Gradient dynamics: Motion near a manifold of quasi-equilibria

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Abstract

In a general Banach space we consider gradient-like dynamical systems with the property that there is a manifold along which solutions move slowly compared to attraction in the transverse direction. Conditions are given on the energy (or Lyapunov functional) that ensure solutions starting near the manifold stay near for a long time or even for ever. The abstract results are then used to show the super slow motion of interfaces for the vector Allen-Cahn and Cahn-Morral systems.

1 Introduction

The dynamics of a gradient system is obviously determined by the geometric structure or, as some authors like to say, by the *landscape* of the graph $\mathcal{G}^J$ of the energy functional $J : H \to \mathbb{R}$. In certain cases, for instance in singular perturbation problems, J depends on a small parameter $\epsilon > 0$ and, for $\epsilon << 1$, $\mathcal{G}^J$ exhibits special features that have peculiar dynamical counterparts. A quite striking phenomenon in this context is the occurrence of *Slow Motion* (see, for instance, [15], [18], [4] and [30]). The geometric structure of $\mathcal{G}^J$ responsible for this phenomenon can qualitatively be described as follows: There exists a manifold $\mathcal{M} \subset H$ of low energy states relative to some neighborhood, that is, the energy rapidly grows when moving away from $\mathcal{M}$. Moreover $\mathcal{M}$ is a set of *quasi-equilibria* in the sense that variations of J along $\mathcal{M}$ are small compared to variations away from $\mathcal{M}$. We formulate hypotheses that correspond to a quantitative description of $\mathcal{G}^J$ in a neighborhood of $\mathcal{M}$ and show that, provided a certain condition is satisfied, if the initial condition $u_0 \in H$ is sufficiently close to $\mathcal{M}$ and the value of $J(u_0)$ is of the order of the typical values of J on $\mathcal{M}$, then the solution $t \to u(t)$ of the gradient system

$$(1.1) \qquad \begin{cases} u_t = -\nabla J(u), \\ u(0) = u_0, \end{cases}$$

remains near $\mathcal{M}$ for a long time or even for ever. A similar point of view was already adopted in [25] where an abstract theorem that relates the structure of $\mathcal{G}^J$ to the existence of slow motion has been proved. Our main results, Theorems 2.1 and 2.2 below, are

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abstract developments of some ideas first considered in [4] in the context of slow motion of bubbles in Cahn-Hilliard and conserved Allen-Cahn dynamics. We do not insist that the motion along $\mathcal{M}$ be super slow for our general result, but that can be inferred if conditions are placed upon the variation of J along $\mathcal{M}$. Thus, our main results only concern positive invariance of a neighborhood of $\mathcal{M}$ and dynamics being then driven by the variation of the energy along $\mathcal{M}$.

In fact, we will consider a slightly more general situation in which J is merely a Lyapunov function for an evolutionary equation

$$\begin{cases} u' = F(u) \\ \\ u(0) = u_0 \in H. \end{cases} \tag{1.2}$$

We assume that H is a Banach space and F is such that (1.2) has a unique continuous solution $t \to u(t) \in H$ existing on $[0,T)$ for some $T > 0$. We assume that $J : H \to \mathbb{R}$ is continuous and

$$\text{if } t_1, t_2 \in [0,T) \text{ with } \ t_1 \geq t_2 \text{ then } \ J(u(t_1)) \leq J(u(t_2)). \tag{1.3}$$

Let $\|u\|$ denote the norm of u in H. We develop this point of view in Section 2.1 where we formulate the assumptions on the geometry of $\mathcal{G}^J$ and prove two abstract theorems: Theorem 2.1 that says that a solution that starts near $\mathcal{M}$ with energy of the order of the energy on $\mathcal{M}$ is trapped near $\mathcal{M}$ until it gets close to $\partial\mathcal{M}$ or, forever, if $\partial\mathcal{M}$ is empty. On the other hand Theorem 2.2 provides a lower bound on the time needed to reach $\partial\mathcal{M}$.

In the remaining part of the paper we apply the results of Section 2 to two different situations. Our intention is to show how different problems fit perfectly into the abstract framework developed in Section 2.1. This also substantiates our view that Theorem 2.1 captures the essential features of the various phenomena of slow motion discussed in the literature. In Section 3 we consider a finite-dimensional approximation of the geometric evolution of a small *droplet* which is contained in a planar region Ω and slides on $\partial\Omega$ (see Theorem 3.1). In Section 4 we study layers dynamics for the Allen-Cahn and Cahn-Morral systems. These are gradient systems of the same functional with respect to two different inner products. In spite of this, since Theorem 2.1 is only based on the geometry of $\mathcal{G}^J$, we can treat the two problems in a unified way. Theorem 4.1, the main result of Section 4, establishes the exponentially slow motion of layers in solutions to the vector Allen-Cahn and Cahn-Morral systems in one space dimension. Our proof of Theorems 3.1 and 4.1 follows a precise path and consists essentially in the systematic verification of the assumptions of Theorem 2.1.

Slow motion of layers for the scalar Allen-Cahn equation was first described in [24] and then analyzed in [15], [18], [16], [17] and [29] with a geometric approach based on linearization and invariant manifold theory. In [13] and [14] the problem was reconsidered by a variational approach in the spirit of Γ-convergence. The geometric approach was also used in [2], [9] and [10] to analyze layer dynamics in the context of the one-dimensional Cahn-Hilliard equation. The technique in [13] was extended in [19] to show exponential slow motion of layers for the Cahn-Morral system. Slow dynamics of layers for the vector Allen-Cahn equation was studied in [11] in the case the minima of W are nondegenerate and in [12] for the degenerate case. The approach in [11] and [12] is variational and bears some similarity with [13] and [19] but uses local energy estimates derived from the parabolic equation (4.2). Our analysis requires more restrictive assumptions but the main result, the implication (4.13) in Theorem 4.1 is merely a geometric property of the graph $\mathcal{G}^J$ of the functional (4.1) that applies indifferently to both (4.2) and (4.3) and makes no use of the corresponding parabolic dynamics.

Slow motion appears also in higher space dimensions. Slow motion of an almost spherical interface dividing a domain into two regions where $u(t, u_0)$ is near to one or the other of the two minima of W was studied in [6], [30] and [4] for the Cahn-Hilliard and mass-conserving Allen-Cahn equation. Other problems that exhibit metastability and slow dynamics are considered in [7], [30] and [22].

2 Persistence of the dynamics near $\mathcal{M}$.

Throughout we assume that $\mathcal{M}$ is a smooth embedded manifold with or without boundary and assume that there is $\delta_0 > 0$ such that

$$|J(v_1) - J(v_2)| \leq \delta_0, \quad \text{for } \; v_1, v_2 \in \mathcal{M}. \tag{2.1}$$

For $\eta > 0$ small we denote by $\mathcal{N}^\eta$ the $\eta-$neighborhood of $\mathcal{M}$ defined by

$$\mathcal{N}^\eta \equiv \begin{cases} \{u \in H : d(u, \mathcal{M}) < \eta\}, & \text{if } \partial\mathcal{M} = \emptyset, \\ \\ \{u \in H : d(u, \mathcal{M}) < \eta, \; d_X(u, \partial\mathcal{M}) > d\}, & \text{if } \partial\mathcal{M} \neq \emptyset, \end{cases} \tag{2.2}$$

where $d > 0$ is fixed and small and d_X is a distance function, possibly with respect to a different space, X.

The situation we have in mind is that X is a separable Hilbert space, $H \subset X$ is a dense subspace which is itself Hilbert. We denote by $\langle , \rangle$ both the inner product in X and the pairing of H and its dual space H^*, $H \subset X \subset H^*$. One may take F to be semilinear, of the familiar form $F(u) = Au + f(u)$, where $A \in \mathcal{L}(H, H^*)$ is a bounded linear operator, typically a uniformly elliptic operator, and $f : H \to X$ is sufficiently smooth. With $D(A)$ the domain of A as an unbounded operator in X, f should be smooth enough such that, for $u_0 \in D(A)$, the solution $t \to u(t, u_0) \in C([0, T]; H)$ and both $u', Au \in C([0, T], X)$. We assume that for some fixed small $\bar{\eta} > 0$ there exists a projection defined on $\mathcal{N}^{\bar{\eta}}$ $u \to v^u \in \mathcal{M}$ that satisfies

$$d(u, \mathcal{M}) \leq \delta \quad \Rightarrow \quad \|u - v^u\| \leq \bar{C}\delta, \;\; u \in \mathcal{N}^{\bar{\eta}}, \; \delta \in [0, \bar{\eta}). \tag{2.3}$$

for some constant $\bar{C} \geq 1$ and allows one to decompose $J(u) - J(v^u)$:

$$J(u) - J(v^u) = L(v^u, u - v^u) + Q(v^u, u - v^u) + N(v^u, u - v^u), \quad \text{for } \; u \in \mathcal{N}^\eta, \tag{2.4}$$

where the name L suggests it is linear, Q suggests it is quadratic, and N stands for higher order nonlinear terms, even though, strictly speaking, these operators need not be linear, etc.. On L, Q, and N we make the following hypotheses:

$\mathbf{H}_1$ There exists $\delta_L > 0$ such that

$$|L(v^u, u - v^u)| \leq \delta_L \|u - v^u\|, \quad \text{for } \; u \in \mathcal{N}^{\bar{\eta}}.$$

$\mathbf{H}_2$ (*$\mathcal{M}$ is a manifold of low energy*) There exists $K_0 > 0$ such that

$$Q(v^u, u - v^u) \geq K_0 \|u - v^u\|^2, \quad \text{for } \; u \in \mathcal{N}^{\bar{\eta}}.$$

H$_3$

$$|N(v^u, u - v^u)| \leq K_1 \|u - v^u\|^\mu, \quad \text{for} \;\; u \in \mathcal{N}^{\bar{\eta}},$$

for some $K_1 \geq 0$ and some $\mu > 2$.

Define

$$\hat{\eta} = \begin{cases} (\frac{K_0}{2K_1})^{\frac{1}{\mu-2}} & \text{if} \;\; K_1 > 0, \\ +\infty, \;\; \text{if} \;\; K_1 = 0, \end{cases} \tag{2.5}$$

and set

$$\eta^* = \min\{\frac{\hat{\eta}}{\bar{C}}, \bar{\eta}\}, \tag{2.6}$$

where $\bar{C}$ is from (2.3).
Our first result is an abstract theorem that captures the essential features of the energy landscape that are remarkably common in singularly perturbed PDEs with variational structure and are at the basis of various phenomena of slow motion that have been discussed in the literature [2], [4], [10], [15], [18]. In section 4 we analyze in detail how the abstract result can be applied to show existence of slow motion in the context of the vector Allen-Cahn and Cahn-Morral dynamics.

Theorem 2.1. *Assume that* $J : H \to \mathbb{R}$, $\mathcal{M}$ *and the projection* $u \to v^u \in \mathcal{M}$, *defined on* $\mathcal{N}^{\bar{\eta}}$, *satisfy hypotheses* **H**$_1$-**H**$_3$. *Assume that* u *satisfies*

$$J(u) < \sup_{v \in \mathcal{M}} J(v) + \delta_1, \tag{2.7}$$

for some $\delta_1 > 0$ *and that* δ_0, δ_1, *and* δ_L *satisfy the condition*

$$\eta_* \equiv \frac{\delta_L}{K_0} + \sqrt{\frac{\delta_L^2}{K_0^2} + 2\frac{\delta_0 + \delta_1}{K_0}} \leq \eta^*. \tag{2.8}$$

Then

(i)

$$u \in \mathcal{N}^{\eta^*} \quad \Rightarrow \quad u \in \mathcal{N}^{\eta_*}, \tag{2.9}$$

i.e., if u *finds itself in the larger neighborhood of* $\mathcal{M}$, $\mathcal{N}^{\eta^*}$, *then it is actually in the smaller neighborhood,* $\mathcal{N}^{\eta_*}$.

(ii) *For any* $u_0 \in \mathcal{N}^{\eta^*}$, *if* $[0, T)$ *is the (positive) maximal interval for which the solution* $u(t, u_0)$ *to (1.2) with initial datum* u_0 *lies in* $\mathcal{N}^{\eta_*}$, *one of the following alternatives prevails*

a) $T = +\infty$ *and* $u(t, u_0) \in \mathcal{N}^{\eta_*}$, *for* $t \in [0, +\infty)$.

b) $T < +\infty$ *and* $\lim_{t \to T} d_X(u(t, u_0), \partial\mathcal{M}) = d$, *the constant in definition (2.2).*

In particular, $T = +\infty$ *if* $\partial\mathcal{M} = \emptyset$.

The idea of the proof can be seen in Figure 1, which illustrates the geometry of $\mathcal{G}^J$ described in Theorem 2.1 and explains how the interplay between the energy and the neighborhood of $\mathcal{M}$ leads to the implication (2.9).

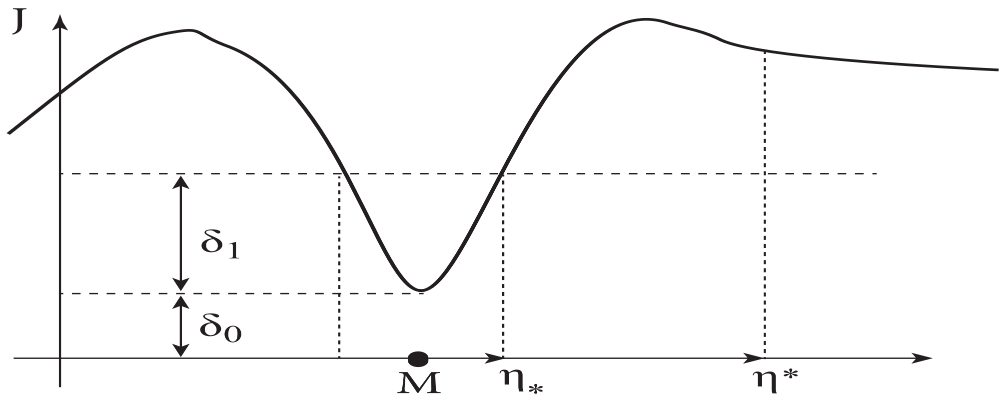

Figure 1: The geometry of $\mathcal{G}^J$ near $\mathcal{M}$

Proof. Take $u \in \mathcal{N}^{\eta^*}$ and set $\eta(u) = \|u - v^u\|$. Note that (2.3) yields

$$\eta(u) \le \bar{C}\eta^* \le \hat{\eta}. \tag{2.10}$$

Observe that hypotheses $\mathbf{H}_1$-$\mathbf{H}_3$ imply

$$-\delta_L \eta(u) + K_0 \eta(u)^2 - K_1 \eta(u)^\mu \le J(u) - J(v^u). \tag{2.11}$$

Also (2.1) and (2.7) yield

$$J(u) - J(v^u) < \sup_{v\in\mathcal{M}} J(v) + \delta_1 - \inf_{v\in\mathcal{M}} J(v) \le \delta_0 + \delta_1. \tag{2.12}$$

Figure 1:

Combining with (2.11) we see that $\eta(u)$ satisfies the inequality

$$-\delta_L \eta(u) + K_0 \eta(u)^2 - K_1 \eta(u)^\mu < \delta_0 + \delta_1,$$

or equivalently

$$\frac{K_0}{2}\eta(u)^2 - \delta_L \eta(u) - (\delta_0 + \delta_1) < K_1 \eta(u)^\mu - \frac{K_0}{2}\eta(u)^2. \tag{2.13}$$

From (2.10) and the definition of $\hat{\eta}$ in (2.5), the expression on the right is non-positive. Hence,

$$\frac{K_0}{2}\eta(u)^2 - \delta_L \eta(u) - (\delta_0 + \delta_1) < 0.$$

But this quadratic in $\eta(u)$ has η_* as its largest zero and consequently, $\eta(u) < \eta_*$. This, of course means $u \in \mathcal{N}^{\eta_*}$.

The proof of (i) is complete. Statement (ii) is an obvious consequence of (i).

□

Remark. Note that for the validity of Theorem 2.1 it is not required that all equilibria of (1.2) in a neighborhood of the basic manifold $\mathcal{M}$ lie on $\mathcal{M}$. In particular δ_L in $\mathbf{H}_1$ is not required to vanish. This is an advantage of our approach since the construction of $\mathcal{M}$ with the property that includes nearby equilibria is rather easy in the case of slow motion for the scalar Allen-Cahn equation [25] but is a nontrivial task in the vector case and for other higher dimension situations.

The next result gives an estimate on the time it takes for a solution to leave a neighborhood of $\mathcal{M}$, in the case that $\partial\mathcal{M} \neq \emptyset$. For this we take H to be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$.

Under the assumptions of Theorem 2.1, if $\partial\mathcal{M}$ is not empty, we know that the only way a solution originating in $\mathcal{N}^{\eta^*}$ and satisfying (2.7) can leave $\mathcal{N}^{\eta_*}$ is through its ends. If we assume that a differentiable functional J is a *Strong Lyapunov function* for (1.2) in the sense that

[SL] $\langle \nabla J(u), F(u) \rangle \leq -c_0 \|F(u)\|_X^\tau$ for some $\tau > 1$, $c_0 > 0$, and all $u \in \mathcal{N}^{\bar{\eta}}$,

we can make a quantitative statement concerning the long-term dynamics and, in particular, establish a lower bound on the time the solution remains in $\mathcal{N}^{\eta_*}$.

