LAYER DYNAMICS FOR THE ONE DIMENSIONAL ε-DEPENDENT CAHN-HILLIARD / ALLEN-CAHN EQUATION

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Abstract. We study the dynamics of the one-dimensional ε-dependent Cahn-Hilliard / Allen-Cahn equation within a neighborhood of an equilibrium of N transition layers, that in general does not conserve mass. Two different settings are considered which differ in that, for the second, we impose a mass-conservation constraint in place of one of the zero-mass flux boundary conditions at $x = 1$. Motivated by the study of Carr and Pego on the layered metastable patterns of Allen-Cahn in [11], and by this of Bates and Xun in [6] for the Cahn-Hilliard equation, we implement an N-dimensional, and a mass-conservative $N-1$-dimensional manifold respectively; therein, a metastable state with N transition layers is approximated. We then determine, for both cases, the essential dynamics of the layers (ode systems with the equations of motion), expressed in terms of local coordinates relative to the manifold used. In particular, we estimate the spectrum of the linearized Cahn-Hilliard / Allen-Cahn operator, and specify wide families of ε-dependent weights $\delta(\varepsilon)$, $\mu(\varepsilon)$, acting at each part of the operator, for which the dynamics are stable and rest exponentially small in ε. Our analysis enlightens the role of mass conservation in the classification of the general mixed problem into two main categories where the solution has a profile close to Allen-Cahn, or, when the mass is conserved, close to the Cahn-Hilliard solution.

1. Introduction

1.1. **The equation.**

In this paper, we examine the dynamics of the Cahn-Hilliard / Allen-Cahn equation

$$u_t = -\delta(\varepsilon)\left(\varepsilon^2 u_{xx} - f(u)\right)_{xx} + \mu(\varepsilon)(\varepsilon^2 u_{xx} - f(u)), \quad x \in (0,1), \quad t > 0, \tag{1.1}$$

in a neighborhood of a layered equilibrium parameterized by a small positive constant ε.

The nonlinearity $f(u) = F'(u)$ is the derivative of a double equal-well potential F taking a non-degenerate global minimum value zero at $u = \pm 1$, where

$$F(\pm 1) = f(\pm 1) = 0, \tag{1.2}$$

$$f'(\pm 1) > 0, \tag{1.3}$$

$$F(u) > 0 \quad \text{for} \quad u \in (-1, 1). \tag{1.4}$$

We define, for simplicity,

$$f(u) := u^3 - u,$$

which is a typical example for a potential $F(u) := \frac{1}{4}(u^2 - 1)^2$. However, many of the results are valid for more general nonlinearities.

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The initial condition

$$u_0(x;\varepsilon) =: u(x,0),\ x \in (0,1),$$

is assumed layered with respect to ε which stands as a measure of the layers width corresponding to a time scale proportional to $e^{C\varepsilon^{-1}}$ for $C > 0$, and therefore, to very long times as $\varepsilon \to 0$ where the solution is expected to change very slowly; see in [11] for the analogous considerations on the Allen-Cahn equation.

We introduce the positive constant $\delta(\varepsilon) > 0$ and the non-negative one $\mu(\varepsilon) \geq 0$ in order to control the coexistence of the 2 operators in terms of ε. Moreover, we impose the presence of the Cahn-Hilliard part in the combined model as $\delta(\varepsilon) \neq 0$, while for $\delta(\varepsilon) := 1$ and $\mu(\varepsilon) := 0$ the Cahn-Hilliard equation stands as a special case.

Equation (1.1) is a gradient flow for the associated free energy with respect to an ε-weighted metric. In particular, the standard Allen-Cahn equation is written as

$$u_t = (\Delta - \varepsilon^{-2} I)u =: \mathcal{A}^\varepsilon(u),$$

the Cahn-Hilliard equation, after rescaling, as

$$u_t = (-\varepsilon\Delta)(\Delta - \varepsilon^{-2} I)u = (-\varepsilon\Delta)(\mathcal{A}^\varepsilon(u)),$$

while (1.1), as

$$u_t = (\varepsilon^2(-\delta(\varepsilon)\Delta + \mu(\varepsilon)I))(\Delta - \varepsilon^{-2} I)u = (\varepsilon^2(-\delta(\varepsilon)\Delta + \mu(\varepsilon)I))(\mathcal{A}^\varepsilon(u)),$$

and the ε-weighted metric is given by

$$< f, g >_\varepsilon := (f, (\varepsilon^2(-\delta(\varepsilon)\Delta + \mu(\varepsilon)I))^{-1} g),$$

for $(\cdot,\cdot)$ the $L^2((0,1))$ inner product; see also the discussion in [19].

A main result of our work is the analysis of the spectrum of the linearized operator, where a crucial spectral condition

$$\varepsilon^2 \mu(\varepsilon) \geq \mathcal{O}(\delta(\varepsilon)),$$

is determined. Considering the CH/AC equation (1.1), in higher dimensions, and for a specific choice of the coefficients $\mu(\varepsilon) := \varepsilon^{-2}$, $\delta(\varepsilon) := \mathcal{O}(1)$, which we stress that satisfy the above inequality, motion by mean curvature was derived on the sharp interface limit $\varepsilon \to 0$, in [19], as in the Allen-Cahn equation limiting dynamics. The sharp interface limit problem of the multi-dimensional Allen-Cahn equation (which is equation (1.1) for $\delta(\varepsilon) := 0$, and $\mu(\varepsilon) := \varepsilon^{-2}$) as $\varepsilon \to 0$, is motion by mean curvature given by $V = \kappa$, where V is the velocity of the interface in normal direction, and κ the mean curvature of the interface. In [19], the authors introduced in the Allen-Cahn equation the Cahn-Hilliard operator with weight $\delta(\varepsilon) = \mathcal{O}(1)$, significantly smaller than ε^{-2} as $\varepsilon \to 0$, and proved that the mixed model exhibits on the limit a qualitatively analogous behavior with velocity proportional to κ. Due to lack of comparison principle for the CH/AC equation, convergence was shown until the first singularity of the limiting evolution occurs, by making, in the proof, a formal asymptotic expansion rigorous with the help of linear stability in the spirit of [1].

For the existence and regularity properties of (1.1), we refer to [20] where the Galerkin method was adapted, while the stochastic version thereof was investigated in [5]. In more detail, in [20], the initial and boundary value problem with Neumann b.c., constant coefficients, for $\varepsilon := 1$, posed in space in $\mathcal{D}$ in dimensions $d = 1, 2, 3$, was written as a system for u and the chemical potential $v := \Delta u - f(u)$. There, it has been proven that if the initial condition u_0 is sufficiently smooth (in $H^1(\mathcal{D})$), then for any $T > 0$ there exists a unique regular solution $(u, v) \in C([0,T]; H^1(\mathcal{D})) \times$

$L^2([0,T];H^1(\mathcal{D}))$. Higher regularity for u in $C([0,T];H^2(\mathcal{D}))$ was derived when $u_0 \in H^2(\mathcal{D})$. The authors in [5] proved local existence and uniqueness for the stochastic problem with non-smooth multiplicative space-time noise with standard Neumann boundary conditions when posed on rectangles in dimensions $d = 1, 2, 3$, by employing the Green's function estimates of the linear part of the operator. Moreover, when the noise diffusion coefficient satisfies a sub-linear growth condition, they proved for $d = 1$ global existence of solution, for $d = 2, 3$ existence of maximal solutions, and also derived space-time path regularity.

1.2. **Physical motivation of the CH/AC equation.** Let us describe first the main lines of the physical motivation of the Cahn-Hilliard / Allen-Cahn model, from [21, 19, 5] and the references therein.

An equation of the form (1.1) has been first analyzed in [19] as a mean field model of the microscopic dynamics associated with adsorption and desorption mechanisms in the context of surface processes; we also refer to [21, 5] for the detailed physical problem presentation. The combined model describes surface diffusion including particle/particle interactions and adsorption and desorption from the surface. It is noticeable that the mobility is completely different from this of Allen-Cahn equation, which implies that the diffusion speeds up the mean curvature flow, ([18, 23]).

More analytically, the CH/AC mixed equation models two surface processes that take place simultaneously. The Cahn-Hilliard operator represents the mass conservative phase separation and surface diffusion in the presence of interacting particles, while the Allen-Cahn operator is related to phase transition and serves as a diffuse interface model for the antiphase boundary coarsening. Surface processes, such as catalysis, chemical vapor deposition and epitaxial growth, typically involve transport and chemistry of precursors in a gas phase; unconsumed reactants and radicals adsorb onto the surface of a substrate where numerous processes may occur concurrently, for instance surface diffusion, reactions and desorption back to the gas phase.

The mathematical tools employed in the statistical mechanics models of surface processes are Interacting Particle Systems (IPS), which are Markov processes set on a lattice corresponding to a solid surface. Typical examples are the Ising-type systems, describing in the microscopic level the evolution of an order parameter at each lattice site. The mesoscopic model in study is derived in [21] from microscopic lattice models when the local mean field limit is considered. The energy of the system is given by a Hamiltonian involving the interparticle potential which is assumed even, rapidly decaying at infinity, and non-negative i.e. the interactions of the particles are attractive. The assumption that the potential is non-negative is an important one from a physical point of view, since it implies that clusters of particles are energetically preferred to totally disordered structures. This is translated, in the mathematical statement of equation (1.1), to the condition $\delta(\varepsilon) > 0$ posed on the Cahn-Hilliard operator coefficient.

At large space/time scales and for long range potentials, it turns out that the small scale fluctuations of the Ising systems are suppressed and an almost deterministic pattern emerges. In [19], the macroscopic cluster evolution laws and transport structure have been rigorously derived. Therein, a space-time diffusive scaling in ε was applied that described the long-time behavior of large clusters. Random fluctuations, when included in the mesoscopic model, appear as stochastic higher order corrections. In [5], the authors derived the stochastic non-linear equation version of this model by inserting in the CH/AC equation a multiplicative space-time white noise with a diffusion coefficient of linear growth stemming from the free energy and thermal fluctuations.

The deterministic Allen-Cahn equation was proposed in [3] as a model for the dynamics of interfaces of crystal structures in alloys. As far as the one-dimensional case is concerned, the limiting behaviour was analyzed in [11, 14]. After a very short time, generation of many very steep transition layers is observed. These well developed transition layers then start to move very slowly, and each time a pair of transition layers meet, the two layers annihilate each other, and thus the number of layers decreases gradually. Although those collision-annihilation process takes place rather quickly, the motion of layers between the collisions is extremely slow and the solution exhibits a metastable pattern. The situation is quite different in the multi-dimensional case, where such metastable patterns hardly appear because of the curvature effect on the motion of the interface as $\varepsilon \to 0$; for rigorous justification of singular limits, see for example in [9], [12, 13], [22]. The deterministic Cahn-Hilliard equation, proposed by [10], describes the evolution of transitions (mass transfer) during the phase separation of alloys. In the case of only two layers, the exponentially slow dynamics have been studied in [2], where a one-dimensional invariant manifold of slowly moving states was constructed. More details of the phenomenon and the motion towards the boundary can be found in [17], [16].

1.3. **Main results.** As it has been observed in [5], the operator at the right-hand side of the equation (1.1) is strongly parabolic in the sense of Petrovskiĭ and the bi-Laplacian since existing ($\delta(\varepsilon) > 0$) dominates, resulting to regularity properties identical to the Cahn-Hilliard equation (at least in the stochastic setting). However, the sharp interface limit of the deterministic equation may exhibit a different profile closer to this of the Allen-Cahn, [19]. The above, ignites a special interest on the scaling of the chosen parameters $\delta(\varepsilon)$, $\mu(\varepsilon)$, and the motivation of a further investigation of their influence on the dynamics of the layers.

It is well known that the Cahn-Hilliard equation with the standard Neumann boundary conditions for u and its Laplacian is mass conservative in the sense that the integral of the solution in space is time independent. In contrast, the Allen-Cahn equation with Neumann or Dirichlet boundary is not satisfying such a property unless a non-local integral term is added, which is the case for the mixed equation as well; this is not considered in this work, however it consists a future plan in progress the detailed investigation of the dynamics for such a version, i.e., (1.1) with the extra integral term.

Our main aim is to obtain the equations of motion and estimate the dynamics of a fixed number of layers, when ε is sufficiently small, for the combined model (1.1), and in dimension one. For this, when the initial and boundary value problem involves the Neumann conditions, and so mass conservation is not holding true, the solution will be approximated into the manifold constructed and effectively used for the Allen-Cahn equation in the classical result of Carr and Pego, [11]. Then, by imposing mass conservation, not through the pde but replacing one only of the b.c. with an integral one, we will apply the mass conserving manifold of Bates and Xun, [6, 7], which has been proposed for the integrated Cahn-Hillliard equation. There, the derived initial and boundary value problem for the integrated equation is identical to this of [6, 7], when $\mu(\varepsilon) := 0$. We note that the problem is of fourth order, since $\delta(\varepsilon)$ is not vanishing, while in dimension one the boundary consists of only two well separated points where four boundary conditions are applied on, and could be therefore of different type. In both cases we will determine the ode systems of the dynamics, and investigate the main order terms with respect to the order in ε of $\delta(\varepsilon)$ and $\mu(\varepsilon)$, and stability.

The general approach for deriving the equations of motion, as a system of odes, consists of specifying the approximate solution into a proper approximate manifold with a residual orthogonal to a set of approximate tangent vectors to the manifold. Differentiating in time the orthogonality condition then yields the system describing the dynamics of the layers.

Let u be the solution of (1.1), with the standard Neumann conditions, $u_x = u_{xxx} = 0$ at $x = 0, 1$, (non mass conserving case). Given a configuration

$$h = (h_1, \ldots, h_N),$$

of exactly N layer positions, we will construct a function $u^h = u^h(x)$ approximating a metastable state of u with N transition layers. Here, we will use the parameterization of Carr and Pego, [11], for the approximate manifold.

More precisely, the function u^h will almost satisfy the steady-state problem of (1.1), that is $A_\varepsilon(u^h)$ is very small, where A_ε is the operator given by the right-hand side of (1.1)

$$A_\varepsilon(u) := -\delta(\varepsilon)\big(\varepsilon^2 u_{xx} - f(u)\big)_{xx} + \mu(\varepsilon)\big(\varepsilon^2 u_{xx} - f(u)\big) =: -\delta(\varepsilon)A_{1,\varepsilon}(u) + \mu(\varepsilon)A_{2,\varepsilon}(u), \tag{1.5}$$

for

$$A_{1,\varepsilon}(u) := \Big(\varepsilon^2 u_{xx} - f(u)\Big)_{xx},$$

the negative of the Cahn-Hilliard operator, and

$$A_{2,\varepsilon}(u) := \varepsilon^2 u_{xx} - f(u),$$

the Allen-Cahn operator.

We shall then define the set of admissible layer positions by

$$\Omega_\rho = \Big\{ h = (h_1, \ldots, h_N) : \ \frac{\varepsilon}{2\rho} < h_1 < \cdots < h_N < 1 - \frac{\varepsilon}{2\rho}, \quad \text{and} \quad \min_{2 \le j \le N} (h_j - h_{j-1}) > \frac{\varepsilon}{\rho} \Big\},$$

for some ρ small and independent of ε, which will be described in detail in the next section. Moreover, we shall specify the N-dimensional manifold of approximate steady states

$$\mathcal{M} := \{u^h : h \in \Omega_\rho\}.$$

The residual v of the approximation is defined as orthogonal to N approximate tangent vectors to $\mathcal{M}$ at u^h.

Section 2.5 presents the equations of motion through the ode system (2.30) for the positions h_i, $i = 1, \cdots, N$. The spectrum of the linearized CH/AC operator is investigated at Section 2.6, and as well the positive definition of the induced bilinear form, when applied on the residual v if $\varepsilon^2\mu(\varepsilon) \ge C_0\delta(\varepsilon)$ for some $C_0 \ge C_{\min} > 0$ sufficiently large and specified through the supremum in $(0, 1)$ of $|\varepsilon^2(f'(u^h))_{xx}| = \mathcal{O}(1)$, see Main Theorem 2.4. Then, Main Theorem 2.8 at Section 2.7 estimates the velocities $\dot{h}_i$ of the layers; at this technical part, we followed the approach of Bates and Xun, [6].

Finally at Section 2.8, after a rather extensive calculus and using the spectral condition of the linearized operator, we specify a wide class of $\mu(\varepsilon)$, $\delta(\varepsilon)$, for which the dynamics are stable, and exponentially small in ε, cf. (2.46), (2.119), and Main Theorem 2.11 for the attractiveness and the slow evolution of states within the slow channel defined by (2.113).

The second part of this manuscript is devoted, at Section 3, to the mass conserving layer dynamics, and the strategy applied is analogous to this in [6].

Let M be a fixed mass in $(-1,1)$. Restricting one degree of freedom we impose a mass conservation property, and define the second approximate $(N-1)$-dimensional manifold, which is a submanifold of $\mathcal{M}$, by

$$\mathcal{M}_1 := \Big\{u^h \in \mathcal{M} : \int_0^1 u^h(x)\,dx = M\Big\},$$

and further define the manifold

$$\tilde{\mathcal{M}} := \Big\{\tilde{u}^h :\ u^h \in \mathcal{M}_1,\ \ \tilde{u}^h(x) = \int_0^x u^h(y)\,dy\Big\}.$$

We impose the mass conservation condition $\int_0^1 u(x,t)dx = \int_0^1 u_0(x)dx =$ fixed, in place of the b.c. $u_{xxx}(1,t)=0$. Then, for the integrated equation, we derive the CH/AC initial and boundary value problem given by (IACH)-(IBC1). In the Appendix we discuss the well posedness of the mass-conserving problem and derive a priori estimates by using the corresponding energy functional.

In Section 3.2, we specify the ode system for the equations of motion of the $N-1$ layers and estimate their dynamics, in the mass conservative case, see in Theorem 3.1.

Finally, at Section 3.4, we prove the Main Theorem 3.2, establishing attractiveness of the manifold and stability of the dynamics, again for a wide class of $\mu(\varepsilon)$, $\delta(\varepsilon)$ for which the dynamics are stable, and exponentially small in ε.

We have also included, at the end, an Appendix where we collected and proved various estimates used throughout the text.

2. Non mass conserving layer dynamics

We supplement (1.1) with the standard Neumann b.c. on u and u_{xx}, and consider the following initial and boundary value problem

$$\begin{aligned} &u_t = -\delta(\varepsilon)\left(\varepsilon^2 u_{xx} - f(u)\right)_{xx} + \mu(\varepsilon)(\varepsilon^2 u_{xx} - f(u)), \quad x\in(0,1), \quad t>0,\\ &u_x = u_{xxx} = 0 \quad \text{at} \quad x=0,1, \quad t>0,\\ &u(x,0) = u_0(x,\varepsilon), \quad x\in(0,1). \end{aligned} \tag{2.1}$$

Let us point out that (2.1) does not conserve mass for any $\mu(\varepsilon)\neq 0$ since in general

$$\partial_t\int_0^1 u(x,t)dx = \int_0^1 u_t(x,t)dx = \int_0^1 \mu(\varepsilon)(\varepsilon^2 u_{xx}(x,t) - f(u(x,t)))dx = -\mu(\varepsilon)\int_0^1 f(u(x,t))dx,$$

is not the zero function, while the solutions of the boundary problem

$$\begin{aligned} &\varepsilon^2 u_{xx} - f(u) = 0, \quad 0<x<1,\\ &u_x = 0 \quad \text{at} \quad x=0,1, \end{aligned} \tag{2.2}$$

are obviously steady states of (2.1); (2.2) follows by setting $u_t = 0$ at the equation of (2.1) and integrating in space twice using the boundary conditions.

We assume, therefore, in the context of Section 2, that $\mu(\varepsilon)>0$ for all $\varepsilon>0$.

