On the existence of solution for a Cahn-Hilliard / Allen-Cahn equation

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Abstract

In this manuscript, we consider a Cahn-Hilliard/Allen-Cahn equation is introduced in [19]. We give an existence of the solution, slightly improved from [20]. We also present a stochastic version of this equation in [2].

1 Introduction

We consider a scalar Cahn-Hilliard/Allen-Cahn equation;

$$u_t = -D\Delta(\Delta u - f'(u)) + (\Delta u - f'(u)) \quad \text{in } U \times [0,T), \tag{1}$$

with

$$\begin{cases} u(x,0) = u_0(x) & \text{in } U, \\ \frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0 & \text{on } \partial U \times [0,T), \end{cases} \tag{2}$$

where U is a smooth bounded domain in $\mathbb{R}^d$, ν is the unit normal on ∂U, $D > 0$ is a diffusion constant and f is a quartic bistable potential which has zeros at ± 1. In this paper for simplicity, we set $f(s) := (1 - s^2)^2$.

We are interested in mathematical properties of (1) and we improve existence of solution. Additionally, we consider a stochastic version of this equation and also give an existence and regularity of solution for the stochastic problem.

This equation (1) is introduced by Karali and Katsoulakis [19] as a simplification of a mesoscopic model for multiple microscopic mechanism in surface processes. Surface process had been modeled using continuum-type reaction diffusion models. These modelings are under assumption of uniform adsorptive layer in space. Even more, in natural phenomenon, it is necessary to consider the detailed interactions between particles and treat them phenomenologically. In [21], they introduced a generalization of mesoscopic theory developed in

[17]. As a specific example, they dealt with a combination of Arrhenius adsorption/desorption dynamics, metropolis surface diffusion and simple unimolecular reaction. For this phenomena, the mesoscopic equation is described by

$$u_t - D\nabla\cdot[\nabla u - \beta u(1-u)\nabla J_m * u] - [k_a p(1-u) - k_d u \exp(-\beta J_d * u)] + k_r u = 0 \quad (3)$$

where u is a coverage, $D > 0$ is a diffusion constant, k_r is a reaction constant, k_d is a desorption constant, k_a is an adsorption constant, p (constant) is a partial pressure of the gaseous species, J_d and J_m are intermolecular potentials for surface desorption and migration. Near critical temperature and in case of $k_r = 0$, by rescaling in space, identifying potentials J_d and J_m as a radial approximation of Dirac distribution and dropping down high order term of its Taylar expansion, they derived the CH/AC equation (1), which still retains its fundamental structure. For more details for the modeling, we refer to Sec 1.3 in [19].

2 Deterministic Problem

In [19] they considered the ε-scaled problem;

$$u_t^\varepsilon = -\varepsilon^2 D\Delta(\Delta u^\varepsilon - \frac{f'(u^\varepsilon)}{\varepsilon^2}) + (\Delta u^\varepsilon - \frac{f'(u^\varepsilon)}{\varepsilon^2}) \quad (4)$$

and studied the limit evolution as ε tends to 0. For the Allen-Cahn equation or the Cahn-Hilliard equation, respectively, there are several studies about the singular limit as ε tends to 0. It is well-known that the limit evolution of the Allen-Cahn equation is a mean curvature flow, which is proved in the several methods, formally by Fife in [10], Rubinstein, Sternberg and Keller in [27], from the viscosity solution by Evans and Spruck in [9] and Chen, Giga and Goto in [6], in the sense of Brakke's motion [4] by Ilmanen in [18]. For the Cahn-Hilliard equation, it is proved that the limit evolution (in different scaling from ours) is the Mullins-Sekerka model, which was formally proved in [26] and rigorously in [3].

For CH/AC equation (4), they showed that the limit evolution is also mean curvature flow but with a different coefficient;

$$V = \mu\sigma\kappa \quad (5)$$

where V is a normal velocity and κ is a mean curvature of the limit interface, σ is a surface tension given by $\sigma = \int_{-1}^{1} \sqrt{f(s)/2}\, ds$ and μ is a mobility constant given by

$$\mu = 2(\int_{\mathbb{R}} \chi q'\, dx)^{-1}, \quad (6)$$

where q is a solution of the ODE;

$$-q'' + f'(q) = 0 \text{ in } \mathbb{R} \quad \text{and} \quad q(\pm\infty) = \pm 1, \quad (7)$$

which is known as a function used in order to describe a transition profile of the Allen-Cahn equation and χ is a solution of the ODE;

$$-D\chi'' + \chi = q' \text{ in } \mathbb{R} \quad \text{and} \quad \chi(\pm\infty) = 0. \quad (8)$$

We remark that the mobility is completely different from the one of the Allen-Cahn equation ($V = \kappa$), and it holds that $\mu\sigma \geq 1$ by a simple calculation, which implies that it speeds up the mean curvature flow.