Theorem 2.2. *Assume that the hypotheses of Theorem 2.1 hold. Assume that* $[0,T) \ni t \to u(t,u_0)$ *is differentiable with* $u' \in X$ *and that* J *is a Strong Lyapunov function for (1.2), i.e.,* **[SL]** *holds. Then each* $t \in (0,T)$ *provides an upper bound for the displacement in the* X*-norm:*

$$\|u(t) - u_0\|_X \leq t^{1/\tau^*} \Big(\frac{\delta_0 + \delta_1}{c_0} \Big)^{1/\tau}, \tag{2.14}$$

where τ^* *is the conjugate of* τ. *In particular, if* $\partial\mathcal{M} \neq \emptyset$,

$$T \geq \frac{(d_X(u_0, \partial\mathcal{M}) - d)^{\tau^*} c_0^{\tau^*-1}}{(\delta_0 + \delta_1)^{\tau^*-1}}.$$

Proof. We have

$$\begin{aligned} \|u(t) - u_0\|_X &\leq \int_0^t \|u'\|_X \leq t^{1/\tau^*} (\int_0^t \|u'\|_X^\tau)^{1/\tau} = t^{1/\tau^*} (\int_0^t \|F(u(s))\|_X^\tau ds)^{1/\tau} \\ &\leq t^{1/\tau^*} \Big(\frac{-1}{c_0} \int_0^t \langle \nabla J(u), F(u) \rangle ds \Big)^{1/\tau} = t^{1/\tau^*} \Big(\frac{-1}{c_0} \int_0^t \langle \nabla J(u), u_t u' \rangle ds \Big)^{1/\tau} \\ &= t^{1/\tau^*} (\frac{J(u_0) - J(u(t))}{c_0})^{1/\tau} \leq t^{1/\tau^*} (\frac{\delta_0 + \delta_1}{c_0})^{1/\tau}, \end{aligned} \tag{2.15}$$

by (2.12). Hence,

$$t \geq \|u(t) - u_0\|_X^{\tau^*} \Big(\frac{c_0}{\delta_0 + \delta_1} \Big)^{\tau^*-1}. \tag{2.16}$$

The other inequality follows from Theorem 2.1. $\square$

3 An example of dynamics on $\mathbb{R}^N$

Our first application of Theorem 2.1 is to a discrete version of the geometric evolution of a small *droplet* along the boundary of a plane region.
Let $\Omega \subset \mathbb{R}^2$ be a bounded smooth domain and let $\Omega^\epsilon \subset \Omega$, $0 < \epsilon << 1$, be a small region such that $\Gamma^\epsilon = \partial\Omega^\epsilon \cap \bar{\Omega}$ is a simple smooth, almost semicircular, arc which intersects $\partial\Omega$ at right angles at the end points, see Fig. 3. We consider a finite-dimensional approximation of the geometric evolution $t \to \Omega_t^\epsilon$ of a given initial droplet Ω_0^ϵ under the gradient dynamics associated to the constrained functional

$$\begin{aligned} &J(\Omega^\epsilon) = L(\Gamma^\epsilon), \\ &A(\Omega^\epsilon) = \epsilon^2 \frac{\pi}{2}, \end{aligned} \tag{3.1}$$

where L and A denote *length* and *area*, respectively.

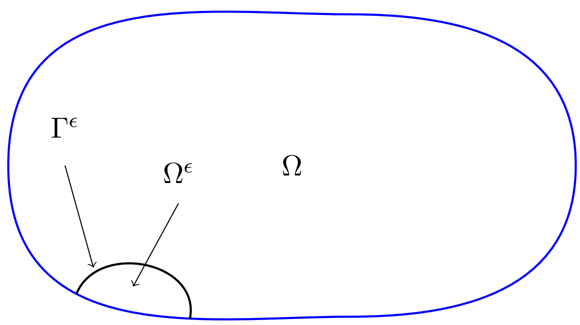

Figure 2: The droplet Ω^ϵ on the boundary of Ω.

Let $p : [0, L(\mathcal{C})) \to \mathbb{R}^2$ be a representation of a simple closed curve $\mathcal{C} \subset \partial\Omega$ with parameter being arclength $s \in [0, L(\mathcal{C}))$. Given s, the map

$$y \to h(y, s) = p(y_1 + s) + n(y_1 + s)y_2, \tag{3.2}$$

where $n(s)$ is the unit normal to $\partial\Omega$ at $p(s)$ pointing inside Ω, is a diffeomorphism of a neighborhood of the origin in $\overline{\mathbb{R}^2_+} = \{y \in \mathbb{R}^2 : y_2 \geq 0\}$ onto a corresponding neighborhood of $p(s)$ in $\overline{\Omega}$. This follows from the inverse mapping theorem and

$$\begin{aligned} &\frac{\partial h}{\partial y}(y, s) = \begin{pmatrix} \dot{p}_1(y_1 + s)(1 - k(y_1 + s)y_2) & -\dot{p}_2(y_1 + s) \\ \dot{p}_2(y_1 + s)(1 - k(y_1 + s)y_2) & \dot{p}_1(y_1 + s) \end{pmatrix} \\ &\Rightarrow \det(\frac{\partial h}{\partial y}) = 1 - k(y_1 + s)y_2, \end{aligned} \tag{3.3}$$

where $k(s)$ is the curvature of $\partial\Omega$ at $p(s)$, taken positive for a circle. Given $s \in [0, L(\mathcal{C}))$ and a point $x \in \Omega$ in a small neighborhood of $p(s)$ the equation $x = h(y, s)$ uniquely defines a vector $y = y(s)$ that we call the vector of the coordinates of x with respect to $p(s)$. If, in particular, we choose $s = s_x$ with s_x such that $p(s_x)$ is the (unique) orthogonal projection of x on $\partial\Omega$ we have

$$y_1(s_x) = 0, \quad y_2(s_x) = (x - p(s_x)) \cdot n(s_x).$$

For any $s \in [0, L(\mathcal{C}))$ the coordinates $y_i(s)$ of x with respect to $p(s)$ are determined by the condition

$$p(s_x) + y_2(s_x)n(s_x) = x = p(y_1(s) + s) + y_2(s)n(y_1(s) + s).$$

This and the uniqueness of $p(s_x)$ imply $p(s_x) = p(y_1(s) + s)$ and therefore

$$y_1(s) = s_x - s, \quad \text{and} \quad y_2(s) = y_2(s_x).$$

Given $s \in [0, L(\mathcal{C}))$ and a map $r \in C^1([0, \pi]; \mathbb{R})$ that satisfies

$$r_\theta(0) = r_\theta(\pi) = 0 \tag{3.4}$$

the arc

$$\Gamma^\epsilon(s, r) = \{h(y(\theta, \epsilon), s) : y(\theta, \epsilon) = \epsilon(1 + \epsilon r(\theta))(\cos\theta, \sin\theta)^\top, \ \theta \in [0, \pi]\}, \tag{3.5}$$

is contained in $\overline{\Omega}$ and intersects $\partial\Omega$ at right angles at the end points. Moreover,

$$\Gamma^\epsilon(s, r) = \overline{\partial\Omega^\epsilon(s, r) \cap \Omega},$$

where $\Omega^\epsilon(s, r) \subset \Omega$ is an almost semicircular region with approximate center at $p(s)$ and radius ϵ. The representation (3.5) of $\Gamma^\epsilon(s, r)$ is not unique but depends on s. We show below that there is a unique choice of s which allows one to represent $\Gamma^\epsilon(s, r)$ with r satisfying

$$\int_0^\pi r(\theta)\cos\theta d\theta = 0. \tag{3.6}$$

We assume that r satisfies (3.6) and therefore we have a unique s and a unique representation of $\Gamma^\epsilon(s, r)$. The uniqueness of s is basic for the definition of the manifold $\mathcal{M}$ and the associated projection for the case at hand. Indeed we show that for each $s \in [0, L(\mathcal{C}))$ and small $\epsilon > 0$ there is a unique constant $\bar{r}_0(s, \epsilon) \in \mathbb{R}$ such that $(3.1)_2$ holds with $\Omega^\epsilon = \Omega^\epsilon(s, \bar{r}_0(s, \epsilon))$. Now set

$$\mathcal{M} = \{\Gamma^\epsilon(s) = h(y, s) : y = \epsilon(1 + \epsilon\bar{r}_0(s, \epsilon))(\cos\theta, \sin\theta)^\top \in C([0, \pi], \mathbb{R}^2), \ s \in [0, L(\mathcal{C}))\}. \tag{3.7}$$

More generally we prove that the constraint $(3.1)_2$ determines the average $r_0 = \frac{1}{\pi}\int_0^\pi r(\theta)d\theta$ of r and therefore that each $\Gamma^\epsilon(s, r)$ of the form (3.5) is actually identified by s and the map $\rho = r - r_0(\rho, s, \epsilon)$ and we refer to the pair (s, ρ), where ρ has zero average and satisfies (3.6), as the *coordinates* of $\Gamma^\epsilon(s, r)$ and write $\Gamma^\epsilon(s, \rho)$ instead of $\Gamma^\epsilon(s, r)$. Determining the evolution of $\Gamma^\epsilon(s, \rho)$ is equivalent to determining the evolution of the pair (s, ρ). After having established all this we compute $L(\Gamma^\epsilon(s, \rho))$ as a function of (s, ρ) and restrict to finite-dimensional subspaces. We fix an integer $N \geq 2$ and take

$$\rho = \sum_{n=2}^N c_n \cos n\theta. \tag{3.8}$$

We denote by $\|\cdot\|$ the $L^2(0, \pi)$ norm and let L^2_N be the subspace of $L^2(0, \pi)$ given by functions of the form (3.8). Fix $\alpha \in (0, \frac{1}{2})$ and set

$$\mathcal{N}^{\bar{\eta}(\alpha)} = \{\Gamma^\epsilon(s, \rho) : s \in [0, L(\mathcal{C})), \ \rho \in L^2_N, \ \|\rho\| < \bar{\eta}(\alpha) = \frac{\bar{c}}{\epsilon^{1-\alpha}}\} \tag{3.9}$$

for some $\bar{c} > 0$. We can now state the main result of this section

Theorem 3.1. *Given $N \geq 2$ there exists $\epsilon_N > 0$ such that for $\epsilon \in (0, \epsilon_N)$, if $\Gamma^\epsilon(s_0, \rho_0) \subset \mathcal{N}^{\bar{\eta}(\alpha)}$ satisfies*

$$L(\Gamma^\epsilon(s_0, \rho_0)) \leq \max_{s \in [0, L(\mathcal{C}))} L(\Gamma^\epsilon(s)) + \bar{C}\epsilon^2$$

and if $(s, \rho) : [0, T) \to [0, L(\mathcal{C})) \times L^2_N$ is the solution to the problem

$$\begin{cases} (\dot{s}, \dot{\rho}) = -\nabla_{(s,\rho)} L(\Gamma^\epsilon(s, \rho)) \\ (s(0), \rho(0)) = (s_0, \rho_0), \end{cases}$$

then $T = +\infty$ and

$$\Gamma^\epsilon(s(t), \rho(t)) \in \mathcal{N}^{\bar{\eta}(\frac{1}{2})}, \quad \text{for} \quad t \in [0, +\infty).$$

The next lemma asserts that (3.6) leads to a unique representation for an interface Γ.

Lemma 3.2. *Given $\bar{s} \in [0, L(\mathcal{C}))$ and C^1 function $\bar{r} : [0, \pi] \to \mathbb{R}$, there exists a unique $s \in [0, L(\mathcal{C}))$ in an ϵ-neighborhood of $\bar{s}$ and function $r : [0, \pi] \to \mathbb{R}$ satisfying (3.6) such that*

$$\Gamma^\epsilon(s, r) = \Gamma^\epsilon(\bar{s}, \bar{r}).$$

Proof. Set $\bar{y}(\vartheta, \epsilon) = \epsilon(1 + \epsilon\bar{r}(\vartheta))(\cos\vartheta, \sin\vartheta)^\top$. To determine the function $r(\cdot, s, \epsilon) : [0, \pi] \to \mathbb{R}$ defined by (3.5) for s in a neighborhood of $\bar{s}$, we solve

$$\begin{aligned} p(\bar{y}_1(\vartheta) + \bar{s}) + n(\bar{y}_1(\vartheta) + \bar{s})\bar{y}_2(\vartheta) &= h(\bar{y}(\vartheta), \bar{s}) \\ &= x \\ &= h(y(\theta, s), s) = p(y_1(\theta, s) + s) + n(y_1(\theta, s) + s)y_2(\theta, s). \end{aligned} \tag{3.10}$$

for each $\vartheta \in [0, \pi]$. We obtain

$$y_1(\theta, s) = \bar{y}_1(\vartheta) + \bar{s} - s, \quad y_2(\theta, s) = \bar{y}_2(\vartheta)$$

and, if we define $\sigma = \frac{s - \bar{s}}{\epsilon}$ and write $r(\theta, \sigma)$ in place of $r(\theta, \bar{s} + \epsilon\sigma, \epsilon)$, we have

$$\begin{aligned} (1 + \epsilon r(\theta, \sigma)) \cos\theta &= (1 + \epsilon\bar{r}(\vartheta)) \cos\vartheta - \sigma, \\ (1 + \epsilon r(\theta, \sigma)) \sin\theta &= (1 + \epsilon\bar{r}(\vartheta)) \sin\vartheta. \end{aligned}$$

This implies

$$\begin{aligned} \cos\theta &= \frac{(1 + \epsilon\bar{r}(\vartheta)) \cos\vartheta - \sigma}{\sqrt{((1 + \epsilon\bar{r}(\vartheta)) \cos\vartheta - \sigma)^2 + (1 + \epsilon\bar{r}(\vartheta))^2 \sin\vartheta^2}}, \\ (1 + \epsilon r(\theta, \sigma))^2 &= ((1 + \epsilon\bar{r}(\vartheta)) \cos\vartheta - \sigma)^2 + (1 + \epsilon\bar{r}(\vartheta))^2 \sin\vartheta^2 \end{aligned} \tag{3.11}$$

and these determine the maps $\theta(\vartheta, \sigma)$ and $r(\theta, \sigma)$.

We observe that $r(\theta, \sigma)$ satisfies (3.6) if and only if

$$\int_0^\pi (1 + \epsilon r(\theta, \sigma)) \cos\theta d\theta = 0. \tag{3.12}$$

From (3.11) we derive

$$\theta_\vartheta = \frac{(1+\epsilon\bar r(\vartheta))^2 - (\epsilon\bar r_\theta(\vartheta)\sin\vartheta + (1+\epsilon\bar r(\vartheta))\cos\vartheta)\sigma}{((1+\epsilon\bar r(\vartheta))\cos\vartheta - \sigma)^2 + (1+\epsilon\bar r(\vartheta))^2\sin\vartheta^2} \tag{3.13}$$

and we can rewrite (3.12) in the form

(3.14)

$$\begin{aligned} F(\epsilon\bar r,\sigma) &\equiv \int_0^\pi ((1+\epsilon\bar r(\vartheta))\cos\vartheta - \sigma)\theta_\vartheta d\vartheta \\ &= \int_0^\pi ((1+\epsilon\bar r(\vartheta))\cos\vartheta - \sigma)\frac{(1+\epsilon\bar r(\vartheta))^2 - (\epsilon\bar r_\theta(\vartheta)\sin\vartheta + (1+\epsilon\bar r(\vartheta))\cos\vartheta)\sigma}{((1+\epsilon\bar r(\vartheta))\cos\vartheta - \sigma)^2 + (1+\epsilon\bar r(\vartheta))^2\sin\vartheta^2} d\vartheta = 0. \end{aligned}$$

We have $F(0,0)=0$ and $D_2F(0,0) = -\frac{\pi}{2}$, therefore the implicit function theorem provides the existence of $C_0>0$ and $\delta>0$ such that, for each $\bar r$ satisfying $\|\bar r\|_{C^1(0,\pi)} < \frac{C_0}{\epsilon}$ there exists a unique $\sigma \in (-\delta,\delta)$ that solves (3.14). This concludes the proof. □

Next we show that the area constraint in (3.1) uniquely determines the average value of r. In the following we use the notation

$$a = a(\theta,s,\epsilon) = 1 + \epsilon r(\theta,s)$$

and

$$C = \cos\theta, \quad S = \sin\theta.$$

With this notation we have

$$\begin{aligned} y_1 &= y_1(\theta) = \epsilon a C, \quad y_{1,\theta} = \epsilon(a_\theta C - aS), \\ y_2 &= y_2(\theta) = \epsilon a S, \quad y_{2,\theta} = \epsilon(a_\theta S + aC), \end{aligned}$$

and using (3.3) we find

(3.15)

$$\begin{aligned} A(\Omega^\epsilon) &= \int_{\Omega^\epsilon} dx = \int_{h^{-1}(\Omega^\epsilon,s)} \left|\frac{\partial h(y,s)}{\partial y}\right| dy = \int_{h^{-1}(\Omega^\epsilon,s)} (1 - k(y_1+s)y_2) dy_1 dy_2 \\ &- \int_0^\pi y_{1,\theta} d\theta \int_0^{y_2(\theta)} (1-k(y_1+s)y_2)dy_2 = -\int_0^\pi \Big(y_2(\theta) - \frac{1}{2}k(y_1+s)(y_2(\theta))^2\Big) y_{1,\theta} d\theta \\ &= -\epsilon^2 \int_0^\pi aS(a_\theta C - aS)(1 - \frac{\epsilon}{2}k(\epsilon aC + s)aS)d\theta = \frac{\epsilon^2}{2}\int_0^\pi \Big(a^2 + \epsilon(a^2 a_\theta S^2 C - a^3 S^3)k(\epsilon aC+s)\Big)d\theta. \end{aligned}$$

From (3.15) we see that the constraint $A(\Omega^\epsilon) = \epsilon^2\frac{\pi}{2}$ is equivalent to

$$\int_0^\pi \Big(\frac{a^2-1}{\epsilon} + (a^2 a_\theta S^2 C - a^3 S^3)k(\epsilon aC + s)\Big)d\theta = 0. \tag{3.16}$$

We first analyze (3.16) for $r = \bar r_0(s,\epsilon)$, a constant in θ. Then, with $\bar a = 1 + \epsilon \bar r_0$,

$$\Phi_0(\bar r_0,\epsilon) \equiv \int_0^\pi \Big(\frac{\bar a^2 - 1}{\epsilon} - \bar a^3 S^3 k(\epsilon \bar a C + s)\Big)d\theta = 0. \tag{3.17}$$

We have $\Phi_0(\bar r_0, 0) = \int_0^\pi (2\bar r_0 - S^3 k(s))d\theta$ and therefore

$$\bar r_0(s,0) = \frac{2}{3\pi}k(s).$$

On the other hand $D_1\Phi_0(\bar r_0(s,0),0) = 2\pi$ and so the implicit function theorem assures the existence of $\epsilon_0 > 0$ and $\delta > 0$ such that for each $\epsilon \in [0,\epsilon_0)$, (3.17) has a unique solution $\bar r_0(s,\epsilon) \in (\frac{2}{3\pi}k(s)-\delta, \frac{2}{3\pi}k(s)+\delta)$. By compactness of $\partial\Omega$ we can take ϵ_0 and δ to be independent of s.

