass to a subsequence $\varepsilon_n \to 0$, $n \to +\infty$, such that $w_n \to w_0$ in $C^1_{loc}(\mathbb{R})$ as $n \to +\infty$. Moreover,

$$M\,(w_0) = -(w_0)_{\xi\xi} + (2U_1 - a_r(r_0)\xi - b_r(r_0)\xi)\,w_0 = 2\,(U_1 - a_r(r_0)\xi)\,(U_1 - b_r(r_0)\xi) = 2(U_1)_{\xi\xi},$$

$|w_0(\xi)| \le C,\ \xi \in \mathbb{R}$, and it follows that $w_0 \in L^2(\mathbb{R})$. On the other hand, it is easy to check that $M\left(U_1 - \xi(U_1)_\xi\right) = 3(U_1)_{\xi\xi},\ \xi \in \mathbb{R}$. Hence, by Remark 3.4 and Proposition 3.6, we deduce that $w_0 = \frac{2}{3}\left(U_1 - \xi(U_1)_\xi\right)$. The uniqueness of the limit implies the assertion of the lemma, and the proof is complete.

4. The non-radial linearized operator on the radial corner layered solution u_-

In the rest of the paper, we will assume that $N \geq 2$. In this section we will study the linearization of (1), in the general class (not necessarily radial), at the radial solution u_-. In particular, we will estimate the asymptotic behavior of the eigenvalues that are closest to zero, as $\varepsilon \to 0$. We will call such eigenvalues critical.

We consider the eigenvalue problem

$$-\varepsilon^2 \Delta\Psi + Q_-(|x|)\Psi = \Lambda\Psi \text{ in } B_1, \ \ \partial_\nu \Psi = 0 \text{ on } \partial B_1, \tag{126}$$

where $Q_- = 2u_- - a - b$ (recall that Q_- satisfies the hypotheses of (102)). Here Ψ is not assumed to be radially symmetric. It is well known that (126) has a sequence of eigenvalues $\Lambda_1 < \Lambda_2 \leq \Lambda_3 \leq \cdots$, with Λ_1 the principal eigenvalue whose corresponding eigenfunction Ψ_1 can be chosen positive, and $\Lambda_k \to +\infty$ as $k \to +\infty$. Moreover, Ψ_1 is radially symmetric and therefore $\Lambda_1 = \lambda_1$ (defined in Theorem 3.29). Any other eigenvalue Λ_k corresponds to a finite number of linearly independent sign-changing eigenfunctions which span a finite-dimensional space Y_k. Note that we have $Y_1 = \text{Span}\{\Psi_1\}$. Denote $m_k = \dim(Y_k)$, and suppose $\Lambda_j < 0$, $\Lambda_{j+1} \geq 0$; then

$$M_\varepsilon = \sum_{k=1}^{j} m_k \tag{127}$$

is called the Morse index of u_-.

4.1. Separation of variables

For studying (126), we make use of polar coordinates

$$x = (r, \theta), \ \ r = |x|, \ \ \theta \in S^{N-1},$$

and the Laplace-Beltrami operator $\Delta_{S^{N-1}}$ on the unit sphere S^{N-1}. We have

$$\Delta = \partial_{rr} + \frac{N-1}{r}\partial_r + \frac{1}{r^2}\Delta_{S^{N-1}}.$$

It is well known that the eigenvalues of $-\Delta_{S^{N-1}}$ are $\tau_k = (k-1)(k+N-3)$, $k = 1, 2, \cdots$, and that the eigenfunctions corresponding to τ_k span the space of homogeneous and harmonic polynomials of degree $k-1$, which we denote by $\mathcal{H}_{k-1}$. Moreover, the following orthogonal decomposition holds

$$L^2\left(S^{N-1}\right) = \bigoplus_{k\geq 1} \mathcal{H}_{k-1}, \ \text{ and } \dim\left(\mathcal{H}_{k-1}\right) = \frac{(2k+N-4)(k+N-4)!}{(k-1)!(N-2)!}. \tag{128}$$

By Lemma 3.3 in [19], we know that the pair (Λ, Ψ), Ψ nontrivial, solves (126) if and only if there exists a pair (Λ, A), A nontrivial, that solves

$$\begin{cases} -\varepsilon^2 A_{rr} - \varepsilon^2 \frac{N-1}{r} A_r + \left(\varepsilon^2 \frac{\tau_k}{r^2} + Q_-(r)\right) A = \Lambda A \ \text{ in } (0,1), \\ A \in C^2\left((0,1]\right) \cap C\left([0,1]\right), \ \ A_r(1) = 0, \end{cases} \tag{129}$$

for some $k = 1, 2, \cdots$. Furthermore,

$$\Psi(x) = A(|x|)\Theta\left(\frac{x}{|x|}\right) \text{ for some } \Theta \in \mathcal{H}_{k-1}. \tag{130}$$

4.2. The critical eigenvalues of the general singular radial problem

As in [19], for later applications, we consider a more general problem

$$\begin{cases} -\varepsilon^2 A_{rr} - \varepsilon^2 \frac{N-1}{r} A_r + \left(\varepsilon^\alpha \frac{\tau}{r^2} + Q_-(r)\right) A = \Lambda A \ \text{ in } (0,1), \\ A \in C^2\left((0,1]\right) \cap C\left([0,1]\right), \quad A_r(1) = 0, \end{cases} \tag{131}$$

where $\tau > 0$ and $\alpha \in \left[\frac{2}{3}, 2\right]$. It has been shown in [19], [60] that if A solves (131) (recall that $\tau > 0$), then $A(0) = 0$ and

$$A(r)r^{-\gamma} \to \beta, \ \ A_r(r)r^{1-\gamma} \to \frac{\varepsilon^{\alpha-2}\tau\beta}{\gamma + N - 2} \ \text{ as } r \to 0, \tag{132}$$

for some $\beta \neq 0$, where

$$\gamma = \frac{1}{2}\left(2 - N + \sqrt{(N-2)^2 + 4\varepsilon^{\alpha-2}\tau}\right).$$

Despite of the fact that (131) is a singular eigenvalue problem, we can still show the existence of a "principal" eigenvalue.

Lemma 4.1. *Given $\tau^* > 0$ (independent of ε), there exists $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0)$, $\tau \in (0, \tau^*]$, $\alpha \in \left[\frac{2}{3}, 2\right]$, problem (131) has a solution pair $(\Lambda_1^{\varepsilon,\tau,\alpha}, A_1^{\varepsilon,\tau,\alpha})$ with $A_1^{\varepsilon,\tau,\alpha}(r) > 0$ in $(0,1]$, $\|A_1^{\varepsilon,\tau,\alpha}\|_{L^\infty(0,1)} = 1$,*

$$\left(A_1^{\varepsilon,\tau,\alpha}\right)_r > 0 \ \ \text{in } (0, r_0 - C\varepsilon^{\frac{2}{3}}); \ \ \left(A_1^{\varepsilon,\tau,\alpha}\right)_r < 0 \ \ \text{in } (r_0 + C\varepsilon^{\frac{2}{3}}, 1),$$

and

$$-C\varepsilon^{\frac{2}{3}} \leq \Lambda_1^{\varepsilon,\tau,\alpha} \leq C\varepsilon^{\frac{2}{3}},$$

for some constant $C > 0$ depending only τ^*.

Moreover, if (Λ_*, A_*) is another solution pair of (131), with $A_*(r) > 0$ in $(0,1]$, then $\Lambda_* = \Lambda_1^{\varepsilon,\tau,\alpha}$ and A_* is a constant multiple of $A_1^{\varepsilon,\tau,\alpha}$.

Proof. This is essentially Lemma 3.4 in [19], where the heterogeneous Allen-Cahn equation was treated. We will adapt their proof to our present situation because it will be the basis for showing that $\Lambda_1^{\varepsilon,\tau,\alpha}$ is differentiable, with respect to $\varepsilon > 0$, in Lemma 4.3 below. In turn, this differentiability property will be required in Section 5 dealing with the bifurcation problem.

For small $\eta > 0$ (independent of ε), let us consider the auxiliary problem over $(\eta, 1)$,

$$-\varepsilon^2 A_{rr} - \varepsilon^2 \frac{N-1}{r} A_r + \left(\varepsilon^\alpha \frac{\tau}{r^2} + Q_-(r)\right) A = \Lambda A, \ \ A(\eta) = 0, \ A_r(1) = 0. \tag{133}$$

This is a regular eigenvalue problem, and let us denote its first eigenvalue by Λ_η, and by A_η the corresponding eigenfunction such that

$$A_\eta > 0 \text{ in } (\eta, 1] \text{ and } \|A_\eta\|_{L^\infty(\eta,1)} = 1. \tag{134}$$

From the variational characterization

$$\Lambda_\eta = \inf_{v \in D_\eta - \{0\}} \int_{\eta<|x|<1} \left[\varepsilon^2 |\nabla v|^2 + \left(\varepsilon^\alpha \frac{\tau}{|x|^2} + Q_-(|x|)\right) v^2\right] dx \bigg/ \int_{\eta<|x|<1} v^2 dx, \tag{135}$$

where $D_\eta = \{v \in W_r^{1,2}(\eta < |x| < 1) \ : \ v = 0 \text{ on } |x| = \eta, \ \partial_\nu v = 0 \text{ on } \partial B_1\}$, we easily see that Λ_η varies continuously, and is strictly increasing, with respect to η. By (104) and (135), certainly

$$\Lambda_\eta \geq -C\varepsilon^{\frac{2}{3}}, \ \ \varepsilon \in (0, \varepsilon_1) \ \ (\varepsilon_1, \ C > 0 \text{ independent of } \varepsilon, \eta, \tau, \alpha). \tag{136}$$

Next we use Φ_1, defined in (113) (with $\psi_1 = \psi_{1-}$), as a test function in (135) to obtain an upper bound for Λ_η. We have

$$\int_{\eta<|x|<1}\left[\varepsilon^2|\nabla\Phi_1|^2+\left(\varepsilon^\alpha\frac{\tau}{|x|^2}+Q_-(|x|)\right)\Phi_1^2\right]dx=$$

$$\int_{B_1}\left[\varepsilon^2|\nabla\Phi_1|^2+Q_-(|x|)\Phi_1^2\right]dx+\varepsilon^\alpha\tau\int_{r_0-2\delta}^{r_0+2\delta}\psi_1^2\left(\frac{r-r_0}{\varepsilon^{\frac{2}{3}}}\right)\zeta^2(r)r^{N-3}dr\le$$

$$(\mathbb{L}_-(\Phi_1),\Phi_1)_{L^2(B_1)}+\varepsilon^{\alpha+\frac{2}{3}}\tau\int_{-\frac{2\delta}{\varepsilon^{\frac{2}{3}}}}^{\frac{2\delta}{\varepsilon^{\frac{2}{3}}}}\psi_1^2(\xi)(r_0+\varepsilon^{\frac{2}{3}}\xi)^{N-3}d\xi\le$$

$$\mu_{1-}\varepsilon^{\frac{2}{3}}\|\Phi_1\|^2_{L^2(B_1)}+\|\mathbb{L}_-(\Phi_1)-\mu_{1-}\varepsilon^{\frac{2}{3}}\Phi_1\|_{L^2(B_1)}\|\Phi_1\|_{L^2(B_1)}+\varepsilon^{\alpha+\frac{2}{3}}\tau\left(r_0^{N-3}\|\psi_1\|^2_{L^2(\mathbb{R})}+o(1)\right).$$