Besides, focusing on a dependence of the diffusion constant $D > 0$, in [20] they showed that solutions of (1) converge to a solution of the Allen-Cahn equation as D tends to 0 under some technical assumptions.

Concerning the Allen-Cahn structure, we rewrite (1) with (2) to the following form;

$$\begin{cases} u_t = (1 - D\Delta)v & \text{in } U \times [0, T), \\ v = \Delta u - f'(u) & \text{in } U \times [0, T), \\ u(x, 0) = u_0(x) & \text{in } U, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial U. \end{cases} \tag{9}$$

For the diffused interface problem, we usually consider the free energy functional given by

$$E(u) := \int_U \frac{|\nabla u|^2}{2} + f(u)\, dx. \tag{10}$$

For a pair of solution (u, v) of (1) it holds that

$$\begin{aligned} \frac{d}{dt}E(u) &= -\int_U (\Delta u - f'(u))u_t\, dx = -\int_U v(-D\Delta v + v)\, dx \\ &= -\int_U D|\nabla v|^2 + v^2\, dx \leq 0, \end{aligned} \tag{11}$$

and the equation (4) is a gradient flow for the free energy functional $E(u)$ with respect to the metric $(f, g) = (f, (1 - D\Delta)^{-1}g)_{L^2(U)}$.

Here we provide an existence of the solution, especially in dimension $d = 1, 2, 3$, slightly improving the result obtained in [20].

Notation. We set the initial energy $E_0 := E(u_0)$, which is well-defined for $u_0 \in H^1(U)$ in $d = 1, 2, 3$. We set

$$H^2_{bc} := \left\{ u \in H^2(U) \mid \frac{du}{d\nu} = 0 \ \ on \ \ \partial U \right\} \tag{12}$$

and

$$H^4_{bc} := \left\{ u \in H^4(U) \mid \frac{du}{d\nu} = \frac{d\Delta u}{d\nu} = 0 \ \ on \ \ \partial U \right\}. \tag{13}$$

We remark that norms on H^2_{bc} which is given by

$$\{\|\Delta u\|^2_{L^2(U)} + \eta\|u\|^2_{L^2(U)}\}^{\frac{1}{2}} \quad \text{for any} \quad \eta > 0 \tag{14}$$

are equivalent to H^2-norm. Similarly, norms on H^4_{bc} given by

$$\{\|\Delta^2 u\|^2_{L^2(U)} + \eta\|u\|^2_{L^2(U)}\}^{\frac{1}{2}} \quad \text{for any} \quad \eta > 0 \tag{15}$$

are equivalent to H^4-norm, referred to [25].

Theorem 2.1. *Suppose the initial data $u_0 \in H^1(U)$, then there exists a solution u of the initial boundary problem* (1) *with* (2) *satisfying*

$$u \in C([0, T]; H^1(U)) \cap L^2(0, T; H^2_{bc}) \cap L^4(U \times (0, T)) \quad for \ \ all \ \ T > 0. \tag{16}$$

Additionally, the function v satisfies $v \in L^2(0,T;H^1(U))$.

Moreover if the initial data $u_0 \in H^2(U)$, then

$$u \in C([0,T];H^2_{bc}) \cap L^2(0,T;H^4_{bc}) \quad for\ \ all\ \ T>0. \tag{17}$$

Remark 1. The same claim also holds for a rectangular domain under a periodic boundary condition for u and its derivatives up to the 3rd.