We now analyze (3.16) for a general $r \in C^1([0,\pi],\mathbb{R})$. Define $r_0 = \frac{1}{\pi}\int_0^\pi r d\theta$, set $\gamma = r_0 - \bar r_0$, write $r = \bar r_0 + \gamma + \rho$, and observe that $\int_0^\pi \rho d\theta = 0$. In the following we denote by $\tilde C$ a generic constant that may change from line to line and by $g_i(\epsilon z, \theta, s, \epsilon)$ certain smooth functions. By subtracting (3.17) from (3.16) and multiplying by ϵ we get a fixed point problem for the quantity $\epsilon\gamma$:

$$\begin{aligned}\epsilon\gamma &= -\frac{\epsilon^2}{2}\Big(2\bar r_0\gamma + \gamma^2 + \frac{1}{\pi}\int_0^\pi \rho^2 d\theta\Big)\\ &+ \frac{\epsilon}{2\pi}\int_0^\pi \Big(g_1(\epsilon(\gamma+\rho),\theta,s,\epsilon) + \epsilon\rho_\theta g_2(\epsilon(\gamma+\rho),\theta,s,\epsilon)\Big)d\theta \equiv \Phi(\epsilon\gamma,\rho,s,\epsilon),\end{aligned} \tag{3.18}$$

where $g_1(0,\theta,s,\epsilon) = 0$. Under the assumption that $\epsilon(|\gamma| + \|\rho\|_{W^{1,1}([0,\pi])}) \le \bar c\epsilon^\alpha$ for some constant $\bar c$, $\Phi(\cdot,\rho,s,\epsilon)$ is a contraction map for $\epsilon\gamma$ in $[-\epsilon^\alpha,\epsilon^\alpha]$, with contraction factor $K = \tilde C\epsilon^\alpha$. Note that $\|\rho_\theta\|_{L^1([0,\pi])}$ is equivalent to $\|\rho\|_{W^{1,1}([0,\pi])}$ since $\rho \in L^2_N$ has mean value zero. Let $\epsilon\gamma_j = \Phi(\epsilon\gamma_{j-1},\rho,s,\epsilon), j = 1,\ldots$ with $\epsilon\gamma_0 = \Phi(0,\rho,s,\epsilon)$. It is routine to check that

$$|\gamma_0| \le \tilde C\epsilon^\alpha\|\rho\|_{W^{1,1}}, \quad |\gamma_1 - \gamma_0| \le \tilde C\epsilon^{1+\alpha}\|\rho\|_{W^{1,1}}$$

and therefore that the solution $\gamma = \gamma(\rho,s,\epsilon)$ of (3.18) satisfies

$$\begin{aligned}&|\gamma - \gamma_0| \le \frac{1}{1-K}|\gamma_1 - \gamma_0|\\ &\text{and so}\\ &|\gamma| \le \tilde C\epsilon^\alpha\|\rho\|_{W^{1,1}} + \frac{\tilde C\epsilon^{1+\alpha}}{(1-\tilde C\epsilon^\alpha)}\|\rho\|_{W^{1,1}} \le \tilde C\epsilon^\alpha\|\rho\|_{W^{1,1}}.\end{aligned} \tag{3.19}$$

So, if we set

$$\tilde\rho = \gamma(\rho,s,\epsilon) + \rho, \tag{3.20}$$

we have the estimate

$$\|\tilde\rho\| \le \tilde C\|\rho\|. \tag{3.21}$$

Next we compute the length $L(\Gamma^\epsilon(s,r))$ of $\Gamma^\epsilon(s,r)$ and estimate the difference $L(\Gamma^\epsilon(s,r)) - L(\Gamma^\epsilon(s))$. With $y = \epsilon(1+\epsilon r(\theta,s))(\cos\theta,\sin\theta)^\top$, from (3.2) and (3.5) we have

$$\begin{aligned}L(\Gamma^\epsilon(s,r)) &= \int_0^\pi |(\dot p(y_1+s) + y_2\dot n(y_1+s))y_{1,\theta} + n(y_1+s)y_{2,\theta}|d\theta\\ &= \int_0^\pi \sqrt{(1-k(y_1+s)y_2)^2|y_{1,\theta}|^2 + |y_{2,\theta}|^2}d\theta\\ &= \int_0^\pi \sqrt{(|y_{1,\theta}|^2 + |y_{2,\theta}|^2) - (1-(1-k(y_1+s)y_2)^2)|y_{1,\theta}|^2}d\theta\\ &= \epsilon\int_0^\pi \sqrt{(a^2+a_\theta^2) - (1-(1-\epsilon k(\epsilon aC+s)aS)^2)(aS-a_\theta C)^2}d\theta\\ &= \epsilon\int_0^\pi \sqrt{1+\xi}d\theta,\end{aligned} \tag{3.22}$$

where

$$\xi = a^2 - 1 + a_\theta^2 - (1 - (1 - \epsilon k(\epsilon a C + s) a S)^2)(aS - a_\theta C)^2.$$

From $\sqrt{1+\xi} = 1 + \frac{1}{2}\xi - \frac{1}{8}\xi^2 + \frac{3}{16}\int_0^\xi \frac{(\xi-t)^2}{(1+t)^{\frac{5}{2}}}dt$, (3.22) becomes

(3.23)

$$\begin{aligned}
&L(\Gamma^\epsilon(s,r)) = \epsilon\pi + \frac{\epsilon}{2}\int_0^\pi \Big(a^2 - 1 + a_\theta^2 - (1 - (1 - \epsilon k(\epsilon a C + s) a S)^2)(aS - a_\theta C)^2\Big) d\theta \\
&- \frac{\epsilon}{8}\int_0^\pi \Big(a^2 - 1 + a_\theta^2 - (1 - (1 - \epsilon k(\epsilon a C + s) a S)^2)(aS - a_\theta C)^2\Big)^2 d\theta + \frac{3\epsilon}{16}\int_0^\pi \int_0^\xi \frac{(\xi-t)^2}{(1+t)^{\frac{5}{2}}} dt d\theta \\
&= \epsilon\pi \\
&+ \frac{\epsilon}{2}\int_0^\pi \Big(- \epsilon a^2 S^2 (a_\theta C - aS) k(\epsilon a C + s) + a_\theta^2 - (1 - (1 - \epsilon k(\epsilon a C + s) a S)^2)(aS - a_\theta C)^2\Big) d\theta \\
&- \frac{\epsilon}{8}\int_0^\pi \Big(a^2 - 1 + a_\theta^2 - (1 - (1 - \epsilon k(\epsilon a C + s) a S)^2)(aS - a_\theta C)^2\Big)^2 d\theta \\
&+ \frac{3\epsilon}{16}\int_0^\pi \int_0^\xi \frac{(\xi-t)^2}{(1+t)^{\frac{5}{2}}} dt d\theta \\
&= \epsilon\pi + \frac{\epsilon}{2}(I_1 + I_2 + I_3),
\end{aligned}$$

where we have also used (3.16). Note that I_1, I_2 and I_3 depend on (s, ρ). We let $\bar{I}_1, \bar{I}_2$ and $\bar{I}_3$ be the values of I_1, I_2 and I_3, respectively, computed on the basic manifold, that is, for $\rho = 0$. With this notation we have

$$L(\Gamma^\epsilon(s,r)) - L(\Gamma^\epsilon(s)) = \frac{\epsilon}{2}\sum_{i=1}^3 (I_i - \bar{I}_i). \tag{3.24}$$

We let $\|\rho\|$ and $\|\rho\|_\infty$ be the L^2 and the L^∞ norms of $\rho \in L_N^2$ and observe that

$$\begin{aligned}
&\|\rho_\theta\|_{L^1} \leq N\|\rho\|, \\
&\|\rho\|_\infty \leq \sqrt{N}\|\rho\|, \\
&\text{and} \\
&\|\rho_\theta\|_\infty \leq N^{\frac{3}{2}}\|\rho\|.
\end{aligned} \tag{3.25}$$

In the following estimates we will systematically use the above inequalities, with N fixed, to control terms containing powers of ρ and ρ_θ.
For $\Gamma^\epsilon(s,\rho) \in \mathcal{N}^{\bar{\eta}(\alpha)}$ we compute

(3.26)

$$\begin{aligned}
&I_1 = \int_0^\pi \Big(- \epsilon a^2 S^2 (a_\theta C - aS) k(\epsilon a C + s) + a_\theta^2 - (1 - (1 - \epsilon k(\epsilon a C + s) a S)^2)(aS - a_\theta C)^2\Big) d\theta \\
&= \epsilon \int_0^\pi (-k(\epsilon a C + s) aS + \epsilon k^2(\epsilon a C + s) a^2 S^2) a^2 S^2 d\theta + \epsilon^2 \int_0^\pi \rho_\theta^2 d\theta \\
&+ \epsilon \int_0^\pi a_\theta \Big(- a^2 S^2 C k(\epsilon a C + s) + (-2k(\epsilon a C + s) aS + \epsilon k^2(\epsilon a C + s) a^2 S^2)(-2aSC + a_\theta C^2)\Big) d\theta \\
&= \epsilon^2 \int_0^\pi \rho_\theta^2 d\theta + \epsilon(I_{1,1} + I_{1,2}).
\end{aligned}$$

To estimate $I_{1,1}$ recall that $a = 1 + \epsilon(\bar{r}_0 + \tilde{\rho})$ with $\tilde{\rho} \equiv \gamma(\rho, s, \epsilon) + \rho$ and observe that we can write

$$I_{1,1} = \int_0^\pi g(\epsilon\tilde{\rho}, \theta, s, \epsilon) d\theta, \tag{3.27}$$

where $g(z, \theta, s, \epsilon)$ is a smooth function. Note that $g(0, \theta, s, \epsilon)$ is the integrand evaluated at $r = \bar{r}_0$, that is, evaluated on the basic manifold (3.7). Since $\Gamma^\epsilon(s, \rho) \in \mathcal{N}^{\bar{\eta}(\alpha)}$, the derivative of g with respect to z is bounded, therefore,

$$I_{1,1} = \int_0^\pi g(\epsilon\tilde{\rho}, \theta, s, \epsilon) d\theta = \int_0^\pi g(0, \theta, s, \epsilon) d\theta + \mathrm{O}(\epsilon\|\tilde{\rho}\|_\infty). \tag{3.28}$$

We also note that $g(0, \theta, s, \epsilon) = (-k(\epsilon\bar{a}C + s)\bar{a}S + \epsilon k^2(\epsilon\bar{a}C + s)\bar{a}^2S^2)\bar{a}^2S^2 = -k(s)S^3 + \mathrm{O}(\epsilon)$ and therefore

$$\bar{I}_{1,1} = \int_0^\pi g(0, \theta, s, \epsilon) d\theta = -\frac{4}{3}k(s) + \mathrm{O}(\epsilon). \tag{3.29}$$

Since the integrand of $I_{1,2}$ is the product of $a_\theta = \epsilon\rho_\theta$ and a function which, for $\Gamma^\epsilon(s, \rho) \in \mathcal{N}^{\bar{\eta}(\alpha)}$, is bounded together with its derivative, we have

$$|I_{1,2}| \leq \tilde{C}\epsilon\|\rho_\theta\|_\infty. \tag{3.30}$$

From (3.25), (3.28) and (3.30) we obtain

$$\begin{aligned} I_1 &= \bar{I}_1 + \epsilon^2 \int_0^\pi \rho_\theta^2 d\theta + \mathrm{O}(\epsilon^2(\|\tilde{\rho}\|_\infty + \|\rho_\theta\|_\infty)) \\ &= \bar{I}_1 + \epsilon^2 \int_0^\pi \rho_\theta^2 d\theta + \mathrm{O}(\epsilon^2\|\rho\|), \end{aligned} \tag{3.31}$$

where $\bar{I}_1$ is I_1 computed on $\mathcal{M}$, that is, for $\rho = 0$. To estimate I_2 and I_3 we begin by estimating $\bar{\xi}$ and the difference $\xi - \bar{\xi}$ for $\Gamma^\epsilon(s, \rho) \in \mathcal{N}^{\bar{\eta}(\alpha)}$, where $\bar{\xi}$ is ξ computed at $\rho = 0$. With $\bar{a} = 1 + \epsilon\bar{r}_0$ we have

(3.32)

$$\begin{aligned} \xi - \bar{\xi} &= 2\epsilon(1 + \epsilon\bar{r}_0)\tilde{\rho} + \epsilon^2(\tilde{\rho}^2 + \rho_\theta^2) \\ &+ \epsilon k(\epsilon aC + s)aSC(2 - \epsilon k(\epsilon aC + s)aS)(2\epsilon aS\rho_\theta - \epsilon^2\rho_\theta^2 C) \\ &- \epsilon\Big(k(\epsilon aC + s)aS(2 - \epsilon k(\epsilon aC + s)aS)a^2S^2 - k(\epsilon\bar{a}C + s)\bar{a}S(2 - \epsilon k(\epsilon\bar{a}C + s)\bar{a}S)\bar{a}^2S^2\Big) \\ &= 2\epsilon(1 + \epsilon\bar{r}_0)\tilde{\rho} + \epsilon^2(\tilde{\rho}^2 + \rho_\theta^2) + R_1 + R_2. \end{aligned}$$

Since $\Gamma^\epsilon(s, \rho) \in \mathcal{N}^{\bar{\eta}(\alpha)}$, a is bounded, and since k and k' are bounded we have

$$|R_1| \leq \tilde{C}(\epsilon^2\|\rho_\theta\|_\infty + \epsilon^3\|\rho_\theta\|_\infty^2) \leq \tilde{C}\epsilon^2\|\rho\|. \tag{3.33}$$

From (3.32) it follows that $R_2 = 0$ for $a - \bar{a} = \epsilon\tilde{\rho} = 0$ and therefore we see that R_2 is a smooth function of the form $g(\epsilon\tilde{\rho}, \theta, s, \epsilon)$ with $g(0, \theta, s, \epsilon) = 0$. From this it follows that

$$|R_2| \leq \tilde{C}\epsilon^2\|\tilde{\rho}\|. \tag{3.34}$$

Therefore, using (3.21), we see that $\Gamma^\epsilon(s, \rho) \in \mathcal{N}^{\bar{\eta}(\alpha)}$ implies

$$|\xi - \bar{\xi}| \leq \tilde{C}\epsilon^\alpha.$$

This and the fact that

$$|\bar{\xi}| = |\bar{a}^2 - 1 - (1-(1-\epsilon k(\epsilon\bar{a}C+s)\bar{a}S)^2)\bar{a}^2S^2| \le \tilde{C}\epsilon. \tag{3.35}$$

imply that for $\Gamma^\epsilon(s,\rho)\in\mathcal{N}^{\bar{\eta}(\alpha)}$ we have

$$|\xi| \le \tilde{C}\epsilon^\alpha. \tag{3.36}$$

From (3.25), (3.32), (3.33) and (3.34) it follows that

$$\begin{aligned}\xi &= \bar{\xi} + 2\epsilon\tilde{\rho} + \epsilon^2(\tilde{\rho}^2+\rho_\theta^2) + \epsilon^2\mathrm{O}(\|\rho\|)\\ &= \bar{\xi} + 2\epsilon\tilde{\rho} + \epsilon^2\mathrm{O}(\|\rho\|+\|\rho\|^2).\end{aligned} \tag{3.37}$$

Using that together (3.35) and $\Gamma^\epsilon(s,\rho)\in\mathcal{N}^{\bar{\eta}(\alpha)}$ imply $|\bar{\xi}+2\epsilon\rho| = \mathrm{O}(\epsilon^\alpha)$,

$$\begin{aligned}\xi^2 &= \bar{\xi}^2 + 4\epsilon^2\tilde{\rho}^2 + 4\epsilon\bar{\xi}\tilde{\rho} + 2\epsilon^2(\bar{\xi}+2\epsilon\rho)\mathrm{O}(\|\rho\|+\|\rho\|^2)\\ &+ \epsilon^4\mathrm{O}(\|\rho\|^2+\|\rho\|^4)\\ &= \bar{\xi}^2 + 4\epsilon^2\tilde{\rho}^2 + 4\epsilon\bar{\xi}\tilde{\rho}\\ &+ \mathrm{O}(\epsilon^{2+\alpha}(\|\rho\|+\|\rho\|^2) + \epsilon^4\|\rho\|^4).\end{aligned} \tag{3.38}$$

From (3.38), using also (3.35) and (3.19), we deduce

$$\begin{aligned}I_2 &= -\frac{1}{4}\int_0^\pi \xi^2 d\theta = -\frac{1}{4}\int_0^\pi \bar{\xi}^2 d\theta - \epsilon^2\int_0^\pi \rho^2 d\theta - \epsilon^2\pi\gamma^2 - \epsilon\gamma\int_0^\pi \bar{\xi}d\theta - \epsilon\int_0^\pi \bar{\xi}\rho d\theta\\ &+ \mathrm{O}(\epsilon^{2+\alpha}(\|\rho\|+\|\rho\|^2) + \epsilon^4\|\rho\|^4)\\ &= -\frac{1}{4}\int_0^\pi \bar{\xi}^2 d\theta - \epsilon^2\int_0^\pi \rho^2 d\theta\\ &+ \mathrm{O}(\epsilon^{2+\alpha}(\|\rho\|+\|\rho\|^2)) + \mathrm{O}(\epsilon^2\|\rho\|),\end{aligned} \tag{3.39}$$

where we have also used $\int_0^\pi \rho d\theta = 0$.