In view of (114), (115), and (135), we derive that

$$\Lambda_\eta\le\mu_{1-}\varepsilon^{\frac{2}{3}}+\varepsilon^\alpha r_0^{-2}\tau+C\varepsilon^{\frac{4}{3}}+\tau o(\varepsilon^\alpha)\le C_*\varepsilon^{\frac{2}{3}},\quad \varepsilon\in(0,\varepsilon_1)\quad (C_*>0 \text{ depends only on } \tau^*), \tag{137}$$

where ε_1 is independent of η,τ,α. Therefore, it follows from (103) and (137) that

$$\varepsilon^\alpha\frac{\tau}{r^2}+Q_-(r)-\Lambda_\eta\ge c\varepsilon^{\frac{2}{3}}\ \text{ in } (\eta,r_0-C_*\varepsilon^{\frac{2}{3}})\cup(r_0+C_*\varepsilon^{\frac{2}{3}},1),\ \ \varepsilon\in(0,\varepsilon_*), \tag{138}$$

for some new $C_*>L$, $\varepsilon_*>0$ depending only on τ^*, and $c>0$ independent of ε, η, τ, α. So, from (133), (134), (138), we obtain that $\left(r^{N-1}(A_\eta)_r\right)_r>0$ in $(\eta,r_0-C_*\varepsilon^{\frac{2}{3}})\cup(r_0+C_*\varepsilon^{\frac{2}{3}},1)$, and, since $(A_\eta)_r(\eta)>0$, $(A_\eta)_r(1)=0$, we infer that

$$(A_\eta)_r>0\ \text{ in } (\eta,r_0-C_*\varepsilon^{\frac{2}{3}});\ \ (A_\eta)_r<0\ \text{ in } (r_0+C_*\varepsilon^{\frac{2}{3}},1),\ \ \varepsilon\in(0,\varepsilon_*)\quad (\varepsilon_*,\ C_*>0 \text{ depend only on } \tau^*). \tag{139}$$

By (133), (134), (136), (137), and standard elliptic estimates, we can find a subsequence $\eta_j\to 0,\ j\to+\infty$, such that $A_{\eta_j}\to A_0$ in $C^1_{loc}((0,1])$, and $\Lambda_{\eta_j}\to\Lambda_0$ as $j\to+\infty$. Furthermore, we have

$$-\varepsilon^2(A_0)_{rr}-\varepsilon^2\frac{N-1}{r}(A_0)_r+\left(\varepsilon^\alpha\frac{\tau}{r^2}+Q_-(r)\right)A_0=\Lambda_0A_0\ \text{ in } (0,1),\ \ (A_0)_r(1)=0,\ \ A_0\in C^2((0,1]),$$

$-C\varepsilon^{\frac{2}{3}}\le\Lambda_0\le C\varepsilon^{\frac{2}{3}}$, $\varepsilon\in(0,\varepsilon_0)$, ($\varepsilon_0$, $C>0$ depend only on τ^*). From (134), (139), and the above equation, we obtain that $\|A_0\|_{L^\infty(0,1)}=1$, $A_0>0$ in $(0,1]$, and $(A_0)_r\ge 0$ in $(0,r_0-C\varepsilon^{\frac{2}{3}})$; $(A_0)_r\le 0$ in $(r_0+C\varepsilon^{\frac{2}{3}},1)$, $\varepsilon\in(0,\varepsilon_0)$, ($\varepsilon_0$, $C>0$ depend only on τ^*). It follows that $A_0\in C([0,1])$, and thus satisfies (131). We have proven the existence part of the lemma, with $\Lambda_1^{\varepsilon,\tau,\alpha}=\Lambda_0$ and $A_1^{\varepsilon,\tau,\alpha}=A_0$.

It remains to show uniqueness. Suppose that (Λ_*,A_*) and (Λ,A) are two pairs of solutions of (131), as described in the statement of the lemma, with $\Lambda_*\ne\Lambda$. By virtue of (132), the behavior of A_*, A for r near 0 allows us to use integration by parts to obtain

$$\int_0^1\left(r^{N-1}A_r\right)_rA_*dr=\int_0^1\left(r^{N-1}(A_*)_r\right)_rAdr.$$

Therefore we can multiply the equation of (Λ,A) by $r^{N-1}A_*$, the equation of (Λ_*,A_*) by $r^{N-1}A$, subtract, and integrate over $(0,1)$, to arrive at

$$(\Lambda-\Lambda_*)\int_0^1A_*(r)A(r)r^{N-1}dr=0.$$

But this is impossible since $A_*(r)$, $A(r)>0$ in $(0,1]$. This proves that $\Lambda_*=\Lambda$. Then, the uniqueness theorem of initial value problems for ordinary differential equations implies that $A_*(r)=\frac{A_*(1)}{A(1)}A(r)$, $r\in[0,1]$.

The proof of the lemma is complete.

The following proposition concerns the asymptotic behavior of $\left(\Lambda_1^{\varepsilon,\tau,\alpha},\ A_1^{\varepsilon,\tau,\alpha}\right)$ as $\varepsilon\to 0$.

Proposition 4.2. *Given $\tau^*>0$ (independent of ε), we have*

$$A_1^{\varepsilon,\tau,\alpha}(r_0+\varepsilon^{\frac{2}{3}}\xi)\to\psi_{1-}\ \text{ in } C^1_{loc}(\mathbb{R})\ \text{ and }\ \Lambda_1^{\varepsilon,\tau,\alpha}=\mu_{1-}\varepsilon^{\frac{2}{3}}+r_0^{-2}\tau\varepsilon^\alpha+O\left(\varepsilon^{\frac{4}{3}}\right),$$

as $\varepsilon\to 0$, uniformly in $\tau\in(0,\tau^*]$ and $\alpha\in\left[\frac{2}{3},2\right]$; where $\psi_{1-}>0$, $\mu_{1-}<0$ were defined in Proposition 3.6.

Proof. Note that $\tilde{A}_1(\xi) = A_1^{\varepsilon,\tau,\alpha}(r_0 + \varepsilon^{\frac{2}{3}}\xi)$ satisfies

$$-(\tilde{A}_1)_{\xi\xi} - \varepsilon^{\frac{2}{3}}\frac{N-1}{r_0+\varepsilon^{\frac{2}{3}}\xi}(\tilde{A}_1)_\xi + \left(\varepsilon^{\alpha-\frac{2}{3}}\frac{\tau}{(r_0+\varepsilon^{\frac{2}{3}}\xi)^2} + \tilde{Q}_-(\xi)\right)\tilde{A}_1 = \varepsilon^{-\frac{2}{3}}\Lambda_1^{\varepsilon,\tau,\alpha}\tilde{A}_1, \tag{140}$$

$\xi \in \left(-\frac{r_0}{\varepsilon^{\frac{2}{3}}}, \frac{1-r_0}{\varepsilon^{\frac{2}{3}}}\right)$, where $\tilde{Q}_-$ is as in (105).

Suppose that $\varepsilon \to 0$, $\tau_\varepsilon \in (0,\tau^*]$, and $\alpha_\varepsilon \in \left[\frac{2}{3}, 2\right]$. Then, thanks to the properties of $A_1^{\varepsilon,\tau,\alpha}$, $\Lambda_1^{\varepsilon,\tau,\alpha}$ we established in Lemma 4.1, relation (106), and the standard compactness argument, we can pass to a subsequence $\varepsilon_n \to 0$, $n \to +\infty$, such that

$$\tilde{A}_{1,n} \to \tilde{A}_{1,0} \text{ in } C^1_{loc}(\mathbb{R}), \quad \varepsilon_n^{-\frac{2}{3}}\Lambda_1^{\varepsilon_n,\tau_n,\alpha_n} \to \tilde{\Lambda}_1^0 \text{ and } \varepsilon_n^{\alpha_n-\frac{2}{3}}\tau_n \to c_0 \text{ as } n \to +\infty, \tag{141}$$

for some $\tilde{A}_{1,0} \in C^1(\mathbb{R})$, $\tilde{\Lambda}_1^0 \in \mathbb{R}$, $c_0 \in [0,\tau^*]$. Furthermore, we have

$$-(\tilde{A}_{1,0})_{\xi\xi} + (2U_{1-}(\xi) - a_r(r_0)\xi - b_r(r_0)\xi)\tilde{A}_{1,0} = \left(\tilde{\Lambda}_1^0 - \frac{c_0}{r_0^2}\right)\tilde{A}_{1,0}, \quad \xi \in \mathbb{R},$$

$\tilde{A}_{1,0}(\xi) > 0$, $\xi \in \mathbb{R}$, $\|\tilde{A}_{1,0}\|_{L^\infty(\mathbb{R})} = 1$, and it follows that $\tilde{A}_{1,0} \in L^2(\mathbb{R})$. Hence, we infer that $\tilde{A}_{1,0} = \psi_{1-}$, and $\tilde{\Lambda}_1^0 = \mu_{1-} + r_0^{-2}c_0$. By the uniqueness of the limit (of $\{\tilde{A}_1\}$), we deduce that

$$\tilde{A}_1 \to \psi_{1-} \text{ in } C^1_{loc}(\mathbb{R}) \text{ as } \varepsilon \to 0, \text{ uniformly in } \tau \in (0,\tau^*] \text{ and } \alpha \in \left[\frac{2}{3}, 2\right]. \tag{142}$$

This proves the first assertion of the proposition.