Proof. **(STEP1)** The proof is by a usual Galerkin method. First we consider the case of the initial value $u_0 \in H^1(U)$. Let $\{\lambda_i\}_{i\in\mathbb{N}}$ be eigenvalues and $\{\phi_i\}_{i\in\mathbb{N}}$ be eigenfunctions of Laplacian under the Neumann boundary condition

$$-\lambda_i\phi_i = \Delta\phi_i \quad \text{in } U, \qquad \frac{\partial\phi_i}{\partial\nu} = 0 \quad \text{on } \partial U \quad \text{for } i = 1,2,\cdots. \tag{18}$$

We can assume that the first eigenvalue $\lambda_1 = 0$ and the normalization condition $(\phi_i,\phi_j)_{L^2(U)} = \delta_{ij}$ for $0 = \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots$ without loss of generality. For every $N \in \mathbb{N}$ we consider the following function u^N defined by the Galerkin ansatz

$$u^N(x,t) = \sum_{i=1}^{N} a_i^N(t)\phi_i(x), \tag{19}$$

$$\int_U u_t^N\phi_j + D\Delta u^N\Delta\phi_j - Df'(u^N)\Delta\phi_j - \Delta u^N\phi_j + f'(u^N)\phi_j\,dx = 0 \tag{20}$$

for $j = 1,\cdots,N$, and

$$u^N(x,0) = \sum_{i=1}^{N}(u_0,\phi_i)_{L^2(U)}\ \phi_i(x). \tag{21}$$

This yields the following initial value problem of ODE for $a_j^N(t)$ for $j = 1,\cdots,N$

$$\frac{d}{dt}a_j^N(t) + D\lambda_j^2 a_j^N(t) + D\lambda_j(f'(u^N),\phi_j)_{L^2(U)} + \lambda_j a_j^N + (f'(u^N),\phi_j)_{L^2(U)} = 0, \tag{22}$$

with

$$a_j^N(0) = (u_0,\phi_j)_{L^2(U)}. \tag{23}$$

By the standard argument of ODE, this initial value problem has a local solution. We want to show that a global solution $\{a_j^N\}_j^N$ exists on $(0,T)$ for any $T>0$.

By multiplying $\phi_j u^N$ for each $j = 1,\cdots,N$ by both side of (22), taking $\sum_{j=1}^N$ and integrating, we have

$$\begin{aligned}\frac{d}{dt}\int_U |u^N|^2\,dx + &\int_U D|\Delta u^N|^2\,dx + D(\nabla(f'(u^N)),\nabla u^N)_{L^2(U)} \\ &+ \int_U |\nabla u^N|^2\,dx + (f'(u^N),u^N)_{L^2(U)} = 0.\end{aligned} \tag{24}$$

Since $\nabla(f'(u^N)) = f''(u^N)\nabla u^N = 12(u^N)^2\nabla u^N - 4\nabla u^N$, we have

$$D(\nabla(f'(u^N)),\nabla u^N)_{L^2(U)} = 12D\int_U (u^N)^2|\nabla u^N|^2\,dx - 4D\int_U |\nabla u^N|^2\,dx. \tag{25}$$

Similarly, since $f'(u^N) = 4(u^N)^3 - 4u^N$, we have

$$(f'(u^N), u^N)_{L^2(U)} = 4 \int_U |u^N|^4 - |u^N|^2 \, dx. \tag{26}$$

Thus by (24), (25) and (26), we have

$$\begin{aligned} &\frac{d}{dt} \int_U |u^N|^2 \, dx + \int_U D|\Delta u^N|^2 + |\nabla u^N|^2 + |u^N|^4 \, dx \\ &\leq 4 \int_U |u^N|^2 \, dx + 4D \int_U |\nabla u^N|^2 \, dx. \end{aligned} \tag{27}$$

For the second term of RHS of (27), by interpolation and the equivalence of the norm (14), we have

$$\begin{aligned} 4D \int_U |\nabla u^N|^2 \, dx &\leq cD \|u^N\|_{L^2(U)} \|u^N\|_{H^2(U)} \\ &\leq cD \|u^N\|_{L^2(U)} \{ \int_U |\Delta u^N|^2 \, dx + \|u^N\|^2_{L^2(U)} \}^{\frac{1}{2}} \\ &\leq cD \int_U |u^N|^2 \, dx + \frac{D}{2} \int_U |\Delta u^N|^2 \, dx. \end{aligned} \tag{28}$$

By (27) and (28), we have

$$\frac{d}{dt} \int_U |u^N|^2 \, dx + \int_U \frac{D}{2} |\Delta u^N|^2 + |\nabla u^N|^2 + |u^N|^4 \, dx \leq c \int_U |u^N|^2 \, dx. \tag{29}$$

Thus by Gronwall's inequality and by the definition of $a_j^N(0)$ in (23), we have

$$\int_U |u^N|^2 \, dx \leq c(T) \int_U |u^N(x, 0)|^2 \, dx \leq c \int_U |u_0|^2 \, dx. \tag{30}$$

for an arbitrary fixed $T > 0$. Thus by (30) and (29), we obtain uniform bounds $L^\infty(0, T; L^2(U))$, $L^2(0, T; H^2_{bc})$ and $L^4(U \times (0, T))$ norm of u^N.