Remark 2.1. *We observe that the free energy of the problem analyzed in this section is decreasing in time.*

Indeed, the relevant to the scaling of the standard Allen-Cahn operator

$$\varepsilon^2\Delta u - f(u),$$

free energy functional is defined as follows

$$E(u) := \int_0^1 \Big(\frac{\varepsilon^2|\nabla u|^2}{2} + F(u)\Big)dx. \tag{2.3}$$

Multiplying both sides of the equation of the i.b.v.p (2.1) *with* $\varepsilon^2\Delta u - f(u)$*, integrating in space, and using the boundary conditions, we derive*

$$\begin{aligned}(u_t, \varepsilon^2\Delta u - f(u)) =& -\delta(\varepsilon)(\Delta\big(\varepsilon^2\Delta u - f(u)\big), \varepsilon^2\Delta u - f(u)) + \mu(\varepsilon)\|\varepsilon^2\Delta u - f(u)\|^2 \\ =&\delta(\varepsilon)\|\nabla\big(\varepsilon^2\Delta u - f(u)\big)\|^2 + \mu(\varepsilon)\|\varepsilon^2\Delta u - f(u)\|^2,\end{aligned} \tag{2.4}$$

where here and for the rest of the manuscript, $(\cdot,\cdot)$ *denotes the* $L^2((0,1))$ *inner product, and* $\|\cdot\|$ *the induced* $L^2((0,1))$ *norm.*

Differentiating in time, (2.3) *yields*

$$\begin{aligned}\frac{\partial E(u)}{\partial t} =& \int_0^1 \Big(\varepsilon^2\nabla u\nabla u_t + F'(u)u_t\Big)dx \\ =& \int_0^1 \Big(\varepsilon^2\nabla u\nabla u_t + f(u)u_t\Big)dx \\ =& \int_0^1 \Big(-\varepsilon^2\Delta u u_t + f(u)u_t\Big)dx \\ =& -(u_t, \varepsilon^2\Delta u - f(u)).\end{aligned} \tag{2.5}$$

So, by (2.4) *and* (2.5)*, we obtain the free energy decreasing property for the combined model*

$$\frac{\partial E(u)}{\partial t} = -\delta(\varepsilon)\|\nabla\big(\varepsilon^2\Delta u - f(u)\big)\|^2 - \mu(\varepsilon)\|\varepsilon^2\Delta u - f(u)\|^2 \leq 0, \tag{2.6}$$

since $\delta(\varepsilon) > 0$ *and* $\mu(\varepsilon) \geq 0$*.*

Besides the three homogeneous equilibria $u = \pm 1$ and $u = u_0$ for the zero u_0 of f in $(-1,1)$ ($u = 0$ for odd f as in our special case), problem (2.2) has non-constant solutions for all sufficiently small ε; see e.g. [17]. More precisely, if $\varepsilon_{n+1} \leq \varepsilon < \varepsilon_n$ with $\varepsilon_i := (-f'(0))^{1/2}/2\pi i$, $i = 1, 2, \ldots,$ then problem (2.2) has exactly n pairs of non-constant solutions $u^{\pm}_{\varepsilon_i}$, $1 \leq i \leq n$ ($u^-_{\varepsilon_i} = -u^+_{\varepsilon_i}$ if f is odd). For each i, the equilibria $u^-_{\varepsilon_i}, u^+_{\varepsilon_i}$ have exactly i zeros at $x = 1/2i, 3/2i, \ldots, 1 - 1/2i$. The two solutions for $i = 1$ are monotone, and the other solutions for $i \geq 2$ are oscillating taken as rescaled reflections and periodic extensions of monotone solutions, they correspond to periodic orbits around the origin on the phase plane and we speak of solutions with i internal *transition* layers.

Since the internal transition layers of stationary solutions must have periodic spacing, the solutions of (1.1) which reach patterns that are nearly piecewise constant, say with N transition layers, but not periodic, they are close to a stationary state but they are not solutions of the steady state problem and we do not expect them to remain at these patterns. We will concern with these solutions yet not with their end-state but rather with their dynamics as long as they remain at these "metastable" N-layered patterns.

2.1. **The approximate manifold.** First we note that assumption (1.3) ensures the existence of an $a > 0$ such that $f'(u) > 0$ for $|u \pm 1| < a$, so let us fix such an $a > 0$.

We will follow the strategy of the pioneering works of Carr and Pego [11], and Bates and Xun [7] for the construction of the approximate manifold solutions; we refer also to the work of

Antonopoulou, Blömker, Karali [4] for the Cahn-Hilliard equation with noise where this approach was applied effectively in the stochastic setting as well.

For initial condition $u_0(x;\varepsilon) = u(x,0)$ close to the manifold $\mathcal{M}$, we will approximate the profile of a metastable state of the solution u of (2.1) with N transition layers by piecing together the stationary solutions of (1.1) satisfying the following boundary value Dirichlet problem for the bistable equation,

$$\begin{cases} \varepsilon^2 \phi_{xx} - f(\phi) = 0, & |x| < \dfrac{\ell}{2} + \varepsilon, \\ \phi\Big(\mp \dfrac{\ell}{2}\Big) = 0, \end{cases} \tag{2.7}$$

with $\ell > 0$ denoting the distance between two successive layer positions.

Remark 2.2. *We summarize briefly the properties of the solutions of* (2.7)*, as established in* [11]*, cf. Prop. 2.1. therein:*

There exists $\rho_0 > 0$ such that if $\varepsilon/\ell < \rho_0$, then

(i) *a unique solution $\phi_\varepsilon(x,\ell,+1)$ of* (2.7) *exists, with*

$$\phi_\varepsilon(x,\ell,+1) > 0 \quad for \quad |x| < \ell/2, \qquad and \qquad |\phi_\varepsilon(0,\ell,+1) - 1| < a,$$

(ii) *a unique solution $\phi_\varepsilon(x,\ell,-1)$ of* (2.7) *exists, with*

$$\phi_\varepsilon(x,\ell,-1) < 0 \quad for \quad |x| < \ell/2, \qquad and \qquad |\phi_\varepsilon(0,\ell,-1) + 1| < a.$$

Moreover, the functions ϕ_ε are smooth and depend on ε and ℓ only through the ratio ε/ℓ.

2.2. **The approximate solution.** Let us consider a smooth cut-off function satisfying

$$\chi : \mathbb{R} \to [0,1] \qquad \text{with} \quad \chi(x) = 0 \quad \text{for} \quad x \le -1, \qquad \text{and} \quad \chi(x) = 1 \quad \text{for} \quad x \ge 1. \tag{2.8}$$

Given a choice of admissible layer positions $h = (h_1, \dots, h_N) \in \Omega_\rho$, let

$$\begin{aligned} &\ell_j = h_j - h_{j-1} && \text{for} \quad j = 2,3,\dots,N, && \text{and} \qquad \ell_1 = 2h_1, \;\; \ell_{N+1} = 2(1-h_N), \\ &m_j = \frac{h_{j-1}+h_j}{2} && \text{for} \quad j = 2,3,\dots,N, && \text{and} \qquad m_1 = 0, \;\; m_{N+1} = 1, \\ &I_j = [m_j,\, m_{j+1}] && \text{for} \quad j = 1,2,\dots,N. \end{aligned} \tag{2.9}$$

We define the approximate solution u^h for any $x \in I_j$, by

$$\begin{aligned} u^h(x) = &\Big[1 - \chi\Big(\frac{x-h_j}{\varepsilon}\Big)\Big]\, \phi_\varepsilon(x - m_j,\, \ell_j,\, (-1)^j) \\ &+ \chi\Big(\frac{x-h_j}{\varepsilon}\Big)\, \phi_\varepsilon(x - m_{j+1},\, \ell_{j+1},\, (-1)^{j+1}). \end{aligned} \tag{2.10}$$

In order to ease notation, we suppress the dependence of u^h on ε and omit hereafter the subscript in ϕ_ε by simply writing ϕ, and define

$$\phi^j(x) := \phi_\varepsilon(x - m_j,\, h_j - h_{j-1},\, (-1)^j), \tag{2.11}$$

for ϕ_ε the steady states presented analytically in Remark 2.2.

Moreover, we define

$$\chi^j(x) := \chi\Big(\frac{x-h_j}{\varepsilon}\Big). \tag{2.12}$$

The profile of u^h is presented at Figure 2.1.

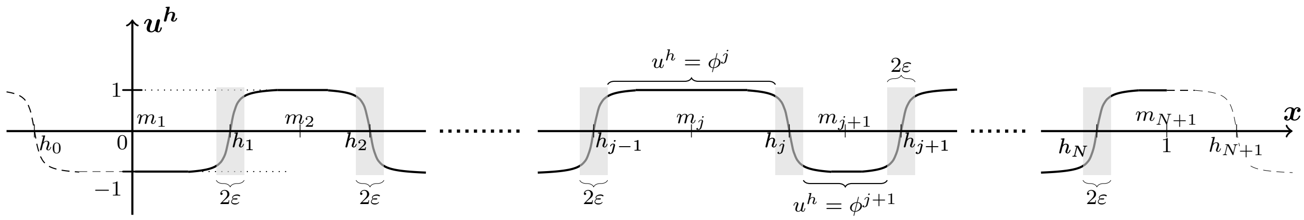

FIGURE 2.1. Given a configuration $h = (h_1, \cdots, h_N)$ of N layer positions, we construct u^h by piecing together steady state solutions of (2.7). We set $u^h = (1-\chi^j)\phi^j + \chi^j\phi^{j+1}$ on $[h_j - \varepsilon, h_j + \varepsilon]$ (shaded areas). Note that $h_0 = -h_1$, $h_{N+1} = 2 - h_N$, and $m_1 = 0, m_{N+1} = 1$.

2.3. **Properties of u^h.** Note that u^h is a smooth function of x and h. In particular, for $x \in [m_j,\, m_{j+1}]$, we have

$$u^h_x = \begin{cases} \phi^j_x, & m_j \le x \le h_j - \varepsilon, \\ \chi^j_x \left(\phi^{j+1} - \phi^j\right) + \left(1 - \chi^j\right)\phi^j_x + \chi^j\,\phi^{j+1}_x, & |x - h_j| < \varepsilon, \\ \phi^{j+1}_x, & h_j + \varepsilon \le x \le m_{j+1}, \end{cases} \tag{2.13}$$

and so

$$u^h_{xx} = \begin{cases} \phi^j_{xx}, & m_j \le x \le h_j - \varepsilon, \\ \chi^j_{xx}\left(\phi^{j+1} - \phi^j\right) + 2\chi^j_x\left(\phi^{j+1}_x - \phi^j_x\right) + (1-\chi^j)\phi^j_{xx} + \chi^j\phi^{j+1}_{xx}, & |x - h_j| < \varepsilon, \\ \phi^{j+1}_{xx}, & h_j + \varepsilon \le x \le m_{j+1}, \end{cases} \tag{2.14}$$

where, from the definition of ϕ^j (see (2.7)),

$$\varepsilon^2\,\phi^j_{xx} = f(\phi^j), \qquad \text{on} \quad [h_{j-1} - \varepsilon,\, h_j + \varepsilon]. \tag{2.15}$$

It is then straightforward to see that u^h satisfy the bistable equation

$$\mathscr{L}^b(u^h) = 0, \qquad \text{for} \quad |x - h_j| \ge \varepsilon, \;\; j = 1, 2, \ldots, N. \tag{2.16}$$

where $\mathscr{L}^b$ is the bistable operator

$$\mathscr{L}^b(u) := A_{2,\varepsilon}(u) = \varepsilon^2 u_{xx} - f(u). \tag{2.17}$$

Notice also that by reflecting the solutions $\phi(\cdot, \ell, \pm 1)$ of (2.7) about the origin we can show that they are even in x, and thus

$$\phi_x(0, \ell, \pm 1) = 0, \tag{2.18}$$

which together with (2.10) and (2.8) yields

$$u^h_x(m_j) = 0,\; j = 1, \ldots, N+1.$$

So, we obtain

$$u^h_x(0) = u^h_x(1) = 0, \tag{2.19}$$

and, therefore,

$$\frac{\partial}{\partial x} f(u^h) = 0 \qquad \text{at} \quad x = 0, 1. \tag{2.20}$$

Moreover, we note that

$$u^h_j \sim -u^h_x, \qquad \text{as} \quad r \to 0, \qquad \text{uniformly on} \quad I_j := [m_j,\, m_{j+1}],$$

see Remark 4.9 for more details thereof.

2.4. **The coordinate system.** Following [11], we introduce a local coordinate system

$$u \mapsto (h, \upsilon),$$

in a tubular neighborhood of the approximate manifold $\mathcal{M}$, defined by the decomposition

$$u(x,t) = u^{h(t)}(x) + \upsilon(x,t), \tag{2.21}$$

with $u^h \in \mathcal{M}$, and υ satisfying the orthogonality condition

$$\langle \upsilon, \tau^h_j \rangle := \int_0^1 \upsilon\, \tau^h_j \, dx = 0, \qquad j = 1, \ldots, N, \tag{2.22}$$

where τ^h_j are approximate tangent vectors to $\mathcal{M}$ at u^h.

More precisely, let

$$\gamma^j(x) := \chi\big(\tfrac{x - m_j - \varepsilon}{\varepsilon}\big) \Big[1 - \chi\big(\tfrac{x - m_{j+1} + \varepsilon}{\varepsilon}\big)\Big], \tag{2.23}$$

which yields that

$$\gamma^j(x) = \begin{cases} 0, & x \notin (m_j,\, m_{j+1}), \\ 1, & x \in [m_j + 2\varepsilon,\, m_{j+1} - 2\varepsilon]. \end{cases}$$

The approximate tangent vectors are then defined through $\gamma^j(x)$ by

$$\tau^h_j(x) \;=\; \gamma^j(x)\, u^h_x(x) \tag{2.24}$$

which are smooth functions of x and h.

Considering differentiation, we introduce the notation

$$\tau^h_{j,k} := \frac{\partial \tau^h_j}{\partial h_k} \qquad \text{and} \qquad \tau^h_{j,x} := \frac{\partial \tau^h_j}{\partial x},$$

and observe that $\tau^h_{j,x} = 0$ at $x = 0, 1$, for $j = 1, \ldots, N$.

2.5. **Equations of motion.** For a classical solution $u = u(x,t)$ of (1.1), we will establish the ODEs system for the dynamics of (h, υ), which is defined by (2.21)-(2.22).

This first order ODE system with unknowns the positions coordinates $h_k(t)$, $k = 1, \cdots, N$ for each one of the N layers (fronts) will be derived by differentiating in time the orthogonality condition (2.22). We will insert the Cahn-Hilliard / Allen-Cahn equation into the differentiated condition and then we shall use linearization which will be given by the linear combination through the weights $\delta(\varepsilon)$, $\mu(\varepsilon)$ of the C-H and A-C linearized operators respectively.

We differentiate (2.22), with respect to t, to get

$$\big\langle \partial_t \upsilon \,,\, \tau^h_j \big\rangle + \big\langle \upsilon \,,\, \partial_t \tau^h_j \big\rangle = 0, \qquad j = 1, \ldots, N.$$

Using (2.21), and the substituting u_t by the equation (1.1), together with the definition (1.5) of the operator A_ε, we obtain

$$\begin{aligned}\partial_t v =&\partial_t(u-u^h)\\ =&-\delta(\varepsilon)\Big(\varepsilon^2 u_{xx}-f(u)\Big)_{xx}+\mu(\varepsilon)(\varepsilon^2 u_{xx}-f(u))-\partial_t u^h)-\partial_t u^h\\ =&A_\varepsilon u-\partial_t u^h = A_\varepsilon u-\sum_{k=1}^N(\partial_{h_k} u^h)\dot{h}_k,\end{aligned}$$

to arrive at the system

$$\sum_{k=1}^N a_{jk}\,\dot{h}_k \;=\; \big\langle A_\varepsilon(u^h+v)\;,\;\tau_j^h\,\big\rangle, \qquad j=1,2,\ldots,N, \tag{2.25}$$

where

$$a_{jk} := \big\langle u_k^h\;,\;\tau_j^h\big\rangle \;-\; \big\langle v\;,\;\tau_{j,k}^h\,\big\rangle, \qquad j,k=1,2,\ldots,N. \tag{2.26}$$

In the above, the subscripts k indicate differentiation w.r.t. h_k, i.e.,

$$u_k^h := \frac{\partial u^h}{\partial h_k} \qquad \text{and} \qquad \tau_{j,k}^h := \frac{\partial \tau_j^h}{\partial h_k}.$$

We expand

$$A_\varepsilon(u^h+v) \;=\; A_\varepsilon(u^h)\;+\;L_\varepsilon^h(v)\;-\;\delta(\varepsilon)\left(f^h\,v^2\right)_{xx}\;+\;\mu(\varepsilon)f^h\,v^2, \tag{2.27}$$

where $L_\varepsilon^h(v)$ is the linearization of A_ε at u^h, i.e.,

$$\begin{aligned}L_\varepsilon^h(v) :=& -\delta(\varepsilon)\Big(\varepsilon^2 v_{xx}-f'(u^h)v\Big)_{xx}+\mu(\varepsilon)(\varepsilon^2 v_{xx}-f'(u^h)v)\\ =:& -\delta(\varepsilon)L_{1,\varepsilon}^h(v)+\mu(\varepsilon)L_{2,\varepsilon}^h(v),\end{aligned} \tag{2.28}$$

and

$$f^h(x) := \int_0^1 (\tau-1)\; f''(u^h+\tau v)\;d\tau. \tag{2.29}$$

Using (2.27), then the system (2.25) is written as

$$\begin{aligned}\sum_{k=1}^N a_{jk}\,\dot{h}_k \;=&\; \big\langle A_\varepsilon(u^h)\;,\;\tau_j^h\,\big\rangle\;+\;\big\langle L_\varepsilon^h(v)\;,\;\tau_j^h\,\big\rangle\\ &-\;\delta(\varepsilon)\,\big\langle\big(f^h\,v^2\big)_{xx}\;,\;\tau_j^h\,\big\rangle\;+\;\mu(\varepsilon)\big\langle f^h\,v^2\;,\;\tau_j^h\,\big\rangle\\ =&\;-\delta(\varepsilon)\,\big\langle A_{1,\varepsilon}(u^h)+L_{1,\varepsilon}^h(v)+\big(f^h\,v^2\big)_{xx}\;,\;\tau_j^h\big\rangle\\ &+\mu(\varepsilon)\big\langle A_{2,\varepsilon}(u^h)+L_{2,\varepsilon}^h(v)+f^h\,v^2\;,\;\tau_j^h\big\rangle\\ =&\;-\delta(\varepsilon)\,\big\langle A_{1,\varepsilon}(u^h),\tau_j^h\big\rangle+\mu(\varepsilon)\big\langle A_{2,\varepsilon}(u^h),\tau_j^h\big\rangle\\ &-\delta(\varepsilon)\,\big\langle L_{1,\varepsilon}^h(v)+\big(f^h\,v^2\big)_{xx}\;,\;\tau_j^h\big\rangle\\ &+\mu(\varepsilon)\big\langle L_{2,\varepsilon}^h(v)+f^h\,v^2\;,\;\tau_j^h\big\rangle,\end{aligned} \tag{2.30}$$

for $j=1,2,\ldots,N$.

Remark 2.3. *We note that as in the Allen-Cahn case, υ-independent exponentially small terms in the dynamics* (2.30) *will be derived by the term*

$$\mu(\varepsilon)\big\langle A_{2,\varepsilon}(u^h),\tau_j^h\big\rangle,$$

due to the second order operator there. Of course $\mu(\varepsilon)$ will influence the result.

More specifically, as in [11] *Lemma 3.3., when ρ in the definition of Ω_ρ is sufficiently small, we observe that*

$$\begin{aligned}\mu(\varepsilon)\big\langle A_{2,\varepsilon}(u^h)\ ,\ \tau_j^h\big\rangle =& -\mu(\varepsilon)\int_{h_j-\varepsilon}^{h_j+\varepsilon}(\varepsilon^2u^h_{xx}-f(u^h))u^h_x dx\\ =&\mu(\varepsilon)[F(u^h)-\frac{1}{2}\varepsilon^2(u_x^h)^2]_{h_j-\varepsilon}^{h_j+\varepsilon}=:\mu(\varepsilon)(a^{j+1}-a^j),\end{aligned} \tag{2.31}$$

where the difference $a^{j+1}-a^j$ is exponentially small in ε.

In our case, we also have the υ-independent term

$$\big\langle A_{1,\varepsilon}(u^h),\tau_j^h\big\rangle,$$

stemming from the Cahn-Hilliard part, which will be shown exponentially small as well.

Moreover, we apply (2.21) to (1.1) to get

$$\upsilon_t = A_\varepsilon(u^h+\upsilon)-\sum_{j=1}^N u_j^h\,\dot h_j. \tag{2.32}$$

As above, we expand in (2.32) the term $A_\varepsilon(u^h+\upsilon)$, according to (2.27), to get

$$\upsilon_t\ =\ A_\varepsilon(u^h)\ +\ L^h_\varepsilon(\upsilon)\ -\ \delta(\varepsilon)\left(f^h\,\upsilon^2\right)_{xx}\ +\ \mu(\varepsilon)f^h\,\upsilon^2\ -\ \sum_{j=1}^N u_j^h\,\dot h_j \tag{2.33}$$

or discriminating between the CH and AC parts,

$$\begin{aligned}\upsilon_t\ =&\ -\delta(\varepsilon)\left[A_{1,\varepsilon}(u^h)\ +\ L^h_{1,\varepsilon}(\upsilon)\ +\ \left(f^h\,\upsilon^2\right)_{xx}\right]\\ &+\ \mu(\varepsilon)\left[A_{2,\varepsilon}(u^h)\ +\ L^h_{2,\varepsilon}(\upsilon)\ +\ f^h\,\upsilon^2\right]\ -\ \sum_{j=1}^N u_j^h\,\dot h_j.\end{aligned} \tag{2.34}$$

According to Proposition 2.3 in [11], there exist $\rho_2>0$, constants A_0, C and $b(\rho)=o(1)$ as $\rho\to 0^+$ such that if $h\in\Omega_\rho$ with $\rho<\rho_2$, then:

(i) for $j=1,2,\cdots,N$,

$$\big(A_0-b(\rho)\big)^2\varepsilon^{-1}\ \le\ \big\langle u_j^h\ ,\ \tau_j^h\big\rangle\ \le\ \big(A_0+b(\rho)\big)^2\varepsilon^{-1} \tag{2.35}$$

$$\|\tau^h_{jj}\|\le C\varepsilon^{-3/2}\qquad\text{and}\qquad \|\tau^h_{jj}\|_{L^1}\ \le\ C\varepsilon^{-1} \tag{2.36}$$

(ii) for $j\neq k$,

$$\big|\big\langle u_j^h\ ,\ \tau_k^h\big\rangle\big|\ \le\ b(\rho)\varepsilon^{-1}, \tag{2.37}$$

$$\|\tau^h_{kj}\|\le b(\rho)\varepsilon^{-3/2}\qquad\text{and}\qquad \|\tau^h_{kj}\|_{L^1}\ \le\ b(\rho)\varepsilon^{-1}. \tag{2.38}$$

2.6. **The spectrum of the linearized operator and transverse coercivity.** A key point in the analysis of the equations of motion (2.30),(2.33) is the spectral analysis of the linear operator L_ε^h on $v = u - u^h$, the part of the solution transverse to the manifold $\mathcal{M}$.