We multiply (140) by ψ_{1-}, and integrate over $\left(-\frac{\delta}{\varepsilon^{\frac{2}{3}}}, \frac{\delta}{\varepsilon^{\frac{2}{3}}}\right)$, to find

$$-\int_{-\frac{\delta}{\varepsilon^{\frac{2}{3}}}}^{\frac{\delta}{\varepsilon^{\frac{2}{3}}}}(\tilde{A}_1)_{\xi\xi}\psi_{1-} - \varepsilon^{\frac{2}{3}}\int_{-\frac{\delta}{\varepsilon^{\frac{2}{3}}}}^{\frac{\delta}{\varepsilon^{\frac{2}{3}}}}\frac{N-1}{r_0+\varepsilon^{\frac{2}{3}}\xi}(\tilde{A}_1)_\xi\psi_{1-} + \int_{-\frac{\delta}{\varepsilon^{\frac{2}{3}}}}^{\frac{\delta}{\varepsilon^{\frac{2}{3}}}}\left(\varepsilon^{\alpha-\frac{2}{3}}\frac{\tau}{(r_0+\varepsilon^{\frac{2}{3}}\xi)^2} + \tilde{Q}_-(\xi)\right)\tilde{A}_1\psi_{1-} = \varepsilon^{-\frac{2}{3}}\Lambda_1^{\varepsilon,\tau,\alpha}\int_{-\frac{\delta}{\varepsilon^{\frac{2}{3}}}}^{\frac{\delta}{\varepsilon^{\frac{2}{3}}}}\tilde{A}_1\psi_{1-}.$$

Recalling the definition of (μ_{1-}, ψ_{1-}) (see also (148) below), it is convenient to integrate by parts the first integral in the above relation (the boundary terms are of order $O\left(e^{-\frac{c}{\varepsilon}}\right)$, by (111) and Lemma 4.1). We can now pass to the limit $\varepsilon \to 0$ in the resulting identity, thanks to (106), (111), (142) and Lebesgue's dominated convergence theorem. We conclude that the second assertion of the proposition holds as well.

The proof of the proposition is complete.

A formal calculation, based on the second assertion of Proposition 4.2, predicts the following lemma.

Lemma 4.3. *Given $\tau^* > 0$ (independent of ε), if $\tau \in (0,\tau^*]$, $\alpha \in \left[\frac{2}{3}, 2\right]$, then $\Lambda_1^{\varepsilon,\tau,\alpha}$ is C^1 with respect to $\varepsilon \in (0,\varepsilon_0)$. Moreover,*

$$\frac{\partial}{\partial\varepsilon}\Lambda_1^{\varepsilon,\tau,\alpha} = \frac{2}{3}\mu_{1-}\varepsilon^{-\frac{1}{3}} + \alpha r_0^{-2}\tau\varepsilon^{\alpha-1} + o\left(\varepsilon^{-\frac{1}{3}}\right) \quad \textit{as } \varepsilon \to 0,$$

uniformly in $\tau \in (0,\tau^*]$ and $\alpha \in \left[\frac{2}{3}, 2\right]$.

Proof. Because (131) is a singular eigenvalue problem, we will again make use of the regularized problem (133). Since Λ_η is a simple eigenvalue of (133) (according to the definition in [14]), by a result of [14], we know that (Λ_η, A_η) depend smoothly on $\varepsilon \in (0,\varepsilon_0)$, (recall that ε_0 depends only on τ^*). In particular, it follows that A_η is a C^1 map from $(0,\varepsilon_0)$ to $C^2([\eta, 1])$.

For simplifying notation in this proof, we will write $(\Lambda, A) = (\Lambda_\eta, A_\eta)$, $\dot{\Lambda} = \frac{\partial}{\partial\varepsilon}\Lambda$, $\dot{A} = \frac{\partial}{\partial\varepsilon}A$, $r \in [\eta, 1]$, $\varepsilon \in (0,\varepsilon_0)$, and $(\Lambda_1, A_1) = (\Lambda_1^{\varepsilon,\tau,\alpha}, A_1^{\varepsilon,\tau,\alpha})$, $r \in [0,1]$, $\varepsilon \in (0,\varepsilon_0)$.

Differentiating (133) with respect to $\varepsilon \in (0,\varepsilon_0)$, we derive that

$$-\varepsilon^2\dot{A}_{rr} - \varepsilon^2\frac{N-1}{r}\dot{A}_r + \left(\varepsilon^\alpha\frac{\tau}{r^2} + Q_-\right)\dot{A} - \Lambda\dot{A} = 2\varepsilon^{-1}Q_-A + (2-\alpha)\varepsilon^{\alpha-1}\frac{\tau}{r^2}A - 2\varepsilon^{-1}\Lambda A - 2\dot{u}_-A + \dot{\Lambda}A,$$

$r \in (\eta, 1)$, and $\dot{A}(\eta) = \dot{A}_r(1) = 0$. Multiplying both sides of the above equation by $r^{N-1}A$, integrating by parts over $(\eta, 1)$, and using (133), we arrive at

$$-\dot{\Lambda}\int_\eta^1 A^2 r^{N-1} = 2\varepsilon^{-1}\int_\eta^1 Q_- A^2 r^{N-1} + (2-\alpha)\varepsilon^{\alpha-1}\tau\int_\eta^1 A^2 r^{N-3} - 2\varepsilon^{-1}\Lambda\int_\eta^1 A^2 r^{N-1} - 2\int_\eta^1 \dot{u}_- A^2 r^{N-1}. \quad (143)$$

Let us fix an arbitrary compact interval $J \subset (0, \varepsilon_0)$. From the proof of Lemma 4.1, we have that $\Lambda \to \Lambda_1$ as $\eta \to 0$, uniformly in $\varepsilon \in J$. Thus Λ_1 is continuous in $\varepsilon \in J$. Next we want to show that $\dot{\Lambda}$ converges, pointwise in $(0, \varepsilon_0)$, as $\eta \to 0$. We will make use of the fact that $A \to A_1$ in $C^1_{loc}((0, 1])$ as $\eta \to 0$ (from the proof of Lemma 4.1), together with the bounds: $0 < A \le 1$, $r \in (\eta, 1]$, and $0 < A \le Dr^{\gamma'}$, $r \in (\eta, d]$ for some positive constants γ', d, D independent of η (this follows from (133), (134), and a standard barrier argument). Now employing Lebesgue's dominated convergence theorem, via (143), we find that

$$\dot{\Lambda} \to -\frac{2\varepsilon^{-1}\int_0^1 Q_- A_1^2 r^{N-1} + (2-\alpha)\varepsilon^{\alpha-1}\tau\int_0^1 A_1^2 r^{N-3} - 2\varepsilon^{-1}\Lambda_1\int_0^1 A_1^2 r^{N-1} - 2\int_0^1 \dot{u}_- A_1^2 r^{N-1}}{\int_0^1 A_1^2 r^{N-1}} \quad \text{as } \eta \to 0, \quad \varepsilon \in (0, \varepsilon_0),$$

(note that if $N \ge 3$, then the bound $|A| \le 1$ suffices in order to pass to the limit). Hence Λ_1 is differentiable with respect to $\varepsilon \in J$, and

$$\frac{\partial}{\partial\varepsilon}\Lambda_1 = -\frac{2\varepsilon^{-1}\int_0^1 Q_- A_1^2 r^{N-1} + (2-\alpha)\varepsilon^{\alpha-1}\tau\int_0^1 A_1^2 r^{N-3} - 2\varepsilon^{-1}\Lambda_1\int_0^1 A_1^2 r^{N-1} - 2\int_0^1 \dot{u}_- A_1^2 r^{N-1}}{\int_0^1 A_1^2 r^{N-1}}, \quad \varepsilon \in J. \quad (144)$$

Since J was an arbitrary compact interval of $(0, \varepsilon_0)$, and the righthand side of (144) is continuous in $\varepsilon > 0$ (recall Lemma 3.30), we conclude that $\Lambda_1 \in C^1((0, \varepsilon_0))$.

Before we proceed any further, let us note that as in [10], [19], we have

$$0 < A(r) \le \exp\left(-c\frac{|r - r_0|}{\varepsilon^{\frac{2}{3}}}\right), \ r \in (\eta, 1] \text{ and } 0 < A_1(r) \le \exp\left(-c\frac{|r - r_0|}{\varepsilon^{\frac{2}{3}}}\right), \ r \in (0, 1], \ \ \varepsilon \in (0, \varepsilon_0), \quad (145)$$

where ε_0, $c > 0$ depend only on τ^* (recall (138)).

Note that, via Lemma 3.30, Proposition 4.2, and (145),

$$\int_0^1 \dot{u}_- A_1^2 r^{N-1} = \varepsilon^{\frac{1}{3}}\int_{-\frac{r_0}{\varepsilon^{\frac{2}{3}}}}^{\frac{1-r_0}{\varepsilon^{\frac{2}{3}}}} \varepsilon^{\frac{1}{3}}(\dot{u}_-)(r_0 + \varepsilon^{\frac{2}{3}}\xi)A_1^2(r_0 + \varepsilon^{\frac{2}{3}}\xi)(r_0 + \varepsilon^{\frac{2}{3}}\xi)^{N-1} = \frac{2}{3}r_0^{N-1}\varepsilon^{\frac{1}{3}}\int_{-\infty}^{+\infty}\left(U_{1-} - \xi\,(U_{1-})_\xi\right)\psi_{1-}^2 + o\left(\varepsilon^{\frac{1}{3}}\right)$$

as $\varepsilon \to 0$. Similarly,

$$\int_0^1 A_1^2 r^{N-1} = \varepsilon^{\frac{2}{3}} r_0^{N-1}\int_{-\infty}^{+\infty}\psi_{1-}^2 + o\left(\varepsilon^{\frac{2}{3}}\right) \quad \text{as } \varepsilon \to 0,$$

$$\int_0^1 A_1^2 r^{N-3} = \varepsilon^{\frac{2}{3}} r_0^{N-3}\int_{-\infty}^{+\infty}\psi_{1-}^2 + o\left(\varepsilon^{\frac{2}{3}}\right) \quad \text{as } \varepsilon \to 0,$$

$$\int_0^1 Q_- A_1^2 r^{N-1} = \varepsilon^{\frac{4}{3}} r_0^{N-1}\int_{-\infty}^{+\infty}\left(2U_{1-} - a_r(r_0)\xi - b_r(r_0)\xi\right)\psi_{1-}^2 + o\left(\varepsilon^{\frac{4}{3}}\right) \quad \text{as } \varepsilon \to 0, \quad \text{(recall (106)).}$$

In view of the above, Proposition 4.2, and (144), we obtain that

$$\frac{\partial}{\partial\varepsilon}\Lambda_1 = \left(-2\frac{\int_{-\infty}^{+\infty}\left(2U_{1-} - a_r(r_0)\xi - b_r(r_0)\xi\right)\psi_{1-}^2}{\int_{-\infty}^{+\infty}\psi_{1-}^2} + \alpha\varepsilon^{\alpha-\frac{2}{3}}\tau r_0^{-2} + 2\mu_{1-} + \frac{4}{3}\frac{\int_{-\infty}^{+\infty}\left(U_{1-} - \xi\,(U_{1-})_\xi\right)\psi_{1-}^2}{\int_{-\infty}^{+\infty}\psi_{1-}^2}\right)\varepsilon^{-\frac{1}{3}} + o\left(\varepsilon^{-\frac{1}{3}}\right) \quad (146)$$

as $\varepsilon \to 0$.