Since $\|u^N\|_{L^2(U)} = \sum_{i=1}^N (a_i^N(t))^2$, by the bound of $\|u^N\|_{L^\infty(0,T;L^2(U))}$, we obtain a priori bound of a_j^N for $j = 1, \cdots, N$. Thus the ODE (22) and (23) have a global solution.

Next, we set $b_j^N(t)$ and $v^N(x, t)$ such as

$$b_j^N = -\lambda_j a_j^N(t) - (f'(u^N), \phi_j)_{L^2(U)} \tag{31}$$

and

$$v^N(x, t) = \sum_{j=1}^N b_j^N(t) \phi_j(x). \tag{32}$$

By the definition of v^N and b_j^N, we have for $t \in (0, T]$

$$\int_0^t \int_U D|\nabla v^N|^2 + |v^N|^2 \, dx \, dt + E(u^N(t)) = E(u^N(0)) \leq E_0. \tag{33}$$

Thus we obtain uniform bounds of $\|u^N\|_{L^\infty(0,T;H^1(U))}$ and $\|v^N\|_{L^2(0,T;H^1(U))}$.

Let Π_N be a projection of $L^2(U)$ onto $span\{\phi_1, \cdots, \phi_N\}$. For all $\zeta \in L^2(0,T;H^1(U))$ by (22), (31) and (32) we have

$$
\begin{aligned}
|\int_0^T \int_U \partial_t u^N \zeta \, dxdt| &= |\int_0^T \int_U \partial_t u^N \Pi_N \zeta \, dxdt| \\
&= |\int_0^T \int_U -D\nabla v^N \nabla \Pi_N \zeta + \int_0^T \int_U v^N \Pi_N \zeta \, dxdt| \\
&\leq c\|v^N\|_{L^2(0,T;H^1(U))} \|\zeta\|_{L^2(0,T;H^1(U))}.
\end{aligned} \tag{34}
$$

Thus we obtain a uniform bound of $\|\partial_t u^N\|_{L^2(0,T;(H^1(U))^*)}$. Together with the bounds of $L^4(U \times (0,T))$ and $L^2(0,T;H^2_{bc})$ norm of u^N, by compactness results in [22], there exist (u,v) and a subsequence, which we denote $\{u^N\}$ and $\{v^N\}$ again, such that

$$u^N \to u \quad \text{weak} - * \text{ in } L^\infty(0,T;H^1(U)), \tag{35}$$

$$u^N \to u \quad \text{weakly in } L^2(0,T;H^2_{bc}) \text{ } and \text{ } L^4((0,T) \times U), \tag{36}$$

$$u^N \to u \quad \text{strongly in } C([0,T];L^2(U)), \tag{37}$$

$$u^N_t \to u_t \quad \text{weakly in } L^2(0,T;\{H^1(U)\}^*), \tag{38}$$

$$u^N \to u \quad \text{strongly in } L^2(U \times (0,T)) \text{ and a.e. in } U \times (0,T) \tag{39}$$

and

$$v^N \to v \quad \text{weakly in } L^2(0,T;H^1(U)) \tag{40}$$

as N tends to ∞.

Consequently, we can pass to the limit in (19), (20) and (21) and the pair of (u,v) satisfies the equation. For the convergence of the initial value $u^N(0)$, by the strong convergence of u^N in $C([0,T];L^2(U))$, $u^N(0)$ converges to u_0 in $L^2(U)$. Thus we have that $u(0) = u_0$. Then the first claim of the theorem holds.