We consider the eigenvalue problem

$$\text{(EVP)}\qquad \begin{cases} L_\varepsilon^h(\phi) := -\delta(\varepsilon)\big(\varepsilon^2\phi'' - f'(u^h)\phi\big)'' + \mu(\varepsilon)\big(\varepsilon^2\phi'' - f'(u^h)\phi\big) = \lambda(\varepsilon)\phi, \quad 0 < x < 1,\\ \phi'(0) = \phi'(1) = 0,\\ \phi'''(0) = \phi'''(1) = 0, \end{cases}$$

and the associated quadratic form

$$\begin{aligned} \tilde{\mathcal{A}}_\varepsilon[\phi] \;&:=\; -\langle L_\varepsilon^h(\phi)\,,\,\phi\rangle \\ &= \int_0^1 \Big[\delta(\varepsilon)\Big(\varepsilon^2\phi_{xx}^2 + f'(u^h)\,\phi_x^2 - \tfrac{1}{2}\big(f'(u^h)\big)_{xx}\phi^2\Big) + \mu(\varepsilon)\Big(\varepsilon^2\phi_x^2 + f'(u^h)\phi^2\Big)\Big]\,dx, \end{aligned} \tag{2.39}$$

where we have applied integration by parts. We note that the operator $L_\varepsilon^h : \mathcal{D}(L_\varepsilon^h) \to L^2(0,1)$, with domain

$$\mathcal{D}(L_\varepsilon^h) := \Big\{\phi \in H^4(0,1) :\ \phi'(0) = \phi'(1) = 0 = \phi'''(0) = \phi'''(1)\Big\},$$

is not symmetric in $L^2(0,1)$. However (EVP) may be recast into a self-adjoint form, as the form $\tilde{\mathcal{A}}_\varepsilon$ it will be seen in the sequel to be lower semibounded and it is also closable since it is associated with the symmetric operator $S_\varepsilon^h : \mathcal{D}(S_\varepsilon^h) \to L^2(0,1)$, defined by

$$S_\varepsilon^h(\phi) := -\delta(\varepsilon)\Big(\varepsilon^2\phi'''' - \big(f'(u^h)\phi'\big)' - \frac{1}{2}\big(f'(u^h)\big)_{xx}\phi\Big) + \mu(\varepsilon)\Big(\varepsilon^2\phi'' - f'(u^h)\phi\Big), \tag{2.40}$$

with domain $\mathcal{D}(S_\varepsilon^h) \equiv \mathcal{D}(L_\varepsilon^h)$, in the sense that

$$\langle L_\varepsilon^h(\phi)\,,\,\phi\rangle = -\tilde{\mathcal{A}}_\varepsilon[\phi] = \langle S_\varepsilon^h(\phi)\,,\,\phi\rangle, \qquad \text{for any} \quad \phi \in \mathcal{D}(L_\varepsilon^h). \tag{2.41}$$

We can then consider the self-adjoint extension (Friedrichs extension) of S_ε^h associated with the closure of $\tilde{\mathcal{A}}_\varepsilon$ which we still denote by S_ε^h. The spectrum of S_ε^h turns out to consist of a sequence of real eigenvalues

$$\cdots \le \lambda_{N+1} \le \lambda_N \le \cdots \le \lambda_1,$$

satisfying, when $\varepsilon^2\mu(\varepsilon) \ge \mathcal{O}(\delta(\varepsilon))$,

$$\lambda_k \le C\mu(\varepsilon),$$

and in particular for $k = N+1, N+2, \cdots$

$$\lambda_k \le -C = -C(\varepsilon) < 0,$$

for some $C = C(\varepsilon)$ which will be specified. To this aim, we will use the variational characterization of the eigenvalues for S_ε^h,

$$-\lambda_{N+1} := \max_{\substack{V\\ \dim V = N}} \min_{\phi \in V^\perp} \frac{\tilde{\mathcal{A}}_\varepsilon[\phi]}{\|\phi\|^2}, \tag{2.42}$$

where the maximum is taken over the linear N-dimensional subspaces V of

$$\big\{\phi \in H^2(0,1) :\ \phi'(0) = \phi'(1) = 0\big\}. \tag{2.43}$$

Let us decompose the Rayleigh quotient in (2.42) into the Cahn-Hilliard and the Allen-Cahn part, i.e.

$$R(\phi) := \frac{\langle -S^h_\varepsilon(\phi), \phi\rangle}{\|\phi\|^2} = \delta(\varepsilon)R_1(\phi) + \mu(\varepsilon)R_2(\phi), \tag{2.44}$$

for

$$\begin{aligned} R_1(\phi) &:= \frac{\displaystyle\int_0^1 (\varepsilon^2\phi^2_{xx} + f'(u^h)\,\phi^2_x - \tfrac{1}{2}\big(f'(u^h)\big)_{xx}\phi^2)\,dx}{\|\phi\|^2},\\ R_2(\phi) &:= \frac{\overbrace{\displaystyle\int_0^1 (\varepsilon^2\phi^2_x + f'(u^h)\phi^2)\,dx}^{\langle L^h_{2,\varepsilon}\phi,\phi\rangle}}{\|\phi\|^2}. \end{aligned} \tag{2.45}$$

In the sequel we will assume that $\delta(\varepsilon) > 0$ and $\mu(\varepsilon) > 0$ satisfy the condition

$$\varepsilon^2\mu(\varepsilon) \geq C_0\delta(\varepsilon) \qquad \forall\varepsilon > 0, \tag{2.46}$$

for some $C_0 \geq C_{\min} > 0$, where $C_{\min}$ depends on the supremum of $|\varepsilon^2(f'(u^h))_{xx}| = \mathcal{O}(1)$ in $(0, 1)$ and can be determined implicitly in the context of the proof of following theorem.

We shall show that when $\delta(\varepsilon), \mu(\varepsilon)$ satisfy (2.46), then

$$-\lambda_{N+1} \geq C(\varepsilon) = \mathcal{O}(\mu(\varepsilon) - C_{\min}\delta(\varepsilon)\varepsilon^{-2}) > 0.$$

For our purpose, we state the result in the variational form. The main implication for us is the transverse coercivity (cf. Lemma 2.6), the coercivity of L^h_ε on $v = u - u^h$ which recall that is approximately orthogonal to the tangent space (see (2.22)).

Theorem 2.4. *Let $\delta(\varepsilon) > 0$ and $\mu(\varepsilon) > 0$ satisfying* (2.46)*. Then there exist $\Lambda, \rho_2 > 0$ such that for $h \in \Omega_\rho$ with $\rho \leq \rho_2$ it holds that*

$$\min_{\phi\in V^\perp} \frac{-\langle S^h_\varepsilon(\phi)\,,\,\phi\rangle}{\|\phi\|^2} \geq \eta(\varepsilon) > 0,$$

for $V = \operatorname{span}\{\tau^h_i,\ i = 1, 2, \ldots, N\}$, with τ^h_i are the approximate tangent vectors defined in (2.24)*, and*

$$\eta(\varepsilon) := \Lambda(\mu(\varepsilon) - C_{\min}\delta(\varepsilon)\varepsilon^{-2}). \tag{2.47}$$

Proof. Remind that $f(u) = u^3 - u$, and so $f'(u) = 3u^2 - 1 \geq -1$.

Moreover, it holds that

$$|(f'(u^h))_{xx}| \leq C\varepsilon^{-2} \qquad \text{in} \quad [0, 1],$$

since there

$$|u^h_x| \leq C\varepsilon^{-1}, \qquad |u^h_{xx}| \leq C\varepsilon^{-2},$$

see in Appendix relations (4.34), and (4.35), while u^h is bounded.

Using that $\delta(\varepsilon) > 0$, $\mu(\varepsilon) > 0$, and the above, we get for any $\phi \in V^{\perp}$

$$
\begin{aligned}
R(\phi) :=& \frac{\langle -S^h_\varepsilon(\phi), \phi\rangle}{\|\phi\|^2} \\
=& \delta(\varepsilon) \frac{\displaystyle\int_0^1 (\varepsilon^2 \phi_{xx}^2 + f'(u^h)\phi_x^2 - \tfrac{1}{2}\big(f^{'}(u^h)\big)_{xx}\phi^2) dx}{\|\phi\|^2} \\
&+ \mu(\varepsilon) \frac{\int_0^1 (\varepsilon^2 \phi_x^2 + f'(u^h)\phi^2) dx}{\|\phi\|^2} \\
\geq& - \delta(\varepsilon) \frac{\displaystyle\int_0^1 \phi_x^2 dx}{\|\phi\|^2} + \mu(\varepsilon) \frac{\int_0^1 (\varepsilon^2 \phi_x^2 + f'(u^h)\phi^2) dx}{\|\phi\|^2} \\
&- \delta(\varepsilon) \frac{\displaystyle\int_0^1 \tfrac{1}{2}\big(f^{'}(u^h)\big)_{xx}\phi^2 dx}{\|\phi\|^2} \\
\geq& - \delta(\varepsilon) \frac{\displaystyle\int_0^1 \phi_x^2 dx}{\|\phi\|^2} + \mu(\varepsilon)\Lambda \frac{\int_0^1 (\varepsilon^2 \phi_x^2 + \phi^2) dx}{\|\phi\|^2} \\
&- \frac{\delta(\varepsilon)}{2} \sup_{(0,1)} |(f^{'}(u^h))_{xx}| \\
\geq& \frac{\displaystyle\int_0^1 [-\delta(\varepsilon) + \mu(\varepsilon)\Lambda\varepsilon^2]\phi_x^2 dx}{\|\phi\|^2} + \mu(\varepsilon)\Lambda - \sup_{(0,1)} |\varepsilon^2 (f^{'}(u^h))_{xx}| \frac{\delta(\varepsilon)}{2}\varepsilon^{-2} \\
=& \Lambda\Big(\mu(\varepsilon)\varepsilon^2 - \frac{1}{\Lambda}\delta(\varepsilon)\Big)\frac{\|\phi_x\|^2}{\|\phi\|^2} + \Lambda\varepsilon^{-2}\Big(\mu(\varepsilon)\varepsilon^2 - \frac{C_1}{2\Lambda}\delta(\varepsilon)\Big)
\end{aligned}
\tag{2.48}
$$

where we applied the Main Theorem 4.2 (i) of [11], at pg. 538, and defined

$$C_1 := \sup_{(0,1)} |\varepsilon^2 (f^{'}(u^h))_{xx}| = \mathcal{O}(1).$$

More specifically, we used that for some $\Lambda > 0$ and $\rho_2 > 0$

$$\int_0^1 (\varepsilon^2 \phi_x^2 + f'(u^h)\phi^2) dx \geq \Lambda \int_0^1 (\varepsilon^2 \phi_x^2 + \phi^2) dx$$

for $h \in \Omega_\rho$ with $\rho \leq \rho_2$. Here, Λ satisfies

$$\Lambda \leq \frac{\Lambda_*}{\Lambda_* + 2},$$

(see [11], at pg. 541, and use the specific f), with Λ_* arbitrary in $(0, \lambda_*)$ for some $\lambda_* > 0$ such that

$$\lambda_* \leq \min f'(\pm 1) = 2,$$

see [11], at pg. 536, for the detailed definition.

Therefore, in (2.48), if we choose $\mu(\varepsilon), \delta(\varepsilon)$ such that

$$\varepsilon^2 \mu(\varepsilon) \geq C_0 \delta(\varepsilon),$$

for

$$C_0 \geq C_{\min} := \max\Big\{\frac{1}{\Lambda},\ \frac{C_1}{2\Lambda}\Big\} + \beta^2,$$

for $\beta = \mathcal{O}(1) \neq 0$ as small, we obtain

$$\frac{\langle -S_\varepsilon^h(\phi), \phi\rangle}{\|\phi\|^2} \geq C(\varepsilon) = \Lambda(\mu(\varepsilon) - C_{\min}\delta(\varepsilon)\varepsilon^{-2}) > 0,$$

for any $\phi \in V^\perp$, and thus the result. □

The next theorem, gives an upper bound for the spectrum.

Theorem 2.5. *Let $\delta(\varepsilon) > 0$ and $\mu(\varepsilon) > 0$ satisfying* (2.46),

$$\varepsilon^2\mu(\varepsilon) \geq C_0\delta(\varepsilon) \qquad \forall \varepsilon > 0,$$

for $C_0 \geq C_{\min} > 0$ as defined in Theorem 2.4. Then, for any k it holds that

$$\lambda_k \leq C\mu(\varepsilon),$$

for some $C > 0$.

Proof. Using that $\delta(\varepsilon) > 0$, $\mu(\varepsilon) > 0$, and $f(u) = u^3 - u$, $f'(u) = 3u^2 - 1 \geq -1$, we get for any ϕ, and since $C_0 \geq \frac{1}{\Lambda} > 1$

$$\begin{aligned}
R(\phi) :=& \frac{\langle -S_\varepsilon^h(\phi), \phi\rangle}{\|\phi\|^2} \\
=& \delta(\varepsilon)\frac{\displaystyle\int_0^1 (\varepsilon^2\phi_{xx}^2 + f'(u^h)\phi_x^2 - \tfrac{1}{2}\big(f'(u^h)\big)_{xx}\phi^2)dx}{\|\phi\|^2} \\
&+ \mu(\varepsilon)\frac{\int_0^1(\varepsilon^2\phi_x^2 + f'(u^h)\phi^2)dx}{\|\phi\|^2} \\
\geq& \delta(\varepsilon)\frac{\displaystyle\int_0^1 (-\phi_x^2 - \tfrac{1}{2}\big(f'(u^h)\big)_{xx}\phi^2)dx}{\|\phi\|^2} \\
&+ \mu(\varepsilon)\frac{\int_0^1(\varepsilon^2\phi_x^2 - \phi^2)dx}{\|\phi\|^2} \\
=& \frac{\displaystyle\int_0^1 (-\delta(\varepsilon) + \mu(\varepsilon)\varepsilon^2)\phi_x^2 dx}{\|\phi\|^2} - \frac{\displaystyle\int_0^1 [\mu(\varepsilon) + \delta(\varepsilon)\tfrac{1}{2}\big(f'(u^h)\big)_{xx}]\phi^2 dx}{\|\phi\|^2} \\
\geq& -\frac{\displaystyle\int_0^1 [\mu(\varepsilon) + \delta(\varepsilon)\tfrac{1}{2}\big(f'(u^h)\big)_{xx}]\phi^2 dx}{\|\phi\|^2} \\
\geq& -\mu(\varepsilon) - C_1\delta(\varepsilon)\tfrac{1}{2}\varepsilon^{-2}.
\end{aligned} \tag{2.49}$$

This yields that for all $k = 1, \cdots, N$

$$\lambda_k \leq \mu(\varepsilon) + C_1\delta(\varepsilon)\tfrac{1}{2}\varepsilon^{-2} \leq C\mu(\varepsilon).$$

□

For $\upsilon \in C^2([0,1])$ with $\upsilon_x = 0$ at $x = 0, 1$, we define the forms

$$\tilde{\mathcal{A}}_\varepsilon[\upsilon] := \int_0^1 \Big[\delta(\varepsilon)\Big(\varepsilon^2 \upsilon_{xx}^2 + f'(u^h)\,\upsilon_x^2 - \tfrac{1}{2}\left(f'(u^h)\right)_{xx}\upsilon^2\Big) + \mu(\varepsilon)\Big(\varepsilon^2\upsilon_x^2 + f'(u^h)\upsilon^2\Big)\Big]\,dx, \tag{2.50}$$

(see (2.40), (2.39)) and

$$\tilde{\mathcal{B}}_\varepsilon[\upsilon] := \int_0^1 \Big[\delta(\varepsilon)\varepsilon^2\upsilon_{xx}^2 + \big(\delta(\varepsilon) + \mu(\varepsilon)\varepsilon^2\big)\upsilon_x^2 + \big(\delta(\varepsilon) + \mu(\varepsilon)\big)\upsilon^2\Big]\,dx. \tag{2.51}$$

Lemma 2.6. *There is a $\rho_0 > 0$ such that if $0 < \rho < \rho_0$ and $h \in \Omega_\rho$, then for any $\upsilon \in C^2$ with $\upsilon_x = 0$ at $x = 0, 1$ and $\langle \upsilon, \tau_j^h\rangle = 0$, $j = 1, \cdots, N$,*

$$\tilde{\mathcal{B}}_\varepsilon[\upsilon] \le C\tilde{\mathcal{A}}_\varepsilon[\upsilon] \tag{2.52}$$

for some positive constant C, and

$$\eta(\varepsilon)\tilde{\mathcal{A}}_\varepsilon[\upsilon] \;\le\; \|S_\varepsilon^h(\upsilon)\|^2. \tag{2.53}$$

Proof. By Theorem 4.2 in [11] there exists $\Lambda > 0$ such that

$$\int_0^1 \varepsilon^2\upsilon_x^2 + f'(u^h)\upsilon^2\,dx \ge \Lambda\int_0^1 \varepsilon^2\upsilon_x^2 + \upsilon^2. \tag{2.54}$$

For such Λ, and $c := \max\{c_1, c_2\}$ for positive constants of the uniform bounds

$$|\big(f'(u^h)\big)_{xx}| \le c_1\varepsilon^{-2} \qquad \text{and} \qquad |f'(u^h)| \le c_2 \qquad \text{and} \qquad [0,1],$$

we have

$$\begin{aligned}
\tilde{\mathcal{A}}_\varepsilon[\upsilon] &:= \int_0^1 \delta(\varepsilon)\Big(\varepsilon^2\upsilon_{xx}^2 + f'(u^h)\,\upsilon_x^2 - \tfrac{1}{2}\left(f'(u^h)\right)_{xx}\upsilon^2\Big) + \mu(\varepsilon)\Big(\varepsilon^2\upsilon_x^2 + f'(u^h)\upsilon^2\Big)\,dx \\
&= \int_0^1 \delta(\varepsilon)\varepsilon^2\upsilon_{xx}^2\,dx + \delta(\varepsilon)\int_0^1 f'(u^h)\,\upsilon_x^2 - \tfrac{1}{2}\left(f'(u^h)\right)_{xx}\upsilon^2\,dx + \mu(\varepsilon)\int_0^1 \varepsilon^2\upsilon_x^2 + f'(u^h)\upsilon^2\,dx \\
&\ge \int_0^1 \delta(\varepsilon)\varepsilon^2\upsilon_{xx}^2\,dx - c\,\delta(\varepsilon)\int_0^1 \upsilon_x^2 + \varepsilon^{-2}\upsilon^2\,dx + \mu(\varepsilon)\,\Lambda\int_0^1 \varepsilon^2\upsilon_x^2 + \upsilon^2\,dx \\
&= \int_0^1 \delta(\varepsilon)\varepsilon^2\upsilon_{xx}^2 + \big(\mu(\varepsilon)\,\Lambda\,\varepsilon^2 - c\,\delta(\varepsilon)\big)\upsilon_x^2 + \big(\mu(\varepsilon)\,\Lambda - c\,\delta(\varepsilon)\varepsilon^{-2}\big)\upsilon^2\,dx \\
&\ge C\int_0^1 \delta(\varepsilon)\varepsilon^2\upsilon_{xx}^2 + \big(\delta(\varepsilon) + \mu(\varepsilon)\,\varepsilon^2\big)\upsilon_x^2 + \big(\delta(\varepsilon) + \mu(\varepsilon)\big)\upsilon^2\,dx \\
&=: C\,\tilde{\mathcal{B}}_\varepsilon[\upsilon]
\end{aligned}$$

for some positive constant C small enough; the last inequality follows from the assumption (2.46), since

$$\mu(\varepsilon)\,\Lambda\,\varepsilon^2 - c\,\delta(\varepsilon) \ge C\left(\delta(\varepsilon) + \mu(\varepsilon)\,\varepsilon^2\right) \iff \mu(\varepsilon)\,\varepsilon^2 \;\ge\; \frac{c + C}{\Lambda - C}\,\delta(\varepsilon)$$

and

$$\mu(\varepsilon)\,\Lambda - c\,\delta(\varepsilon)\varepsilon^{-2} \ge C\big(\delta(\varepsilon) + \mu(\varepsilon)\big) \iff \mu(\varepsilon)\,\varepsilon^2 \;\ge\; \frac{c + C\varepsilon^2}{\Lambda - C}\,\delta(\varepsilon).$$