The only thing that remains is to calculate the integrals in the above relation. Since ψ_{1-} is even (recall Remark 3.9), certainly

$$\int_{-\infty}^{+\infty}\xi\psi_{1-}^2 = 0 \quad \text{and} \quad \int_{-\infty}^{+\infty}\xi^2\psi_{1-}(\psi_{1-})_\xi = 0. \quad (147)$$

Now, we multiply the relation

$$-(\psi_{1-})_{\xi\xi} + (2U_{1-} - a_r(r_0)\xi - b_r(r_0)\xi)\,\psi_{1-} = \mu_{1-}\psi_{1-} \tag{148}$$

by ψ_{1-}, and integrate by parts over $(-\infty, +\infty)$, to find that

$$\int_{-\infty}^{+\infty} U_{1-}\psi_{1-}^2 = \frac{\mu_{1-}}{2}\int_{-\infty}^{+\infty}\psi_{1-}^2 - \frac{1}{2}\int_{-\infty}^{+\infty}\Big((\psi_{1-})_\xi\Big)^2. \tag{149}$$

Differentiating (148), multiplying the resulting identity by $\xi\psi_{1-}$, then integrating by parts over $(-\infty, +\infty)$, using (147) and (149), we arrive at

$$\int_{-\infty}^{+\infty}\xi(U_{1-})_\xi\psi_{1-}^2 = \int_{-\infty}^{+\infty}\Big((\psi_{1-})_\xi\Big)^2. \tag{150}$$

Now the assertion of the lemma follows at once from (146) via (147), (149), and (150).

The proof of the lemma is complete.

We will need the following rough estimate.

Lemma 4.4. *Given $\tau^* > 0$ (independent of ε), suppose that $\varepsilon \in (0, \varepsilon_0)$, $\tau \in (0, \tau^*]$, $\alpha \in \left[\frac{2}{3}, 2\right]$, and let (Λ, A) be a solution pair of (131) with $\Lambda \neq \Lambda_1^{\varepsilon,\tau,\alpha}$ and $\|A\|_{L^\infty(0,1)} = 1$. Then, for a possibly smaller $\varepsilon_0 > 0$ (independent of τ, α),*

$$\Lambda \geq \frac{\mu_{2-}}{2}\varepsilon^{\frac{2}{3}}, \quad \varepsilon \in (0, \varepsilon_0),$$

where $\mu_{2-} > 0$ was defined in Proposition 3.6.

Proof. We argue by contradiction. Suppose that there exist $\varepsilon_n \to 0$ as $n \to +\infty$, $\tau_n \in (0, \tau^*]$, $\alpha_n \in \left[\frac{2}{3}, 2\right]$, and (Λ_n, A_n) solving (131), with $\varepsilon = \varepsilon_n$, $\tau = \tau_n$, $\alpha = \alpha_n$, such that $\Lambda_n \neq \Lambda_1^{\varepsilon_n,\tau_n,\alpha_n}$, $\|A_n\|_{L^\infty(0,1)} = 1$, and $\Lambda_n < \frac{\mu_{2-}}{2}\varepsilon_n^{\frac{2}{3}}$, $n \geq 1$. In view of (104), we get

$$-C\varepsilon_n^{\frac{2}{3}} \leq \Lambda_n < \frac{\mu_{2-}}{2}\varepsilon_n^{\frac{2}{3}}, \quad C > 0 \text{ independent of } n, \tag{151}$$

(plainly multiply $(131)_n$ by $r^{N-1}A_n$, and integrate by parts over $(0, 1)$ using (132)). Since $\Lambda_n \neq \Lambda_1^{\varepsilon_n,\tau_n,\alpha_n}$, the second assertion of Lemma 4.1 implies that A_n should change sign in $(0, 1)$. By (103), (151), and recalling that $A_n(0) = 0$, $(A_n)_r(1) = 0$, we deduce that all sign changes of A_n, as well as the maxima of $|A_n|$, take place in $(r_0 - C\varepsilon_n^{\frac{2}{3}}, r_0 + C\varepsilon_n^{\frac{2}{3}})$, $C > 0$ independent of n.

Let $\tilde{A}_n(\xi) = A_n(r_0 + \varepsilon_n^{\frac{2}{3}}\xi)$, $\xi \in \left(-\frac{r_0}{\varepsilon_n^{\frac{2}{3}}}, \frac{1-r_0}{\varepsilon_n^{\frac{2}{3}}}\right)$. Then, arguing as in the proof of Proposition 4.2, we can pass to a subsequence such that

$$\tilde{A}_n \to \tilde{A}_0 \text{ in } C^1_{loc}(\mathbb{R}), \quad \varepsilon_n^{-\frac{2}{3}}\Lambda_n \to \tilde{\Lambda}_0 \text{ and } \tau_n\varepsilon_n^{\alpha_n - \frac{2}{3}} \to d_0 \text{ as } n \to +\infty.$$

Furthermore, we have

$$-(\tilde{A}_0)_{\xi\xi} + (2U_{1-}(\xi) - a_r(r_0)\xi - b_r(r_0)\xi)\,\tilde{A}_0 = \left(\tilde{\Lambda}_0 - \frac{d_0}{r_0^2}\right)\tilde{A}_0, \quad \xi \in \mathbb{R}.$$

Moreover, the function $\tilde{A}_0$ changes sign in $(-2C, 2C)$, $\|\tilde{A}_0\|_{L^\infty(\mathbb{R})} = 1$, and it follows that $\tilde{A}_0 \in L^2(\mathbb{R})$. On the other hand, since $\tilde{\Lambda}_0 - r_0^{-2}d_0 \leq \tilde{\Lambda}_0 \leq \frac{\mu_{2-}}{2}$, we get that $\tilde{A}_0 = \psi_{1-} > 0$, contradicting the fact that $\tilde{A}_0$ changes sign.

The proof of the lemma is complete.

4.3. The critical eigenvalues of the non-radial operator

We are now in position to give some accurate estimates for the critical eigenvalues of the linearized eigenvalue problem (126).

Theorem 4.5. *Given $\tau^* > 0$ (independent of ε), there exists $\varepsilon_0 > 0$ such that (126) has eigenvalues of the form*

$$\Lambda_1^{\varepsilon,\tau_k,2} = \mu_{1-}\varepsilon^{\frac{2}{3}} + (k-1)(k+N-3)r_0^{-2}\varepsilon^2 + O\left(\varepsilon^{\frac{4}{3}}\right), \quad k = 1, \cdots, K \quad \text{as } \varepsilon \to 0, \tag{152}$$

provided $\tau_K = (K-1)(K+N-3) \le \tau^*\varepsilon^{-\frac{4}{3}}$, $\varepsilon \in (0, \varepsilon_0)$. In the above, the multiplicity m_k of the eigenvalue $\Lambda_1^{\varepsilon,\tau_k,2}$ is given by

$$m_k = \frac{(2k+N-4)(k+N-4)!}{(k-1)!(N-2)!}, \quad 1 \le k \le K,$$

which is the dimension of the space $\mathcal{H}_{k-1}$ of homogeneous and harmonic polynomials of degree $k-1$, and the eigenfunctions associated to $\Lambda_1^{\varepsilon,\tau_k,2}$ are of the form

$$A_1^{\varepsilon,\tau_k,2}(|x|)\Theta\left(\frac{x}{|x|}\right) = \psi_{1-}\left(\frac{|x|-r_0}{\varepsilon^{\frac{2}{3}}}\right)\Theta\left(\frac{x}{|x|}\right) + o(1)\|\Theta\|_{L^\infty(S^{N-1})}, \quad \Theta \in \mathcal{H}_{k-1}, \ \text{uniformly in } \bar{B}_1, \ 1 \le k \le K, \tag{153}$$

as $\varepsilon \to 0$. ($\Lambda_1^{\varepsilon,\tau_k,2}$, $A_1^{\varepsilon,\tau_k,2}$ were defined in Lemma 4.1).

Furthermore, for any integer $K(\varepsilon)$ satisfying

$$\mu_{1-} + r_0^{-2}\tau_{K(\varepsilon)}\varepsilon^{\frac{4}{3}} \le \frac{\mu_{2-}}{4}, \tag{154}$$

the first eigenvalues $\Lambda_1 < \Lambda_2 \le \cdots \le \Lambda_{K(\varepsilon)}$ of (126) are $\Lambda_k = \Lambda_1^{\varepsilon,\tau_k,2}$, $k = 1, \cdots, K(\varepsilon)$, $\varepsilon \in (0, \varepsilon_0)$.

Proof. By Lemma 4.1, there exists $\varepsilon_0 > 0$ such that $\left(\Lambda_1^{\varepsilon,\tau,\frac{2}{3}}, A_1^{\varepsilon,\tau,\frac{2}{3}}\right)$ is a solution pair of (131) for each $\varepsilon \in (0, \varepsilon_0)$, $\tau \in [0, \tau^*]$, and $\alpha = \frac{2}{3}$ (when $\tau = 0$, we have $\Lambda_1^{\varepsilon,0,\frac{2}{3}} = \lambda_{1-}$ as in Theorem 3.29). In other words, $\left(\Lambda_1^{\varepsilon,\varepsilon^{-\frac{4}{3}}\tau,2}, A_1^{\varepsilon,\varepsilon^{-\frac{4}{3}}\tau,2}\right)$ is a solution pair of (131) for each $\varepsilon \in (0, \varepsilon_0)$, $\tau \in [0, \tau^*]$, and $\alpha = \frac{2}{3}$. It follows that $\left(\Lambda_1^{\varepsilon,\tau_k,2}, A_1^{\varepsilon,\tau_k,2}\right)$ are solution pairs of (129) provided $\tau_k\varepsilon^{\frac{4}{3}} \le \tau^*$. By Proposition 4.2, for those k's, the eigenvalue $\Lambda_1^{\varepsilon,\tau_k,2}$ satisfies (152), and, via (145), the associated eigenfunction (of (129)) satisfies $A_1^{\varepsilon,\tau_k,2} = \psi_{1-}\left(\frac{|x|-r_0}{\varepsilon^{\frac{2}{3}}}\right) + o(1)$ as $\varepsilon \to 0$, uniformly in $\bar{B}_1$. From [19], we know that the eigenvalues of (126) are in a one to one correspondence with those of (129), and that the eigenfunctions of (126) corresponding to $\Lambda_1^{\varepsilon,\tau_k,2}$, with $\tau_k\varepsilon^{\frac{4}{3}} \le \tau^*$, $\varepsilon \in (0, \varepsilon_0)$, are of the form $A_1^{\varepsilon,\tau_k,2}(|x|)\Theta\left(\frac{x}{|x|}\right)$, $\Theta \in \mathcal{H}_{k-1}$ (see (128) for the explicit formula of $\dim(\mathcal{H}_{k-1})$). The first assertion of the theorem follows readily.