To evaluate I_3 we need an estimate of the difference

$$\begin{aligned}D(\xi,\bar{\xi}) &= \int_0^\xi \frac{(\xi-t)^2}{(1+t)^{\frac{5}{2}}}dt - \int_0^{\bar{\xi}} \frac{(\bar{\xi}-t)^2}{(1+t)^{\frac{5}{2}}}dt\\ &= \int_0^{\bar{\xi}} \frac{(\xi-t)^2-(\bar{\xi}-t)^2}{(1+t)^{\frac{5}{2}}}dt + \int_{\bar{\xi}}^\xi \frac{(\xi-t)^2}{(1+t)^{\frac{5}{2}}}dt\\ &= \int_0^{\bar{\xi}} \frac{(\xi+\bar{\xi}-2t)(\xi-\bar{\xi})}{(1+t)^{\frac{5}{2}}}dt + \int_{\bar{\xi}}^\xi \frac{(\xi-t)^2}{(1+t)^{\frac{5}{2}}}dt.\end{aligned} \tag{3.40}$$

Therefore from (3.40), using also (3.35), (3.36) and $|t| \le \max\{|\xi|,|\bar{\xi}|\}$, it follows that

$$\begin{aligned}|D(\xi,\bar{\xi})| &\le \tilde{C}(\epsilon^{1+\alpha}|\xi-\bar{\xi}| + |\xi-\bar{\xi}|^3)\\ &\le \tilde{C}\epsilon^{2+\alpha}(\|\rho\|+\|\rho\|^2).\end{aligned} \tag{3.41}$$

From (3.41) we finally conclude

$$I_3 = \bar{I}_3 + \mathrm{O}(\epsilon^{2+\alpha}(\|\rho\|+\|\rho\|^2)), \tag{3.42}$$

where we have used (3.37). We also have from (3.35)

$$|\bar{I}_3| \le 2|\bar{\xi}|^3 \le \tilde{C}\epsilon^3. \tag{3.43}$$

Using (3.26), (3.29), (3.39), (3.43) and $|\int_0^\pi \bar{\xi}^2 d\theta| \le \tilde{C}\epsilon^2$ it follows that

$$L(\Gamma^\epsilon(s)) = \epsilon\pi + \frac{\epsilon}{2}(\bar{I}_1 + \bar{I}_2 + \bar{I}_3) = \epsilon\pi - \frac{2}{3}\epsilon^2 k(s) + \mathrm{O}(\epsilon^3) \tag{3.44}$$

and therefore

$$|L(\Gamma^\epsilon(s_1)) - L(\Gamma^\epsilon(s_2))| \le \tilde{C}\epsilon^2 =: \delta_0, \tag{3.45}$$

giving condition (2.1) with $J := L$. We now examine (2.4) and show that assumptions $\mathbf{H}_1$-$\mathbf{H}_3$ hold.
From (3.44), (3.23), (3.31), (3.39) and (3.42), it follows that

$$L(\Gamma^\epsilon(s,\rho)) - L(\Gamma^\epsilon(s)) = \mathrm{O}(\epsilon^3\|\rho\|) + \frac{\epsilon^3}{2}\int_0^\pi (\rho_\theta^2 - \rho^2)d\theta + \mathrm{O}(\epsilon^{3+\alpha}\|\rho\|^2). \tag{3.46}$$

On the basis of the abstract Theorem 2.1, we can interpret the various terms in (3.46) as follows:

$$|L(s,\rho)| \le \tilde{C}\epsilon^3\|\rho\| \quad \Rightarrow \quad \delta_L = \tilde{C}\epsilon^3, \tag{3.47}$$

$$\begin{aligned} Q(s,\rho) &\ge \frac{\epsilon^3}{2}\sum_{n=2}^{N}(n^2-1)c_n^2 - \tilde{C}\epsilon^{3+\alpha}\|\rho\|^2 \\ &\ge \frac{\epsilon^3}{2}(3 - \tilde{C}\epsilon^\alpha)\|\rho\|^2 \ge \epsilon^3\|\rho\|^2, \quad \Rightarrow \quad K_0 = \epsilon^3, \end{aligned} \tag{3.48}$$

$$\text{and} \quad N(s,\rho) = 0, \quad \text{which gives} \quad K_1 = 0. \tag{3.49}$$

We take our initial bubble Γ_0^ϵ such that

$$L(\Gamma_0^\epsilon) < \max_{s\in[0,L(\partial\Omega))} L(\Gamma^\epsilon(s)) + C_1\epsilon^2 \quad \Rightarrow \quad \delta_1 = C_1\epsilon^2. \tag{3.50}$$

From (3.49) and (2.6) it follows that

$$\eta^* = \bar{\eta}(\alpha) = \frac{\bar{c}}{\epsilon^{1-\alpha}}.$$

From this, (3.45), (3.47), (3.48), and (3.50), it follows that the condition (2.8) in Theorem 2.1 reads

(3.51)

$$\eta_* = \frac{\delta_L}{K_0} + \sqrt{\frac{\delta_L^2}{K_0^2} + 2\frac{\delta_0+\delta_1}{K_0}} \le \tilde{C}(1 + \sqrt{1 + 1/\epsilon}\,) < \frac{\bar{c}}{\epsilon^{1-\alpha}}.$$

which, provided $\epsilon > 0$ is sufficiently small, is satisfied for $\alpha \in (0, \frac{1}{2})$. We can now invoke Theorem 2.1 to conclude the assertions in Theorem 3.1.

4 Slow motion for the vector Allen-Cahn equation and the Cahn-Morral system

In this section we consider the Allen-Cahn energy $J : W^{1,2}([0,1],\mathbb{R}^m) \to \mathbb{R}$

$$J(u) := J(u,(0,1)) = \int_0^1 (\frac{\epsilon^2}{2}|u_x|^2 + W(u))\,dx. \tag{4.1}$$

and the associated gradient dynamics governed by the parabolic systems

$$\begin{cases} u_t = \epsilon^2 u_{xx} - W_u(u),\ t > 0,\ x \in (0,1), \\ u_x = 0,\ x = 0,1, \\ u(0) = u_0, \end{cases} \tag{4.2}$$

and

$$\begin{cases} u_t = -(\epsilon^2 u_{xx} - W_u(u))_{xx},\ t > 0,\ x \in (0,1), \\ u_x = u_{xxx} = 0,\ x = 0,1, \\ u(0) = u_0. \end{cases} \tag{4.3}$$

We refer to (4.2) as the vector Allen-Cahn equation and to (4.3) as the Cahn-Morral system. System (4.2) is the L^2-gradient flow of (4.1) while (4.3) is the gradient flow in $H_0^{-1}((0,1);\mathbb{R}^m)$, the Hilbert space of the maps $v \in H^{-1}((0,1);\mathbb{R}^m)$ that satisfy $\int_0^1 v dx = 0$. The inner product in $H_0^{-1}((0,1);\mathbb{R}^m)$ is defined by

$$\langle f, g\rangle_{H_0^{-1}} \equiv \int_0^1 (-D)^{-\frac{1}{2}} f \cdot (-D)^{-\frac{1}{2}} g dx \tag{4.4}$$

where $D : H_0^{-1}((0,1);\mathbb{R}^m) \to H_0^1((0,1);\mathbb{R}^m)$[1] is the operator $f \to v$ defined by

$$\begin{cases} v_{xx} = f,\ \ x \in (0,1),\ \ \int_0^1 f dx = 0, \\ v_x = 0,\ \ x = 0,1, \\ \int_0^1 v dx = 0. \end{cases}$$

We assume

h$_1$) $W : \mathbb{R}^m \to \mathbb{R}$ is a smooth potential that satisfies

$$0 = W(a) < W(u), \quad \text{for } a \in A \text{ and } u \in \mathbb{R}^m \setminus A.$$

where $A \subset \mathbb{R}^m$ is a discrete set with at least two elements. Moreover $a \in A$ is nondegenerate, in the sense that the quadratic form $D^2W(a)$ is strictly positive definite.

h$_2$) $\liminf_{|u|\to+\infty} W(u) > 0$.

For $\epsilon << 1$, in a time interval of length O(1), the evolution of the solution $t \to u(t,u_0)$ of (4.2) is mainly dictated by the kinetic equation $u_t = -W_u(u)$ and $u(t,u_0)$ moves to a neighborhood of A in the subset of the spatial domain $(0,1)$ where $u_0(x)$ lies in the basin of attraction of A, with respect to this kinetic equation. Depending on the structure of u_0 this leads to an intermediate state where $u(t,u_0)$ has a *layered* shape. That is, there

[1] Here H_0^1 is the subspace of H^1 of maps with zero average.

is a certain number N of thin intervals of width $\mathrm{O}(\epsilon)$, $I_1, \ldots, I_N \subset (0,1)$ across which $u(t, u_0)$ jumps from a neighborhood of $a_i \in A$ to a neighborhood $a_j \in A$ for some distinct $a_i, a_j \in A$. In the complement $(0,1) \setminus \cup_j I_j$, $u(t, u_0)$ remains close to A. Once $u(t, u_0)$ has achieved this layered shape its energy is essentially concentrated in the layers and changes only a very small amount with the layers' positions. Therefore the layered structure is a kind of metastable state which persists for a very long time during which the layers move until two of more of them collide and annihilate. This type of metastable dynamics of layers occurs also for the Cahn-Hilliard equation in one space dimension and for the vector version of Cahn-Hilliard, the Cahn-Morral system (4.3). However the route leading to a layered structure is more complex with respect to the simple evolution sketched above for the Allen-Cahn equation.

All these slow motion phenomena are consequences of the fact that the geometry of the graph $\mathcal{G}^J$ of J for $\epsilon << 1$ has the structure described in the abstract Theorem 2.1. To substantiate this statement we discuss in detail layer dynamics for the systems (4.2) and (4.3). We take Theorem 2.1 as a paradigm for our proof and proceed step by step with the definition of $\mathcal{M}$ the associated projection and the computation of the quantities $\delta_0, K_0, K_1, \ldots$ necessary to check the validity of the condition (2.8) in Theorem 2.1.

Under assumptions $\mathbf{h}_1$) and $\mathbf{h}_2$), if $A = \{a_1, a_2\}$, $a_1 \neq a_2$ (see [26], [23] [31],[27],[8] and [5]) there exists a connecting orbit $\bar{u} : \mathbb{R} \to \mathbb{R}^m$ between a_1 and a_2 that is a solution of

$$\begin{cases} u'' = W_u(u), & s \in \mathbb{R} \\ \lim_{s \to -\infty} u(s) = a_1, \ \lim_{s \to +\infty} u(s) = a_2. \end{cases} \tag{4.5}$$

The connection map $\bar{u}$ is characterized as a minimizer of the functional

$$J_{\mathbb{R}}(u) = \int_{\mathbb{R}} (\frac{\epsilon^2}{2}|u_x|^2 + W(u))\,dx, \quad \text{that is,} \quad J_{\mathbb{R}}(\bar{u}) = \min_{u \in \mathcal{A}} J_{\mathbb{R}}(u),$$

where $\mathcal{A} = \{u \in W^{1,2}_{loc}(\mathbb{R}, R^m), \lim_{x \to \pm\infty} \bar{u}(x) = \pm a\}$.
In the general case we assume that there exists $a_j \in A, j = 1, \ldots, N+1$, $a_j \neq a_{j+1}$, $j = 1, \ldots, N$ and connections $\bar{u}_j, j = 1, \ldots, N$ that minimize $J_{\mathbb{R}}(u)$ on $\mathcal{A} = \{u \in W^{1,2}_{loc}(\mathbb{R}, R^m), \lim_{s \to -\infty} \bar{u}(s) = a_j, \lim_{s \to +\infty} \bar{u}(s) = a_{j+1}\}$.

The fact that a_j is a nondegenerate zero of W implies the existence of constants k, K such that

$$\begin{cases} |\bar{u}_j(s) - a_j| \leq K e^{ks}, & s \leq 0 \\ |\bar{u}_j(s) - a_{j+1}| \leq K e^{-ks}, & s \geq 0 \end{cases} \tag{4.6}$$

This and elliptic regularity imply that we can also assume

$$|\bar{u}_j'(s)|, |\bar{u}_j''(s)| \leq K e^{-k|s|}. \tag{4.7}$$

We also make the following generic assumption

$\mathbf{h}_3$) The zero eigenvalue of the operator $\mathcal{L}_j : W^{2,2}(\mathbb{R}; \mathbb{R}^m) \to L^2(\mathbb{R}; \mathbb{R}^m)$ defined by

$$\mathcal{L}_j \varphi = -\varphi'' + W_{uu}(\bar{u}_j)\varphi$$

is simple and therefore there is $\beta > 0$ such that $\sigma(\mathcal{L}_j) \subset \{0\} \cup \{\lambda > \beta\}$.

In the scalar case this is automatically satisfied and is related to the monotonicity of the connection $\bar{u}$. In the vector case the situation is more involved: the connection may not be unique [3] and the kernel of $\mathcal{L}$ may be m-dimensional.

4.1 The manifold $\mathcal{M}$

Define

$$\Xi = \Xi(\rho) := \Big\{\xi \in R^N : \quad 0 < \xi_1 < \cdots < \xi_N < 1, \text{ such that} \\ \xi_{j+1} - \xi_j > \rho,\ j = 1, \ldots, N-1,\ \xi_1 > \rho/2,\ 1 - \xi_N > \rho/2\Big\},$$

where $\rho \in (0, 1/N)$ is a small fixed number. Given $\xi \in \Xi$, set $\xi_0 = -\xi_1$, $\xi_{N+1} = 2 - \xi_N$ and let $\hat{\xi}_j := \frac{\xi_{j+1}+\xi_j}{2}$, $j = 0, \ldots, N$ be the mid points.

Definition. For each $\xi \in \Xi$ we define the function u^ξ by

$$u^\xi(x) = \sum_{j=1}^{N} \Big(\bar{u}_j(\frac{x - \xi_j}{\epsilon}) - a_j\Big) + a_1 \tag{4.8}$$

In the special case where $A = \{-a, +a\}$ we have

$$u^\xi(x) = \sum_{j=1}^{N} \left[\bar{u}\left(\frac{(-1)^{j+1}(x - \xi_j)}{\epsilon}\right) + (-1)^{j+1}a\right] - a \tag{4.9}$$

where $\bar{u}$ is as in (4.5).

The manifold $\mathcal{M}$ is defined by,

$$\mathcal{M} := \{u^\xi : \xi \in \Xi\} \tag{4.10}$$

and, with $d > 0$ to be specified later, define the neighborhood $\mathcal{N}^\eta$ of $\mathcal{M}$ by

$$\mathcal{N}^\eta = \{u \in W^{1,2}_\epsilon : d_{W^{1,2}_\epsilon}(u, \mathcal{M}) < \eta,\ d_{L^2}(u, \partial\mathcal{M}) > d\}, \tag{4.11}$$

where d_{L^2} is the distance in the L^2 sense and $d_{W^{1,2}_\epsilon}$ denotes the distance in the sense of the $\|\cdot\|_{W^{1,2}_\epsilon}$ norm

$$\|u\|^2_{W^{1,2}_\epsilon} \equiv \epsilon^2 \|u_x\|^2_{L^2} + \|u\|^2_{L^2}. \tag{4.12}$$

We can now state the main results of this section. We denote by $\langle \cdot, \cdot \rangle$ the standard inner product in $L^2((0,1); \mathbb{R}^m)$ and by $\|\cdot\|$ the associated norm. We use the notation $\mathrm{O}(e^{-\frac{k\rho}{2\epsilon}})$ to denote a quantity q that satisfies a bound of the form $|q| \le C_\epsilon e^{-\frac{k\rho}{2\epsilon}}$ for some constant $C_\epsilon > 0$ that may depend algebraically on ϵ.

Theorem 4.1. *Let $H = W^{1,2}((0,1); \mathbb{R}^m)$ with norm $\|u\|_{W^{1,2}_\epsilon}$. Assume that $W : \mathbb{R}^m \to \mathbb{R}$ satisfies $\mathbf{h}_1$)-$\mathbf{h}_3$). Let $\Xi \ni \xi \to u^\xi$ be defined by (4.8). Take any $\delta_1 = O(e^{-\frac{k\rho}{2\epsilon}})$. Then there are $\epsilon_0 > 0$ and constants $C, C' > 0$ independent of $\epsilon \in (0, \epsilon_0)$ such that, with $d = C'\epsilon^{\frac{1}{2}}$ in (4.11), one has*

$$u_0 \in \mathcal{N}^{C\epsilon^3} \Rightarrow u(t, u_0) \in \mathcal{N}^{e^{-\frac{k\rho}{2\epsilon}}}, \quad \textit{for } t \in [0, T), \tag{4.13}$$

where $t \to u(t, u_0)$ is the solution of (4.2) or (4.3) and either

(i) $T = +\infty$

or

(ii) $\lim_{t\to T} d_{L^2}(u(t,u_0),\partial\mathcal{M}) = d$ *and*

$$T \geq e^{\frac{k\rho}{2\epsilon}}(d_X(u_0,\partial\mathcal{M}) - C'\epsilon^{\frac{1}{2}})^2,$$

where $X = L^2$ for (4.2) and $X = H^{-1}$ for (4.3).

Proof. We divide the proof in several Lemmas.

Lemma 4.2. [*Computation of δ_0*]
Let $\xi, \tilde{\xi} \in \Xi$ and $u^\xi, u^{\tilde{\xi}} \in \mathcal{M}$, then

$$|J(u^\xi) - J(u^{\tilde{\xi}})| \leq \delta_0 \tag{4.14}$$

where $\delta_0 = e^{-\frac{k\rho}{2\epsilon}}$.