(STEP 2) Next we consider the case of the initial value $u_0 \in H^2(U)$. Adding to the previous calculation, we consider the bound of $\sup_{t\in(0,T)} \|\Delta u^N\|_{L^2(U)}$. By multiplying $\phi_j \Delta^2 u^N$ for $j = 1, \cdots, N$ by both side of (22), taking $\sum_{j=1}^N$ and integrating, and by the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
&\frac{d}{dt}\int_U |\Delta u^N|^2 dx + \int_U \frac{D}{2}|\Delta^2 u^N|^2 + D|\Delta u^N|^2 \, dx \\
&\leq \int_U D|\Delta f'(u^N)|^2 \, dx + c\int_U |\Delta u^N|^2 \, dx.
\end{aligned} \tag{41}
$$

For the first term of (41) we claim that

$$\int_U D|\Delta f'(u^N)|^2 \, dx \leq \frac{D}{4}\int_U |\Delta^2 u^N|^2 \, dx + c\int_U |\Delta u^N|^2 \, dx + c. \tag{42}$$

Indeed, since $\Delta f'(u^N) = f'''(u^N)|\nabla u^N|^2 + f''(u^N)\Delta u^N$, we have

$$
\begin{aligned}
&\int_U |\Delta f'(u^N)|^2 \, dx \\
&\leq c\int_U |u^N|^2|\nabla u^N|^4 \, dx + c\int_U (1 + |u^N|^4)|\nabla u^N|^2 \, dx \\
&\leq c\|u^N\|^2_{L^\infty(U)}\|\nabla u^N\|^4_{L^4(U)} + c\|u^N\|^4_{L^\infty(U)}\int_U |\Delta u^N|^2 \, dx + c\int_U |\Delta u^N|^2 \, dx.
\end{aligned} \tag{43}
$$

For the term $\|\nabla u^N\|_{L^4(U)}^4$, by the Sobolev inequality, interpolation and (15), we have

$$\|\nabla u^N\|_{L^4(U)} \le c\|u^N\|_{H^{1+\frac{d}{4}}(U)} \le c\|u^N\|_{H^1(U)}^{1-\frac{d}{12}}\|u^N\|_{H^4(U)}^{\frac{d}{12}} \le c(\|\Delta^2 u^N\|_{L^2(U)}^2 + 1)^{\frac{d}{24}}. \tag{44}$$

For the term $\int_U |\Delta u^N|^2\, dx$, by interpolation and (15) we have

$$\|\Delta u^N\|_{L^2(U)} \le \|u^N\|_{H^2(U)} \le c\|u^N\|_{H^1(U)}^{\frac{2}{3}}\|u^N\|_{H^4(U)}^{\frac{1}{3}} \le c(1 + \|\Delta^2 u^N\|_{L^2(U)}^2)^{\frac{1}{6}}. \tag{45}$$

Thus by (43), (44) and (45) we have

$$\begin{aligned}\int_U |\Delta f'(u^N)|^2\, dx \le & c\|u^N\|_{L^\infty(U)}^2(1 + \|\Delta^2 u^N\|_{L^2(U)}^2)^{\frac{d}{6}} \\ & + c\|u^N\|_{L^\infty(U)}^4(1 + \|\Delta^2 u^N\|_{L^2(U)}^2)^{\frac{1}{3}} + c\int_U |\Delta u^N|^2\, dx.\end{aligned} \tag{46}$$

For the norm $\|u^N\|_{L^\infty(U)}$, in $d = 1$, we have

$$\|u^N\|_{L^\infty(U)} \le c\|u^N\|_{H^1(U)} \le c\|u^N\|_{L^\infty(0,T;H^1(U))}. \tag{47}$$

In $d = 2$, since $H^{1+\varepsilon}(U) \subset L^\infty(U)$ for all $\varepsilon > 0$, by taking $\varepsilon = \frac{1}{6}$ and interpolation, we have

$$\|u^N\|_{L^\infty(U)} \le c\|u^N\|_{H^{1+\varepsilon}(U)} \le c\|u^N\|_{H^1(U)}^{1-\varepsilon}\|u^N\|_{H^4(U)}^{\varepsilon} \le c(1+\|\Delta^2 u^N\|_{L^2(U)}^2)^{\frac{1}{12}}. \tag{48}$$

In $d = 3$, since ∂U is smooth we can use Agmon's inequality and by (45) we have

$$\|u^N\|_{L^\infty(U)} \le c\|u^N\|_{H^1(U)}^{\frac{1}{2}}\|u^N\|_{H^2(U)}^{\frac{1}{2}} \le c(1 + \|\Delta^2 u^N\|_{L^2(U)}^2)^{\frac{1}{12}}. \tag{49}$$

Thus together with (43), (44), (45), (46), (47) and (49) by the Cauchy-Schwarz inequality, (42) holds. Thus by (41) and (42), we have

$$\frac{d}{dt}\int_U |\Delta u^N|^2 dx + \int_U \frac{D}{4}|\Delta^2 u^N|^2 + D|\Delta u^N|^2\, dx \le c\int_U |\Delta u^N|^2\, dx + c. \tag{50}$$