On the other hand, Lemma 4.4 implies that there exists a possibly smaller $\varepsilon_0 > 0$ such that if Λ is an eigenvalue of (126) with $\Lambda \ne \Lambda_1^{\varepsilon,\tau_k,2}$, $\tau_k\varepsilon^{\frac{4}{3}} \le \tau^*$, $\varepsilon \in (0, \varepsilon_0)$, then $\Lambda \ge \frac{\mu_{2-}}{2}\varepsilon^{\frac{2}{3}}$. Hence, we infer that $\Lambda_1^{\varepsilon,\tau_k,2}$, $\tau_k\varepsilon^{\frac{4}{3}} \le \tau^*$ are the only eigenvalues of (126) that could be less than $\frac{\mu_{2-}}{2}\varepsilon^{\frac{2}{3}}$, if $\varepsilon \in (0, \varepsilon_0)$. Now, if $K(\varepsilon)$ is an integer as in (154), by (152), we find that $\Lambda_1^{\varepsilon,\tau_{K(\varepsilon)},2} < \frac{\mu_{2-}}{3}$, $\varepsilon \in (0, \varepsilon_0)$, for a possibly smaller $\varepsilon_0 > 0$. So, certainly $\Lambda_k := \Lambda_1^{\varepsilon,\tau_k,2}$, $k = 1, \cdots, K(\varepsilon)$ are the first eigenvalues of (126), $\varepsilon \in (0, \varepsilon_0)$. (The monotonicity of $\Lambda_1^{\varepsilon,\tau,\alpha}$ with respect to τ follows by working as in the second part of the proof of Lemma 4.1). We conclude that the second assertion of the theorem holds as well.

The proof of the theorem is complete.

4.3.1. Eigenvalues crossing zero

In the following corollary, we will show that the eigenvalues of (126) grow from a negative number, and eventually cross zero transversely (with nonzero speed), as $\varepsilon \to 0$. This property will be used in Section 5 for showing that non-radial solutions of (1) bifurcate from the unstable radially symmetric solution branch, as $\varepsilon \to 0$.

Corollary 4.6. *If $k \in \mathbb{N}$ is such that*

$$\tau_k \in \left(\frac{|\mu_{1-}|}{2}r_0^2\varepsilon^{-\frac{4}{3}}, \left(|\mu_{1-}| + \frac{\mu_{2-}}{4}\right)r_0^2\varepsilon^{-\frac{4}{3}}\right), \tag{155}$$

then there exists

$$\varepsilon_k = \left(\frac{r_0^2|\mu_{1-}|}{(k-1)(k+N-3)}\right)^{\frac{3}{4}} + o\left(k^{-\frac{3}{2}}\right) \quad as \;\; k \to +\infty,$$

such that the eigenvalue $\Lambda_k = \Lambda_1^{\varepsilon_k,\tau_k,2}$ *of (126), with* $\varepsilon = \varepsilon_k$*, satisfies* $\Lambda_k = 0$*, provided* $\varepsilon > 0$ *is sufficiently small.*

Moreover,

$$\frac{\partial}{\partial\varepsilon}\Lambda_k = \frac{4}{3}|\mu_{1-}|^{\frac{3}{4}} r_0^{-\frac{1}{2}} \left((k-1)(k+N-3)\right)^{\frac{1}{4}} + o\left(k^{\frac{1}{2}}\right) \;\; as \;\; k \to +\infty. \tag{156}$$

Proof. Let $\tau^* = \left(|\mu_{1-}| + \frac{\mu_{2-}}{4}\right) r_0^2$. By virtue of Theorem 4.5, there exists $\varepsilon_0 > 0$ such that the first eigenvalues of (126) are $\Lambda_k = \Lambda_1^{\varepsilon,\tau_k,2}$, $k = 1, \cdots, K$, provided $\tau_K = (K-1)(K+N-3) \leq \tau^*\varepsilon^{-\frac{4}{3}}$, $\varepsilon \in (0, \varepsilon_0)$.

For $\varepsilon \in (0, \varepsilon_0)$, and k any integer satisfying (155), we define

$$g_{\varepsilon,k}(\epsilon) = \Lambda_1^{\epsilon,\tau_k,2}, \;\; \epsilon \in (0, \varepsilon_0).$$

Note that, by Lemma 4.3, we have $g_{\varepsilon,k} \in C^1\left((0, \varepsilon_0)\right)$. We claim that, for any small $d > 0$ (independent of ε), if $\varepsilon_0 > 0$ is chosen smaller, there exists

$$\varepsilon_k \in \left(\left((|\mu_{1-}| - d) r_0^2 \tau_k^{-1}\right)^{\frac{3}{4}}, \; \left((|\mu_{1-}| + d) r_0^2 \tau_k^{-1}\right)^{\frac{3}{4}}\right)$$

such that $g_{\varepsilon,k}(\varepsilon_k) = 0$. Indeed, let $\underline{\epsilon_k} = \left((|\mu_{1-}| - d) r_0^2 \tau_k^{-1}\right)^{\frac{3}{4}}$, then by Proposition 4.2,

$$g_{\varepsilon,k}(\underline{\epsilon_k}) = \Lambda_1^{\underline{\epsilon_k},\tau_k,2} = \Lambda_1^{\underline{\epsilon_k},(|\mu_{1-}|-d)r_0^2\underline{\epsilon_k}^{-\frac{4}{3}},2} = \Lambda_1^{\underline{\epsilon_k},(|\mu_{1-}|-d)r_0^2,\frac{2}{3}} = -d\underline{\epsilon_k}^{\frac{2}{3}} + o\left(\underline{\epsilon_k}^{\frac{2}{3}}\right) < 0,$$

provided $\varepsilon_0 > 0$ is sufficiently small (note that $c\varepsilon \leq \underline{\epsilon_k} \leq C\varepsilon$). Similarly we find that $g_{\varepsilon,k}(\bar{\epsilon}_k) > 0$, where $\bar{\epsilon}_k = \left((|\mu_{1-}| + d) r_0^2 \tau_k^{-1}\right)^{\frac{3}{4}}$, and the desired claim follows. The first assertion of the corollary now follows at once.

By Lemma 4.3 and Theorem 4.5, we deduce that

$$\frac{\partial}{\partial\varepsilon}\Lambda_k = \frac{2}{3}\mu_{1-}\varepsilon^{-\frac{1}{3}} + 2\tau_k r_0^{-2}\varepsilon + o(\varepsilon^{-\frac{1}{3}}) \;\text{ as }\; \varepsilon \to 0, \;\; \text{if } \tau_k \leq \tau^*\varepsilon^{-\frac{4}{3}}, \; \varepsilon \in (0, \varepsilon_0).$$

Substituting $\varepsilon = \varepsilon_k$ in the above relation, we conclude that the second assertion of the corollary holds as well.

The proof of the corollary is complete.

Remark 4.7. *The above corollary indicates that for showing existence of an unstable corner layered solution of the general problem (16)–(19) one has to overcome resonance phenomena as in [18], [45], [46], [47], [48] and [57].*

4.3.2. Morse index of u_-

In the following corollary we provide an asymptotic estimate for the Morse index M_ε of u_- as $\varepsilon \to 0$.

Corollary 4.8. *The Morse index* M_ε *of* u_- *satisfies*

$$\lim_{\varepsilon\to 0} \frac{M_\varepsilon}{\varepsilon^{-\frac{2}{3}(N-1)}} = \left(\frac{r_0^2|\mu_{1-}|}{4\pi}\right)^{\frac{N-1}{2}} \frac{\left|S^{N-1}\right|}{\Gamma\left(\frac{N+1}{2}\right)}.$$

Proof. We adapt the proof of [19]. From Theorem 4.5, we infer that there exists an integer

$$k_\varepsilon = r_0|\mu_{1-}|^{\frac{1}{2}}\varepsilon^{-\frac{2}{3}} + o\left(\varepsilon^{-\frac{2}{3}}\right) \;\text{ as }\; \varepsilon \to 0, \tag{157}$$

such that the eigenvalues Λ_i, $i \geq 1$, of (126) satisfy $\Lambda_1 < \Lambda_2 \leq \cdots \leq \Lambda_{k_\varepsilon} < 0 \leq \Lambda_{k_\varepsilon+1} \leq \cdots$. Furthermore, the multiplicity m_i of Λ_i is equal to the dimension of the space $\mathcal{H}_{i-1}$ of homogeneous and harmonic polynomials of degree $i-1$. Hence, recalling (127), we see that the Morse index of u_- is given by

$$M_\varepsilon = \sum_{i=1}^{k_\varepsilon} \dim(\mathcal{H}_{i-1}) = N(k_\varepsilon), \;\; \text{where } \; N(\kappa) = \sum_{i=1}^{\kappa} \dim(\mathcal{H}_{i-1}) = \sharp\{\tau_i \;:\; \tau_i \leq \tau_\kappa\}. \tag{158}$$

Consequently, by (157), (158), we derive that

$$M_\varepsilon = N(\tau_{k_\varepsilon}) = N\left(r_0^2|\mu_{1-}|\varepsilon^{-\frac{4}{3}} + o\left(\varepsilon^{-\frac{4}{3}}\right)\right) \text{ as } \varepsilon \to 0.$$

On the other hand, from *Weyl's asymptotic formula* [63, Thm. 3.1], we know that

$$\lim_{\kappa\to+\infty} \frac{N(\kappa)}{\kappa^{\frac{N-1}{2}}} = \frac{\left|S^{N-1}\right|}{\Gamma\left(\frac{N+1}{2}\right)(4\pi)^{\frac{N-1}{2}}}.$$

The assertion of the corollary now follows readily.

5. Non-radial bifurcations from the radial corner layered solution u_-

We will make use of the mapping F and the function spaces X, Z introduced in the proof of Lemma 3.30. We seek non-radial solutions of (1) in the form

$$u = u_-(\varepsilon) + \phi, \quad \phi \in X.$$

In terms of ϕ, problem (1) becomes

$$G(\phi,\varepsilon) = 0, \quad \text{where } G(\phi,\varepsilon) = F\left(u_-(\varepsilon) + \phi, \varepsilon\right). \tag{159}$$

In view of Lemma 3.30, clearly $G : X \times (0,\varepsilon_0) \to Z$ is C^2,

$$G(0,\varepsilon) = 0, \quad \varepsilon \in (0,\varepsilon_0),$$

and

$$G_\phi(0,\varepsilon)w = F_u(u_-(\varepsilon),\varepsilon)w = -\varepsilon^2\Delta w + (2u_- - a - b)w, \quad w \in X, \ \varepsilon \in (0,\varepsilon_0). \tag{160}$$

Furthermore, it is a standard fact that G is a nonlinear Fredholm operator with respect to $\phi \in X$ for all $\varepsilon \in (0,\varepsilon_0)$, and a potential operator from X to Z for all $\varepsilon \in (0,\varepsilon_0)$ (see for instance [41]).