Regarding inequality (2.53), we use the estimate

$$0 < C(\varepsilon) \leq \frac{\tilde{\mathcal{A}}_\varepsilon[v]}{\|v\|^2} \qquad \forall v \text{ with } v_x(0) = v_x(1) = 0 \text{ and } \langle v, \tau_j^h \rangle = 0, \quad j = 1, \cdots, N, \tag{2.55}$$

for $C(\varepsilon) = \eta(\varepsilon) := \Lambda(\mu(\varepsilon) - C_{\min}\delta(\varepsilon)\varepsilon^{-2})$, to get immediately

$$\big|\tilde{\mathcal{A}}_\varepsilon[v]\big| \ = \ |\langle S_\varepsilon^h(v)\,,\,v\rangle| \ \leq \ \|S_\varepsilon^h(v)\|\,\|v\| \overset{(2.55)}{\leq} \frac{1}{\sqrt{C}} \cdot \|S_\varepsilon^h(v)\| \cdot \big|\tilde{\mathcal{A}}_\varepsilon[v]\big|^{1/2} \tag{2.56}$$

and hence (2.53) follows. ☐

2.7. **Flow near layered equilibria.** The main result of this section is Theorem 2.8 regarding the motion of the layers. For $v \in C^1[0,1]$ with $v_x(0) = v_x(1) = 0$, we introduce the norm

$$\mathcal{B}_\varepsilon[v] := \int_0^1 \left[\varepsilon^2 v_x^2 + v^2\right] dx \tag{2.57}$$

and we will study the orbit $u(x,t) = u^{h(t)}(x) + v(x,t)$, of (1.1) as long as

$$\mathcal{B}_\varepsilon[v] \leq C\varepsilon = \mathcal{O}(\varepsilon). \tag{2.58}$$

Lemma 2.7. *For $v \in C^1[0,1]$ we have*

$$\|v\|_{L^\infty}^2 \leq \frac{1+\varepsilon}{\varepsilon}\mathcal{B}_\varepsilon[v]. \tag{2.59}$$

Proof. Let $x_1 \in [0,1]$ be such that

$$v^2(x_1) = \|v\|_{L^\infty}^2 \tag{2.60}$$

and let $x_2 \neq x_1$ be such that

$$v^2(x_2) \leq \mathcal{B}_\varepsilon[v]. \tag{2.61}$$

We can assume that $x_2 \leq x_1$ without loss of generality, for otherwise we would consider the reflection of v about $\frac{1}{2}$ which would then satisfy this assumption with $\|\cdot\|_{L^\infty}$, $\mathcal{B}_\varepsilon[\cdot]$ remaining invariant.

Integrating the inequality

$$\varepsilon\frac{d}{dx}v^2 = 2\varepsilon v v_x \leq \varepsilon^2 v_x^2 + v^2$$

on $[x_2, x_1]$, we obtain

$$\varepsilon v^2(x_1) - \varepsilon v^2(x_2) \leq \mathcal{B}_\varepsilon[v] \tag{2.62}$$

hence (2.59) results upon the substitution of (2.60)-(2.61) into (2.62). ☐

For the coefficients a_{jk} of $\dot{h}_k$ in the LHS of (2.25), defined in (2.26), we introduce the matrices $S(h), \hat{S}(h,v)$,

$$S(h) = \big(S(h)\big)_{jk} := \big\langle u_k^h\,,\,\tau_j^h\big\rangle \qquad \text{and} \qquad \hat{S}(h,v) = \big(\hat{S}(h,v)\big)_{jk} := \big\langle v\,,\,\tau_{j,k}^h\,\big\rangle. \tag{2.63}$$

According to Lemma 3.1 of [11], there exist $\sigma_1, \rho_1 > 0$ such that if $\sigma < \sigma_1$ and $\rho < \rho_1$ and $(h,v) \in \mathscr{S}_{\rho,\sigma}$, then the matrices $S(h)$ and $S(h) - \hat{S}(h,v)$ are invertible with

$$\|S^{-1}\| \leq 4A_0^{-2}\varepsilon \qquad \text{and} \qquad \|(S-\hat{S})^{-1}\| \leq 8A_0^{-2}\varepsilon. \tag{2.64}$$

Here, $\|\cdot\|$ denotes the matrix norm induced by the vector norm $\|h\| = \max_j |h_j|$ on $\mathbb{R}^N$, and A_0 is the constant appearing in (2.35).

Theorem 2.8. *There exists $\rho_2 > 0$ and a constant $C > 0$, such that*

$$
\begin{aligned}
|\dot{h}_i| \leq & C\delta(\varepsilon)\Big(\varepsilon^{-1}\alpha(r) \,+\, \varepsilon^{-5/2}\mathcal{B}_\varepsilon^{1/2}[\upsilon] \,+\, \varepsilon^{-3}\mathcal{B}_\varepsilon[\upsilon]\Big) \\
& + C\mu(\varepsilon)\Big(\varepsilon\alpha(r) \,+\, \varepsilon^{1/2}\mathcal{B}_\varepsilon^{1/2}[\upsilon]\big[\alpha(r)+\beta(r)\big] \,+\, \varepsilon^{-1}\mathcal{B}_\varepsilon[\upsilon]\Big),
\end{aligned}
\tag{2.65}
$$

as long as $h \in \Omega_\rho$, with $\rho < \rho_2$, and the orbit $u(x,t) = u^{h(t)}(x) + \upsilon(x,t)$ of (1.1) remains sufficiently close to $\mathcal{M}$ so that $\mathcal{B}_\varepsilon[\upsilon] = \mathcal{O}(\varepsilon)$.

Proof. The RHS of (2.30) with

$$\tau_j^h(x) = -\gamma^j(x)u_x^h(x),\; j = 1,2,\dots,N,$$

is written as

$$
\begin{aligned}
& -\delta(\varepsilon)\left\langle A_{1,\varepsilon}(u^h) + L_{1,\varepsilon}^h(\upsilon) + \big(f^h\,\upsilon^2\big)_{xx} \,,\, \gamma^j u_x^h\right\rangle \\
& + \mu(\varepsilon)\big\langle A_{2,\varepsilon}(u^h) + L_{2,\varepsilon}^h(\upsilon) + f^h\,\upsilon^2 \,,\, \gamma^j u_x^h\big\rangle.
\end{aligned}
\tag{2.66}
$$

We will consider the first summand in (2.66) that comes from the CH part, and shall estimate the terms

$$
\underbrace{\big\langle A_{1,\varepsilon}(u^h) \,,\, \gamma^j u_x^h\big\rangle}_{T_1} \quad + \quad \underbrace{\big\langle L_{1,\varepsilon}^h(\upsilon) \,,\, \gamma^j u_x^h\big\rangle}_{T_2} \quad + \quad \underbrace{\big\langle \big(f^h\,\upsilon^2\big)_{xx} \,,\, \gamma^j u_x^h\big\rangle}_{T_3},
\tag{2.67}
$$

for $j = 1,2,\dots,N,$ where f^h is given in (2.29); here, recall the notation

$$A_{1,\varepsilon}(u) = \big(\varepsilon^2 u_{xx} - f(u)\big)_{xx}, \qquad \text{and} \qquad L_{1,\varepsilon}^h(\upsilon) = \big(\varepsilon^2 \upsilon_{xx} - f'(u^h)\upsilon\big)_{xx}.$$

Estimate of T_1:
For all j, we have

$$\mathscr{L}^b(\phi^j) = 0,$$

which in view of the definition (2.10), implies that

$$A_{1,\varepsilon}(u^h) = 0 \quad \text{for} \quad |x - h_j| \geq \varepsilon.$$

Using that

$$\gamma^j(x) = 1 \quad \text{for} \quad |x - h_j| \leq \varepsilon,$$

we get

$$T_1 := \big\langle A_{1,\varepsilon}(u^h) \,,\, \gamma^j u_x^h\big\rangle = \int_{h_j-\varepsilon}^{h_j+\varepsilon} \Big(\underbrace{\varepsilon^2 u_{xx}^h - f(u^h)}_{\mathscr{L}^b(u^h)}\Big)_{xx} u_x^h\, dx. \tag{2.68}$$

In the above, we integrate by parts twice, and obtain

$$T_1 = \int_{h_j-\varepsilon}^{h_j+\varepsilon} \mathscr{L}^b(u^h)\, u_{xxx}^h\, dx. \tag{2.69}$$

We apply (4.37) together with (4.43) in (2.69), and derive

$$|T_1| \,\leq\, C\,\varepsilon^{-2}\,\alpha(r). \tag{2.70}$$

Estimate of T_2:
Using $\tau_j^h := \gamma^j u_x^h$ we integrate by parts four times the first term and twice the second one, in the definition of T_2, to get

$$\begin{aligned} T_2 &:= \big\langle L^h_{1,\varepsilon}(\upsilon)\ ,\ \tau_j^h \big\rangle \\ &= -\varepsilon^2 \big\langle \upsilon_{xxxx}\ ,\ \tau_j^h\big\rangle\ +\ \big\langle (f'(u^h)\upsilon)_{xx}\ ,\ \tau_j^h\big\rangle \\ &= -\varepsilon^2 \big\langle \upsilon\ ,\ (\tau_j^h)_{xxxx}\big\rangle\ +\ \big\langle \upsilon\ ,\ f'(u^h)(\tau_j^h)_{xx}\big\rangle \\ &= -\varepsilon^2 \Big[\big\langle \upsilon\ ,\ \gamma^j (u_x^h)_{xxxx}\big\rangle\ +\ 4\big\langle \upsilon\ ,\ \gamma^j_x (u_x^h)_{xxx}\big\rangle\ +\ 6\big\langle \upsilon\ ,\ \gamma^j_{xx} (u_x^h)_{xx}\big\rangle \\ &\qquad +\ 4\big\langle \upsilon\ ,\ \gamma^j_{xxx} (u_x^h)_{x}\big\rangle\ +\ \big\langle \upsilon\ ,\ \gamma^j_{xxxx} u_x^h\big\rangle\Big] \qquad (2.71a) \\ &\quad +\ \big\langle \upsilon\ ,\ \gamma^j f'(u^h) u^h_{xxx}\big\rangle\ +\ 2\big\langle \upsilon\ ,\ \gamma^j_x f'(u^h) u^h_{xx}\big\rangle\ +\ \big\langle \upsilon\ ,\ \gamma^j_{xx} f'(u^h) u^h_{x}\big\rangle. \qquad (2.71b) \end{aligned}$$

Here, we have used that $\tau_j^h, (\tau_j^h)_x, (\tau_j^h)_{xx}$ vanish at $x = 0, 1$.

We now proceed to pointwise estimates for the terms involving u^h in (2.71), within the interval $[m_j, m_{j+1}]$ for a fixed yet arbitrary $j = 1, \ldots, N$.

First notice that

$$\varepsilon^2 (u_x^h)_{xx}\ -\ f'(u^h) u_x^h\ =\ \frac{d}{dx}\mathscr{L}^b(u^h), \tag{2.72}$$

and by (2.16),

$$\frac{d}{dx}\mathscr{L}^b(u^h) = 0 \qquad \text{except in} \quad [h_j - \varepsilon,\, h_j + \varepsilon], \tag{2.73}$$

while, in view of (2.4),

$$\gamma^j_x = \gamma^j_{xx} = \gamma^j_{xxx} = 0 \qquad \text{in} \quad [m_j + 2\varepsilon,\, m_{j+1} - 2\varepsilon] \supset [h_j - \varepsilon,\, h_j + \varepsilon]. \tag{2.74}$$

Therefore the last term in (2.71b) is canceled out by the last one in (2.71a), and (2.71) becomes

$$\begin{aligned} T_2 = -\varepsilon^2 \Big[&\big\langle \upsilon\ ,\ \gamma^j (u_x^h)_{xxxx}\big\rangle\ +\ 4\big\langle \upsilon\ ,\ \gamma^j_x (u_x^h)_{xxx}\big\rangle\ +\ 5\big\langle \upsilon\ ,\ \gamma^j_{xx} (u_x^h)_{xx}\big\rangle \\ &+\ 4\big\langle \upsilon\ ,\ \gamma^j_{xxx} (u_x^h)_{x}\big\rangle\ +\ \big\langle \upsilon\ ,\ \gamma^j_{xxxx} u_x^h\big\rangle\Big] \qquad (2.71\text{a}') \\ &+\ \big\langle \upsilon\ ,\ \gamma^j f'(u^h) u^h_{xxx}\big\rangle\ +\ 2\big\langle \upsilon\ ,\ \gamma^j_x f'(u^h) u^h_{xx}\big\rangle. \qquad (2.71\text{b}') \end{aligned}$$

We have

$$\Big|\frac{d^n \gamma^j}{dx^n}\Big| \le C\varepsilon^{-n}. \tag{2.75}$$

Moreover, differentiating (2.15) twice, and then using (2.75), (2.74), (4.29), we get for $|x - m_j| < 2\varepsilon$,

$$\varepsilon^2 |\gamma^j_x (u^h_x)_{xxx}|\ =\ \varepsilon^2\, |\gamma^j_x|\, |f''(u^h)(u^h_x)^2\ +\ f'(u^h) u^h_{xx}| \le C\varepsilon^{-1}\beta(r). \tag{2.76}$$

By (2.75), (2.10), (2.15) and for $x \in [m_j, m_j + 2\varepsilon] \cup [m_{j+1} - 2\varepsilon, m_{j+1}]$, we derive

$$\varepsilon^2 |\gamma^j_{xx} (u^h_x)_{xx}|\ =\ \varepsilon^2\, |\gamma^j_{xx}|\, |f'(u^h) u^h_x| \le C\varepsilon^{-1}\beta(r). \tag{2.77}$$

Using (2.75), (2.74) and the estimate (4.29) presented in the Appendix, we get

$$\varepsilon^2 |\gamma^j_{xxx} (u^h_x)_{x}|\ =\ |\gamma^j_{xxx}|\, |f(u^h)| \le C\varepsilon^{-3}\beta(r), \tag{2.78}$$

and

$$|\gamma_x^j\, f'(u^h)\, u_{xx}^h| \le C\varepsilon^{-3}\beta^2(r). \tag{2.79}$$

By (2.75), (2.13), (2.74) and the estimate (4.27) (see Appendix), we obtain

$$\varepsilon^2|\gamma_{xxxx}^j u_x^h| \le C\varepsilon^{-3}\beta(r). \tag{2.80}$$

We will see that the dominant asymptotics of T_2 comes from the term

$$\big\langle \upsilon\ ,\ \gamma^j\big(-\varepsilon^2(u_x^h)_{xxxx} + f'(u^h)u_{xxx}^h\big)\big\rangle, \tag{2.81}$$

which is the leading term of T_2. This term as we will see has order $\mathcal{O}(\varepsilon^{-5/2}\|\upsilon\|)$.

We may proceed as previously, to get pointwise estimates on $[m_j, m_j+2\varepsilon] \cup [m_{j+1}-2\varepsilon, m_{j+1}]$ wherein the contribution is in minor order in the asymptotics of T_2, while, as we shall see, the main order comes from the term

$$T_{2,1} \ := \ -\varepsilon^2(u_x^h)_{xxxx} + f'(u^h)u_{xxx}^h \qquad \text{in} \quad [m_j+2\varepsilon,\, m_{j+1}-2\varepsilon]. \tag{2.82}$$

Differentiating the third derivative of u^h, given by (4.36), twice, we obtain

$$\frac{\partial^5 u^h}{\partial x^5} = \begin{cases} \phi_{xxxxx}^j, & \text{for} \quad m_j \le x \le h_j - \varepsilon, \\[2ex] \chi_{xxxxx}^j\left(\phi^{j+1}-\phi^j\right) + 5\chi_{xxxx}^j\left(\phi_x^{j+1}-\phi_x^j\right) + 10\chi_{xxx}^j\left(\phi_{xx}^{j+1}-\phi_{xx}^j\right) \\ +10\chi_{xx}^j\left(\phi_{xxx}^{j+1}-\phi_{xxx}^j\right) + 10\chi_x^j\left(\phi_{xxxx}^{j+1}-\phi_{xxxx}^j\right) \\ +(1-\chi^j)\phi_{xxxxx}^j + \chi^j\phi_{xxxxx}^{j+1}, & \text{for} \quad |x-h_j|<\varepsilon, \\[2ex] \phi_{xxxxx}^{j+1}, & \text{for} \quad h_j+\varepsilon \le x \le m_{j+1}. \end{cases} \tag{2.83}$$

We use first (2.83), (4.30), (4.31), (4.32), (4.33), (4.36), and get

$$T_{2,1} \ = \ \mathcal{O}\big(\varepsilon^{-3}\alpha(r)\big) \ - \ \varepsilon^2\left[(1-\chi^j)\phi_{xxxxx}^j + \chi^j\phi_{xxxxx}^{j+1}\right] \ + \ f'(u^h)\Big[(1-\chi^j)\phi_{xxx}^j + \chi^j\phi_{xxx}^{j+1}\Big], \tag{2.84}$$

and then, after differentiating (2.15) three times, we obtain

$$\begin{aligned} T_{2,1} \quad = \quad & \mathcal{O}\big(\varepsilon^{-3}\alpha(r)\big) \\ & -(1-\chi^j)\, f'(\phi^j)\phi_{xxx}^j \ - \ \chi^j\, f'(\phi^{j+1})\phi_{xxx}^{j+1} \ + \ f'(u^h)\Big[(1-\chi^j)\phi_{xxx}^j + \chi^j\phi_{xxx}^{j+1}\Big] \qquad \text{(2.85a)} \\ & -(1-\chi^j)\left(f'(\phi^j)\right)_{xx}\phi_x^j \ - \ \chi^j\left(f'(\phi^{j+1})\right)_{xx}\phi_x^{j+1} \\ & -2(1-\chi^j)\big(f'(\phi^j)\big)_x\phi_{xx}^j \ - \ 2\chi^j\big(f'(\phi^{j+1})\big)_x\phi_{xx}^{j+1} \\ \overset{(2.15)}{=\!=} \quad & \mathcal{O}\big(\varepsilon^{-3}\alpha(r)\big) \ - \ (2.85\text{a}) \ - \ (1-\chi^j)\Big[f^{(3)}(\phi^j)(\phi_x^j)^3 \ + \ \varepsilon^{-2}f''(\phi^j)f(\phi^j)\phi_x^j\Big] \\ & -\chi^j\Big[f^{(3)}(\phi^{j+1})(\phi_x^{j+1})^3 \ + \ \varepsilon^{-2}f''(\phi^{j+1})f(\phi^{j+1})\phi_x^{j+1}\Big] \\ & -2\varepsilon^{-2}\Big[(1-\chi^j)f''(\phi^j)f(\phi^j)\phi_x^j \ + \ \chi^j f''(\phi^{j+1})f(\phi^{j+1})\phi_x^{j+1}\Big]. \end{aligned}$$

So, this yields

$$\begin{aligned} T_{2,1} &= \mathcal{O}\big(\varepsilon^{-3}\alpha(r)\big) - (2.85\text{a}) - \Big[(1-\chi^j)f^{(3)}(\phi^j)(\phi^j_x)^3 + \chi^j f^{(3)}(\phi^{j+1})(\phi^{j+1}_x)^3\Big] \\ &\qquad - 3\varepsilon^{-2}\Big[(1-\chi^j)f''(\phi^j)f(\phi^j)\phi^j_x + \chi^j f''(\phi^{j+1})f(\phi^{j+1})\phi^{j+1}_x\Big] \\ &= \mathcal{O}\big(\varepsilon^{-3}\alpha(r)\big) - (2.85\text{a}) - f^{(3)}(\phi^j)(\phi^j_x)^3 - 3\varepsilon^{-2}f''(\phi^j)f(\phi^j)\phi^j_x \\ &\quad + \chi^j\Big[f^{(3)}(\phi^j)(\phi^j_x)^3 - f^{(3)}(\phi^{j+1})(\phi^{j+1}_x)^3\Big] \qquad (2.86\text{a}) \\ &\quad + 3\,\varepsilon^{-2}\,\chi^j\Big[f''(\phi^j)f(\phi^j)\phi^j_x - f''(\phi^{j+1})f(\phi^{j+1})\phi^{j+1}_x\Big]. \qquad (2.86\text{b}) \end{aligned}$$