By applying Gronwall's inequality again, we have

$$\sup_{t\in(0,T)} \int_U |\Delta u^N|^2\, dx \le c\int_U |\Delta u^N(0)|^2\, dx + c \le c\int_U |\Delta u_0|^2\, dx + c. \tag{51}$$

By (50) and (51), we obtain uniform bounds of $\|u^N\|_{L^\infty(0,T;H^2_{bc})}$ and $\|u^N\|_{L^2(0,T;H^4_{bc})}$. Thus we can take a subsequence satisfying

$$u^N \to u \quad \text{weak} - * \ \text{ in } \ L^\infty(0,T;H^2_{bc}) \tag{52}$$

and

$$u^N \to u \quad \text{weakly in } \ L^2(0,T;H^4_{bc}) \tag{53}$$

adding to the previous convergence from (35) to (39) as N tends to ∞. Therefore, all the claim of the theorem holds.

□

Remark 2. Even better estimates and regularity of the solutions will be obtained by following up the semi-group theory. See the forthcoming paper [14].

3 Stochastic problem

Next, we discuss a stochastic version of this model. We consider the following CH/AC equation with a multiplicative noise;

$$\begin{cases} u_t = -D\Delta(\Delta u - f'(u)) + (\Delta u - f'(u)) + \sigma(u)\dot{W} & \text{in } U \times [0,T), \\ u(x,0) = u_0(x) & \text{in } U, \\ \frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0 & \text{on } \partial U \times [0,T), \end{cases} \tag{54}$$

where $\sigma(\cdot)$ is a bounded and Lipschitz function and W is a space-time white noise (for the noise we refer to [28]), u_0 is in $L^q(U)$ for $q \in [4, +\infty]$. For the class of U we will mention details in section 3.1.

The first motivation to consider this stochastic model is presented in [2]. Here we will explain the other interesting motivation, that is, a switching problem of a stochastically perturbed Allen-Cahn equation, which was studied in [8], [15], [16] and etc.. For a deterministic Allen-Cahn equation, there are two stable states ± 1. If we consider the Allen-Cahn equation with a white noise (remark that it is with an additive noise), it is known that the switching between deterministic two stable states ± 1 rarely occurs with a small probability by the influence of the noise. The probability is determined through the minimization problem of its action functional within the Wentzell-Freidlin theory of large deviations [13] in [11]. The mathematical analysis of the singular limit of the action functional is also an interesting topic from view of calculus of variation and there are several analysis in [16], [30], [23], [24] and etc..

Since the singular evolution in a deterministic CH/AC equation is a mean curvature flow (of course, the mobility constant is different) similarly to the one for the Allen-Cahn equation, we can expect the same terminology and also analogy holds for a stochastic CH/AC equation. Also from the other aspect, it seems possible to consider the action functional from relation to the optimal control theory [12]

Here, we concentrate on discussing the existence of the solution of (54). For a stochastic Cahn-Hilliard equation, the existence of solution was proved in [7] with an additive noise ($\sigma = 1$). In [5], they proved it for a multiplicative noise and also proved the existence of density within Malliavin calculus. In [1], they proved the existence for a generalized stochastic Cahn-Hilliard equation in general convex or Lipschitz domains.

For a mathematical formulation, let us define a weak solution u of the equation (54), if u satisfies the following;

$$\begin{aligned} &\int_U (u(x,t) - u_0(x))\varphi(x)\, dx \\ &= \int_0^t \int_U -D\Delta^2\varphi(x)u(x,s) + \Delta\varphi(x)\{Df'(u(x,s)) + u(x,s)\} - \varphi(x)f'(u(x,s))\, dxds \\ &\quad + \int_0^t \int_U \varphi(x)\sigma(u(x,s))\, W(dx,ds) \end{aligned} \tag{55}$$

for all $\varphi \in C^4(U)$ with $\frac{\partial}{\partial \nu}\varphi = \frac{\partial}{\partial \nu}\Delta\varphi = 0$ on ∂U.

Notation. For the stochastic integral $\int_0^t \int_U \cdots\ W(dy,ds)$, we use the same notation in [1]. The measure $W(dx,ds)$ induced by the one-dimensional $(d+1)$-parameter Wiener process (d for space variables and 1 for time variable) $W :=$

$\{W(x,t)|\ t \in [0,T],\ x \in U\}$ on the probability space $(\Omega, \mathcal{F}, P)$ in the set of the $\mathcal{F}_t$-adapted processes $\{W(x,s)|s \le t, x \in U\}$.