We say that bifurcation from the trivial branch $\phi = 0$ takes place at $\varepsilon = \bar\varepsilon > 0$ if every neighborhood of $(0,\bar\varepsilon)$ in $X \times (0,\varepsilon_0)$ contains a nontrivial solution (ϕ,ε), $\phi \neq 0$, of $G(\phi,\varepsilon) = 0$.

Remark 5.1. *Note that the bifurcating solutions are non-radial since the solution u_- is radially non-degenerate.*

5.1. Topological bifurcation from the radial corner layered solution u_-

It is easy to check that the only possible values of $\bar\varepsilon$ for which bifurcation is possible must satisfy Kernel $\left\{G_\phi(0,\bar\varepsilon)\right\} \neq 0$. On the other hand, utilizing the *potential structure* of the problem, we will show that the reciprocal also holds true:

Theorem 5.2. *If $k \in \mathbb{N}$ is such that $\tau_k \in \left(\frac{|\mu_{1-}|}{2} r_0^2 \varepsilon^{-\frac{4}{3}}, \left(|\mu_{1-}| + \frac{\mu_{2-}}{4}\right) r_0^2 \varepsilon^{-\frac{4}{3}}\right)$, $\varepsilon \in (0,\varepsilon_0)$, then $(0,\varepsilon_k)$, as defined in Corollary 4.6, is a bifurcation point of $G(\phi,\varepsilon) = 0$ in the following sense: $(0,\varepsilon_k)$ is a cluster point of nontrivial non-radial solutions $(\phi,\varepsilon) \in X \times (0,\varepsilon_0)$, $\phi \neq 0$, of $G(\phi,\varepsilon) = 0$.*

Proof. We know from Theorem 4.5 and Corollary 4.6 that, for ε, k as in the statement of the theorem, $\Lambda_k = 0$ is an isolated eigenvalue of $G_\phi(0,\varepsilon_k)$, and the corresponding kernel has dimension m_k. Furthermore, from the second assertion of Corollary 4.6, we infer that 0 is a locally hyperbolic equilibrium of $G_\phi(0,\varepsilon)$ for $\varepsilon \in (\varepsilon_k-\delta,\varepsilon_k)\cup(\varepsilon_k,\varepsilon_k+\delta)$, and some small $\delta = \delta(k) > 0$, in the sense that $G_\phi(0,\varepsilon)$ has no spectral point on the imaginary axis for $\varepsilon \in (\varepsilon_k-\delta,\varepsilon_k)\cup(\varepsilon_k,\varepsilon_k+\delta)$. Moreover, the crossing number $\chi\left(G_\phi(0,\varepsilon),\varepsilon_k\right)$ of the family $G_\phi(0,\varepsilon)$ at $\varepsilon = \varepsilon_k$ through 0 is nonzero, in the sense that the Morse index of $G_\phi(0,\varepsilon)$ for $\varepsilon \in (\varepsilon_k-\delta,\varepsilon_k)$ is strictly greater than the Morse index of $G_\phi(0,\varepsilon)$, for $\varepsilon \in (\varepsilon_k,\varepsilon_k+\delta)$ (actually it increases by m_k), see [41, pg. 212] for these definitions. In view of the above and Remark 5.1, in order to establish the assertion of the theorem, it is sufficient to apply the local bifurcation result for potential operators of [40] (see also [41, Theorem II.7.3]).

The proof of the theorem is complete.

5.2. Equivariant bifurcation from the radial corner layered solution u_-

In this subsection, following [28] and [54], we will show that (159) has nontrivial solutions by using an equivariant bifurcation theory.

Let $\mathbf{O}(N)$ denote the orthogonal group in $\mathbb{R}^N$ (see [11]). We define an $\mathbf{O}(N)$-action on Z by

$$(\xi \cdot \phi)(x) = \phi(\xi^{-1}x), \quad \phi \in Z, \; \xi \in \mathbf{O}(N), \tag{161}$$

where $\xi^{-1}x$ is the matrix multiplication. It is easy to see that the mapping $G(\cdot, \varepsilon) : X \to Z$ is $\mathbf{O}(N)$-equivariant, namely,

$$G(\xi \cdot \phi, \varepsilon) = \xi \cdot G(\phi, \varepsilon), \quad \phi \in X, \; \xi \in \mathbf{O}(N).$$

The linearization of $G(\phi, \varepsilon) = 0$ around the trivial branch $\phi = 0$ is the linear operator $G_\phi(0, \varepsilon)$ in (160). Corollary 4.6 says that, for $k \in \mathbb{N}$ and $\varepsilon \in (0, \varepsilon_0)$ such that $\tau_k \in \left(\frac{|\mu_{1-}|}{2} r_0^2 \varepsilon^{-\frac{4}{3}}, \left(|\mu_{1-}| + \frac{\mu_{2-}}{4}\right) r_0^2 \varepsilon^{-\frac{4}{3}}\right)$, there exists $\varepsilon_k \in (c\varepsilon, C\varepsilon)$ such that the kth eigenvalue of $G_\phi(0, \varepsilon_k) = F_u\left(u_-(\varepsilon_k), \varepsilon_k\right)$ satisfies $\Lambda_k = 0$. It is a general fact [29, pg. 304] that the linear operator $G_\phi(0, \varepsilon_k)$ is $\mathbf{O}(N)$-equivariant, and that its kernel and range are $\mathbf{O}(N)$-invariant. Actually, by Theorem 4.5,

$$\text{Kernel}\left\{G_\phi(0, \varepsilon_k)\right\} = \text{Span}\left\{A_1^{\varepsilon_k, \tau_k, 2}(|x|)\Theta\left(\frac{x}{|x|}\right), \; \Theta \in \mathcal{H}_{k-1}\right\},$$

where $A_1^{\varepsilon_k, \tau_k, 2}$ was defined in Lemma 4.1, and $\mathcal{H}_{k-1}$ is the space of harmonic and homogeneous polynomials of degree $k-1$.

We will set up (159) for an application of the equivariant branching lemma due to Cicogna and Vanderbauwhede (see [12], [65]). Following [11, Chapter 2], [29, Chapter VII], we will first reduce the infinite dimensional problem (159) to a finite dimensional one. This reduction is called the Lyapunov-Schmidt reduction (with symmetry). According to the standard L^2-inner product, X, Z are decomposed as

$$X = E_k \oplus \mathcal{M}, \quad Z = \mathcal{N} \oplus F_k, \tag{162}$$

where

$$E_k = \text{Kernel}\left\{G_\phi(0, \varepsilon_k)\right\}, \;\; \mathcal{M} = E_k^\perp \;\text{ and }\; \mathcal{N} = \text{Range}\left\{G_\phi(0, \varepsilon_k)\right\}, \;\; F_k = \mathcal{N}^\perp.$$

(Note that $\dim(E_k) = \dim(F_k) = \dim(\mathcal{H}_{k-1})$). By Lemma 2.3.1 in [11], we can choose the projection $P : Z \to \mathcal{N}$ associated with the decomposition (162) to be $\mathbf{O}(N)$-equivariant. Now, problem (159) becomes equivalent to

$$\begin{cases} \text{(a)} \quad PG(p + w, \varepsilon_k + \mu) = 0, \\ \text{(b)} \quad (I - P)G(p + w, \varepsilon_k + \mu) = 0, \end{cases} \quad p \in E_k, \; w \in \mathcal{M}, \tag{163}$$

where $\mu \in (-\varepsilon_k, \varepsilon_0 - \varepsilon_k)$ is our bifurcation parameter. Because of the invertibility of

$$PG_\phi(0, \varepsilon_k) : \mathcal{M} \to \mathcal{N},$$

the implicit function theorem gives rise to a solution of (163)(a) as $w = w(p, \mu)$, in a neighborhood of $(p, \mu) = (0, 0)$, which satisfies

$$w(0, \mu) = 0 \;\text{ and }\; w_p(0, 0) = 0. \tag{164}$$

Then, substituting the function $w = w(p, \mu)$ into (163)(b), we obtain the bifurcation equation

$$\mathcal{G}(p, \mu) = 0, \tag{165}$$

where $\mathcal{G}(\cdot, \mu) : E_k \to F_k$ is defined by

$$\mathcal{G}(p, \mu) = (I - P)G\left(p + w(p, \mu), \varepsilon_k + \mu\right). \tag{166}$$

It is known [11, Chapter 2], [29, Chapter VII] that $\mathcal{G}(\cdot, \mu)$ is also $\mathbf{O}(N)$-equivariant. By virtue of (164), the bifurcation problem (165) also has the trivial branch $(p, \mu) = (0, \mu)$ which corresponds to the one of (159). Nontrivial solutions of (165) thus correspond to non-radial solutions of (159) which are as symmetric as nonzero elements of $\mathcal{H}_{k-1}$. We will now show that nontrivial solutions of (165) bifurcate from $(p, \mu) = (0, 0)$ by utilizing the following equivariant branching lemma.

Proposition 5.3. *([12], [65], and Chapter 2 of [11], Chapter XIII of [30]). Let $\boldsymbol{O}(N)$ be acting on E_k and F_k as in (161).*

Assume:

(a) Fix $(\boldsymbol{O}(N)) := \{p \in E_k \ : \ \xi \cdot p = p \ \forall \ \xi \in \boldsymbol{O}(N)\} = \{0\}$,

(b) Ξ is an isotropy subgroup of $\boldsymbol{O}(N)$ such that dimFix$(\Xi) = 1$ in E_k,

(c) $\mathcal{G}(p,\mu) = 0$ is the bifurcation equation (165) in Fix(Ξ), and

$$\mathcal{G}_{p\mu}(0,0)(p_k) \neq 0,$$

where $p_k \in$ Fix(Ξ) is nonzero.

Then there exists a smooth nontrivial branch of solutions $(p,\mu) = (tp_k, \mu_k(t))$, $\mu_k(0) = 0$, to the equation $\mathcal{G}(p,\mu) = 0$ for t near zero.