Proof. We have

$$J(u^\xi) = \sum_{i=1}^{N} \int_{[\hat{\xi}_{i-1},\hat{\xi}_i]} (\frac{\epsilon^2}{2}|u^\xi_x|^2 + W(u^\xi))dx. \tag{4.15}$$

The change of variable $x = s + \xi_i$ gives

$$\int_{[\hat{\xi}_{i-1},\hat{\xi}_i]} (\frac{\epsilon^2}{2}|u^\xi_x|^2 + W(u^\xi))dx = \int_{-\frac{\xi_i-\xi_{i-1}}{2}}^{\frac{\xi_{i+1}-\xi_i}{2}} (\frac{\epsilon^2}{2}|u^\xi_x(s+\xi_i)|^2 + W(u^\xi(s+\xi_i)))ds. \tag{4.16}$$

$$\begin{aligned} u^\xi(s+\xi_i) = \bar{u}_i(\frac{s}{\epsilon}) \quad &+ \sum_{1\leq j<i} \Big(\bar{u}_j(\frac{s+\xi_i-\xi_j}{\epsilon}) - a_{j+1}\Big) \\ &+ \sum_{i<j\leq N} \Big(\bar{u}_j(\frac{s+\xi_i-\xi_j}{\epsilon}) - a_j\Big). \end{aligned} \tag{4.17}$$

From (4.8) we also have

$$u^\xi_x(s+\xi_i) = \frac{1}{\epsilon}\bar{u}'_i(\frac{s}{\epsilon}) + \sum_{j\neq i} \frac{1}{\epsilon}\bar{u}'_j(\frac{s+\xi_i-\xi_j}{\epsilon}). \tag{4.18}$$

For $s \in (-\frac{\xi_i-\xi_{i-1}}{2}, \frac{\xi_{i+1}-\xi_i}{2})$ and $j \neq i$, one has $|s+\xi_i-\xi_j| \geq \rho/2$. Therefore from (4.6), (4.17) and (4.18) it follows

$$\begin{aligned} &u^\xi(s+\xi_i) = \bar{u}_i(\tfrac{s}{\epsilon}) + \mathrm{O}(e^{-k\frac{\rho}{2\epsilon}}), \\ &u^\xi_x(s+\xi_i) = \tfrac{1}{\epsilon}\bar{u}'_i(\tfrac{s}{\epsilon}) + \mathrm{O}(\tfrac{1}{\epsilon}e^{-k\frac{\rho}{2\epsilon}}) \ \text{ for } \ s \in (-\tfrac{\xi_i-\xi_{i-1}}{2}, \tfrac{\xi_{i+1}-\xi_i}{2}). \end{aligned} \tag{4.19}$$

Now observe that (4.6) implies

$$\begin{aligned}(4.20)\qquad &\int_{-\frac{\xi_i-\xi_{i-1}}{2}}^{\frac{\xi_{i+1}-\xi_i}{2}} \Big(\frac{1}{2}|\bar{u}_i'(\frac{s}{\epsilon})|^2 + W(\bar{u}_i(\frac{s}{\epsilon}))\Big)ds \\ &= \epsilon\int_{\mathbb{R}} \Big(\frac{1}{2}|\bar{u}_i'(x)|^2 + W(\bar{u}_i(x))\Big)dx + \mathrm{O}(\epsilon e^{-k\frac{\rho}{\epsilon}}).\end{aligned}$$

From this and the estimates (4.19) it follows that

$$\int_{-\frac{\xi_i-\xi_{i-1}}{2}}^{\frac{\xi_{i+1}-\xi_i}{2}} (\frac{\epsilon^2}{2}|u_x^\xi(s+\xi_i)|^2 + W(u^\xi(s+\xi_i)))ds = \epsilon\bar{J}_i + \mathrm{O}(e^{-k\frac{\rho}{2\epsilon}}),$$

where $\bar{J}_i = \int_{\mathbb{R}}\Big(\frac{1}{2}|\bar{u}_i'(x)|^2 + W(\bar{u}_i(x))\Big)dx$ and thus from (4.15) we obtain

$$J(u^\xi) = \epsilon\sum_1^N \bar{J}_j + \mathrm{O}(e^{-k\frac{\rho}{2\epsilon}}), \quad \text{for } \ \xi\in\Xi.$$

which shows that we can take

$$\delta_0 = \mathrm{O}(e^{-k\frac{\rho}{2\epsilon}})$$

in (4.14). □

In the following lemma we collect some properties of the map $\Xi\ni\xi\to u^\xi$ defined above.

Lemma 4.3. *There exists $q_0>0$ and $\epsilon_0>0$ such that, for $\epsilon\in(0,\epsilon_0)$, the condition*

$$(4.21)\qquad |\xi-\hat{\xi}|\le q_0\epsilon$$

implies

$$(4.22)\qquad \frac{c_1}{\epsilon^{\frac{1}{2}}}|\xi-\hat{\xi}| \le \|u^\xi - u^{\hat{\xi}}\|_{L^2} \le \frac{C_1}{\epsilon^{\frac{1}{2}}}|\xi-\hat{\xi}|,$$

and

$$(4.23)\qquad \|u_x^\xi - u_x^{\hat{\xi}}\|_{L^2} \le \frac{C_2}{\epsilon^{\frac{3}{2}}}|\xi-\hat{\xi}|$$

for some positive constants c_1, C_1 and C_2.

Proof. We only sketch the proof of (4.22). The proof of (4.23) is similar. We have

$$(4.24)\qquad u^{\hat{\xi}}(x) - u^\xi(x) = \int_0^1 u_\xi^{\xi+t(\hat{\xi}-\xi)}(x)\cdot(\hat{\xi}-\xi)dt.$$

Observe that from (4.8) it follows that

$$
\begin{aligned}
&\int_0^1 u_\xi^{\xi+t(\hat\xi-\xi)}(x)dt\cdot(\hat\xi-\xi) = -\frac{1}{\epsilon}\sum_{j=1}^N\int_0^1 \bar u_j'\Big(\frac{x-\xi_j-t(\hat\xi_j-\xi_j)}{\epsilon}\Big)dt(\hat\xi_j-\xi_j)\\
&= -\frac{1}{\epsilon}\sum_{j=1}^N \bar u_j'\Big(\frac{x-\xi_j}{\epsilon}\Big)(\hat\xi_j-\xi_j)\\
&-\frac{1}{\epsilon}\sum_{j=1}^N\int_0^1\Big(\bar u_j'\Big(\frac{x-\xi_j-t(\hat\xi_j-\xi_j)}{\epsilon}\Big)-\bar u_j'\Big(\frac{x-\xi_j}{\epsilon}\Big)\Big)dt(\hat\xi_j-\xi_j)\\
&= -\frac{1}{\epsilon}\sum_{j=1}^N \bar u_j'\Big(\frac{x-\xi_j}{\epsilon}\Big)(\hat\xi_j-\xi_j)\\
&+\int_0^1\frac{t}{\epsilon^2}\int_0^1\sum_{j=1}^N \bar u_j''\Big(\frac{x-\xi_j-\tau t(\hat\xi_j-\xi_j)}{\epsilon}\Big)d\tau dt(\hat\xi_j-\xi_j)^2.
\end{aligned} \tag{4.25}
$$

To compute the $L^2((0,1);\mathbb{R}^m)$ norm of $u^{\hat\xi}-u^\xi$ we estimate the $L^1((0,1);\mathbb{R}^m)$ norms of products of the functions on the right hand side of (4.25). Note that

$$
\begin{aligned}
&\frac{1}{\epsilon^2}\int_0^1 \bar u_j'\Big(\frac{x-\xi_j}{\epsilon}\Big)\cdot\bar u_i'\Big(\frac{x-\xi_i}{\epsilon}\Big)dx(\hat\xi_j-\xi_j)(\hat\xi_i-\xi_i)\\
&=\Big(\frac{\delta_{ij}}{\epsilon}\int_{\mathbb{R}}|\bar u_j'|^2ds+\mathrm{O}(e^{-\frac{k\rho}{2\epsilon}})\Big)(\hat\xi_j-\xi_j)(\hat\xi_i-\xi_i).
\end{aligned} \tag{4.26}
$$

From (4.7) and the assumption (4.21) it follows that

$$
\begin{aligned}
&|\bar u_j''\Big(\frac{x-\xi_j-\tau t(\hat\xi_j-\xi_j)}{\epsilon}\Big)|\le Ke^{-k\frac{|x-\xi_j-\tau t(\hat\xi_j-\xi_j)|}{\epsilon}}\\
&\le Ke^{-k\frac{|x-\xi_j|-|\hat\xi_j-\xi_j|}{\epsilon}}\le Ke^{kq_0}e^{-k\frac{|x-\xi_j|}{\epsilon}}
\end{aligned} \tag{4.27}
$$

and therefore

$$
\begin{aligned}
&\frac{1}{\epsilon^3}\Big|\int_0^1\bar u_j'\Big(\frac{x-\xi_j}{\epsilon}\Big)\cdot\int_0^1 t\int_0^1\bar u_i''\Big(\frac{x-(\xi_i+\tau t(\hat\xi_i-\xi_i))}{\epsilon}\Big)d\tau dt dx\Big||\hat\xi_j-\xi_j||\hat\xi_i-\xi_i|^2\\
&\le\frac{1}{\epsilon^3}\int_0^1K^2e^{kq_0}e^{-k\frac{|x-\xi_j|+|x-\xi_i|}{\epsilon}}dx|\hat\xi_j-\xi_j||\hat\xi_i-\xi_i|^2\\
&=\frac{1}{\epsilon^2}(\delta_{ij}\frac{K^2e^{kq_0}}{2k}+\mathrm{O}(e^{-\frac{k\rho}{2\epsilon}}))|\hat\xi_j-\xi_j||\hat\xi_i-\xi_i|^2\\
&\le\frac{1}{\epsilon}(\delta_{ij}q_0\frac{K^2e^{kq_0}}{2k}+\mathrm{O}(e^{-\frac{k\rho}{2\epsilon}}))|\hat\xi_j-\xi_j||\hat\xi_i-\xi_i|.
\end{aligned} \tag{4.28}
$$

From (4.25), (4.26), (4.28) and a similar estimate for terms containing products of the $\bar u_j''$ we conclude

$$
\|u^{\hat\xi}-u^\xi\|^2-\frac{1}{\epsilon}\sum_{j=1}^N\int_{\mathbb{R}}|\bar u_j'|^2ds|\hat\xi_j-\xi_j|^2=\mathrm{O}(q_0+e^{-\frac{k\rho}{2\epsilon}})|\hat\xi-\xi|^2
$$

and, by taking $q_0>0$ sufficiently small, (4.22) follows. □

4.2 Existence of the projection

In this section we consider a more local version of $\mathcal{M}$ and $\mathcal{N}_{L^2}^{\eta}$. Namely, we define

$$\mathcal{M}(\xi_0) = \{u^{\xi} : \xi \in B_{q_0\epsilon}(\xi_0)\}, \qquad B_{q_0\epsilon}(\xi_0) = \{\xi : |\xi - \xi_0| < q_0\epsilon\}$$

where $\xi_0 \in \Xi$ and q_0 is as in Lemma 4.3. Set

$$\mathcal{N}_{L^2}^{\eta}(\xi_0) = \{u \in W^{1,2} : d_{L^2}(u, \mathcal{M}(\xi_0)) < \eta,\ d_{L^2}(u, \partial\mathcal{M}(\xi_0)) > c\epsilon\},$$

where $c > 0$ is to be chosen later. Note that for $\epsilon > 0$ small, one has $\mathcal{N}_{L^2}^{\eta}(\xi_0) \neq \emptyset$. Indeed, from (4.22) we have

$$\|u^{\xi_0} - u^{\zeta}\| \geq \frac{c_1}{\epsilon^{\frac{1}{2}}}|\xi - \zeta| = c_1 q_0 \epsilon^{\frac{1}{2}} \geq c\epsilon, \quad \text{for} \quad \zeta \in \partial B_{q_0\epsilon}(\xi_0)$$

and therefore $u^{\xi_0} \in \mathcal{N}_{L^2}^{\eta}(\xi_0)$.

We show that, for $\eta > 0$ sufficiently small, if $u \in \mathcal{N}_{L^2}^{\eta}(\xi_0)$, then there is a unique $u^{\xi} \in \mathcal{M}(\xi_0)$ such that

$$\|u - u^{\xi}\|_{L^2} = \delta := \inf_{|\zeta - \xi_0| < q_0\epsilon} \|u - u^{\zeta}\|_{L^2}. \tag{4.29}$$

Indeed, with $\langle \cdot, \cdot \rangle$ the standard inner product in $L^2((0,1); \mathbb{R}^m)$, we have

Lemma 4.4. *Take $\eta = a\epsilon$. If $a > 0$ is sufficiently small, then for each $u \in \mathcal{N}_{L^2}^{\eta}(\xi_0)$ there is a unique $\xi \in B_{q_0\epsilon}(\xi_0)$ such that $u - u^{\xi}$ satisfies (4.29). Moreover,*

(i) *ξ is a smooth function of $u \in \mathcal{N}_{L^2}^{\eta}(\xi_0)$ and*

$$\langle u - u^{\xi}, u_{\xi_i}^{\xi} \rangle = 0, \quad i = 1, \ldots, N$$

where $u_{\xi_i}^{\xi}$ is the derivative of u^{ξ} with respect to ξ_i.

(ii) *There exists $\bar{C} > 0$, independent of $\epsilon > 0$, such that for each $\gamma \in (0, a\epsilon)$,*

$$d_{W_\epsilon^{1,2}}(u, \mathcal{M}(\xi_0)) < \gamma \quad \textit{implies} \quad \|u - u^{\xi}\|_{W_\epsilon^{1,2}} < \bar{C}\gamma.$$

Proof. 1. The map $B_{q_0\epsilon} \ni \xi \to u^{\xi} \in L^2$ is continuous. Therefore there exists $\xi \in \overline{B_{q_0\epsilon}}$ that satisfies (4.29).

2. $\xi \in B_{q_0\epsilon}$. From (4.22), for $\zeta \in \partial B_{q_0\epsilon}$, it follows that

$$\begin{aligned}|\xi - \zeta| &\geq \frac{\epsilon^{\frac{1}{2}}}{C_1}\|u^{\xi} - u^{\zeta}\|_{L^2} \\ &\geq \frac{\epsilon^{\frac{1}{2}}}{C_1}(\|u - u^{\zeta}\|_{L^2} - \|u - u^{\xi}\|_{L^2}) \geq \frac{\epsilon^{\frac{1}{2}}}{C_1}(c - a)\epsilon\end{aligned}$$

and therefore $\xi \notin \partial B_{q_0\epsilon}$ provided $a < c$.

3. (i) follows from 2. and a standard argument.

4. There is a unique $\xi \in B_{q_0\epsilon}$ satisfying (4.29). If ξ and $\hat{\xi}$ solve (4.29), from the parallelogram identity we derive

$$4\delta^2 = 2\|u - u^{\xi}\|_{L^2}^2 + 2\|u - u^{\hat{\xi}}\|_{L^2}^2 = \|2u - u^{\xi} - u^{\hat{\xi}}\|_{L^2}^2 + \|u^{\xi} - u^{\hat{\xi}}\|_{L^2}^2 \tag{4.30}$$

and therefore

$$
\begin{aligned}
&\delta^2 = \|u - \frac{u^{\xi}+u^{\hat{\xi}}}{2}\|_{L^2}^2 + \frac{1}{4}\|u^{\xi}-u^{\hat{\xi}}\|_{L^2}^2 \quad \Rightarrow \\
&\|u - \frac{u^{\xi}+u^{\hat{\xi}}}{2}\|_{L^2} = \delta\sqrt{1-\frac{1}{4\delta^2}\|u^{\xi}-u^{\hat{\xi}}\|_{L^2}^2} \leq \delta - \frac{1}{8\delta}\|u^{\xi}-u^{\hat{\xi}}\|_{L^2}^2.
\end{aligned} \tag{4.31}
$$

This and

$$
\|u-u^{\frac{\xi+\hat{\xi}}{2}}\|_{L^2} - \|\frac{u^{\xi}+u^{\hat{\xi}}}{2} - u^{\frac{\xi+\hat{\xi}}{2}}\|_{L^2} \leq \|u - \frac{u^{\xi}+u^{\hat{\xi}}}{2}\|_{L^2}
$$

imply

$$
\|u-u^{\frac{\xi+\hat{\xi}}{2}}\|_{L^2} \leq \delta - \frac{1}{8\delta}\|u^{\xi}-u^{\hat{\xi}}\|_{L^2}^2 + \|\frac{u^{\xi}+u^{\hat{\xi}}}{2} - u^{\frac{\xi+\hat{\xi}}{2}}\|_{L^2}. \tag{4.32}
$$

To estimate $\|\frac{u^{\xi}+u^{\hat{\xi}}}{2} - u^{\frac{\xi+\hat{\xi}}{2}}\|_{L^2}$ we observe that, for each $x \in [0,1]$ one has

$$
\begin{aligned}
&u^{\frac{\xi+\hat{\xi}}{2}} - \frac{u^{\xi}+u^{\hat{\xi}}}{2} = \frac{1}{2}((u^{\frac{\xi+\hat{\xi}}{2}} - u^{\xi}) + (u^{\frac{\xi+\hat{\xi}}{2}} - u^{\hat{\xi}})) \\
&= \frac{1}{4}\int_0^1 (u_{\xi}^{\xi+\frac{s}{2}(\hat{\xi}-\xi)} - u_{\xi}^{\xi+(1-\frac{s}{2})(\hat{\xi}-\xi)})ds\cdot(\hat{\xi}-\xi) \\
&= \frac{1}{4}\int_0^1 (1-s)\int_0^1 u_{\xi\xi}^{\xi+(1-\frac{s}{2}+t(1-s))(\hat{\xi}-\xi)}dtds(\hat{\xi}-\xi)\cdot(\hat{\xi}-\xi).
\end{aligned}
$$

Since $|\hat{\xi}-\xi| < 2q_0\epsilon$, proceeding as in the proof of Lemma 4.3 and reducing the value of q_0 if necessary, we obtain

$$
\|\frac{u^{\xi}+u^{\hat{\xi}}}{2} - u^{\frac{\xi+\hat{\xi}}{2}}\|_{L^2} \leq \frac{C_3}{\epsilon^{\frac{3}{2}}}|\hat{\xi}-\xi|^2.
$$

Inserting this into (4.32) and using (4.22) yields

$$
\|u-u^{\frac{\xi+\hat{\xi}}{2}}\|_{L^2} \leq \delta - (\frac{c_1^2}{8a\epsilon^2} - \frac{C_3}{\epsilon^{\frac{3}{2}}})|\hat{\xi}-\xi|^2, \tag{4.33}
$$

since $\delta \leq \eta = a\epsilon$. The claim follows from (4.33) which contradicts the minimality of δ if $\epsilon > 0$ is sufficiently small and if $\hat{\xi} \neq \xi$. This complete the proof of (i).