3.1 Green's function

We use Green's function for operator $-D\Delta^2$, referred to [7] and [1]. First we consider the Neumann Laplacian operator $A = -\Delta$ on $D(A) := \{u \in H^2(U)|\frac{\partial}{\partial \nu}u = 0 \text{ on } \partial U\}$, which we introduced in the proof of theorem 2.1 for a smooth domain. The eigenvalue problem for A, that is,

$$Au = \lambda u \quad \text{in } U, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial U \tag{56}$$

admits a countable set of eigenvalues as U is open, bounded and connected. As a property, any eigenvalues are real and non-negative. There exists an orthonomal basis in $L^2(U)$ consisting on eigenfunctions $\{\phi_1, \phi_2, \phi_3, \cdots\}$ corresponding to eigenvalues $0 = \lambda_1 < \lambda_2 \le \lambda_3 < \cdots$ of A. ϕ_0 related to $\lambda_0 = 0$ is obviously a constant function $\phi_1 = |U|^{-\frac{1}{2}}$. As a fact, $\lambda_i \to +\infty$ as $i \to \infty$.

Let $S(t) := e^{-DA^2 t}$ be a semi-group generated by the operator $A^2 u := \sum_{i=2}^{\infty} \lambda_i^2 u_i \phi_i$, where $u := \sum_{i=1}^{\infty} u_i \phi_i$. Then the convolution semigroup is defined by

$$S(t)u(x) := \sum_{i=2}^{\infty} e^{-D\lambda_i^2 t}(u, \phi_i)_{L^2}\phi_i(x)$$

for any $u(x)$ in $L^2(U)$ with the associated Green's function given by

$$G^D(x,y,t) := \sum_{i=1}^{+\infty} e^{-D\lambda_i^2 t}\phi_i(x)\phi_i(y). \tag{57}$$

We remark that if we consider only existence of solution, we can extend the class of U to a more general domain as far as if some estimates of Green's function for $-D\Delta^2$ hold, since the geometry of the boundary is related to Green's function. More specifically, when U is an arbitrary rectangle for $d = 1,2,3$, or more generally for $d = 1,2$ when U is a piece-wise smooth convex domain or a smooth Lipschitz domain, we can extend the existence result. In $d = 3$, U must satisfy also the minimum eigenfunctions growth, which is true for rectangles.

For more analysis of a density within Malliavin calculus, we need more detailed information of Green's function, namely, we have to restrict $U = (0,\pi)^d$ and use an explicit form of G^D.

3.2 Mild solution

By using the Green's function G^D, we can write down the equation as integral form;

$$
\begin{aligned}
u(x,t) = &\int_U u_0(y) G^D(x,y,t)\, dy \\
&+ \int_0^t \int_U \Delta G^D(x,y,t-s)\{Df'(u(y,s)) + u(y,s)\}\, dyds \\
&- \int_0^t \int_U G^D(x,y,t-s) f'(u(y,s))\, dyds \\
&+ \int_0^t \int_U G^D(x,y,t-s)\sigma(u(y,s))\, W(dy,ds)
\end{aligned}
\tag{58}
$$

for $x \in U$ and $t \in [0,T]$. We remark that a solution of (58), which is so-called a mild solution, is equivalent to a weak solution of (55).

As a recent progress, we obtained the following existence of solution and its regularity in [2];

Theorem 3.1. *There exists a unique process* $u = \{u(x,t); (x,t) \in U \times [0,T]\}$ *in* $L^\infty([0,T], L^q(U))$ *which is* $\mathcal{F}_t$*-measurable for* (x,t) *in* $U \times [0,T]$ *and satisfies the equation* (58)*. Moreover, if* u_0 *is continuous, then the solution of* (58) *has almost surely continuous trajectories. If* u_0 *is* α*-Hölder continuous for* $0 < \alpha < 1$*, then the trajectories of the solutions* (58) *are almost surely* β_1*-continuous in space and almost surely* β_2*-continuous in time, with* $\beta_1 \le \alpha$, $\beta_1 < (2 - \frac{d}{2})$ *and* $\beta_2 \le \frac{\alpha}{4}$, $\beta_2 < \frac{1}{2}(1 - \frac{d}{4})$.

Proof. See [2]. □

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