In order to apply this beautiful result, we need to verify that the three conditions (a), (b) and (c) above are satisfied for our present situation. Condition (a) has to do with the way in which the Lie group $\mathbf{O}(N)$ acts on E_k, and hence is independent of the mapping $\mathcal{G}$. In our case, $\mathbf{O}(N)$ acts via (161), and the action of $\mathbf{O}(N)$ on the unit sphere by $\xi \cdot x = \xi x$ $(\xi \in \mathbf{O}(N), \ x \in S^{N-1})$ is transitive. Therefore, elements of E_k which are fixed by all $\xi \in \mathbf{O}(N)$ are functions of $r = |x|$ only, namely, radially symmetric ones. On the other hand, the only radially symmetric element of E_k is zero. Hence, condition (a) is fulfilled. Condition (b) is also strictly related to the action of $\mathbf{O}(N)$ on E_k, and thus is independent of $\mathcal{G}$. To show that (b) is satisfied, we need to classify the isotropy subgroups of $\mathbf{O}(N)$ whose fixed point subspace in E_k has dimension one. It has been shown in Subsection 3.1 of [53] that, when $N = 2$, the dihedral group $\mathbf{D}_{k-1}$ of degree $2(k-1)$ is the only maximal isotropy subgroup of $\mathbf{O}(2)$ whose fixed point subspace in E_k is one-dimensional. Moreover, this one-dimensional subspace is spanned by

$$p_k = A_1^{\varepsilon_k,\tau_k,2}(|x|)\cos\left((k-1)\frac{x}{|x|}\right). \tag{167}$$

Therefore, condition (b) in Proposition 5.3 is fulfilled with $\Xi = \mathbf{D}_{k-1}$. Furthermore, it has been shown in Subsection 3.2 of [53] that condition (c) in Proposition 5.3 is equivalent to

$$\frac{\partial}{\partial\varepsilon}\Lambda_k|_{\varepsilon=\varepsilon_k} \neq 0,$$

(see also the proof of the first part of Theorem 1.16 in [14]). In view of (156), we infer that the above relation holds, since $\tau_k \in \left(\frac{|\mu_{1-}|}{2} r_0^2 \varepsilon^{-\frac{4}{3}}, \ \left(|\mu_{1-}| + \frac{\mu_{2-}}{4}\right) r_0^2 \varepsilon^{-\frac{4}{3}}\right)$, provided we chose $\varepsilon_0 > 0$ sufficiently small. Consequently, when $N = 2$, all the conditions in Proposition 5.3 are satisfied with $\Xi = \mathbf{D}_{k-1}$ and p_k as in (167).

We conclude that the following theorem holds.

Theorem 5.4. *Suppose that $N = 2$, and $k \in \mathbb{N}$ is such that $\tau_k \in \left(\frac{|\mu_{1-}|}{2} r_0^2 \varepsilon^{-\frac{4}{3}}, \ \left(|\mu_{1-}| + \frac{\mu_{2-}}{4}\right) r_0^2 \varepsilon^{-\frac{4}{3}}\right)$, $\varepsilon \in (0,\varepsilon_0)$, then $(0,\varepsilon_k)$ (defined in Corollary 4.6) is a bifurcation point of $G(\phi,\varepsilon) = 0$ in the following sense: There exists a smooth nontrivial branch of solutions $(\phi,\varepsilon) = \left(tp_k + O(t^2), \varepsilon_k(t)\right)$, $\varepsilon_k(0) = \varepsilon_k$ (p_k as in (167)), to the equation $G(\phi,\varepsilon) = 0$ for t near zero. Moreover, the symmetry group of $\phi_k(\cdot)$ is $\boldsymbol{D}_{k-1}$.*

Remark 5.5. *We emphasize that the only place where $N = 2$ was used was in the verification of condition (b) in Proposition 5.3. Therefore, Theorem 5.4 extends to any dimension $N \geq 3$ as soon as one identifies the isotropy subgroups of $\boldsymbol{O}(N)$ whose fixed point subspace in E_k is one dimensional.*

5.2.1. Multiple bifurcation

The ball B_1 of $\mathbb{R}^N$, $N \geq 2$, is invariant under many group actions. By considering suitable symmetries of some homogeneous and harmonic polynomials, we can derive results on multiple non-radial bifurcation.

Let us consider the subspaces

$$\tilde{X} = \{\phi \in X \ : \ \phi(x_1, \cdots, x_N) = \phi\left(\xi \cdot (x_1, \cdots, x_{N-1}), x_N\right), \text{ for any } \xi \in \mathbf{O}(N-1)\},$$

$$\tilde{Z} = \{\phi \in Z \ : \ \phi(x_1, \cdots, x_N) = \phi\left(\xi \cdot (x_1, \cdots, x_{N-1}), x_N\right), \text{ for any } \xi \in \mathbf{O}(N-1)\},$$

where $X,\ Z$ were defined in the proof of Lemma 3.30. Clearly the mapping

$$\tilde{G}(\phi, \varepsilon) = G(\phi, \varepsilon), \ \ \phi \in \tilde{X}, \ \varepsilon \in (0, \varepsilon_0),$$

satisfies $\tilde{G} : \tilde{X}\times(0, \varepsilon_0) \to \tilde{Z}$, and is C^2. Furthermore, we have $\tilde{G}(0, \varepsilon) = 0, \ \varepsilon \in (0, \varepsilon_0)$, and the linear operator $\tilde{G}_\phi(0, \varepsilon)$: $\tilde{X} \to \tilde{Z}$ is Fredholm of index zero, for every $\varepsilon \in (0, \varepsilon_0)$. By Proposition 5.2 in [58], we know that the subspace V_k, spanned by the functions of $\mathcal{H}_{k-1}$ which are $\mathbf{O}(N-1)$ invariant, is one-dimensional. So, let $V_k = \text{Span}\{v_k\},\ k \geq 1$, for some nonzero $v_k \in \mathcal{H}_{k-1}$. Hence, if $k,\ \varepsilon_k$ are as in Corollary 4.6, we have

$$\text{Ker}\left\{\tilde{G}_\phi(0, \varepsilon_k)\right\} = \text{Span}\{q_k\}, \ \text{ where } \ q_k = A_1^{\varepsilon_k, \tau_k, 2}(|x|)v_k \in \tilde{X}. \tag{168}$$

Moreover, relation (156) implies that

$$\tilde{G}_{\phi\varepsilon}(0, \varepsilon_k)(q_k) \cap \text{Range}\left\{\tilde{G}_\phi(0, \varepsilon_k)\right\} = 0,$$

(see the proof of the first part of Theorem 1.16 in [14]). Thus, all the conditions of the Crandall-Rabinowitz bifurcation theorem from a simple eigenvalue [41, pg. 15] are satisfied for $\tilde{G} : \tilde{X} \times (0, \varepsilon_0) \to \tilde{Z}$. We conclude that the following theorem holds.

Theorem 5.6. *Suppose that $N \geq 2$, and $k \in \mathbb{N}$ is such that $\tau_k \in \left(\frac{|\mu_{1-}|}{2} r_0^2 \varepsilon^{-\frac{4}{3}}, \left(|\mu_{1-}| + \frac{\mu_{2-}}{4}\right) r_0^2 \varepsilon^{-\frac{4}{3}}\right)$, $\varepsilon \in (0, \varepsilon_0)$, then $(0, \varepsilon_k)$ (defined in Corollary 4.6) is a bifurcation point of $G(\phi, \varepsilon) = 0$ in the following sense: There exists a smooth nontrivial branch of solutions $(\tilde{\phi}, \tilde{\varepsilon}) = \left(tq_k + O(t^2), \tilde{\varepsilon}_k(t)\right)$, $\tilde{\varepsilon}_k(0) = \varepsilon_k$, $\tilde{\phi}(t) \in \tilde{X}$ (q_k as in (168)), to the equation $G(\phi, \varepsilon) = 0$ for t near zero.*

Let us now consider the subgroup $\Xi_h \subseteq \mathbf{O}(N)$ defined by

$$\Xi_h = \mathbf{O}(h) \times \mathbf{O}(N-h) \ \text{ for } 1 \leq h \leq \left[\frac{N}{2}\right].$$

In [59] it was shown that if k is odd, then the space $\mathcal{H}_{k-1}$, restricted to the functions invariant by the action of Ξ_h, has dimension one. Moreover, if we get a nontrivial bifurcating solution ϕ of (159) which is invariant with respect to the action of two groups Ξ_{h_1} and Ξ_{h_2}, then ϕ must be radial and this is not possible (recall Remark 5.1). Hence, solutions which are invariant with respect to the action of different groups Ξ_h are actually distinct. Therefore, repeating the arguments leading to the previous theorem, restricted to the subspace of functions which are invariant with respect to the action of Ξ_h, $1 \leq h \leq \left[\frac{N}{2}\right]$, we derive the existence of $\left[\frac{N}{2}\right]$ distinct smooth solution branches of (159) bifurcating from $(0, \varepsilon_k)$, k odd and sufficiently large.

Remark 5.7. *Standard tools of bifurcation theory should allow one to perform a detailed local analysis near the bifurcation points. What is the global behavior of the solution branches is a very interesting question. Based on paper [2] we expect that, at least when $N = 2$, the non-radial branches can be continued for $\varepsilon > 0$ arbitrarily small, and reach non-radial solutions of (1) of the form $u_+ - \psi$ where ψ is a suitable superposition of scaled ground states of*

$$\Delta U - V_+(x)U + U^2 = 0, \ \ (x, y) \in \mathbb{R}^2, \ \ U \in L^2(\mathbb{R}^2),$$

where $V_+ > 0$ is as in (55), (56). (Existence of ground states for the above equation has been proven in [44]).

Remark 5.8. *Bifurcations of non-symmetric solutions of some classes of singularly perturbed elliptic equations have been considered in [3], [15], [28], [32], [43], [51], [53], and [54].*

Acknowledgment. We would like to thank Prof. M. Kowalczyk for stimulating discussions. The first author is supported by a Marie Curie International Reintegration Grant within the 7th European Community Framework Programme, MIRG-CT-2007-200526, and partially supported by the FP7-REGPOT-2009-1 project "Archimedes Center for Modeling, Analysis and Computation".

[1] Alikakos, N. D., Bates, P. W., Cahn, J. W., Fife, P. C., Fusco, G., and Tanoglu, G. B., *Analysis of the corner layer problem in anisotropy*, Discrete Contin. Dyn. Syst. **6** (2006), 237–255.

[2] Allain, G., and Beaulieu, A., *Singly periodic solutions of a semilinear equation*, Ann. Inst. H. Poincaré Anal. Non Linaire **26** (2009), 1277–1297.

[3] Ambrosetti, A., Malchiodi, A., and Ni, W.-M, *Singularly perturbed elliptic equations with symmetry: existence of solutions concentrating on spheres. I.*, Comm. Math. Phys. **235** (2003), 427–466.

[4] Bender, C. M., and Orszag, S. A., Advanced Mathematical Methods for Scientists and Engineers: Asymptotic Methods and Perturbation Theory, Springer-Verlag, New York, 1999.

[5] Berestycki, H., and Lions, P. L., *Some applications of the method of sub- and supersolutions*, Lecture Notes in Math. **782**, Springer, Berlin, 1980, pp. 16–41.

[6] Butuzov, V. F., *On the stability and domain of attraction of asymptotically non-smooth stationary solutions to a singularly perturbed parabolic equation*, Computational Mathematics and Mathematical Physics **46** (2006), 413–424.

[7] Butuzov V. F., Nefedov N. N., and Schneider K. R., *Singularly perturbed elliptic problems in the case of exchange of stabilities*, J. Differential Equations **169** (2001), 373–395.

[8] Caffarelli, L. A., and Roquejoffre, J.-M., *Uniform Hölder estimates in a class of elliptic systems and applications to singular limits in models for diffusion flames*, Arch. Ration. Mech. Anal. **183** (2007), 457–487.