To prove (ii) note that $d_{W_\epsilon^{1,2}}(u, \mathcal{M}(\xi_0)) < \gamma \leq a\epsilon$ implies $d_{L^2}(u, \mathcal{M}(\xi_0)) < \gamma$ and therefore

$$
\|u-u^{\xi}\|_{L^2} < \gamma. \tag{4.34}
$$

Fix $\zeta \in B_{q_0\epsilon}(\xi_0)$ such that

$$
\|u-u^{\zeta}\|_{W_\epsilon^{1,2}} < 2\gamma. \tag{4.35}
$$

Then we have

$$
\begin{aligned}
&2\gamma > \|u-u^{\zeta}\|_{W_\epsilon^{1,2}} \geq \|u-u^{\zeta}\|_{L^2} \geq \|u^{\zeta}-u^{\xi}\|_{L^2} - \|u-u^{\xi}\|_{L^2} \\
&\Rightarrow \qquad \|u^{\zeta}-u^{\xi}\|_{L^2} < 3\gamma.
\end{aligned} \tag{4.36}
$$

Observe that (4.23) implies

$$\|u^\xi - u^\zeta\|_{W_\epsilon^{1,2}} \leq \frac{C_4}{\epsilon^{\frac{1}{2}}}|\xi - \zeta|. \tag{4.37}$$

From this, (4.22) and (4.36) it follows that

$$\|u^\xi - u^\zeta\|_{W_\epsilon^{1,2}} \leq 3\frac{c_1}{C_4}\gamma \tag{4.38}$$

and therefore

$$\|u - u^\xi\|_{W_\epsilon^{1,2}} \leq \|u - u^\zeta\|_{W_\epsilon^{1,2}} + \|u^\xi - u^\zeta\|_{W_\epsilon^{1,2}} \leq (2 + 3\frac{c_1}{C_4})\gamma.$$

The proof is complete. □

Set

$$\mathcal{N}^{a\epsilon}(\xi_0) = \{u \in W^{1,2} : d_{W_\epsilon^{1,2}}(u, \mathcal{M}(\xi_0)) < a\epsilon,\ d_{L^2}(u, \mathcal{M}(\xi_0)) > c\epsilon\}.$$

On the basis of Lemma 4.4, each $u \in \mathcal{N}^{a\epsilon}(\xi_0)$ can be decomposed in a unique way in the form

$$u = u^\xi + \psi, \tag{4.39}$$

where $\xi \in B_{q_0\epsilon}(\xi_0)$ is as in Lemma 4.4 and $\psi = u - u^\xi$ satisfies the orthogonality condition (i). The decomposition (4.39) brings about a decomposition of the energy difference $J(u) - J(u^\xi)$ that corresponds to the (2.4) in the abstract Theorem 2.1. We can indeed write

$$\begin{aligned}
&J(u) - J(u^\xi) = J(u^\xi + \psi) - J(u^\xi)\\
&= \int_0^1 \Big(\frac{\epsilon^2}{2}(2u_x^\xi \cdot \psi_x + |\psi_x|^2) + W(u^\xi) + \psi) - W(u^\xi)\Big)dx\\
&= \int_0^1 \Big(\epsilon^2 u_x^\xi \cdot \psi_x + W_u(u^\xi)\psi\Big)dx\\
&+ \int_0^1 \frac{1}{2}\Big(\epsilon^2|\psi_x|^2) + W_{uu}(u^\xi)\psi \cdot \psi)\Big)dx\\
&+ \int_0^1 \Big(W(u^\xi + \psi) - W(u^\xi) - W_u(u^\xi)\psi - \frac{1}{2}W_{uu}(u^\xi)\psi \cdot \psi\Big)dx\\
&= L(u^\xi, \psi) + Q(u^\xi, \psi) + N(u^\xi, \psi)
\end{aligned} \tag{4.40}$$

with obvious identification of linear, quadratic, and higher order terms L, Q and N.
Based on the decomposition (4.40), we proceed to estimate the constants δ_L, K_0, K_1, and μ for the case at hand.

Lemma 4.5. *We have*

$$|L(u^\xi, \psi)| \leq \delta_L \|\psi\|_{W_\epsilon^{1,2}}, \tag{4.41}$$

where $\delta_L = O(e^{-\frac{k\rho}{2\epsilon}})$.

Proof. Note that

$$
\begin{aligned}
L(u^\xi,\psi) &= \int_0^1 (\epsilon^2 u_x^\xi \psi_x + W_u(u^\xi)\psi)dx \\
&= \epsilon^2(u_x^\xi(1)\psi(1) - u_x^\xi(0)\psi(0)) + \int_0^1 (-\epsilon^2 u_{xx}^\xi + W_u(u^\xi))\psi dx \\
&= \epsilon^2(u_x^\xi(1)\psi(1) - u_x^\xi(0)\psi(0)) + \sum_{i=1}^N \int_{[\hat{\xi}_{i-1},\hat{\xi}_i]} (-\epsilon^2 u_{xx}^\xi + W_u(u^\xi))\psi dx.
\end{aligned}
\tag{4.42}
$$

We have

$$
\begin{aligned}
&|\epsilon^2(u_x^\xi(1)\psi(1) - u_x^\xi(0)\psi(0))| \\
&\le C_\epsilon e^{-\frac{k\rho}{2\epsilon}}(|\psi(0)| + |\psi(1)|) \le C_\epsilon e^{-\frac{k\rho}{2\epsilon}} \|\psi\|_{W_\epsilon^{1,2}}.
\end{aligned}
\tag{4.43}
$$

The change of variables $x = s + \xi_i$ gives

$$
\int_{[\hat{\xi}_{i-1},\hat{\xi}_i]} (-\epsilon^2 u_{xx}^\xi + W_u(u^\xi))\psi dx =
\tag{4.44}
$$

$$
\int_{-\frac{\xi_i-\xi_{i-1}}{2}}^{\frac{\xi_{i+1}-\xi_i}{2}} (-\epsilon^2 u_{xx}^\xi(s+\xi_i) + W_u(u^\xi(s+\xi_i)))\psi(s+\xi_i))ds
$$

The same argument leading to (4.19) yields

$$
u_{xx}^\xi(s+\xi_i) = \frac{1}{\epsilon^2}\bar{u}_i''(\frac{s}{\epsilon}) + \mathrm{O}(\frac{1}{\epsilon^2}e^{-k\frac{\rho}{2\epsilon}}) \text{ for } s \in (-\frac{\xi_i - \xi_{i-1}}{2}, \frac{\xi_{i+1} - \xi_i}{2}).
\tag{4.45}
$$

This and (4.5) imply

$$
\begin{aligned}
&-\epsilon^2 u_{xx}^\xi(s+\xi_i) + W_u(u^\xi(s+\xi_i)) \\
&= -\bar{u}_i''(\frac{s}{\epsilon}) + W_u(\bar{u}_i(\frac{s}{\epsilon})) + \mathrm{O}(e^{-k\frac{\rho}{2\epsilon}}) \\
&= \mathrm{O}(e^{-k\frac{\rho}{2\epsilon}}), \quad s \in (-\frac{\xi_i - \xi_{i-1}}{2}, \frac{\xi_{i+1} - \xi_i}{2}).
\end{aligned}
\tag{4.46}
$$

□

To evaluate the constant K_0 needed to apply Theorem 2.1 to the case at hand we need to analyze the operator $\mathcal{L}^\xi = -\epsilon^2\frac{d^2}{dx^2} + W_{uu}(u^\xi) : W^{2,2}([0,1];\mathbb{R}^m) \to L^2([0,1];\mathbb{R}^m)$ that appears when $J(u)$ is expanded around $u^\xi \in \mathcal{M}$. The eigenvalues and the eigenvectors of $\mathcal{L}^\xi$ are the solutions (λ, ϕ) to

$$
\begin{cases}
\mathcal{L}^\xi\varphi = -\epsilon^2\varphi_{xx} + W_{uu}(u^\xi)\varphi = \lambda\varphi, & 0 < x < 1 \\
\varphi' = 0, & x = 0, 1.
\end{cases}
\tag{4.47}
$$

As in the scalar case we expect that $\mathcal{L}^\xi$ has N exponentially small eigenvalues $\lambda_1, \dots, \lambda_N$ with corresponding eigenvectors lying approximately in the subspace span$\{u_{\xi_1}^\xi, \dots, u_{\xi_N}^\xi\}$ tangent to $\mathcal{M}$ at u^ξ.

Proposition 4.6. *There exist $\epsilon_0 > 0$ and $\lambda > 0$ such that, for $\epsilon \in (0, \epsilon_0]$, the eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots$ of the operator $\mathcal{L}^\xi$ satisfy*

$$
\begin{aligned}
&|\lambda_n| \leq Ce^{-\frac{k\rho}{2\epsilon}}, \ \ n = 1, \ldots, N, \\
&\lambda_n \geq \lambda, \ \ n \geq N+1,
\end{aligned}
\tag{4.48}
$$

To complement the information for the operator $\mathcal{L}^\xi$ given in the proposition above, we show that the subspace of the first N (normalized) eigenfunctions $\varphi_1, \ldots, \varphi_N$ of $\mathcal{L}^\xi$ is well approximated by $\text{span}\{u^\xi_{\xi_1}, \ldots, u^\xi_{\xi_N}\}$.

Proposition 4.7. *Let $\lambda_1 \leq \ldots \leq \lambda_N$ the first N eigenvalues of $\mathcal{L}^\xi$ and let $\varphi_1, \ldots, \varphi_N$ be the corresponding orthonormal eigenfunctions. Then*

$$
\|\varphi_i - \sum_j c^i_j \frac{u^\xi_{\xi_j}}{\|u^\xi_{\xi_j}\|}\| = O(e^{-\frac{k\rho}{2\epsilon}}), \quad i = 1, \ldots, N,
\tag{4.49}
$$

for some $c^i_j \in \mathbb{R}$ such that $\sum_j (c^i_j)^2 = 1$.

For the proofs of Proposition 4.6 and Proposition 4.7 we refer the reader to Section 5. Set $\Sigma = \text{Span}(\varphi_1, \ldots, \varphi_N)$.

Lemma 4.8. *One has*

$$
Q(u^\xi, \psi) \geq K_0 \|\psi\|^2_{W^{1,2}_\epsilon},
\tag{4.50}
$$

where $K_0 = \frac{\lambda}{2}$ and $\lambda > 0$ is the constant in Proposition 4.6.

Proof. From

$$
\langle \psi, \frac{u^\xi_{\xi_i}}{\|u^\xi_{\xi_i}\|} \rangle_{L^2} = 0, \ \ i = 1, \cdots N.
\tag{4.51}
$$

and $(4.49)_2$ in Proposition 4.7 it follows

$$
\langle \psi, \varphi_i \rangle_{L^2} = \langle \psi, \varphi_i - \sum_j c^i_j \frac{u^\xi_{\xi_j}}{\|u^\xi_{\xi_j}\|} \rangle_{L^2} = \mathrm{O}(\|\psi\| e^{-\frac{k\rho}{2\epsilon}}).
\tag{4.52}
$$

Therefore, if we decompose ψ as

$$
\psi = \psi^\Sigma + \psi^\perp,
$$

where ψ^Σ is the orthogonal projection of ψ on Σ and $\psi^\perp$ the projection on the orthogonal complement, we have by (4.52)

$$
\|\psi^\Sigma\| = \mathrm{O}(\|\psi\| e^{-\frac{k\rho}{2\epsilon}})
\tag{4.53}
$$

and

$$
\|\psi^\perp\| = \|\psi\| (1 - \mathrm{O}(e^{-\frac{k\rho}{2\epsilon}})).
\tag{4.54}
$$

Therefore, using (4.49) and Proposition 4.6 we have, denoting by ψ_i the projection of ψ on the i-th eigenvector φ_i of $\mathcal{L}^\xi$,

$$
\begin{aligned}
&\int_0^1 (\epsilon^2|\psi'|^2 + W_{uu}(u^\xi)\psi\cdot\psi)dx = \sum_{i=1}^{\infty}\lambda_i\|\psi_i\|^2 \\
&= \sum_{i=1}^{N}\lambda_i\|\psi_i\|^2 + \sum_{i=N+1}^{\infty}\lambda_i\|\psi_i\|^2 \qquad (4.55)\\
&\geq \lambda\|\psi^\perp\|^2 - \max_{i\leq N}|\lambda_i|\|\psi^\Sigma\|^2 \geq \frac{\lambda}{2}\|\psi\|^2.
\end{aligned}
$$

Following [15] this estimate can be upgraded to

$$
\int_0^1 (\epsilon^2|\psi'|^2 + W_{uu}(u^\xi)\psi\cdot\psi)dx \geq \frac{\lambda}{2}\|\psi\|^2_{W^{1,2}_\epsilon}. \qquad (4.56)
$$

The proof is complete. □

Lemma 4.9. *One has*

$$
|N(u^\xi,\psi)| \leq K_1\|\psi\|^\mu_{W^{1,2}_\epsilon}, \qquad (4.57)
$$

where $K_1 = \dfrac{C}{\epsilon^3}$ *and* $\mu = 3$.

Proof. Note that

$$
\begin{aligned}
&W(u^\xi(x)+\psi(x)) - W(u^\xi(x)) - W_u(u^\xi(x))\psi(x) - \frac{1}{2}W_{uu}(u^\xi(x))\psi(x)\cdot\psi(x)\\
&= \int_0^1\int_0^1\int_0^1 \sigma_1^2\sigma_2 W_{uuu}(u^\xi(x)+\sigma_1\sigma_2\sigma_3\psi(x))(\psi(x),\psi(x),\psi(x))d\sigma_1 d\sigma_2 d\sigma_3.
\end{aligned}
$$

This

$$
N(u^\xi,\psi) = \int_0^1 \Big(W(u^\xi+\psi) - W(u^\xi) - W_u(u^\xi)\psi - \frac{1}{2}W_{uu}(u^\xi)\psi\cdot\psi\Big)dx
$$

and

$$
\|\psi\|_{L^\infty} \leq C\epsilon^{-1}\|\psi\|_{W^{1,2}_\epsilon}
$$

imply that for $u \in \mathcal{N}^{a\epsilon}$ one has

$$
|N(u^\xi,\psi)| \leq \frac{C}{\epsilon^3}\|\psi\|^3_{W^{1,2}_\epsilon}.
$$

The proof is complete. □

We are now in a position to apply Theorem 2.1. On the basis of the estimates in Lemmas 4.2, 4.5, 4.8, 4.9, and on the assumption on δ_1 in Theorem 4.1, we compute

$$
\begin{aligned}
&\hat{\eta} \geq (\frac{\lambda\epsilon^3}{4C})^{1/(3-2)},\\
&\eta^* \geq \min\{(\frac{\lambda\epsilon^3}{4C\bar{C}})^{1/(3-2)}, a\epsilon\} \geq C\epsilon^3, \qquad (4.58)\\
&\eta_* \leq Ce^{-\frac{k\rho}{2\epsilon}}.
\end{aligned}
$$

The continuity of the map $\overline{\Xi} \ni \xi \to u^\xi$ implies that, given $u_0 \in \mathcal{N}^{C\epsilon^3}$, there exists $\xi_0 \in \overline{\Xi}$ such that

$$\|u_0 - u^{\xi_0}\|_{L^2} = d_{L^2}(u_0, \mathcal{M}) < C\epsilon^3.$$

We claim that, if in (4.11) we take $d = C'\epsilon^{\frac{1}{2}}$ with $C' > 2C_1 q_0$, then the ball $B_{q_0\epsilon}(\xi_0)$ is contained in Ξ. To show this we note that, for $\gamma \in \overline{\Xi} \cap B_{q_0\epsilon}(\xi_0)$, Lemma 4.3 implies

$$\|u^\gamma - u^{\xi_0}\|_{L^2} \leq C_1 q_0 \epsilon^{\frac{1}{2}}.$$

On the other hand for $u^\zeta \in \partial\mathcal{M}$ we have $\|u_0 - u^\zeta\|_{L^2} \geq d$ and therefore from

$$\begin{aligned} d &\leq \|u_0 - u^\zeta\|_{L^2} \leq \|u_0 - u^{\xi_0}\|_{L^2} + \|u^{\xi_0} - u^\gamma\|_{L^2} + \|u^\gamma - u^\zeta\|_{L^2} \\ &\leq \|u^\gamma - u^\zeta\|_{L^2} + C\epsilon^3 + C_1 q_0 \epsilon^{\frac{1}{2}} \end{aligned}$$

we conclude

$$\|u^\gamma - u^\zeta\|_{L^2} \geq d - 2C_1 q_0 \epsilon^{\frac{1}{2}} = (C' - C_1 q_0)\epsilon^{\frac{1}{2}} > 0.$$

Therefore $u^\gamma \in \mathcal{M}$ and $\gamma \in \Xi$. This proves the claim. The fact that $B_{q_0\epsilon}(\xi_0) \subset \Xi$ allows the application of Theorem 2.1 to $\mathcal{M}(\xi_0)$ and $\mathcal{N}^{C\epsilon^3}(\xi_0)$ and, on the basis of (4.58), conclude that $u(t, u_0) \in \mathcal{N}^{e^{-\frac{k\rho}{2\epsilon}}}(\xi_0)$ for $t \in [0, T(\xi_0))$ for some $T(\xi_0) > 0$. It may happen that $T(\xi_0) = +\infty$ in which case $u(t, u_0) \in \mathcal{N}^{e^{-\frac{k\rho}{2\epsilon}}}$ for all $t \geq 0$. If instead $T(\xi_0) < +\infty$ we have two possibilities:

(i) $d_{L^2}(u(T(\xi_0), u_0), \partial\mathcal{M}) = d$ or

(ii) $d_{L^2}(u(T(\xi_0), u_0), \partial\mathcal{M}(\xi_0)) = c\epsilon$ and $d_{L^2}(u(T(\xi_0), u_0), \partial\mathcal{M}) > d$.