To estimate the term (2.85a), we use (4.38)-(4.39) with $\chi := \chi^j$ and

$$F(s) := f'\big(\theta_1(s)\big)\,\theta_2(s), \qquad s\in[0,1], \tag{2.87}$$

where

$$\theta_1(s) := (1-s)\phi^j + s\phi^{j+1}, \qquad \theta_2(s) := (1-s)\phi^j_{xxx} + s\phi^{j+1}_{xxx}, \qquad s\in[0,1], \tag{2.88}$$

so (2.85a) equals to $R(\chi^j)$, (cf. (4.38)-(4.39) for the detailed definition of the remainder R), and hence, it is given by

$$\begin{aligned} (2.85\text{a}) = (1-\chi^j)\Big[&(\phi^{j+1}-\phi^j)^2\int_0^{\chi^j} s\,\theta_2(s)f^{(3)}(\theta_1(s))\,ds \\ &+ 2\,(\phi^{j+1}_{xxx}-\phi^j_{xxx})\,(\phi^{j+1}-\phi^j)\int_0^{\chi^j} s\,f''(\theta_1(s))\,ds\Big] \\ + \chi^j\Big[&(\phi^{j+1}-\phi^j)^2\int_{\chi^j}^1 (1-s)\,\theta_2(s)f^{(3)}(\theta_1(s))\,ds \\ &+ 2\,(\phi^{j+1}_{xxx}-\phi^j_{xxx})\,(\phi^{j+1}-\phi^j)\int_{\chi^j}^1 (1-s)\,f''(\theta_1(s))\,ds.\Big] \end{aligned} \tag{2.89}$$

The above, in view of (4.30a), (4.32) is bounded as follows

$$\big|(2.85\text{a})\big| \le C\varepsilon^{-3}\alpha(r), \qquad \text{for} \quad |x-h_j|<\varepsilon. \tag{2.90}$$

We also exploit (4.23), (4.30), to obtain

$$\begin{aligned} (2.86\text{a}) &= \big|f^{(3)}(\phi^j)(\phi^j_x)^3 \pm f^{(3)}(\phi^j)(\phi^{j+1}_x)^3 - f^{(3)}(\phi^{j+1})(\phi^{j+1}_x)^3\big| \\ &\le \big|f^{(3)}(\phi^j)\big|\big|(\phi^j_x)^3 - (\phi^{j+1}_x)^3\big| + \big|\phi^{j+1}_x\big|^3\big|f^{(3)}(\phi^{j+1}) - f^{(3)}(\phi^j)\big| \\ &\le C\varepsilon^{-2}\big|\phi^j_x - \phi^{j+1}_x\big| + C\varepsilon^{-3}\big|f^{(3)}(\phi^{j+1}) - f^{(3)}(\phi^j)\big| \\ &\le C\varepsilon^{-2}\big|\phi^j_x - \phi^{j+1}_x\big| + C\varepsilon^{-3}\big|\phi^{j+1} - \phi^j\big| \\ &\le C\varepsilon^{-3}\alpha(r), \qquad \text{for} \quad |x-h_j|<\varepsilon. \end{aligned} \tag{2.91}$$

A similar argument yields

$$(2.86\text{b}) \le C\varepsilon^{-3}\alpha(r), \qquad \text{for} \quad |x-h_j|<\varepsilon. \tag{2.92}$$

Combining (4.26), (2.59), (2.71a′), (2.76)-(2.80), (2.86), (2.90), (2.91), and (2.92), we get the final estimate for T_2, given by

$$|T_2| \le C\,\varepsilon^{-5/2}\,\mathcal{B}^{1/2}_\varepsilon[v]. \tag{2.93}$$

Estimate of T_3:
Integrating by parts twice, we obtain

$$T_3 = \langle f^h \upsilon^2 \, , \, \big(\gamma^j u^h_x\big)_{xx} \rangle, \tag{2.94}$$

where f^h is defined in (2.29).

By (2.58), (2.59) and the fact that

$$u^h = \mathcal{O}(1),$$

we get

$$u^h + \tau\upsilon = \mathcal{O}(1), \tag{2.95}$$

and thus, the integrand $|f''(u^h + \tau\upsilon)|$ in the definition of f^h is uniformly bounded; so, using again (2.59), we derive

$$|f^h|\upsilon^2 \leq C\varepsilon^{-1}\mathcal{B}_\varepsilon[\upsilon]. \tag{2.96}$$

Then (2.94), by employing (2.13), (2.14), (2.15), (4.36), (4.26), (2.74), (4.28), (4.29), yields

$$|T_3| \, \leq \, C\,\varepsilon^{-4}\,\mathcal{B}_\varepsilon[\upsilon]. \tag{2.97}$$

Gathering (2.70), (2.93), and (2.97), we arrive at the final estimate

$$|T_1 + T_2 + T_3| \leq C\Big(\varepsilon^{-2}\alpha(r) \, + \, \varepsilon^{-5/2}\mathcal{B}_\varepsilon^{1/2}[\upsilon] \, + \, \varepsilon^{-4}\mathcal{B}_\varepsilon[\upsilon]\Big). \tag{2.98}$$

Let us proceed now with the second summand in (2.66) that comes from the AC part.

In particular, we have to estimate the terms

$$\underbrace{\langle A_{2,\varepsilon}(u^h) \, , \, \gamma^j u^h_x \rangle}_{I_1} \quad + \quad \underbrace{\langle L^h_{2,\varepsilon}(\upsilon) \, , \, \gamma^j u^h_x \rangle}_{I_2} \quad + \quad \underbrace{\langle f^h \, \upsilon^2 \, , \, \gamma^j u^h_x \rangle}_{I_3} \tag{2.99}$$

for $j = 1, 2, \ldots, N,$ where f^h is given in (2.29); here, recall the notation

$$A_{2,\varepsilon}(u) = \varepsilon^2 u_{xx} - f(u) = \mathscr{L}^b(u) \qquad \text{and} \qquad L^h_{2,\varepsilon}(\upsilon) = \varepsilon^2\upsilon_{xx} - f'(u^h)\upsilon.$$

Estimate of I_1 :

For all j, we have

$$\mathscr{L}^b(\phi^j) = 0,$$

which by the definition (2.10), implies that

$$A_{2,\varepsilon}(u^h) = 0 \qquad \text{for} \qquad |x - h_j| \geq \varepsilon.$$

Using the fact that

$$\gamma^j(x) = 1 \qquad \text{for} \qquad |x - h_j| \leq \varepsilon,$$

we get

$$I_1 := \big\langle A_{2,\varepsilon}(u^h)\ ,\ \gamma^j u^h_x \big\rangle = \int_{h_j-\varepsilon}^{h_j+\varepsilon} \underbrace{\big[\varepsilon^2 u^h_{xx} - f(u^h)\big]}_{\mathscr{L}^b(u^h)} u^h_x \, dx. \tag{2.100}$$

We apply (4.34), (4.43) into (2.100), and obtain

$$|I_1| \le C\,\alpha(r). \tag{2.101}$$

See also the analytical calculation at Remark 2.3.

Estimate of I_2 :
With $\tau^h_j := \gamma^j u^h_x$ we integrate by parts twice the first term, to get

$$\begin{aligned} I_2 &:= \big\langle L^h_{2,\varepsilon}(\upsilon)\ ,\ \tau^h_j \big\rangle \\ &= \varepsilon^2 \big\langle \upsilon_{xx}\ ,\ \tau^h_j \big\rangle \ -\ \big\langle f'(u^h)\upsilon\ ,\ \tau^h_j \big\rangle \\ &= \varepsilon^2 \big\langle \upsilon\ ,\ (\tau^h_j)_{xx} \big\rangle \ -\ \big\langle f'(u^h)\upsilon\ ,\ \tau^h_j \big\rangle \\ &= \varepsilon^2 \big\langle \upsilon\ ,\ \gamma^j_{xx} u^h_x \big\rangle \ +\ 2\varepsilon^2 \big\langle \upsilon\ ,\ \gamma^j_x (u^h_x)_x \big\rangle \ +\ \big\langle \upsilon\ ,\ \gamma^j \big[\varepsilon^2 \big(u^h_x\big)_{xx} \ -\ f'(u^h) u^h_x\big] \big\rangle. \end{aligned} \tag{2.102}$$

In the above, we used that $\tau^h_j, (\tau^h_j)_x$ vanish at $x = 0, 1$.

By (2.13), (2.14), (4.27b), (4.29), (2.74), (2.75) we have

$$\big|\varepsilon^2 \gamma^j_{xx} u^h_x\big| \le C\varepsilon^{-1}\beta(r) \qquad \text{and} \qquad \big|\varepsilon^2 \gamma^j_x (u^h_x)_x\big| \le C\varepsilon^{-1}\beta(r). \tag{2.103}$$

Considering third term in (2.102), by (2.72) and (2.73) we have

$$\varepsilon^2 (u^h_x)_{xx} \ -\ f'(u^h) u^h_x \ =\ 0 \qquad \text{except in} \quad [h_j - \varepsilon,\, h_j + \varepsilon]. \tag{2.104}$$

By (2.13), (4.36) we have, for $|x - h_j| < \varepsilon$

$$\begin{aligned} I_{2,1} &:= \gamma^j \big[\varepsilon^2 \big(u^h_x\big)_{xx} \ -\ f'(u^h) u^h_x\big] \\ &= \varepsilon^2 \Big[\chi^j_{xxx}\big(\phi^{j+1} - \phi^j\big) + 3\chi^j_{xx}\big(\phi^{j+1}_x - \phi^j_x\big) + 3\chi^j_x\big(\phi^{j+1}_{xx} - \phi^j_{xx}\big) + (1-\chi^j)\phi^j_{xxx} + \chi^j \phi^{j+1}_{xxx}\Big] \\ &\quad - f'\big(\big(1-\chi^j\big)\,\phi^j + \chi^j\,\phi^{j+1}\big)\big[\chi^j_x\big(\phi^{j+1} - \phi^j\big) + \big(1-\chi^j\big)\,\phi^j_x + \chi^j\,\phi^{j+1}_x\big]. \end{aligned}$$

Using (2.15), it follows that

$$\begin{aligned} I_{2,1} = \varepsilon^2 \Big[\chi^j_{xxx}\big(\phi^{j+1} - \phi^j\big) + 3\chi^j_{xx}\big(\phi^{j+1}_x - \phi^j_x\big) + 3\chi^j_x\big(\phi^{j+1}_{xx} - \phi^j_{xx}\big)\Big] \\ + f'\big(u^h\big)\chi^j_x\big(\phi^{j+1} - \phi^j\big) + I_{2,2}, \end{aligned} \tag{2.105}$$

where

$$I_{2,2} = (1-\chi^j) f'(\phi^j)\phi^j_x + \chi^j f'(\phi^{j+1})\phi^{j+1}_x - f'\big((1-\chi^j)\,\phi^j + \chi^j\,\phi^{j+1}\big)\big[\big(1-\chi^j\big)\,\phi^j_x + \chi^j\,\phi^{j+1}_x\big]. \tag{2.106}$$

To estimate the term $I_{2,2}$, we employ (4.38)-(4.39), for $\chi := \chi^j$, and

$$F(s) \ :=\ f'\big(\theta_1(s)\big)\,\theta_3(s), \qquad s \in [0,1], \tag{2.107}$$

where

$$\theta_1(s) \ :=\ (1-s)\phi^j + s\phi^{j+1}, \qquad \theta_3(s) \ :=\ (1-s)\phi^j_x + s\phi^{j+1}_x, \qquad s \in [0,1], \tag{2.108}$$

to obtain

$$
\begin{aligned}
I_{2,2} =&(1-\chi^j)\Big[(\phi^{j+1}-\phi^j)^2\int_0^{\chi^j} s\,\theta_2(s)f^{(3)}(\theta_1(s))\;ds \\
&+\;2\,(\phi_x^{j+1}-\phi_x^j)\,(\phi^{j+1}-\phi^j)\int_0^{\chi^j} s\,f''(\theta_1(s))\,ds\Big] \\
&+\chi^j\Big[(\phi^{j+1}-\phi^j)^2\int_{\chi^j}^1 (1-s)\,\theta_2(s)f^{(3)}(\theta_1(s))\;ds \\
&+\;2\,(\phi_x^{j+1}-\phi_x^j)\,(\phi^{j+1}-\phi^j)\int_{\chi^j}^1 (1-s)\,f''(\theta_1(s))\,ds\Big].
\end{aligned}
\tag{2.109}
$$

Combining (4.30), (4.31), (4.33), (2.59), (2.102)-(2.105), (2.109), and taking into account that each of the integrals in (2.102) is taken over an interval of length $\mathcal{O}(\varepsilon)$, we conclude that

$$
|I_2| \;\le\; C\,\varepsilon^{-1/2}\,\mathcal{B}_\varepsilon^{1/2}[\upsilon]\,\big[\alpha(r)\;+\;\beta(r)\big]. \tag{2.110}
$$

Estimate of I_3 :

In view of (2.13), (4.26), (2.96), we obtain

$$
|I_3| \;\le\; C\,\varepsilon^{-2}\,\mathcal{B}_\varepsilon[\upsilon]. \tag{2.111}
$$

Gathering together (2.101), (2.110), (2.111), we get

$$
|I_1+I_2+I_3| \le C\Big(\alpha(r)\;+\;\varepsilon^{-1/2}\mathcal{B}_\varepsilon^{1/2}[\upsilon]\,\big[\alpha(r)+\beta(r)\big]\;+\;\varepsilon^{-2}\mathcal{B}_\varepsilon[\upsilon]\Big). \tag{2.112}
$$

Combining (2.30), (2.64), (2.98), and (2.112), the final estimate (2.65) follows. □

Remark 2.9. *In view of the estimate of Main Theorem 2.8 for the dynamics, considering the main order of the Allen-Cahn part (which follows from* (2.112)*) in comparison with this of Carr-Pego* [11]*, let us emphasize the difference in our approach. We estimated separately the contributions of* $(S-\hat{S})^{-1}$ *and the right side of the equations of motion, while the analysis in* [11] *is carried out only after having applied the inverse* $(S-\hat{S})^{-1}$ *of the coefficient matrix on the right side of the system; see in particular the first equations of (3.1) and (3.4) therein. Nevertheless, our result is analogous even if only the Allen-Cahn part is considered.*

The estimate (2.65) *shows that the main order in the dynamics will be given by the contribution of* $B_\varepsilon[\upsilon]$ *at the terms where the exponentially small quantities* $\alpha(r)$, $\beta(r)$ *do not act.*

Using the spectrum of the linearized Cahn-Hilliard / Allen-Cahn operator, and a quite wide class of weights $\delta(\varepsilon)>0$, $\mu(\varepsilon)\ge 0$, *we shall show that for initial data close enough to the manifold* $\mathcal{M}$ *(through a form* $\tilde{\mathcal{A}}_\varepsilon[\upsilon]$*) the layer dynamics will be stable, and will remain exponentially small in* ε *if the initial data are exponentially small. (The form* $\tilde{\mathcal{A}}_\varepsilon[\upsilon]$ *will involve* u^h *and up to second derivatives of* υ*, and as we shall prove, satisfies* $\tilde{\mathcal{A}}_\varepsilon[\upsilon]\ \ge\ c\mathcal{B}_\varepsilon[\upsilon]$*.) This stability profile is in agreement to Sections 1.2-1.3 and (1.2)-(1.3) of* [11]*.*

2.8. **The slow channel.** We define the slow channel for (1.1) to be

$$\Gamma_\rho := \Big\{u(x) : \ u = u^h + \upsilon, \ \ \tilde{\mathcal{A}}_\varepsilon[\upsilon] \leq c\,\gamma(\varepsilon)\,\alpha^2(r)\Big\} \tag{2.113}$$

with

$$\gamma(\varepsilon) := \big(\delta^2(\varepsilon)\varepsilon^{-3} \ + \ \mu^2(\varepsilon)\varepsilon \ + \ \varepsilon^{-4}\big)/\eta(\varepsilon). \tag{2.114}$$

We will study the orbit $u(x,t) = u^{h(t)}(x) + \upsilon(x,t)$ of (IACH) as long as

$$\Big(\frac{\delta^2(\varepsilon)\,\varepsilon^{-7}}{\nu(\varepsilon)^2} \ + \ \frac{\mu^2(\varepsilon)\,\varepsilon^{-3}}{\nu(\varepsilon)^2} \ + \ \frac{\delta(\varepsilon)\,\varepsilon^{-5}}{\nu(\varepsilon)} \ + \ \frac{\delta(\varepsilon)\,\varepsilon^{-2}}{\delta(\varepsilon)+\mu(\varepsilon)\varepsilon^2}\Big)\,\tilde{\mathcal{B}}_\varepsilon[\upsilon] \ll \eta(\varepsilon) \tag{2.115}$$

(conditions (2.119) and (2.115) arise in (2.135)-(2.136) further below); for instance, for the condition (2.115) it suffices to have

$$\tilde{\mathcal{B}}_\varepsilon[\upsilon] \ll \varepsilon^7\eta(\varepsilon).$$

For later use, notice that clearly $\tilde{\mathcal{B}}_\varepsilon \geq (\delta(\varepsilon)+\mu(\varepsilon))\mathcal{B}_\varepsilon$ and the estimate (2.59) directly yields

$$\|\upsilon\|^2_{L^\infty} \leq \frac{1}{\delta(\varepsilon)+\mu(\varepsilon)}\,\frac{1+\varepsilon}{\varepsilon}\,\tilde{\mathcal{B}}_\varepsilon[\upsilon]. \tag{2.116}$$

The following Lemma will be also useful.

Lemma 2.10. *For* $\upsilon \in C^2[0,1]$ *with* $\upsilon_x(0) = \upsilon_x(1) = 0$ *we have*

$$\|\upsilon_x\|_{L^\infty} \leq \delta(\varepsilon)^{-1/2}\,\varepsilon^{-1}\,\tilde{\mathcal{B}}_\varepsilon^{1/2}[\upsilon]. \tag{2.117}$$

Proof. We have

$$|\upsilon_x(x)| = \Big|\int_0^x \upsilon_{xx}\,dy\Big| \leq \|\upsilon_{xx}\| \leq \delta(\varepsilon)^{-1/2}\,\varepsilon^{-1}\,\tilde{\mathcal{B}}_\varepsilon^{1/2}[\upsilon]. \tag{2.118}$$

□

Next, besides the condition (2.46) we assume that the coefficients $\delta(\varepsilon), \mu(\varepsilon)$ satisfy the condition

$$\delta^2(\varepsilon)\,\varepsilon^{-6} \leq c\eta(\varepsilon)\nu(\varepsilon) \tag{2.119}$$

for some $c > 0$ small enough, with $\nu(\varepsilon) := \delta(\varepsilon) + \mu(\varepsilon)$ and the $\eta(\varepsilon)$ given in (2.47).

The result about the attractiveness and the slow evolution of states within the channel (2.113) is stated in the following theorem.

Theorem 2.11. *Let* $u(x,t) = u^{h(t)}(x) + \upsilon(x,t)$ *be an orbit of* (1.1) *starting outside but near the slow channel* (2.113) *in the sense that* $\upsilon(\cdot,0)$ *satisfies condition* (2.115). *Then* $\tilde{\mathcal{B}}_\varepsilon[\upsilon]$ *will decrease exponentially until* u *enters the channel and will remain in the channel following the approximate manifold* $\mathcal{M}$ *with speed* $\mathcal{O}(e^{-c/r})$, *thus staying in the channel for an exponentially long time. It can leave* Γ_ρ *only through the ends of the channel i.e at a time that* $(h_j - h_{j-1})$ *is reduced to* $\frac{\varepsilon}{\rho}$ *for some* j.