[9] Cantrell, R.S., and Cosner, C., Spatial Ecology via Reaction-Diffusion Equations, Wiley Series in Mathematical and Computational Biology, 2003.

[10] Carmona, R., and Simon, B., *Pointwise bounds on eigenfunctions and wave packets in N-body quantum systems*, Comm. Math. Phys. **80** (1981), 59–98.

[11] Chossat, P., and Lauterbach, R., Methods in Equivariant Bifurcations and Dynamical Systems, Advanced series in nonlinear dynamics **15**, World Scientific, 2000.

[12] Cicogna, G., *Symmetry breakdown from bifurcation*, Lett. Nuovo Cimento **31** (1981), 600–602.

[13] Conti, M., Terracini, S., and Verzini, G., *Asymptotic estimates for the spatial segregation of competitive systems*, Adv. Math. **195** (2005), 524–560.

[14] Crandall M. G., and Rabinowitz, P. H., *Bifurcation, perturbation of simple eigenvalues, and linearized stability*, Arch. Rational Mech. Anal. **52** (1973), 161–180.

[15] Dancer, E. N., *New solutions of equations on $\mathbb{R}^n$*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) Vol. **XXX** (2001), 535–563.

[16] Dancer, E.N., and Du, Y., *Competing species equations with diffusion, large interactions, and jumping nonlinearities*, J. Differential Equations, **114** (1994), 434–475.

[17] Dancer, E.N., and Yan, S., *On the superlinear Lazer-McKenna conjecture*, J. Differential Equations **210** (2005), 317–351.

[18] del Pino, M., Kowalczyk, M., and Wei, J., *Concentration on curves for nonlinear Schrödinger equations*, Comm. Pure Appl. Math. **60** (2007), pp. 113–146.

[19] Du, Y., and Nakashima, K., *Morse index of layered solutions to the heterogeneous Allen–Cahn equation*, J. Differential Equations **238** (2007), 87–117.

[20] Dumortier, F., and Roussarie. R., *Geometric singular perturbation theory beyond normal hyperbolicity*, in Multiple-time-scale Dynamical Systems (Minneapolis, MN, 1997), IMA Vol. Math. Appl. **122**, Springer, New York, 2001, pp. 29–63.

[21] Fenichel, N., *Geometric singular perturbation theory for ordinary differential equations*, J. Differential Equations **31** (1979), 53–98.

[22] Fife, P. C., *Semilinear elliptic boundary value problems with small parameters*, Arch. Rational Mech. Anal. **52** (1973), 205–232.

[23] Fife, P. C., *A phase plane analysis of a corner layer problem arising in the study of crystalline grain boundaries*, unpublished preprint, University of Utah.

[24] Fusco, G., and Pignotti, C., *Estimates for fundamental solutions and spectral bounds for a class of Schrödinger operators*, J. Differential Equations **244** (2008), 514–554.

[25] Gallo, C., and Pelinovsky, D., *On the Thomas–Fermi ground state in a harmonic potential*, to appear in Asymptotic Analysis (2011).

[26] Gidas, B., Ni, W. M., and Nirenberg, L., *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. **68** (1979), 209–243.

[27] Gilbarg, D., and Trudinger, N.S., Elliptic Partial Differential Equations of Second Order, second ed., Springer-Verlag, New York, 1983.

[28] Gladiali, F., Grossi, M., Pacella, F., and Srikanth, P.N., *Bifurcation and symmetry breaking for a class of semilinear elliptic equations in an annulus*, preprint (2009).

[29] Golubitsky, M., and Schaeffer, D.G., Singularities and Groups in Bifurcation Theory, Vol. I, Applied mathematical sciences **51**, Springer-Verlag, New York, 1985.

[30] Golubitsky, M., Stewart, I., Schaeffer, D.G., Singularities and Groups in Bifurcation Theory, Vol. II, Applied mathematical sciences **69**, Springer-Verlag, New York, 1988.

[31] Grossi, M., *Radial solutions for the Brezis–Nirenberg problem involving large nonlinearities*, J. Funct. Anal. **254** (2008), 2995–3036.

[32] *Hărăguş*-Courcelle, M., and Pego, R. L., *Travelling waves of the KP equations with transverse modulations*, C. R. Acad. Sci. Paris **328** (1999), 227–232.

[33] Hastings, S. P., and McLeod, J. B., *A boundary value problem associated with the second Painlevé transcendent and the Korteweg-de Vries equation*, Arch. Rat. Mec. Anal. **73** (1980), 31–51.

[34] Hislop, P., and Sigal, I. M., Introduction to Spectral Theory with Applications to Schrödinger Operators, Applied mathematical sciences **113**, Springer-Verlag, New York, 1996.

[35] Ignat, R., and Millot, V., *The critical velocity for vortex existence in a two–dimensional rotating Bose–Einstein condensate*, J. Funct. Anal. **233** (2006), 260–306.

[36] Iida, M., Nakashima, K., and Yanagida, E., *On certain one-dimensionar elliptic systems under different growth conditions at respective infinities*, in Asympiotic Analysis and Singularities, Advanced Studies in Pure Mathematics **47-2**, 2007, pp. 565–572.

[37] Jones, C. K. R. T., *Geometric singular perturbation theory*, in Dynamical Systems (Montecatini Terme, 1994), Lecture Notes in Math. **1609**, Springer, Berlin, 1995, pp. 44–118.

[38] Kabeya, Y., and Tanaka, K., *Uniqueness of positive solutions of semilinear elliptic equations in* $\mathbb{R}^N$ *and Séré's non–degeneracy condition*, Commun. PDEs **24** (1999), 563-598.

[39] Kaper, T. J., *An introduction to geometric methods and dynamical systems theory for singular perturbation problems*, in Analyzing Multiscale Phenomena Using Singular Perturbation Methods (Baltimore, MD, 1998), Proc. Sympos. Appl. Math. **56**, Amer. Math. Soc., Providence, RI, 1999, pp. 85–131.

[40] Kielhöfer, H., *A bifurcation theorem for potential operators*, J. Funct. Anal. **77** (1988), 1–8.

[41] Kielhöfer, H., Bifurcation Theory : An Introduction with Applications to PDEs, Applied mathematical sciences **156**, Springer-Verlag, New York, 2004.

[42] Krupa, M. and Szmolyan, P., *Geometric analysis of the singularly perturbed planar fold*, in Multiple-time-scale Dynamical Systems (Minneapolis, MN, 1997), IMA Vol. Math. Appl. **122**, Springer, New York, 2001, pp. 89–116.

[43] Lin, S.S., *On non-radially symmetric bifurcation in the annulus*, J. Differential Equations **80** (1989), 251–279.

[44] Lorca, S., and Montenegro, M., *Spike solutions of a nonlinear Schrödinger equation with degenerate potential*, J. Math. Anal. Appl. **295** (2004), 276–286.

[45] Mahmoudi, F., Malchiodi, A., and Wei, J., *Transition layer for the heterogeneous Allen–Cahn equation*, Ann. Inst. H. Poincaré Anal. Non Linaire **25** (2008), 609–631.

[46] Mahmoudi, F., Mazzeo, R., and Pacard, F., *Constant mean curvature hypersurfaces condensing along a submanifold*, Geom. Funct. Anal. **16** (2006), 924–958.

[47] Malchiodi, A., and Montenegro, M., *Boundary concentration phenomena for a singularly perturbed elliptic problem*, Comm. Pure Appl. Math. **55** (2002) 1507–1568.

[48] Mazzeo, R., and Pacard, F., *Constant curvature foliations in asymptotically hyperbolic spaces*, to appear in Revista Matematica Iberoamericana (2011).

[49] Matano, H., *Asymptotic behaviour and stability of solutions of semilinear diffusion equations*, Publ. Res. Inst. Math. Sci. **15** (1979), 401-454.

[50] Ni, W.-M., and Takagi, I., *On the shape of least-energy solutions to a semilinear Neumann problem*, Comm. Pure Appl. Math. **44** (1991), 819–851.

[51] Pacard, F., *Radial and non-radial solutions of* $-\Delta u = \lambda f(u)$*, on an annulus of* $\mathbb{R}^n$, $n \geq 3$, J. Differential Equations **101** (1993), 103–138.

[52] Rabinowitz, P. H., *On a class of nonlinear Schrödinger equations*, Z. Angew. Math. Phys. **43** (1992), 270–291.

[53] Sakamoto, K., and Suzuki, H., *Spherically symmetric internal layers for activator-inhibitor systems II. Stability and symmetry breaking bifurcations*, J. Differential Equations **204** (2004), 93–122.

[54] Sakamoto, K., *Infinitely many fine modes bifurcating from radially symmetric internal layers*, Asymptot. Anal. **42** (2005), 55–104.

[55] Schecter, S., *Existence of Dafermos profiles for singular shocks*, J. Differential Equations **205** (2004), 185–210.

[56] Schecter, S., and Sourdis, C., *Heteroclinic orbits in slow-fast Hamiltonian systems with slow manifold bifurcations*, J. Dyn. Diff. Equat. **22** (2010), 629–655.

[57] Shatah, J., and Zeng, C., *Periodic solutions for Halmitonian systems under strong constraining forces*, J. Differential Equations **186** (2002), 572–585.

[58] Smoller, J., and Wasserman, A., *Symmetry–breaking for solutions of semilinear elliptic equations with general boundary conditions*, Commun. Math. Phys. **105** (1986), 415–441.

[59] Smoller, J., and Wasserman, A., *Bifurcation and symmetry-breaking*, Invent. Math. **100** (1990), 63-95.

[60] Smoller, J.A., and Wasserman, A.G., *Symmetry, degeneracy, and uniqueness in semilinear elliptic equations, infinitesimal symmetry-breaking*, J. Funct. Anal. **89** (1990), 364–409.

[61] Sourdis, C., *On the equation* $u'' = u(|u|^p - x)$, $p > 1$, preprint (2011).

[62] Sourdis, C., and Fife, P. C., *Existence of heteroclinic orbits for a corner layer problem in anisotropic interfaces*, Adv. Differential Equations **12** (2007), 623–668.

[63] Taylor, M.E., Partial Differential Equations II, Qualitative Studies of Linear Equations, Applied mathematical sciences **116**, Springer-Verlag, New York, 1996.

[64] Tin, S.-K., Kopell, N., and Jones, C. K. R. T., *Invariant manifolds and singularly perturbed boundary value problems*, SIAM J. Numer. Anal. **31** (1994), 1558–1576.

[65] Vanderbauwhede, A., Local Bifurcation and Symmetry, Research Notes Mathematics **75**, Pitman, Boston, 1982.

[66] Walter, W., Ordinary differential equations, Graduate texts in mathematics **182**, Springer-Verlag, New York, 1998.