If (i) prevails we can identify $T(\xi_0)$ with T in Theorem 4.1. If (ii) prevails we can iterate the procedure after replacing u_0 with $u_{0,1} = u(T(\xi_0), u_0)$ and u^{ξ_0} with the projection $u^{\xi_{0,1}}$ of $u(T(\xi_0), u_0)$ on $\mathcal{M}(\xi_0)$. Continuing in this way the iteration process ends if for some $j \geq 1$ one has either $T(\xi_{0,j}) = +\infty$ or $d_{L^2}(u(T(\xi_{0,j}), u_{0,j}), \partial\mathcal{M}) = d$. This concludes the proof of the first part of Theorem 4.1. It remains to estimate T. From Theorem 2.2 with $\tau = 2$, for $\epsilon \in (0, \epsilon_0)$ for some $\epsilon_0 > 0$ we have

$$T \geq e^{\frac{k\rho}{2\epsilon}} (d_{L^2}(u_0, \partial\mathcal{M}) - d)^2.$$

This completes the proof for the vector Allen-Cahn case. To estimate the time T for the Cahn-Morral system (4.3) we start from

$$T \geq e^{\frac{k\rho}{2\epsilon}} \|u(T, u_0) - u_0\|_{H^{-1}}^2,$$

that follows from Theorem 2.2 with $\tau = 2$ and $X = H^{-1}$. Let $u^{\bar{\xi}}$ be such that

$$\|u(T, u_0) - u^{\bar{\xi}}\|_{L^2} = d.$$

Then we have

$$\begin{aligned} &\|u(T, u_0) - u_0\|_{H^{-1}} \geq \|u^{\bar{\xi}} - u_0\|_{H^{-1}} - \|u(T, u_0) - u^{\bar{\xi}}\|_{H^{-1}} \\ &\geq d_{H^{-1}}(u_0, \partial\mathcal{M}) - d, \end{aligned}$$

where we have used

$$\|v\|_{H^{-1}} \leq \|v\|_{L^2}, \quad \text{for } v \in L^2((0,1); \mathbb{R}^m).$$

The proof of Theorem 4.1 is complete. □

5 Appendix: Eigenvalues and Eigenvectors of $\mathcal{L}^\xi$

5.1 Proof of Proposition 4.6

Proof. The proof is based on the min-max characterization of the eigenvalues of second order selfadjoint operators

$$\lambda_n = \max_{\Sigma_n} \min_{\Sigma_n^\top} \frac{\langle \mathcal{L}^\xi \phi, \phi \rangle}{\langle \phi, \phi \rangle}, \quad n = 1, \ldots \tag{5.1}$$

where $\langle f, g \rangle = \int_0^1 f \cdot g \, \mathrm{d}x$ and $\Sigma_n \subset W^{1,2}([0,1];\mathbb{R}^m)$ is an $n-1$ dimensional subspace and $\Sigma_n^\top \subset W^{1,2}([0,1];\mathbb{R}^m)$ its L^2-orthogonal complement. We associate to (5.1) two auxiliary problems. In the first problem we replace $W^{1,2}([0,1];\mathbb{R}^m)$ with the subspace $U \subset W^{1,2}([0,1];\mathbb{R}^m)$ of maps $\phi : [0,1] \to \mathbb{R}^m$ that vanish at $0, \frac{\xi_1+\xi_2}{2}, \frac{\xi_2+\xi_3}{2}, \ldots, \frac{\xi_{N-1}+\xi_N}{2}, 1$ and denote by $\nu_n,\ n = 1, \ldots$, the eigenvalues given by (5.1) with $\Sigma_n, \Sigma_n^\top \subset U$. In the second problem we regard $W^{1,2}([0,1];\mathbb{R}^m)$ as a subspace of

$$V \equiv \oplus_{j=1}^N W^{1,2}([(\xi_j + \xi_{j-1})/2, (\xi_{j+1} + \xi_j)/2];\mathbb{R}^m), \quad (\xi_0 = -\xi_1,\ \xi_{N+1} = 2 - \xi_N).$$

We let $\mu_n,\ n = 1, \ldots$ the eigenvalues given by (5.1) with $\Sigma_n, \Sigma_n^\top \subset V$.

From (5.1) and the definition of U and V, it follows that

$$\mu_n \leq \lambda_n \leq \nu_n, \quad n = 1, \ldots \tag{5.2}$$

and therefore that we can derive lower and upper bounds for the eigenvalues of $\mathcal{L}^\xi$ by studying the eigenvalues of the auxiliary problems. We remark that in both problems the analysis involves intervals where u^ξ has just one layer. Indeed, in both cases it suffices to consider the Rayleigh quotient

$$\begin{aligned} &\frac{\int_{\frac{\xi_j+\xi_{j-1}}{2}}^{\frac{\xi_{j+1}+\xi_j}{2}} (|\phi_x|^2 + W_{uu}(u^\xi)\phi \cdot \phi) \, \mathrm{d}x}{\int_{\frac{\xi_j+\xi_{j-1}}{2}}^{\frac{\xi_{j+1}+\xi_j}{2}} |\phi|^2 \, \mathrm{d}x} \\ &= \frac{\int_{-\frac{\alpha_j}{\epsilon}}^{\frac{\beta_j}{\epsilon}} (|\psi'|^2 + W_{uu}(\bar{u}_j + \bar{v}_j)\psi \cdot \psi) \, \mathrm{d}r}{\int_{-\frac{\alpha_j}{\epsilon}}^{\frac{\beta_j}{\epsilon}} |\psi|^2 \, \mathrm{d}r}, \quad j = 1, \ldots, N, \end{aligned} \tag{5.3}$$

where $\phi(x) = \psi(\frac{x-\xi_j}{\epsilon})$, $\alpha_j = \frac{\xi_j - \xi_{j-1}}{2}$, $\beta_j = \frac{\xi_{j+1} - \xi_j}{2}$, $\bar{v}_j(r) = u^\xi(\xi_j + \epsilon r) - \bar{u}_j(r)$ with homogeneous Dirichlet or Neumann boundary conditions. Note that the Rayleigh quotient $\mathcal{R}^*$ defined in (5.3) is associated to the operator $\mathcal{L}_j^*$

$$\mathcal{L}_j^* \psi = -\psi'' + W_{uu}(\bar{u}_j + \bar{v}_j)\psi, \quad r \in (-\frac{\alpha_j}{\epsilon}, \frac{\beta_j}{\epsilon}),$$

with Dirichlet or Neumann boundary conditions

To estimate the eigenvalues of $\mathcal{L}_j^*$ we introduce test functions that satisfy the boundary conditions at $-\frac{\alpha_j}{\epsilon}$ and $\frac{\beta_j}{\epsilon}$. Since the argument that we develop is valid for all $j \in \{1, \ldots, N\}$ we temporarily drop the index j in (5.3) and in the following equations. We denote by $\mu_i^*,\ i = 1, \ldots$ ($\nu_i^*,\ i = 1, \ldots$) the eigenvalues of $\mathcal{L}^*$ with Neumann (Dirichlet) boundary conditions. The test maps that we consider are the restrictions of $\bar{u}'$ and $\bar{u}''$ to the interval $[-\frac{\alpha}{\epsilon}, \frac{\beta}{\epsilon}]$ with exponentially small correction designed to satisfy Dirichlet or Neumann conditions at $-\frac{\alpha}{\epsilon}$ and $\frac{\beta}{\epsilon}$. We use the letter τ for test maps satisfying Dirichlet conditions and the letter ω for test maps satisfying Neumann conditions. Let $\langle\cdot,\cdot\rangle^*$ be the inner product in $L^2((-\frac{\alpha}{\epsilon}, \frac{\beta}{\epsilon}); \mathbb{R}^m)$, $\|\cdot\|^*$ the associated norm and define

$$\begin{aligned}\tau_0 &= \bar{u}' - h_0\bar{u}'(-\frac{\alpha}{\epsilon}) - h_1\bar{u}'(\frac{\beta}{\epsilon}),\\ \tau_1 &= \bar{u}'' - h_0\bar{u}''(-\frac{\alpha}{\epsilon}) - h_1\bar{u}''(\frac{\beta}{\epsilon}) + p\tau_0,\end{aligned} \tag{5.4}$$

where $h_0(r) = \frac{\epsilon}{\alpha+\beta}(\frac{\beta}{\epsilon} - r)$, $h_1(r) = \frac{\epsilon}{\alpha+\beta}(\frac{\alpha}{\epsilon} + r)$ and p is determined by the condition $\langle\tau_0, \tau_1\rangle^* = 0$. For Neumann conditions we define

$$\begin{aligned}\omega_0 &= \bar{u}' - g_0\bar{u}''(-\frac{\alpha}{\epsilon}) - h_1\bar{u}''(-\frac{\beta}{\epsilon}),\\ \omega_1 &= \bar{u}'' - g_0\bar{u}'''(-\frac{\alpha}{\epsilon}) - g_1\bar{u}'''(\frac{\beta}{\epsilon}) + q\omega_0,\end{aligned} \tag{5.5}$$

where $g_0(r) = -\frac{\epsilon}{2(\alpha+\beta)}(\frac{\beta}{\epsilon}-r)^2$, $g_1(r) = \frac{\epsilon}{2(\alpha+\beta)}(\frac{\alpha}{\epsilon}+r)^2$ and q is determined by the condition $\langle\omega_0, \omega_1\rangle^* = 0$. From (5.4) and (5.5) it follows that

$$\begin{aligned}&p, q = \mathrm{O}(e^{-\frac{k\rho}{2\epsilon}}),\\ &(\|\tau_0\|^*)^2, (\|\omega_0\|^*)^2 = \int_{\mathbb{R}} |\bar{u}'|^2 + \mathrm{O}(e^{-\frac{k\rho}{2\epsilon}}),\\ &(\|\tau_1\|^*)^2, (\|\omega_1\|^*)^2 = \int_{\mathbb{R}} |\bar{u}''|^2 + \mathrm{O}(e^{-\frac{k\rho}{2\epsilon}}).\end{aligned}$$

A standard computation yields

$$\begin{aligned}&(\|\tau_0\|^*)^2\mathcal{R}^*(\tau_0),\ (\|\omega_0\|^*)^2\mathcal{R}^*(\omega_0) = \int_{\mathbb{R}} \Big(|\bar{u}''|^2 + W_{uu}(\bar{u})\bar{u}'\cdot\bar{u}'\Big) + \mathrm{O}(e^{-\frac{k\rho}{2\epsilon}})\\ &= \int_{\mathbb{R}} \Big(-\bar{u}''' + W_{uu}(\bar{u})\bar{u}'\Big)\cdot\bar{u}' + \mathrm{O}(e^{-\frac{k\rho}{2\epsilon}}) = \mathrm{O}(e^{-\frac{k\rho}{2\epsilon}});\\ &(\|\tau_1\|^*)^2\mathcal{R}^*(\tau_1),\ (\|\omega_1\|^*)^2\mathcal{R}^*(\omega_1) = \int_{\mathbb{R}} \Big(|\bar{u}'''|^2 + W_{uu}(\bar{u})\bar{u}''\cdot\bar{u}''\Big) + \mathrm{O}(e^{-\frac{k\rho}{2\epsilon}}).\end{aligned} \tag{5.6}$$

Lemma 5.1. *We have*

$$-C \leq \lambda_n \leq \mathrm{O}(e^{-\frac{k\rho}{2\epsilon}}),\ \ n = 1, \dots, N,$$

for some constant $C > 0$ *independent of* $\epsilon \in (0, \epsilon_0]$.

Proof. The lower bound follows from the fact that $W_{uu}(\bar{u} + \bar{v})$ is bounded by a constant independent of ϵ. From (5.6) and the bound for $\|\tau_0\|^*$ we have

$$\nu_1^* \leq \mathcal{R}^*(\tau_0) = \frac{\langle\mathcal{L}^*\tau_0, \tau_0\rangle^*}{(\|\tau_0\|^*)^2} \leq \mathrm{O}(e^{-\frac{k\rho}{2\epsilon}}). \tag{5.7}$$

Since this is true for all $j \in \{1, \dots, N\}$ the upper bound follows from (5.2). □

To prove $(4.48)_2$ we observe that

$$\mu_2^* \geq \mu^* \equiv \min_{\substack{\langle\psi,\omega_0\rangle^*=0 \\ \|\psi\|^*=1}} \mathcal{R}^*(\psi) \tag{5.8}$$

and show that μ^* has a positive lower bound independent of ϵ. The existence of the minimum in (5.8) follows from standard arguments based on the fact that from (5.6) and the bound on $W_{uu}(\bar{u}+\bar{v})$ we have

$$\|\psi'\|^* \leq \|\omega_1'\|^* \leq C.$$

This and $\|\psi\|^* = 1$ imply the existence of a minimizing sequence $\{\psi_n\}_n$ that converges weakly in $W^{1,2}([-\frac{\alpha}{\epsilon}, \frac{\beta}{\epsilon}]; \mathbb{R}^m)$ and strongly in $L^2((-\frac{\alpha}{\epsilon}, \frac{\beta}{\epsilon}); \mathbb{R}^m)$ to some $\psi \in W^{1,2}([-\frac{\alpha}{\epsilon}, \frac{\beta}{\epsilon}]; \mathbb{R}^m)$. It follows that ψ is bounded and satisfies $\|\psi\|^* = 1$ and $\langle\psi, \omega_0\rangle^* = 0$. From elliptic theory we can then assume that

$$\|\psi\|_{C^{2+\gamma}([-\frac{\alpha}{\epsilon}, \frac{\beta}{\epsilon}]; \mathbb{R}^m)} \leq C, \tag{5.9}$$

with $C > 0$, $\gamma \in (0,1)$ independent of ϵ and that ψ is a classical solution of

$$-\psi'' + W_{uu}(\bar{u}+\bar{v})\psi - \eta\omega_0 - \mu^*\psi = 0, \quad r \in (-\frac{\alpha}{\epsilon}, \frac{\beta}{\epsilon}), \tag{5.10}$$

with η a Lagrange multiplier determined by the condition $\langle\psi, \omega_0\rangle^* = 0$

$$\eta = -\frac{\langle\mathcal{L}^*\psi, \omega_0\rangle}{(\|\omega_0\|^*)^2} = -\frac{\langle\mathcal{L}^*\omega_0, \psi\rangle}{(\|\omega_0\|^*)^2} = \mathrm{O}(e^{-\frac{k\rho}{2\epsilon}}), \tag{5.11}$$

where we have also used (5.6).

Lemma 5.2. *There exist $\epsilon_0 > 0$ and $\lambda > 0$ such that*

$$\mu^* \geq \lambda \quad \textit{for all } \epsilon \in (0, \epsilon_0].$$

Proof. We first show that, if $\mu^* < c_0$ with c_0 the smallest eigenvalue of the matrix $W_{uu}(a)$, $a \in \{a_1, \ldots, a_{N+1}\}$, then $\psi(r)$ decays exponentially in $|r|$. This is a special instance of a general fact [1]. Note that (5.10) implies

$$(|\psi|^2)'' \geq 2\psi'' \cdot \psi = 2(W_{uu}(\bar{u}+\bar{v}) - \mu^* I)\psi \cdot \psi - 2\eta\omega_0 \cdot \psi. \tag{5.12}$$

From the assumption on c_0 and the fact that $\bar{u}(r)$ converges exponentially to some $a \in \{a_1, \ldots, a_{N+1}\}$ for $r \to \pm\infty$ and the smallness of $\bar{v}$ for $\epsilon > 0$ small we obtain

$$(W_{uu}(\bar{u}+\bar{v}) - \mu^* I)\psi \cdot \psi \geq c^2|\psi|^2, \ \text{ for } |r| \geq r_0, \ \epsilon \in (0, \epsilon_0], \tag{5.13}$$

for some $c > 0$ and $\epsilon_0 > 0$. From (5.11) and (5.5) that implies $\omega_0(r) \leq Ce^{-k|r|}$ it follows

$$2|\eta\omega_0 \cdot \psi| \leq \frac{|\eta|^2 Ce^{-2k|r|}}{c^2} + c^2|\psi|^2.$$

From this (5.13) and (5.12) we obtain

$$(|\psi|^2)'' \geq c^2|\psi|^2 - \tilde{\eta}e^{-c|r|}, \quad |r| \geq r_0, \tag{5.14}$$

where $\tilde{\eta} = \frac{|\eta|^2 C}{c^2} = \mathrm{O}(e^{-\frac{k\rho}{\epsilon}})$ and we have observed that we can assume $0 < c < 2k$. The comparison principle and (5.14) imply that, for $r \in (r_0, \frac{\beta}{\epsilon})$, one has $|\psi|^2(r) \leq y(r)$ where $y : [r_0, \frac{\beta}{\epsilon}] \to \mathbb{R}$ is the solution to the problem

$$\begin{cases} y'' = c^2 y - \tilde{\eta} e^{-cr}, \ \ r \in (r_0, \frac{\beta}{\epsilon}), \\ y(r_0) = y_0, \\ y'(\frac{\beta}{\epsilon}) = 0, \end{cases} \tag{5.15}$$

with y_0 an L^∞ bound for $|\psi|^2$. We have

$$y(r) = Ae^{cr} + Be^{-cr} + \frac{\tilde{\eta}}{2c} r e^{-cr},$$

with

$$\begin{aligned} A &= \frac{l e^{-c\frac{\beta}{\epsilon}} + m e^{-cr_0}}{e^{c(\frac{\beta}{\epsilon} - r_0)} + e^{-c(\frac{\beta}{\epsilon} - r_0)}}, \\ B &= \frac{l e^{c\frac{\beta}{\epsilon}} - m e^{cr_0}}{e^{c(\frac{\beta}{\epsilon} - r_0)} + e^{-c(\frac{\beta}{\epsilon} - r_0)}}, \\ l &= y_0 - \frac{\tilde{\eta}}{2c} r_0 e^{-cr_0}, \\ m &= \frac{\tilde{\eta}}{2c}(\frac{\beta}{\epsilon} - \frac{1}{c}) e^{-c\frac{\beta}{\epsilon}}. \end{aligned}$$

From these expressions and $\tilde{\eta} = \mathrm{O}(e^{-\frac{k\rho}{\epsilon}})$ it follows that

$$A = \mathrm{O}(e^{-\frac{2c\beta}{\epsilon}}) \ \text{and} \ B \leq 2y_0 e^{cr_0}.$$

These estimates imply that there is a constant $C > 0$ independent of $\epsilon \in (0, \epsilon_0]$ such that

$$y(r) \leq Ce^{-cr}, \ \ r \in [r_0, \frac{\beta}{\epsilon}].$$

Since a similar estimate applies to the interval $[-\frac{\alpha}{\epsilon}, -r_0]$ and $|\psi(r)|^2 \leq y_0$ in $[-\frac{\alpha}{\epsilon}, \frac{\beta}{\epsilon}]$ we conclude that

$$|\psi(r)| \leq Ce^{-\frac{c}{2}|r|}, \ \ r \in [-\frac{\alpha}{\epsilon}, \frac{\beta}{\epsilon}], \tag{5.16}$$

for some $C > 0$ independent of $\epsilon \in (0, \epsilon_0]$.