Proof. Applying (2.52) combined with the estimate $\mathcal{B}_\varepsilon \leq \nu(\varepsilon)^{-1}\tilde{\mathcal{B}}_\varepsilon$ into (2.65) we immediately get

$$\begin{aligned}|\dot{h}_i| \leq C\delta(\varepsilon)\big(\varepsilon^{-5/2}\nu(\varepsilon)^{-1/2}\tilde{\mathcal{A}}_\varepsilon^{1/2}[\upsilon] \ + \ \varepsilon^{-3}\nu(\varepsilon)^{-1}\tilde{\mathcal{A}}_\varepsilon[\upsilon]\big) \ + \ C\big(\delta(\varepsilon)+\mu(\varepsilon)\varepsilon^2\big)\varepsilon^{-1}\alpha(r)\\ + \ C\mu(\varepsilon)\big(\big(\alpha(r)+\beta(r)\big)\,\varepsilon^{1/2}\nu(\varepsilon)^{-1/2}\tilde{\mathcal{A}}_\varepsilon^{1/2}[\upsilon] \ + \ \varepsilon^{-1}\nu(\varepsilon)^{-1}\tilde{\mathcal{A}}_\varepsilon[\upsilon]\big).\end{aligned} \tag{2.120}$$

We have to estimate the growth of $\tilde{\mathcal{A}}_\varepsilon[\upsilon(\cdot,t)]$, so that to prove the attractiveness of the slow channel, and then combined with (2.120), we will get an upper bound of the layers' speed within the channel; see (2.139). To this end, we set

$$
\begin{aligned}
I_\varepsilon[\upsilon] &:= \frac{1}{2}\frac{d}{dt}\tilde{\mathcal{A}}_\varepsilon[\upsilon] \\
&= \frac{1}{2}\frac{d}{dt}\big\langle -L^h_\varepsilon(\upsilon),\, \upsilon\big\rangle \\
&= \Big\langle -\frac{1}{2}\frac{\partial}{\partial t}L^h_\varepsilon(\upsilon),\ \upsilon\Big\rangle \;-\; \frac{1}{2}\big\langle L^h_\varepsilon(\upsilon),\ \upsilon_t\big\rangle \qquad (2.121)
\end{aligned}
$$

where, we recall by (2.28),

$$
L^h_\varepsilon(\upsilon) := -\delta(\varepsilon)\underbrace{\big(\varepsilon^2\upsilon_{xx} - f'(u^h)\upsilon\big)_{xx}}_{L^h_{1,\varepsilon}(\upsilon)=\text{the linearized CH part}} + \mu(\varepsilon)\underbrace{\big(\varepsilon^2\upsilon_{xx} - f'(u^h)\upsilon\big)}_{L^h_{2,\varepsilon}(\upsilon)=\text{the linearized AC part}}. \qquad (2.28)
$$

In order to write $I_\varepsilon[\upsilon]$ in a more convenient form, we first notice the pointwise estimate

$$
\begin{aligned}
\frac{\partial}{\partial t}L^h_\varepsilon(\upsilon) &= \frac{\partial}{\partial t}\Big[-\delta(\varepsilon)\big(\varepsilon^2\upsilon_{xx} - f'(u^h)\upsilon\big)_{xx} \;+\; \mu(\varepsilon)\big(\varepsilon^2\upsilon_{xx} - f'(u^h)\upsilon\big)\Big] \\
&= L^h_\varepsilon(\upsilon_t) \;+\; \delta(\varepsilon)\big(\big(f'(u^h)\big)_t\upsilon\big)_{xx} \;-\; \mu(\varepsilon)\big(f'(u^h)\big)_t\upsilon \qquad (2.122)
\end{aligned}
$$

and note also that integration by parts yields

$$
\big\langle L^h_\varepsilon(\upsilon_t),\ \upsilon\big\rangle \;=\; \big\langle \upsilon_t,\ L^h_\varepsilon(\upsilon)\big\rangle \;-\; \delta(\varepsilon)\big\langle \upsilon_t,\ \big(f'(u^h)\upsilon\big)_{xx}\big\rangle \;+\; \delta(\varepsilon)\big\langle \upsilon_t,\ f'(u^h)\upsilon_{xx}\big\rangle \qquad (2.123)
$$

where the boundary terms vanish due to the zero Neumann conditions on $\upsilon_x, \upsilon_{xxx}$ and (2.20).

Therefore, by (2.121), (2.122), (2.123) we get

$$
\begin{aligned}
I_\varepsilon[\upsilon] &= -\big\langle L^h_\varepsilon(\upsilon),\, \upsilon_t\big\rangle \;+\; \frac{\delta(\varepsilon)}{2}\Big\langle \big(f'(u^h)\upsilon\big)_{xx},\ \upsilon_t\Big\rangle \;-\; \frac{\delta(\varepsilon)}{2}\Big\langle f'(u^h)\upsilon_{xx},\ \upsilon_t\Big\rangle \\
&\qquad -\; \frac{\delta(\varepsilon)}{2}\Big\langle \Big(\big(f'(u^h)\big)_t\upsilon\Big)_{xx},\ \upsilon\Big\rangle \;+\; \frac{\mu(\varepsilon)}{2}\Big\langle \big(f'(u^h)\big)_t\upsilon,\ \upsilon\Big\rangle \\
&= T_\varepsilon[\upsilon] \;-\; \frac{\delta(\varepsilon)}{2}\Big\langle \Big(\big(f'(u^h)\big)_t\upsilon\Big)_{xx},\ \upsilon\Big\rangle \;+\; \frac{\mu(\varepsilon)}{2}\Big\langle \big(f'(u^h)\big)_t\upsilon,\ \upsilon\Big\rangle \qquad (2.124)
\end{aligned}
$$

for

$$
T_\varepsilon[\upsilon] \;:=\; -\big\langle S^h_\varepsilon(\upsilon),\, \upsilon_t\big\rangle \qquad (2.125)
$$

with S^h_ε the symmetric operator (corresponding to L^h_ε) given in (2.40).

Regarding the last term in (2.124), we use Lemma 2.7, and notice that the support of each u^h_j is contained in an interval of length 2ε where $|u^h_j| \le c\varepsilon^{-1}$, therefore

$$
\|u^h_j\|_{L^1} = \mathcal{O}(1), \qquad (2.126)
$$

so we obtain

$$\begin{aligned}
\mu(\varepsilon)\Big\langle \big(f'(u^h)\big)_t\upsilon,\ \upsilon\Big\rangle &\le \mu(\varepsilon)\,\|\upsilon\|^2_{L^\infty}\,\|f''(u^h)\|_{L^\infty}\sum_{j=1}^N \|u^h_j\|_{L^1}|\dot h_j| \\
&\le C\,\varepsilon^{-1}\,\mu(\varepsilon)\,\mathcal{B}_\varepsilon[\upsilon]\,\sum_{j=1}^N|\dot h_j| \\
&\le C\,\varepsilon^{-1}\big(\mu^2(\varepsilon)\mathcal{B}^2_\varepsilon[\upsilon]\ +\ \max_j|\dot h_j|^2\big) \\
&\le C\,\varepsilon^{-1}\Big(\frac{\mu^2(\varepsilon)}{\nu(\varepsilon)^2}\,\tilde{\mathcal{B}}^2_\varepsilon[\upsilon]\ +\ \max_j|\dot h_j|^2\Big). \qquad (2.127)
\end{aligned}$$

As for the middle term in (2.124), after integrating by parts and using again Lemma 2.7, it can be similarly seen that

$$\begin{aligned}
\Big\langle\Big(\big(f'(u^h)\big)_t\upsilon\Big)_{xx},\ \upsilon\Big\rangle &= -\Big\langle\big(f'(u^h)\big)_t\upsilon_x,\ \upsilon_x\Big\rangle\ -\ \Big\langle\big(f''(u^h)u^h_x\big)_t\upsilon,\ \upsilon_x\Big\rangle \\
&\le \|\upsilon_x\|^2\,\|f''(u^h)\|_{L^\infty}\sum_{j=1}^N\|u^h_j\|_{L^\infty}\,|\dot h_j| \\
&\quad +\|\upsilon\|_{L^\infty}\,\|\upsilon_x\|_{L^1}\,\|f'''(u^h)\|_{L^\infty}\,\|u^h_x\|_{L^\infty}\sum_{j=1}^N\|u^h_j\|_{L^\infty}\,|\dot h_j| \\
&\quad +\|\upsilon\|_{L^\infty}\,\|\upsilon_x\|_{L^1}\,\|f''(u^h)\|_{L^\infty}\sum_{j=1}^N\|\partial_{h_j}(u^h_x)\|_{L^\infty}\,|\dot h_j| \\
&\le C\,\big(\varepsilon^{-3}+\varepsilon^{-7/2}\big)\,\mathcal{B}_\varepsilon[\upsilon]\sum_{j=1}^N|\dot h_j| \\
&\le C\,\varepsilon^{-7/2}\,\mathcal{B}_\varepsilon[\upsilon]\sum_{j=1}^N|\dot h_j|.
\end{aligned}$$

In the last inequality we applied (4.15), (4.26), (4.30), (4.31) into (3.45) to get

$$\|\partial_{h_j}(u^h_x)\|_{L^\infty}=\mathcal{O}(\varepsilon^{-2}),$$

and additionally (4.13)-(4.14) into (4.53) to get

$$\|u^h_j\|_{L^\infty}=\mathcal{O}(\varepsilon^{-1}).$$

We also used (2.59) and as well the estimate

$$\|\upsilon_x\|_{L^1}<\varepsilon^{-1}\mathcal{B}^{1/2}_\varepsilon[\upsilon].$$

Therefore,

$$\begin{aligned}\delta(\varepsilon)\left\langle\left(\left(f'(u^h)\right)_t \upsilon\right)_{xx},\ \upsilon\right\rangle &\le C\,\delta(\varepsilon)\ \varepsilon^{-3-\frac{1}{2}}\,\mathcal{B}_\varepsilon[\upsilon]\ \sum_{j=1}^{N}|\dot h_j| \\ &\le C\left(\delta^2(\varepsilon)\ \varepsilon^{-6}\,\mathcal{B}_\varepsilon^2[\upsilon]\ +\ \varepsilon^{-1}\ \max_j|\dot h_j|^2\right) \\ &\le C\left(\frac{\delta^2(\varepsilon)\ \varepsilon^{-6}}{\nu(\varepsilon)^2}\ \tilde{\mathcal{B}}_\varepsilon^2[\upsilon]\ +\ \varepsilon^{-1}\ \max_j|\dot h_j|^2\right).\end{aligned} \tag{2.128}$$

We next want to estimate the first term in (2.124). In view of the equation of motion (2.33)

$$\upsilon_t\ =\ A_\varepsilon(u^h)\ +\ L_\varepsilon^h(\upsilon)\ -\ \delta(\varepsilon)\left(f^h\,\upsilon^2\right)_{xx}\ +\ \mu(\varepsilon)f^h\,\upsilon^2\ -\ \sum_{j=1}^{N}u_j^h\,\dot h_j$$

with (see (2.29))

$$f^h(x) := \int_0^1 (\tau-1)\ f''(u^h+\tau\upsilon)\ d\tau$$

we may write $T_\varepsilon[\upsilon]$ as follows,

$$\begin{aligned}T_\varepsilon[\upsilon]\ &=\ -\langle S_\varepsilon^h(\upsilon),\,\upsilon_t\rangle \\ &=\ -\left\langle S_\varepsilon^h(\upsilon)\,,\,L_\varepsilon^h(\upsilon)\right\rangle\ -\ \left\langle S_\varepsilon^h(\upsilon)\,,\,A_\varepsilon(u^h)\right\rangle\ +\ \delta(\varepsilon)\left\langle S_\varepsilon^h(\upsilon)\,,\,\left(f^h\,\upsilon^2\right)_{xx}\right\rangle \\ &\quad -\mu(\varepsilon)\left\langle S_\varepsilon^h(\upsilon)\,,\,f^h\,\upsilon^2\right\rangle\ +\ \sum_{j=1}^{N}\left\langle S_\varepsilon^h(\upsilon)\,,\,u_j^h\right\rangle\dot h_j \\ &=\ -\left\langle S_\varepsilon^h(\upsilon)\,,\,S_\varepsilon^h(\upsilon)\right\rangle\ -\ T_{2,\varepsilon}[\upsilon]\ +\ \delta(\varepsilon)T_{3,\varepsilon}[\upsilon]\ -\ \mu(\varepsilon)\,T_{4,\varepsilon}[\upsilon]\ +\ T_{5,\varepsilon}[\upsilon] \\ &\quad -\delta(\varepsilon)\left[\left\langle S_\varepsilon^h(\upsilon)\,,\,\left(f'(u^h)\right)_x\upsilon_x\right\rangle\ +\ \frac{1}{2}\left\langle S_\varepsilon^h(\upsilon)\,,\,\left(f'(u^h)\right)_{xx}\upsilon\right\rangle\right]\end{aligned}$$

where in in the last equality, we substituted the following relation into the first term of the left side,

$$L_\varepsilon^h(\upsilon)\ =\ S_\varepsilon^h(\upsilon)\ +\ \delta(\varepsilon)\left[\left(f'(u^h)\right)_x\upsilon_x\ +\ \frac{1}{2}\left(f'(u^h)\right)_{xx}\upsilon\right].$$

So, we obtain

$$\begin{aligned}T_\varepsilon[\upsilon] = -\|S_\varepsilon^h(\upsilon)\|^2\ -\ T_{2,\varepsilon}[\upsilon]\ +\ \delta(\varepsilon)T_{3,\varepsilon}[\upsilon]\ -\ \mu(\varepsilon)\,T_{4,\varepsilon}[\upsilon]\ +\ T_{5,\varepsilon}[\upsilon] \\ -\ \delta(\varepsilon)\left[\left\langle S_\varepsilon^h(\upsilon)\,,\,\left(f'(u^h)\right)_x\upsilon_x\right\rangle\ +\ \frac{1}{2}\left\langle S_\varepsilon^h(\upsilon)\,,\,\left(f'(u^h)\right)_{xx}\upsilon\right\rangle\right]\end{aligned} \tag{2.129}$$

for

$$\begin{aligned}T_{2,\varepsilon}[\upsilon]\ &:=\ \left\langle S_\varepsilon^h(\upsilon)\,,\,A_\varepsilon(u^h)\right\rangle, \\ T_{3,\varepsilon}[\upsilon]\ &:=\ \left\langle S_\varepsilon^h(\upsilon)\,,\,\left(f^h\,\upsilon^2\right)_{xx}\right\rangle, \\ T_{4,\varepsilon}[\upsilon]\ &:=\ \left\langle S_\varepsilon^h(\upsilon)\,,\,f^h\,\upsilon^2\right\rangle,\end{aligned}$$

$$T_{5,\varepsilon}[\upsilon] \;:=\; \sum_{j=1}^{N}\Big\langle S^h_\varepsilon(\upsilon)\,,\,u^h_j\Big\rangle\,\dot h_j.$$

We have

$$\big|T_{2,\varepsilon}[\upsilon]\big| = \Big|\Big\langle S^h_\varepsilon(\upsilon)\,,\,A_\varepsilon(u^h)\Big\rangle\Big| \;\le\; \|S^h_\varepsilon(\upsilon)\|\,\|A_\varepsilon(u^h)\| \;\le\; \frac14\|S^h_\varepsilon(\upsilon)\|^2 \;+\; \|A_\varepsilon(u^h)\|^2$$

$$\le \frac14\|S^h_\varepsilon(\upsilon)\|^2 \;+\; C\varepsilon^{-4}\alpha^2(r), \tag{2.129a}$$

where we combined (2.16), (4.35), (4.43) to estimate the term $\|A_\varepsilon(u^h)\|$. Also we have

$$\big(f^h\,\upsilon^2\big)_{xx} = \upsilon^2\int_0^1(\tau-1)\;f^{(4)}(u^h+\tau\upsilon)(u^h_x+\tau\upsilon_x)^2\;d\tau \;+\;\upsilon^2\int_0^1(\tau-1)\;f''(u^h+\tau\upsilon)(u^h_{xx}+\tau\upsilon_{xx})\;d\tau$$
$$+\;4\,\upsilon\,\upsilon_x\int_0^1(\tau-1)\;f^{(3)}(u^h+\tau\upsilon)(u^h_x+\tau\upsilon_x)\;d\tau\;+\;2(\upsilon_{xx}\upsilon+\upsilon_x^2)\int_0^1(\tau-1)\;f''(u^h+\tau\upsilon)\;d\tau$$

therefore

$$\begin{aligned}
\delta(\varepsilon)\big|T_{3,\varepsilon}[\upsilon]\big| &= \delta(\varepsilon)\Big|\Big\langle S^h_\varepsilon(\upsilon)\,,\,\big(f^h\,\upsilon^2\big)_{xx}\Big\rangle\Big|\\
&\le \delta(\varepsilon)\;\|S^h_\varepsilon(\upsilon)\|\;\|\big(f^h\,\upsilon^2\big)_{xx}\|\\
&\le \epsilon\|S^h_\varepsilon(\upsilon)\|^2\;+\;\frac{\delta^2(\varepsilon)}{4\epsilon}\|\big(f^h\,\upsilon^2\big)_{xx}\|^2\\
&\le \epsilon\|S^h_\varepsilon(\upsilon)\|^2\;+\;\frac{C\,\delta^2(\varepsilon)}{\epsilon}\Big[\|\upsilon\|^4_{L^\infty}\big(\frac{1}{\varepsilon^4}+\|\upsilon_x\|^4+\|\upsilon_{xx}\|^2\big)\;+\;\|\upsilon\|^2_{L^\infty}\,\|\upsilon_x\|^2_{L^\infty}\,\big(\frac{1}{\varepsilon^2}+\|\upsilon_x\|^2\big)\\
&\qquad\qquad+\;\|\upsilon\|^2_{L^\infty}\,\|\upsilon_{xx}\|^2\;+\;\|\upsilon_x\|^2_{L^\infty}\,\|\upsilon_x\|^2\Big]\\
&\le \epsilon\|S^h_\varepsilon(\upsilon)\|^2+\frac{C\,\delta^2(\varepsilon)}{\epsilon}\Big(\|\upsilon\|^4_{L^\infty}\frac{1}{\varepsilon^4}+\|\upsilon\|^2_{L^\infty}\,\|\upsilon_{xx}\|^2+\|\upsilon\|^2_{L^\infty}\,\|\upsilon_x\|^2_{L^\infty}\frac{1}{\varepsilon^2}+\|\upsilon_x\|^2_{L^\infty}\,\|\upsilon_x\|^2\Big)
\end{aligned}$$

for some ϵ small enough which is to be determined later on. In the last inequality we used that $\|\upsilon\|^2_{L^\infty}=\mathcal{O}(1)$ as follows by (2.58), (2.59).

Then, applying into the above inequality, the estimates (2.116), (2.117) and the following two estimates that follow directly from the definition (2.51),

$$\|\upsilon_{xx}\|^2 \;\le\; \delta(\varepsilon)^{-1}\,\varepsilon^{-2}\,\tilde{\mathcal{B}}_\varepsilon[\upsilon] \tag{2.130}$$

$$\|\upsilon_x\|^2 \;\le\; (\delta(\varepsilon)+\mu(\varepsilon)\varepsilon^2)^{-1}\,\tilde{\mathcal{B}}_\varepsilon[\upsilon], \tag{2.131}$$

we get

$$\delta(\varepsilon)\,\big|T_{3,\varepsilon}[\upsilon]\big| \le \epsilon\|S^h_\varepsilon(\upsilon)\|^2\;+\;\frac{C}{\epsilon}\Big[\frac{\delta^2(\varepsilon)\,\varepsilon^{-6}}{\nu(\varepsilon)^2}\;+\;\frac{\delta(\varepsilon)\,\varepsilon^{-5}}{\nu(\varepsilon)}\;+\;\frac{\delta(\varepsilon)\,\varepsilon^{-2}}{\delta(\varepsilon)+\mu(\varepsilon)\varepsilon^2}\Big]\;\tilde{\mathcal{B}}^2_\varepsilon[\upsilon]. \tag{2.129b}$$

Also we clearly have

$$\mu(\varepsilon)\big|T_{4,\varepsilon}[\upsilon]\big| =\;\mu(\varepsilon)\Big|\Big\langle S^h_\varepsilon(\upsilon)\,,\,f^h\,\upsilon^2\Big\rangle\Big|\;\le\;\mu(\varepsilon)\|S^h_\varepsilon(\upsilon)\|\;\|f^h\,\upsilon^2\|$$

$$\le\;\epsilon\|S^h_\varepsilon(\upsilon)\|^2\;+\;\frac{\mu^2(\varepsilon)}{4\epsilon}\|f^h\,\upsilon^2\|^2$$

$$\leq \epsilon \|S^h_\varepsilon(v)\|^2 + \frac{C\,\mu^2(\varepsilon)}{\epsilon}\|v\|^2_{L^\infty}\,\|v\|^2$$

$$\leq \epsilon \|S^h_\varepsilon(v)\|^2 + \frac{C}{\epsilon}\,\frac{\mu^2(\varepsilon)}{\nu(\varepsilon)^2}\,\varepsilon^{-1}\,\tilde{\mathcal{B}}^2_\varepsilon[v]. \tag{2.129c}$$

In the last inequality we applied the estimate (2.116) for the term $\|v\|_{L^\infty}$ and the estimate

$$\|v\|^2 \leq \nu(\varepsilon)^{-1}\tilde{\mathcal{B}}_\varepsilon[v] \tag{2.132}$$

which follows immediately from the definition (2.51).

Using (2.126), we get

$$\big|T_{5,\varepsilon}[v]\big| = \left|\sum_{j=1}^{N}\Big\langle S^h_\varepsilon(v)\,,\,u^h_j\Big\rangle\,\dot h_j\right| \leq \|S^h_\varepsilon(v)\|\sum_{j=1}^{N}\|u^h_j\|\cdot|\dot h_j| \leq C\,\|S^h_\varepsilon(v)\|\varepsilon^{-1/2}\Big(\sum_{j=1}^{N}|\dot h_j|\Big)$$

$$\leq \epsilon\,\|S^h_\varepsilon(v)\|^2 + \frac{C}{\epsilon}\,\varepsilon^{-1}\Big(\sum_{j=1}^{N}|\dot h_j|\Big)^2$$

$$\leq \epsilon\,\|S^h_\varepsilon(v)\|^2 + \frac{C}{\epsilon}\,\varepsilon^{-1}\max_j|\dot h_j|^2. \tag{2.129d}$$

As for the last two terms in (2.129), we use (4.34),(4.35) and then (2.131) and (2.132) respectively, to get:

$$\delta(\varepsilon)\Big\langle S^h_\varepsilon(v)\,,\,\big(f'(u^h)\big)_x v_x\Big\rangle \leq \epsilon\,\|S^h_\varepsilon(v)\|^2 + \frac{C\,\delta^2(\varepsilon)}{\epsilon}\,\varepsilon^{-2}\,\|v_x\|^2$$

$$\leq \epsilon\,\|S^h_\varepsilon(v)\|^2 + \frac{C}{\epsilon}\,\frac{\delta^2(\varepsilon)\,\varepsilon^{-2}}{\delta(\varepsilon)+\mu(\varepsilon)\varepsilon^2}\,\tilde{\mathcal{B}}_\varepsilon[v] \tag{2.129e}$$

$$\delta(\varepsilon)\Big\langle S^h_\varepsilon(v)\,,\,\big(f'(u^h)\big)_{xx} v\Big\rangle \leq \epsilon\,\|S^h_\varepsilon(v)\|^2 + \frac{C\,\delta^2(\varepsilon)}{\epsilon}\,\varepsilon^{-4}\,\|v\|^2$$

$$\leq \epsilon\,\|S^h_\varepsilon(v)\|^2 + \frac{C}{\epsilon}\,\frac{\delta^2(\varepsilon)\,\varepsilon^{-4}}{\nu(\varepsilon)}\,\tilde{\mathcal{B}}_\varepsilon[v]. \tag{2.129f}$$

We then apply (2.129a)-(2.129f) into (2.129) and next apply the resulted estimate together with (2.127), (2.128) into (2.124) and use the assumption (2.46) to conclude that

$$\frac{1}{2}\frac{d}{dt}\tilde{\mathcal{A}}_\varepsilon[v] + \big(1-\frac{1}{4}-5\epsilon\big)\,\|S^h_\varepsilon(v)\|^2 \leq C\,\Big[\varepsilon^{-1}\max_j|\dot h_j|^2 + \varepsilon^{-4}\alpha^2(r)\Big] \tag{2.133}$$

$$+ C\Big[\frac{\delta^2(\varepsilon)}{\nu(\varepsilon)}\,\varepsilon^{-4} + \Big(\frac{\mu^2(\varepsilon)\,\varepsilon^{-1}}{\nu(\varepsilon)^2} + \frac{\delta^2(\varepsilon)\,\varepsilon^{-6}}{\nu(\varepsilon)^2} + \frac{\delta(\varepsilon)\,\varepsilon^{-5}}{\nu(\varepsilon)} + \frac{\delta(\varepsilon)\,\varepsilon^{-2}}{\delta(\varepsilon)+\mu(\varepsilon)\varepsilon^2}\Big)\,\tilde{\mathcal{B}}_\varepsilon[v]\Big]\,\tilde{\mathcal{B}}_\varepsilon[v].$$

We apply the estimate (2.65) for the term $\max_j|\dot h_j|^2$ of the RHS of (2.133), and then substitute the estimate $\mathcal{B}_\varepsilon[v] \leq \nu(\varepsilon)^{-1}\tilde{\mathcal{B}}_\varepsilon[v]$ and fix any $\epsilon < \frac{3}{20}$ to get

$$\frac{1}{2}\frac{d}{dt}\tilde{\mathcal{A}}_\varepsilon[v] + C\,\|S^h_\varepsilon(v)\|^2 \leq C\,\big(\delta^2(\varepsilon)\varepsilon^{-3} + \mu^2(\varepsilon)\varepsilon + \varepsilon^{-4}\big)\alpha^2(r) \tag{2.134}$$

$$+ \; C\Big[\frac{\delta^2(\varepsilon)\,\varepsilon^{-6}}{\nu(\varepsilon)} \;+\; \Big(\frac{\delta^2(\varepsilon)\,\varepsilon^{-7}}{\nu(\varepsilon)^2} + \frac{\mu^2(\varepsilon)\,\varepsilon^{-3}}{\nu(\varepsilon)^2} \;+\; \frac{\delta(\varepsilon)\,\varepsilon^{-5}}{\nu(\varepsilon)} \;+\; \frac{\delta(\varepsilon)\,\varepsilon^{-2}}{\delta(\varepsilon)+\mu(\varepsilon)\varepsilon^2}\Big)\,\tilde{\mathcal{B}}_\varepsilon[\upsilon]\Big]\,\tilde{\mathcal{B}}_\varepsilon[\upsilon].$$

Applying the estimates (2.52)-(2.53) into (2.134) we obtain

$$\frac{1}{2}\frac{d}{dt}\tilde{\mathcal{A}}_\varepsilon[\upsilon] \;+\; C\,\eta(\varepsilon)\,\tilde{\mathcal{A}}_\varepsilon[\upsilon] \;\le\; C\,\big(\delta^2(\varepsilon)\varepsilon^{-3} \;+\; \mu^2(\varepsilon)\varepsilon \;+\; \varepsilon^{-4}\big)\alpha^2(r) \tag{2.135}$$
$$+\; C\Big[\frac{\delta^2(\varepsilon)\,\varepsilon^{-6}}{\nu(\varepsilon)} \;+\; \Big(\frac{\delta^2(\varepsilon)\,\varepsilon^{-7}}{\nu(\varepsilon)^2} + \frac{\mu^2(\varepsilon)\,\varepsilon^{-3}}{\nu(\varepsilon)^2} \;+\; \frac{\delta(\varepsilon)\,\varepsilon^{-5}}{\nu(\varepsilon)} \;+\; \frac{\delta(\varepsilon)\,\varepsilon^{-2}}{\delta(\varepsilon)+\mu(\varepsilon)\varepsilon^2}\Big)\,\tilde{\mathcal{B}}_\varepsilon[\upsilon]\Big]\,\tilde{\mathcal{A}}_\varepsilon[\upsilon]$$

and in view of the assumptions (2.46),(2.119) on $\delta(\varepsilon), \mu(\varepsilon)$, as well as the condition (2.115) for $\tilde{\mathcal{B}}_\varepsilon[\upsilon(\cdot,t)]$, we have

$$\frac{\delta^2(\varepsilon)\,\varepsilon^{-6}}{\nu(\varepsilon)} \;+\; \Big(\frac{\delta^2(\varepsilon)\,\varepsilon^{-7}}{\nu(\varepsilon)^2} + \frac{\mu^2(\varepsilon)\,\varepsilon^{-3}}{\nu(\varepsilon)^2} \;+\; \frac{\delta(\varepsilon)\,\varepsilon^{-5}}{\nu(\varepsilon)} \;+\; \frac{\delta(\varepsilon)\,\varepsilon^{-2}}{\delta(\varepsilon)+\mu(\varepsilon)\varepsilon^2}\Big)\,\tilde{\mathcal{B}}_\varepsilon[\upsilon] = o\big(\eta(\varepsilon)\big) \tag{2.136}$$

so (2.135) yields

$$\frac{d}{dt}\tilde{\mathcal{A}}_\varepsilon[\upsilon] \;+\; c\,\eta(\varepsilon)\,\tilde{\mathcal{A}}_\varepsilon[\upsilon] \;\le\; C\,\big(\delta^2(\varepsilon)\varepsilon^{-3} \;+\; \mu^2(\varepsilon)\varepsilon \;+\; \varepsilon^{-4}\big)\,\alpha^2(r). \tag{2.137}$$

Integrating (2.137) we get

$$\begin{aligned} \tilde{\mathcal{A}}_\varepsilon[\upsilon(t)] \;&\le\; \tilde{\mathcal{A}}_\varepsilon[\upsilon(0)]\,e^{-c\eta t} \;+\; C\,\gamma(\varepsilon)\,\alpha^2(r)\big(1-e^{-c\eta t}\big) \\ &\le\; \max\big\{\tilde{\mathcal{A}}_\varepsilon[\upsilon(0)]\;,\;C\,\gamma(\varepsilon)\,\alpha^2(r)\big\} \end{aligned} \tag{2.138}$$

with the $\gamma(\varepsilon)$ given in (2.114). In view of (2.138) and the definition (2.113) of the slow channel Γ_ρ we deduce that the solution u evolves exponentially towards Γ_ρ.

Applying (2.138) into (2.120) we get

$$|\dot{h}_i| \le C(\delta(\varepsilon)+\mu(\varepsilon)\varepsilon^2)\varepsilon^{-1}\alpha(r) \;+\; C\big(\varepsilon^{-2}\delta(\varepsilon) \;+\; \mu(\varepsilon)\big)\varepsilon^{-1}\nu(\varepsilon)^{-1}\,\big(\tilde{\mathcal{A}}_\varepsilon[\upsilon(0)] \;+\; \gamma(\varepsilon)\,\alpha^2(r)\big) \tag{2.139}$$
$$+\; C\,\big[\delta(\varepsilon)\varepsilon^{-3} \;+\; \big(\alpha(r)+\beta(r)\big)\,\varepsilon^{1/2}\mu(\varepsilon)\big]\;\nu(\varepsilon)^{-1/2}\;\big(\tilde{\mathcal{A}}_\varepsilon^{1/2}[\upsilon(0)] \;+\; \gamma^{1/2}(\varepsilon)\,\alpha(r)\big)$$

and since in the slow channel (2.113) we have $\mathcal{A}_\varepsilon[\tilde{\upsilon}(0)] \le c\,\gamma(\varepsilon)\,\alpha^2(r)$, (2.139) becomes

$$|\dot{h}_i| \le C(\delta(\varepsilon)+\mu(\varepsilon)\varepsilon^2)\varepsilon^{-1}\alpha(r) \;+\; C\big(\delta(\varepsilon) \;+\; \mu(\varepsilon)\varepsilon^2\big)\varepsilon^{-3}\nu(\varepsilon)^{-1}\,\gamma(\varepsilon)\,\alpha^2(r) \tag{2.140}$$
$$+\; C\,\big[\delta(\varepsilon)\varepsilon^{-3} \;+\; \big(\alpha(r)+\beta(r)\big)\,\varepsilon^{1/2}\mu(\varepsilon)\big]\;\nu(\varepsilon)^{-1/2}\,\gamma^{1/2}(\varepsilon)\,\alpha(r)$$

which implies that

$$|\dot{h}_i| \le C\;\max\big\{\,\big(\delta(\varepsilon)+\mu(\varepsilon)\varepsilon^2\big)\,\varepsilon^{-1}\,,\;\big(\delta(\varepsilon)+\mu(\varepsilon)\varepsilon^2\big)\varepsilon^{-3}\nu(\varepsilon)^{-1}\,\gamma(\varepsilon)\,\alpha(r)\,,\;\delta(\varepsilon)\,\nu(\varepsilon)^{-1/2}\,\varepsilon^{-3}\,\gamma^{1/2}(\varepsilon)\big\}\;\alpha(r)$$

where $\alpha(r)$ is exponentially small (see the definition (4.19), (4.21) and the estimate (4.2)), so provided that

$$\big(\delta(\varepsilon)+\mu(\varepsilon)\varepsilon^2\big)\,\varepsilon^{-1} \ll \alpha^{-1}\;,\quad \big(\delta(\varepsilon)+\mu(\varepsilon)\varepsilon^2\big)\varepsilon^{-3}\nu(\varepsilon)^{-1}\,\gamma(\varepsilon) \ll \alpha^{-2},\quad \delta(\varepsilon)\,\nu(\varepsilon)^{-1/2}\,\varepsilon^{-3}\,\gamma^{1/2}(\varepsilon) \ll \alpha^{-1},$$

by (2.140) we have

$$|\dot{h}_i| = \mathcal{O}(e^{-c/r})$$

and the solution $\tilde{u}$ stay in the channel for an exponentially long time. □

Remark 2.12. *Let for example*

$$\delta(\varepsilon) = \mathcal{O}(1) \quad \textit{small enough},$$

and

$$\mu(\varepsilon) = \varepsilon^{-3},$$

then

$$n(\varepsilon) = \mathcal{O}(\varepsilon^{-3}),$$

or for example

$$\delta(\varepsilon) = \mathcal{O}(\varepsilon^{3}) \quad \textit{small enough},$$

and

$$\mu(\varepsilon) = \mathcal{O}(1),$$

then

$$n(\varepsilon) = \mathcal{O}(1).$$

In both cases, the conditions (2.46) *and* (2.119) *are satisfied.*

3. Mass conserving layer dynamics

We fix a mass $M \in (-1,1)$, and consider the mixed problem

$$u_t = -\delta(\varepsilon)\left(\varepsilon^2 u_{xx} - f(u)\right)_{xx} + \mu(\varepsilon)\left(\varepsilon^2 u_{xx} - f(u)\right), \tag{ACH}$$

for $0 < x < 1,\ t > 0,$ subject to the boundary conditions

$$u_x(0,\, t) = u_x(1,\, t) = 0, \tag{BC1}$$

$$u_{xxx}(0,\, t) = 0, \tag{BC2}$$

together with the constraint of mass conservation

$$\int_0^1 u(x,t)\,dx = \int_0^1 u(x,0)\,dx =: M, \qquad t > 0, \tag{MC}$$

in place of the Neumann b.c. for u_{xx} at $x = 1$; here, we replaced the fourth b.c. $u_{xxx}(1,t) = 0$, used in the previous Section, by (MC).

As we shall see, (MC), when the integrated version is stated, yields the integrated Cahn-Hilliard/ Allen-Cahn equation with the same b.c. as these of the integrated Cahn-Hilliard proposed and analyzed by Bates and Xun in [6].

We remind that $f(u) := u^3 - u$.

Following [6, 11], we define the first approximate manifold by

$$\mathcal{M} = \{u^h : h \in \Omega_\rho\}, \tag{3.1}$$

with u^h given in (2.10).

Let

$$\mathrm{M}(h) := \int_0^1 u^h(x)\,dx,$$

for $h \in \Omega_\rho$, then, by Lemma 2.1 of [6], $\mathrm{M}(h)$ is a smooth function of h, and

$$\frac{\partial \mathrm{M}}{\partial h_j} = 2(-1)^{j+1} + \mathcal{O}(\varepsilon^{-1}\beta(r)). \tag{3.2}$$

We define the second approximate manifold $\mathcal{M}_1$, as the constant mass sub-manifold of $\mathcal{M}$,

$$\mathcal{M}_1 = \big\{u^h \in \mathcal{M} : \int_0^1 u^h(x)\,dx = M\big\}, \tag{3.3}$$

which will be the proper approximate manifold for the mass-conserving problem (ACH); see also in [6].

It is clear that $\mathcal{M}_1$ is smooth, while by (3.2) and the Implicit Function Theorem, we see that h_N is a smooth function of $h_1, \cdots, h_{N-1}$ if $u^h \in \mathcal{M}_1$. Thus, $\mathcal{M}_1$ can be parameterized by $(h_1, h_2, \cdots, h_{N-1})$.

We set

$$\xi := (\xi_1, \xi_2, \cdots, \xi_{N-1}) \equiv (h_1, h_2, \cdots, h_{N-1}), \tag{3.4}$$

and for $u^h \in \mathcal{M}_1$, we will denote u^h by u^ξ, and define

$$u_j^\xi := \frac{\partial u^\xi}{\partial \xi_j} = u_j^h + \frac{\partial u^h}{\partial h_N}\frac{\partial h_N}{\partial h_j}, \qquad j = 1, 2, \cdots, N-1, \tag{3.5}$$

where u_j^h still stands as a notation for $\partial u^h/\partial h_j$.

3.1. **The coordinate system.** Motivated by [2] and [6], instead of working with the original problem (1.1)-(BC1)-(BC2)-(MC) we will work with the integrated problem.

More precisely, we integrate (1.1), use the conditions at $x = 0$ given in (BC1)-(BC2) and set

$$\tilde{u}(x,t) := \int_0^x u(y,t)\,dy, \tag{3.6}$$

to get the integrated CH/AC equation,

$$\tilde{u}_t \;=\; -\delta(\varepsilon)\big(\varepsilon^2\tilde{u}_{xxx} - W'(\tilde{u}_x)\big)_x \;+\; \mu(\varepsilon)\Big(\varepsilon^2\tilde{u}_{xx} - \int_0^x W'\big(\tilde{u}_x(y,t)\big)\,dy\Big), \tag{IACH}$$

with the boundary conditions, following directly from (3.6), (MC), (BC1) respectively,

$$\tilde{u}(0,\,t) \;= 0, \tag{IBC0}$$

$$\tilde{u}(1,\,t) \;= M, \tag{IMC}$$

$$\tilde{u}_{xx}(0,\,t) = \tilde{u}_{xx}(1,\,t) \;= 0. \tag{IBC1}$$

We may apply standard arguments for establishing the well-posedness of this problem, resulting from the one of the original problem (ACH); we outline the basic points in §4.3 of Appendix.

Here, and for the rest of this section, we have adopted the notation $W'(u) := f(u)$ and $W(u) \equiv F(u)$ introduced in [6], since for the case $\delta(\varepsilon) := 1$, $\mu(\varepsilon) := 0$, the equation (IACH) with the above b.c. coincides exactly with the problem analyzed therein (see pg. 431).

Equivalently, (IACH) is written as

$$\tilde{u}_t = -\delta(\varepsilon)\left(\varepsilon^2\tilde{u}_{xx} - \mathcal{W}(\tilde{u}_x)\right)_{xx} \;+\; \mu(\varepsilon)\left(\varepsilon^2\tilde{u}_{xx} - \mathcal{W}(\tilde{u}_x)\right),$$

for $\,0 < x < 1,\; t > 0,\,$ where $\mathcal{W}$ is obviously given by

$$\mathcal{W}(u)(x,t) := \int_0^x W'\big(u(y,t)\big)\,dy, \tag{3.7}$$

that is $(\mathcal{W}(u))_x = W'(u)$ with $\mathcal{W}(u)(0,t) = 0$.

Let us also denote by A_ε the spatial differential operator at the right side of (IACH), that is

$$\mathrm{A}_\varepsilon(\tilde{u}) := \delta(\varepsilon)\mathrm{A}_{1,\varepsilon}(\tilde{u}) \;+\; \mu(\varepsilon)\mathrm{A}_{2,\varepsilon}(\tilde{u}), \tag{3.8}$$

where $\mathrm{A}_{1,\varepsilon}(\tilde{u}), \mathrm{A}_{2,\varepsilon}(\tilde{u})$ stand for the integrated CH operator and the integrated AC operator respectively,

$$\mathrm{A}_{1,\varepsilon}(\tilde{u}) := -\big(\varepsilon^2 \tilde{u}_{xx} - \mathcal{W}(\tilde{u}_x)\big)_{xx} = -\big(\mathrm{A}_{2,\varepsilon}(\tilde{u})\big)_{xx} \qquad \text{and} \qquad \mathrm{A}_{2,\varepsilon}(\tilde{u}) := \varepsilon^2 \tilde{u}_{xx} - \mathcal{W}(\tilde{u}_x). \tag{3.9}$$

To study the dynamics of (IACH) in a neighborhood of $\mathcal{M}$, we introduce a coordinate system relative to $\mathcal{M}$,

$$\tilde{u} \mapsto (\xi, \tilde{v}),$$

as in [6], in the sense that for a solution $\tilde{u}$ close to $\mathcal{M}$ there exist unique components $\tilde{u}^\xi, \tilde{v}$ such that

$$\tilde{u}(x,t) = \tilde{u}^{\xi(t)}(x) + \tilde{v}(x,t). \tag{3.10}$$

More specifically, the approximate solution $\tilde{u}^\xi$ is in $\mathcal{M}$, and

$$\tilde{v} = \tilde{v}_{xx} = 0 \qquad \text{at} \qquad x = 0, 1, \tag{3.11}$$

with

$$\langle \tilde{v}, E_j \rangle := \int_0^1 \tilde{v}\, E_j \, dx = 0, \qquad j = 1, \dots, N-1, \tag{3.12}$$

where E_j are approximate tangent vectors to $\mathcal{M}$, defined as in [6], by

$$E_j(x) = \bar{\mathrm{w}}_j(x) - Q_j(x), \qquad j = 1, 2, \cdots, N-1, \tag{3.13}$$

with

$$\bar{\mathrm{w}}_j(x) := \tilde{u}^h_j(x) + \tilde{u}^h_{j+1}(x), \tag{3.14}$$

and

$$Q_j(x) := (-\tfrac{1}{6}x^3 + \tfrac{1}{2}x^2 - \tfrac{1}{3}x)\bar{\mathrm{w}}_{jxx}(0) + \tfrac{1}{6}(x^3 - x)\bar{\mathrm{w}}_{jxxx}(1) + x\bar{\mathrm{w}}_j(1), \tag{3.15}$$

so that

$$E_j = (E_j)_{xx} = 0 \qquad \text{at} \qquad x = 0, 1. \tag{3.16}$$

3.2. **Equations of motion.** We proceed next to obtain the odes system describing the motion of $(\xi, \tilde{v})$. To this end, we consider the linearized A_ε at $\tilde{u}^h$,

$$L^h_\varepsilon(\tilde{v}) := \delta(\varepsilon) L^h_{1,\varepsilon}(\tilde{v}) \;+\; \mu(\varepsilon) L^h_{2,\varepsilon}(\tilde{v}), \tag{3.17}$$

with the linearized CH part and the linearized AC part

$$L^h_{1,\varepsilon}(\tilde{v}) := -\big(\varepsilon^2 \tilde{v}_{xx} - L^h_W(\tilde{v}_x)\big)_{xx} \;=\; -\big(L^h_{2,\varepsilon}(\tilde{v})\big)_{xx} \qquad \text{and} \qquad L^h_{2,\varepsilon}(\tilde{v}) := \varepsilon^2 \tilde{v}_{xx} - L^h_W(\tilde{v}_x), \tag{3.18}$$

and L^h_W let the linearized $\mathcal{W}$ at u^h,

$$L^h_W(\tilde{v}_x)(x,t) \;:=\; \int_0^x W''\big(u^{h(t)}(y)\big)\; \tilde{v}_x(y,t)\, dy \tag{3.19}$$

that is $(L^h_W(\tilde{v}_x))_x = W''(u^h)\tilde{v}_x$, with $L^h_W(\tilde{v}_x)(0,t) = 0$.