If the lemma is false there is a sequence $\epsilon_k \to 0^+$ such that

$$\mu_k^* \to 0, \tag{5.17}$$

where here and in the remaining part of the proof we denote by $\mu_k^*, \psi_k, \ldots$ the values of $\mu^*, \psi, \ldots$ corresponding to ϵ_k. In particular, $\langle f, g \rangle_k^* = \int_{-\frac{\alpha}{\epsilon_k}}^{\frac{\beta}{\epsilon_k}} f \cdot g$. We can assume that

$$\begin{aligned} &\|\psi_n\|_k^* = 1, \\ &\langle \psi_k, \omega_{0,k} \rangle_k^* = 0. \end{aligned} \tag{5.18}$$

From (5.9) we can also assume that the sequence ψ_k converges in the C^2 sense in compact intervals to a map $\tilde{\psi}$ that satisfies

$$\mathcal{L}\tilde{\psi} = -\tilde{\psi}'' + W_{uu}(\bar{u})\tilde{\psi} = 0, \tag{5.19}$$

where we have also used (5.17) and (5.11). Moreover on the basis of (5.16) the identities (5.18) are preserved in the limit and, together with (5.19), we have

$$\begin{aligned} &\|\tilde{\psi}\| = 1, \\ \text{and} \quad & \\ &\langle \tilde{\psi}, \frac{\bar{u}'}{\|\bar{u}'\|} \rangle = 0, \end{aligned} \tag{5.20}$$

where $\langle f, g \rangle = \int_{\mathbb{R}} f \cdot g$ and we have also observed that, as $k \to +\infty$, $\omega_{0,k}$ converges point-wise to $\bar{u}'$. From (5.19) and (5.20) it follows that $\tilde{\psi}$ is an eigenvector of $\mathcal{L}$ orthogonal to the eigenvector $\frac{\bar{u}'}{\|\bar{u}'\|}$ and corresponding to the 0 eigenvalue. This contradiction with assumption $\mathbf{h}_3$) completes the proof. □

We are now in a position to complete the proof of Proposition 4.6. Since the lower bound for the second eigenvalue of $\mathcal{L}^*$ established in Lemma 5.2 applies for all $j \in \{1, \dots, N\}$ we have from (5.2)

$$\lambda_n \geq \lambda \text{ for all } n \geq N + 1.$$

It remain to establish the lower bound $\lambda_n \geq \mathrm{O}(e^{-\frac{k\rho}{\epsilon}})$, $n = 1, \dots, N$. On the basis of (5.2) it suffices to show that μ_1^*, the first eigenvalue of $\mathcal{L}^*$ (with Neumann conditions) satisfies a similar bound. For this we invoke Lemma 5.5 with $A = \mathcal{L}^*$ (with Neumann conditions). From Lemma 5.2 and (5.7) we can assume $\delta \geq \frac{\lambda}{2}$. We also have $N = 1$ and $\eta_m = 1$. Moreover from (5.5), using that $-\bar{u}''' + W_{uu}(\bar{u})\bar{u}' \equiv 0$, it follows that

$$\mathcal{L}^*\omega_0 = \mathrm{O}(e^{-\frac{k\rho}{\epsilon}}) = 0\omega_0 + \mathrm{O}(e^{-\frac{k\rho}{\epsilon}}).$$

Therefore we see that the left hand side of (5.35) is $\mathrm{O}(e^{-\frac{k\rho}{\epsilon}})$ and (5.36) yields

$$|\mu_1^* - 0| = |\nu_1^*| = \mathrm{O}(e^{-\frac{k\rho}{\epsilon}}).$$

This concludes the proof of Proposition 4.6. □

5.2 Proof of Proposition 4.7

Proof. The map $u_{\xi_j}^{\xi}(x) = -\frac{1}{\epsilon}\bar{u}_j'(\frac{x-\xi_j}{\epsilon})$ does not satisfy the boundary conditions at $x = 0, 1$ and as a consequence is not in the domain of $\mathcal{L}^{\xi}$. Therefore we introduce the map ϕ_j defined by

$$\begin{aligned} q_j\phi_j &= u_{\xi_j}^{\xi} - g_0 u_{\xi_j x}^{\xi}(0) - g_1 u_{\xi_j x}^{\xi}(1), \\ q_j &= \|u_{\xi_j}^{\xi} - g_0 u_{\xi_j x}^{\xi}(0) - g_1 u_{\xi_j x}^{\xi}(1)\| \end{aligned} \tag{5.21}$$

where $g_0(x) = -\frac{1}{2}(1-x)^2$, $g_1(x) = \frac{1}{2}x^2$.

Lemma 5.3. *Let $\phi_j,\ j=1,\dots,N$ be as before. Then there is a constant $C>0$ such that*

$$\|\mathcal{L}^\xi\phi_j\| \leq \frac{C}{\epsilon^2}e^{-\frac{k\rho}{2\epsilon}},\ j=1,\dots,N. \tag{5.22}$$

Proof. We have

$$\begin{aligned} q_j\mathcal{L}^\xi\phi_j &= \frac{1}{\epsilon}\Big[\bar{u}_j'''(\frac{x-\xi_j}{\epsilon}) - W_{uu}(\bar{u}_j(\frac{x-\xi_j}{\epsilon}))\bar{u}_j'(\frac{x-\xi_j}{\epsilon}) \\ &+ \Big(W_{uu}(u^\xi(x)) - W_{uu}(\bar{u}(\frac{x-\xi_j}{\epsilon}))\Big)\bar{u}'(\frac{x-\xi_j}{\epsilon})\Big] - q_j\mathcal{L}^\xi(g_0u^\xi_{\xi_jx}(0)+g_1u^\xi_{\xi_jx}(1)) \\ &= \frac{1}{\epsilon}\Big(W_{uu}(u^\xi(x)) - W_{uu}(\bar{u}_j(\frac{x-\xi_j}{\epsilon}))\Big)\bar{u}'(\frac{x-\xi_j}{\epsilon}) + q_j\mathrm{O}(\frac{1}{\epsilon^2}e^{-\frac{k\rho}{2\epsilon}}) \end{aligned} \tag{5.23}$$

where we have used $\bar{u}_j''' = W_{uu}(\bar{u}_j)\bar{u}_j'$ and observed that

$$\begin{aligned} |u^\xi_{\xi_jx}(0)| &= \frac{1}{\epsilon^2}|\bar{u}''(-\frac{\xi_j}{\epsilon})| = \mathrm{O}(\frac{1}{\epsilon^2}e^{-\frac{k\rho}{2\epsilon}}), \\ |u^\xi_{\xi_jx}(1)| &= \frac{1}{\epsilon^2}|\bar{u}''(\frac{1-\xi_j}{\epsilon})| = \mathrm{O}(\frac{1}{\epsilon^2}e^{-\frac{k\rho}{2\epsilon}}). \end{aligned} \tag{5.24}$$

Set $x = s+\xi_j$ and note that, for $s\in[-\frac{\xi_j-\xi_{j-1}}{2},\frac{\xi_{j+1}-\xi_j}{2}]$, (4.19) implies

$$\begin{aligned} &W_{uu}(u^\xi(s+\xi_j)) - W_{uu}(\bar{u}_j(\frac{s}{\epsilon})) \\ &= W_{uu}(\bar{u}_j(\frac{s}{\epsilon}) + \mathrm{O}(e^{-\frac{k\rho}{2\epsilon}})) - W_{uu}(\bar{u}_j(\frac{s}{\epsilon})) = \mathrm{O}(e^{-\frac{k\rho}{2\epsilon}}), \end{aligned} \tag{5.25}$$

while, for $s\in[-\xi_j,1-\xi_j]\setminus[-\frac{\xi_j-\xi_{j-1}}{2},\frac{\xi_{j+1}-\xi_j}{2}]$, we have

$$|\bar{u}_j'(\frac{s}{\epsilon})| = \mathrm{O}(e^{-\frac{k\rho}{2\epsilon}}). \tag{5.26}$$

Note also that from (5.21), using also (5.24), it follows that $q_j = \epsilon^{-\frac{1}{2}}C_j$ with $C_j \simeq (\int_{\mathbb{R}}|\bar{u}_j'|^2)^{\frac{1}{2}}$. This and the estimates (5.25) and (5.26) conclude the proof. □

Recall $\Sigma = \mathrm{span}\{\varphi_1,\dots,\varphi_N\}$ and let $\phi_j = \phi_j^\Sigma + \phi_j^\perp$ with $\phi_j^\Sigma\in\Sigma$ and $\phi_j^\perp\in\Sigma^\perp$. Since Σ and $\Sigma^\perp$ are invariant under $\mathcal{L}^\xi$ we have

$$\begin{aligned} &\lambda\|\phi_j^\perp\|^2 \leq \langle\mathcal{L}^\xi\phi_j^\perp,\phi_j^\perp\rangle = \langle\mathcal{L}^\xi\phi_j,\phi_j^\perp\rangle \leq \|\mathcal{L}^\xi\phi_j\|\|\phi_j^\perp\| \leq \frac{C}{\epsilon^2}e^{-\frac{k\rho}{2\epsilon}}\|\phi_j^\perp\| \\ &\Rightarrow\ \|\phi_j^\perp\| \leq \frac{C}{\lambda\epsilon^2}e^{-\frac{k\rho}{2\epsilon}} \end{aligned} \tag{5.27}$$

where we have also used Proposition 4.6 and Lemma 5.3. From (5.27) it follows

$$1-\|\phi_j^\Sigma\| \leq \frac{C_1}{\epsilon^4}e^{-\frac{k\rho}{\epsilon}}. \tag{5.28}$$

We also have

$$\langle\phi_j^\Sigma,\phi_i^\Sigma\rangle = \mathrm{O}(e^{-\frac{k\rho}{2\epsilon}})\ \text{ for }\ i\neq j. \tag{5.29}$$

This follows from (5.27) that implies

$$\langle\phi_j^\Sigma,\phi_i^\Sigma\rangle = \langle\phi_j,\phi_i\rangle + \mathrm{O}(e^{-\frac{k\rho}{2\epsilon}})\ \text{ for }\ i\neq j \tag{5.30}$$

and from $\langle \phi_j, \phi_i \rangle = \delta_{ij} + \mathrm{O}(e^{-\frac{k\rho}{2\epsilon}})$ that follows from (5.21), (5.24) and $\frac{1}{\|u_{\xi_j}\|\|u^{\xi}_{\xi_i}\|}\langle u^{\xi}_{\xi_j}, u^{\xi}_{\xi_i} \rangle = \delta_{ij} + \mathrm{O}(e^{-\frac{k\rho}{2\epsilon}})$. The estimates (5.28) and (5.29) imply that the N vectors $\phi^{\Sigma}_1, \dots, \phi^{\Sigma}_N$ are linearly independent and we have

$$\Sigma = \mathrm{span}\{\phi^{\Sigma}_1, \dots, \phi^{\Sigma}_N\}. \tag{5.31}$$

From (5.28) and (5.29) it follows that there are orthonormal vectors $\tilde{\varphi}_1, \dots, \tilde{\varphi}_N$ such that

$$\begin{aligned} &\Sigma = \mathrm{span}\{\phi^{\Sigma}_1, \dots, \phi^{\Sigma}_N\} = \mathrm{span}\{\tilde{\varphi}_1, \dots, \tilde{\varphi}_N\}, \\ &\|\phi^{\Sigma}_j - \tilde{\varphi}_j\| = \mathrm{O}(e^{-\frac{k\rho}{2\epsilon}}),\ j = 1, \dots, N. \end{aligned} \tag{5.32}$$

The vectors $\tilde{\varphi}_1, \dots, \tilde{\varphi}_N$ can be constructed by applying Gram-Schmidt orthonormalization process to $\phi^{\Sigma}_1, \dots, \phi^{\Sigma}_N$. From $(5.32)_2$, (5.27) and (5.21) it follows that

$$\|\tilde{\varphi}_j - \frac{u^{\xi}_{\xi_j}}{\|u^{\xi}_{\xi_j}\|}\| = \mathrm{O}(e^{-\frac{k\rho}{2\epsilon}}),\ j = 1, \dots, N. \tag{5.33}$$

On the other hand $(5.32)_1$ implies

$$\begin{aligned} &\varphi_i = \sum_j c^i_j \tilde{\varphi}_j, \quad \text{with} \quad c^i_j = \langle \varphi_i, \tilde{\varphi}_j \rangle, \\ &\sum_j (c^i_j)^2 = \|\varphi_i\|^2 = 1. \end{aligned} \tag{5.34}$$

This and (5.33) prove (4.49). The proof is complete. □

5.3 Perturbation of spectra

Lemma 5.4. *We list below some well known results on perturbation of selfadjoint operators [21]. Let H be a Hilbert space and $A : D(A) \subset H \to H$ a selfadjoint operator. Let $\sigma(A)$ denote the spectrum of A. Assume that there is a bounded interval $I \subset \mathbb{R}$, a positive number $\delta > 0$, linearly independent normalized vector $\phi_1, \dots, \phi_N \in D(A)$ and numbers $\mu_1, \dots, \mu_N$ such that*

(i)

$$A\phi_j = \mu_j \phi_j + e_j, \quad j = 1, \dots, N$$

for some error vectors $e_1, \dots, e_N$.

(ii)

$$\mu_j \in I,$$

(iii)

$$(\sigma(A) \setminus I) \cap (I + (-\delta, \delta)) = \emptyset.$$

Then

$$\sup_{\substack{\phi \in E \\ \|\phi\| = 1}} \|\pi^{\top}\phi\| \leq \frac{N^{\frac{1}{2}} \max_j \|e_j\|}{\delta \eta_m^{\frac{1}{2}}},$$

where $E = \mathrm{span}\{\phi_1, \dots, \phi_N\}$, F is the closed subspace associated to I, $\pi^{\top}$ is the projection on the orthogonal complement $F^{\top}$ of F, and η_m is the smallest eigenvalue of the matrix $\Phi = (\langle \phi_i, \phi_j \rangle)$.

Proof. We give the proof under the assumption that the normalized eigenvectors $\{w_h\}_{h\in\mathbb{N}}$ of A are a basis for H. Let $\{\lambda_h\}_{h\in\mathbb{N}}$ be the corresponding eigenvalues.
1. Given $\phi_j \in \{\phi_1, \dots, \phi_N\}$ we have $\phi_j = \sum_h \langle \phi_j, w_h\rangle w_h$ and therefore

$$A\phi_j = \mu_j\phi_j + e_j = \sum_h \lambda_h \langle \phi_j, w_h\rangle w_h = \sum_h \mu_j \langle \phi_j, w_h\rangle w_h + \sum_h \langle e_j, w_h\rangle w_h.$$

From this, for $\lambda_h \notin I$, it follows that

$$\delta^2 \|\pi^\top \phi_j\|^2 \le \sum_{\lambda_j \notin I} (\lambda_h - \mu_j)^2 \langle \phi_j, w_h\rangle^2 = \sum_{\lambda_j \notin I} \langle e_j, w_h\rangle^2 \le \|e_j\|^2.$$

2. For $\phi \in E$ we have $\phi = \sum_j \alpha_j \phi_j$. If $\phi \in E$ is a unit vector then

$$1 = \langle \sum_j \alpha_j \phi_j, \sum_i \alpha_i \phi_i \rangle = \Phi\alpha \cdot \alpha \ge \eta_m |\alpha|^2.$$

This, $\sum_j |\alpha_j| \le N^{\frac{1}{2}} |\alpha|$, and 1. imply

$$\|\pi^\top \phi\| = \|\pi^\top (\sum_j \alpha_j \phi_j)\| \le \sum_j |\alpha_j| \frac{\max_j \|e_j\|}{\delta} \le \frac{N^{\frac{1}{2}} \max_j \|e_j\|}{\delta \eta_m^{\frac{1}{2}}}.$$

□

Lemma 5.5. *Let dim(F) $= M$ and assume that*

(5.35) $$\frac{N^{\frac{1}{2}} \max_j \|e_j\|}{\delta \eta_m^{\frac{1}{2}}} < 1.$$

Then $M \ge N$ and, if $M < \infty$, for each j there exists $\lambda_{h_j} \in I$ such that

(5.36) $$|\lambda_{h_j} - \mu_j| \le \frac{M^{\frac{1}{2}} \max_j \|e_j\|}{(1 - \frac{N \max_j \|e_j\|^2}{\delta^2 \eta_m})^{\frac{1}{2}}}.$$

Proof. Let ϕ_j^F be the orthogonal projection of ϕ_j on F. Then ϕ_j^F can be expressed in the form $\phi_j^F = \sum_{\lambda_h \in I} \langle \phi_j^F, w_h\rangle w_h$ and $A\phi_j^F$ can be written

(5.37) $$A\phi_j^F = \sum_{\lambda_h \in I} \lambda_h \langle \phi_j^F, w_h\rangle w_h = \mu_j \sum_{\lambda_h \in I} \langle \phi_j^F, w_h\rangle w_h + \sum_{\lambda_h \in I} \langle e_j^F, w_h\rangle w_h.$$

From Lemma 5.4 we obtain

$$\|\phi_j^F\|^2 = \|\phi_j\|^2 - \|\pi^\top \phi_j\|^2 \ge (1 - \frac{N \max_j \|e_j\|^2}{\delta^2 \eta_m}).$$

On the other hand, there is $\lambda_{h_j} \in I$ such that

$$\langle \frac{\phi_j^F}{\|\phi_j^F\|}, w_{h_j}\rangle^2 \ge \frac{1}{M}.$$

From these estimates and the h_j-th component of (5.37) we obtain

$$|\lambda_{h_j}-\mu_j| \leq \frac{|\langle e_j^F, w_{h_j}\rangle|}{|\langle \phi_j^F, w_{h_j}\rangle|} = \frac{|\langle e_j^F, w_{h_j}\rangle|}{|\langle \frac{\phi_j^F}{\|\phi_j^F\|}, w_{h_j}\rangle| \|\phi_j^F\|} \leq \frac{\|e_j\|}{|\langle \frac{\phi_j^F}{\|\phi_j^F\|}, w_{h_j}\rangle| \|\phi_j^F\|}$$

$$\leq \frac{M^{\frac{1}{2}}\|e_j\|}{(1-\frac{N\max_j\|e_j\|^2}{\delta^2\eta_m})^{\frac{1}{2}}}.$$

The proof is complete. □

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