We differentiate (3.12), with respect to t, to get

$$\big\langle \partial_t \tilde{v} \;,\; E_j \big\rangle \;+\; \big\langle \tilde{v} \;,\; \partial_t E_j \big\rangle = 0, \qquad j = 1, \dots, N-1, \tag{3.20}$$

with

$$\partial_t \tilde{v} = \partial_t (\tilde{u} - \tilde{u}^\xi) \overset{\text{(IACH)}}{=\!=\!=} \mathrm{A}_\varepsilon(\tilde{u}) - \partial_t \tilde{u}^\xi = \mathrm{A}_\varepsilon(\tilde{u}) - \sum_k \tilde{u}^\xi_k \, \dot{\xi}_k,$$

and $\partial_t E_j = \sum_k E_{j,k}\,\dot{\xi}_k$, hence (3.20) becomes

$$\sum_{k=1}^{N-1} a_{jk}\,\dot{\xi}_k \;=\; \big\langle A_\varepsilon(\tilde{u}^\xi+\tilde{v})\;,\;E_j\,\big\rangle, \qquad j=1,2,\dots,N-1, \tag{3.21}$$

where

$$a_{jk} := \big\langle u_k^\xi\;,\;E_j\big\rangle \;-\; \big\langle \tilde{v}\;,\;E_{j,k}\big\rangle, \qquad j,k=1,2,\dots,N-1, \tag{3.22}$$

and the subscripts k indicate the differentiation with respect to ξ_k,

$$u_k^\xi := \frac{\partial u^\xi}{\partial \xi_k} \qquad \text{and} \qquad E_{j,k} := \frac{\partial E_j}{\partial \xi_k}.$$

We write (3.21) in more useful form by expanding the term

$$\mathrm{A}_\varepsilon(\tilde{u}^\xi+\tilde{v}) = \mathrm{A}_\varepsilon(\tilde{u}^\xi) \;+\; L^h_\varepsilon(\tilde{v}) \;+\; \delta(\varepsilon)\,\big(f^\xi\,\tilde{v}_x^2\big)_x \;+\; \mu(\varepsilon)\int_0^x f^\xi\,\tilde{v}_x^2\,dy, \tag{3.23}$$

where $L^h_\varepsilon(\tilde{v})$ is given in (3.17), and

$$f^\xi(x) := \int_0^1 (1-\tau)\;W'''(\tilde{u}^\xi_x+\tau\tilde{v}_x)\;d\tau, \tag{3.24}$$

to get

$$\sum_{k=1}^{N-1} a_{jk}\,\dot{\xi}_k \;=\; \big\langle \mathrm{A}_\varepsilon(\tilde{u}^\xi)\;,\;E_j\,\big\rangle \;+\; \big\langle L^h_\varepsilon(\tilde{v})\;,\;E_j\,\big\rangle \;+\; \delta(\varepsilon)\,\big\langle \big(f^\xi\,\tilde{v}_x^2\big)_x\;,\;E_j\,\big\rangle \;+\; \mu(\varepsilon)\,\big\langle \int_0^x f^\xi\,\tilde{v}_x^2\,dy\;,\;E_j\,\big\rangle.$$

Discriminating between the (integrated) CH and AC parts (see (3.8), (3.17)), we have

$$\begin{aligned}\sum_{k=1}^{N-1} a_{jk}\,\dot{\xi}_k \;=&\;\delta(\varepsilon)\,\big\langle \mathrm{A}_{1,\varepsilon}(\tilde{u}^\xi)+L^h_{1,\varepsilon}(\tilde{v})+\big(f^\xi\,\tilde{v}_x^2\big)_x\;,\;E_j\big\rangle\\ &+\;\mu(\varepsilon)\,\big\langle \mathrm{A}_{2,\varepsilon}(\tilde{u}^\xi)+L^h_{2,\varepsilon}(\tilde{v})+\int_0^x f^\xi\,\tilde{v}_x^2\,dy\;,\;E_j\big\rangle.\end{aligned} \tag{3.25}$$

Moreover, we apply (3.10) to (IACH), to get

$$\tilde{v}_t = \mathrm{A}_\varepsilon(\tilde{u}^\xi+\tilde{v}) - \sum_{j=1}^{N-1}\tilde{u}^\xi_j\,\dot{\xi}_j. \tag{3.26}$$

As above, we expand in (3.26) the term $\mathrm{A}_\varepsilon(\tilde{u}^\xi+\tilde{v})$, according to (3.23), to get

$$\tilde{v}_t \;=\; \mathrm{A}_\varepsilon(\tilde{u}^\xi) \;+\; L^h_\varepsilon(\tilde{v}) \;+\; \delta(\varepsilon)\,\big(f^\xi\,\tilde{v}_x^2\big)_x \;+\; \mu(\varepsilon)\int_0^x f^\xi\,\tilde{v}_x^2\,dy \;-\; \sum_{j=1}^{N-1}\tilde{u}^\xi_j\,\dot{\xi}_j,$$

while separating the (integrated) CH and AC induced parts, we arrive at

$$\tilde{v}_t \;=\; \delta(\varepsilon)\,\Big[\mathrm{A}_{1,\varepsilon}(\tilde{u}^\xi) \;+\; L^h_{1,\varepsilon}(\tilde{v}) \;+\; \big(f^\xi\,\tilde{v}_x^2\big)_x\Big] + \mu(\varepsilon)\,\Big[\mathrm{A}_{2,\varepsilon}(\tilde{u}^\xi) \;+\; L^h_{2,\varepsilon}(\tilde{v}) \;+\; \int_0^x f^\xi\,\tilde{v}_x^2\,dy\Big] \;-\; \sum_{j=1}^{N-1}\tilde{u}^\xi_j\,\dot{\xi}_j. \tag{3.27}$$

Equations (3.25), (3.27) will be mainly used in the sequel, and for the rest of the section.

3.3. **Flow near layered equilibria.** For $\tilde{v} \in C^2[0,1]$ with $\tilde{v}(0) = \tilde{v}(1) = 0$, we introduce the form

$$B_\varepsilon[\tilde{v}] := \int_0^1 \left[\varepsilon^2 \tilde{v}_{xx}^2 + \tilde{v}_x^2\right] dx. \tag{3.28}$$

We will study the orbit $\tilde{u}(x,t) = \tilde{u}^{\xi(t)}(x) + \tilde{v}(x,t)$ of (IACH) as long as (cf. [6, (80)′] at pg. 448, for an analogous argument)

$$\mu^2(\varepsilon)\,\delta^{-2}(\varepsilon)\,\varepsilon^{-3} + \mu^2(\varepsilon)\,\delta^{-1}(\varepsilon)\,\varepsilon^{-5} + \left(\delta(\varepsilon)\,\varepsilon^{-7} + \varepsilon^{-6} + \mu^2(\varepsilon)\,\delta^{-2}(\varepsilon)\,\varepsilon^{-2}\right) B_\varepsilon[\tilde{v}] = o(1), \tag{3.29}$$

(condition (3.29) arises in (3.75)-(3.76) further below), or sufficiently for

$$\mu^2(\varepsilon)\,\delta^{-2}(\varepsilon)\left(1 + \delta(\varepsilon)\,\varepsilon^{-5}\right) = o(\varepsilon^3), \tag{3.30}$$

and as long as

$$\left(\delta(\varepsilon) + \varepsilon\right) B_\varepsilon[\tilde{v}] = o(\varepsilon^7). \tag{3.31}$$

By Lemma 4.1 in [6], if $\tilde{v} \in C^2[0,1]$ with $\tilde{v}(0) = \tilde{v}(1) = 0$, then the following estimates hold true

$$\|\tilde{v}\|_{L^\infty}^2 \le B_\varepsilon[\tilde{v}], \tag{3.32}$$

$$\|\tilde{v}_x\|_{L^\infty}^2 \le \frac{1+\varepsilon}{\varepsilon}\, B_\varepsilon[\tilde{v}]. \tag{3.33}$$

We prove now the next Main Theorem estimating the dynamics of the layers, in the current mass conservative case.

Theorem 3.1. *There exist $\rho_2 > 0$, and constant $C > 0$, such that, as long as $h \in \Omega_\rho$ with $\rho < \rho_2$ and the orbit $\tilde{u}(x,t) = \tilde{u}^{h(t)}(x) + \tilde{v}(x,t)$ of* (IACH) *remains close to $\mathcal{M}$ so that* (3.29) *holds, the next bound is valid*

$$\begin{aligned} |\dot{\xi}_i| \le & C\delta(\varepsilon)\Big(\varepsilon^{-2}\,\alpha(r) + \varepsilon^{-5}\,\beta(r)\,B_\varepsilon^{1/2}[\tilde{v}] + \varepsilon^{-2}\,B_\varepsilon[\tilde{v}]\Big) \\ & + C\mu(\varepsilon)\Big(\alpha(r) + \varepsilon^{-1} B_\varepsilon^{1/2}[\tilde{v}] + \varepsilon^{-2} B_\varepsilon[\tilde{v}]\Big). \end{aligned} \tag{3.34}$$

Proof. The first summand in the RHS of (3.25) is estimated in Bates-Xun [6, (78)-(80)]. In particular, it holds that

$$\left\langle \mathrm{A}_{1,\varepsilon}(\tilde{u}^\xi) + L_{1,\varepsilon}^h(\tilde{v}) + \left(f^\xi\,\tilde{v}_x^2\right)_x,\ E_j \right\rangle \le C\left(\varepsilon^{-1}\,\alpha(r) + \varepsilon^{-4}\,\beta(r)\,B_\varepsilon^{1/2}[\tilde{v}] + \varepsilon^{-1}\,B_\varepsilon[\tilde{v}]\right). \tag{3.35}$$

Let us estimate the AC originated part

$$\underbrace{\left\langle \mathrm{A}_{2,\varepsilon}(\tilde{u}^\xi)\ ,\ E_j \right\rangle}_{T_1} + \underbrace{\left\langle L_{2,\varepsilon}^h(\tilde{v})\ ,\ E_j \right\rangle}_{T_2} + \underbrace{\left\langle \int_0^x f^\xi\,\tilde{v}_x^2\,dy\ ,\ E_j \right\rangle}_{T_3}. \tag{3.36}$$

We begin with the term T_1. First notice that $E_j = \mathcal{O}(1)$ (cf. [6, (55)]), and therefore

$$\left|\left\langle \mathrm{A}_{2,\varepsilon}(\tilde{u}^\xi)\ ,\ E_j \right\rangle\right| \le C \int_0^1 \mathrm{A}_{2,\varepsilon}(\tilde{u}^\xi)\,dy, \tag{3.37}$$

where we recall that

$$\begin{aligned} \mathrm{A}_{2,\varepsilon}(\tilde{u}^\xi) &:= \varepsilon^2 \tilde{u}_{xx}^\xi - \mathcal{W}(\tilde{u}_x^\xi) = \varepsilon^2 u_x^\xi - \mathcal{W}(u^\xi) \\ &= \int_0^x \mathscr{L}^b(u^h)\,dy, \end{aligned} \tag{3.38}$$

and that $\mathcal{W}$ is defined as

$$\mathcal{W}(u^h)(x) := \int_0^x W'\big(u^h(y)\big)\,dy,$$

while $\mathscr{L}^b$ denotes the bistable operator given in (2.17).

Combining (2.16), (4.43), (3.37), (3.38), we get

$$|T_1| \le C\,\varepsilon\,\alpha(r). \tag{3.39}$$

Considering the term T_2 we have, (for $L_{2,\varepsilon}(\tilde{v})$, see (3.17))

$$\begin{aligned}
T_2 &:= \big\langle L_{2,\varepsilon}(\tilde{v})\ ,\ E_j\big\rangle \\
&= \big\langle \varepsilon^2\tilde{v}_{xx} - L^h_W(\tilde{v}_x)\ ,\ E_j\big\rangle \\
&= \varepsilon^2\big\langle \tilde{v}\ ,\ (E_j)_{xx}\big\rangle - \big\langle L^h_W(\tilde{v}_x)\ ,\ E_j\big\rangle && (3.40\text{a})\\
&= \varepsilon^2\Big\langle \tilde{v}\ ,\ \frac{\partial}{\partial h_j}\tilde{u}^h_{xx}\Big\rangle + \varepsilon^2\Big\langle \tilde{v}\ ,\ \frac{\partial}{\partial h_{j+1}}\tilde{u}^h_{xx}\Big\rangle - \varepsilon^2\big\langle \tilde{v}\ ,\ (Q_j)_{xx}\big\rangle - \big\langle L^h_W(\tilde{v}_x)\ ,\ E_j\big\rangle && (3.40\text{b})\\
&= \underbrace{\varepsilon^2\Big\langle \tilde{v}\ ,\ \frac{\partial}{\partial h_j}u^h_x\Big\rangle}_{T_{2,1}} + \underbrace{\varepsilon^2\Big\langle \tilde{v}\ ,\ \frac{\partial}{\partial h_{j+1}}u^h_x\Big\rangle}_{T_{2,2}} - \underbrace{\varepsilon^2\big\langle \tilde{v}\ ,\ (Q_j)_{xx}\big\rangle}_{T_{2,3}} - \underbrace{\big\langle L^h_W(\tilde{v}_x)\ ,\ E_j\big\rangle}_{T_{2,4}}. && (3.40\text{c})
\end{aligned}$$

In (3.40a) we used the Dirichlet boundary conditions for $E_j, \tilde{v}$ given in (3.11) and (3.16) respectively. In (3.40b) we took into account that u^h is a smooth function, so, we interchanged ∂_{h_j} with ∂_{xx} after applying the definition of E_j by (3.13)-(3.15), and then, in (3.40c) we substituted

$$\tilde{u}^h_x = u^h. \tag{3.41}$$

Let us proceed with the term $T_{2,1}$. In order to apply ∂_{h_j} into u^h_x given in (2.13), we notice first that

$$\chi^j = \chi\big(\frac{x-h_j}{\varepsilon}\big), \qquad m_j = \frac{h_{j-1}+h_j}{2}.$$

Moreover, considering

$$\begin{aligned}
\phi^j(x) &:= \phi\big(x-m_j,\, h_j-h_{j-1},\, (-1)^j\big)\\
&= \phi\big(x-\tfrac{h_{j-1}+h_j}{2},\, h_j-h_{j-1},\, (-1)^j\big), \qquad \text{for} \quad x\in[h_{j-1},\, h_j],
\end{aligned}$$

we use (4.10) to get

$$\begin{aligned}
\frac{\partial}{\partial h_j}\phi^j &= \phi^j_x\,\frac{\partial}{\partial h_j}\Big(\tfrac{h_{j-1}+h_j}{2}\Big) + \phi^j_\ell\,\frac{\partial}{\partial h_j}\big(h_j-h_{j-1}\big)\\
&= -\frac{1}{2}\phi^j_x - \frac{1}{2}\,\mathrm{sgn}(x-m_j)\,\phi^j_x + \mathrm{w}^j\\
&= -\phi^j_x + \mathrm{w}^j \qquad \text{in} \quad I_j := [m_j,\, m_{j+1}],
\end{aligned}$$

and similarly

$$\frac{\partial}{\partial h_j}\phi^{j+1} = -\phi^{j+1}_x - \mathrm{w}^{j+1} \qquad \text{in } I_j, \tag{3.42}$$

with

$$\mathrm{w}^j(x,h_{j-1},h_j) := \mathrm{w}\big(x-m_j, h_j-h_{j-1}, (-1)^j\big).$$

Therefore, we obtain

$$\frac{\partial}{\partial h_j}\big(\phi^{j+1}-\phi^j\big) \;=\; \phi^j_x \;-\; \phi^{j+1}_x \;-\; \mathrm{w}^j \;-\; \mathrm{w}^{j+1}, \tag{3.43}$$

and

$$\frac{\partial}{\partial h_j}\big(\phi^{j+1}_x-\phi^j_x\big) \quad = \quad \frac{\partial}{\partial x}\frac{\partial}{\partial h_j}\big(\phi^{j+1}-\phi^j\big) \tag{3.44a}$$

$$\overset{(3.43)}{=} \quad \frac{\partial}{\partial x}\big(\phi^j_x \;-\; \phi^{j+1}_x \;-\; \mathrm{w}^j \;-\; \mathrm{w}^{j+1}\big) \tag{3.44b}$$

$$= \quad \phi^j_{xx} \;-\; \phi^{j+1}_{xx} \;-\; \mathrm{w}^j_x \;-\; \mathrm{w}^{j+1}_x. \tag{3.44c}$$

We now apply apply ∂_{h_j} to u^h_x given in (2.13), then we use (3.42)-(3.44c), and noticing that

$$\chi^j_x = -\chi^j_{h_j},$$

we get
(3.45)

$$\frac{\partial}{\partial h_j}u^h_x \;=\; \begin{cases} -\phi^j_{xx} \;+\; \mathrm{w}^j_x, & \text{for} \quad m_j\le x\le h_j-\varepsilon, \\ -\chi^j_{xx}\,\big(\phi^{j+1}-\phi^j\big) \;+\; \chi^j_x\big(\phi^j_x \;-\; \phi^{j+1}_x \;-\; \mathrm{w}^j \;-\; \mathrm{w}^{j+1}\big) - \chi^j_x\,\big(\phi^{j+1}_x-\phi^j_x\big) & \\ +\,\chi^j\big(\phi^j_{xx} \;-\; \phi^{j+1}_{xx} \;-\; \mathrm{w}^j_x \;-\; \mathrm{w}^{j+1}_x\big) \;-\; \phi^j_{xx} \;+\; \mathrm{w}^j_x, & \text{for} \quad |x-h_j|<\varepsilon, \\ -\phi^{j+1}_{xx} \;-\; \mathrm{w}^{j+1}_x, & \text{for} \quad h_j+\varepsilon\le x\le m_{j+1}. \end{cases}$$

By (4.15), (4.24), (4.30), (4.31), (3.45), we derive

$$|T_{2,1}| < \varepsilon^2\|\tilde v\|_{L^1}\,\|\partial_{h_j}u^h_x\|_{L^\infty} \le C\,B^{1/2}_\varepsilon[\tilde v]. \tag{3.46}$$

We may see that a similar estimate holds true for the term $|T_{2,2}|$.

The term $|T_{2,3}|$ turns out to be dominated by the other terms (cf. [6, (54)]), and

$$|T_{2,4}| = |\big\langle L^h_W(\tilde v_x)\;,\;E_j\big\rangle| \le \|L^h_W(\tilde v_x)\|_{L^1}\;\|E_j\|_{L^\infty} \le C\|\tilde v_x\|_{L^1} \le C\,B^{1/2}_\varepsilon[\tilde v]. \tag{3.47}$$

Moreover, we have

$$|T_3| \le C\varepsilon^{-1}B_\varepsilon[\tilde v]. \tag{3.48}$$

Combining (3.25), (3.35), (3.39), (3.46), (3.47), and (3.48), and taking into account the fact that the matrix $\varepsilon(a_{ij})^{-1}$ is uniformly bounded as $\varepsilon\to 0$, as shown in [6, p. 448], we derive (3.34). □

3.4. **The slow channel.** For $\tilde v\in C^2([0,1])$ with $\tilde v=\tilde v_{xx}=0$ at $x=0,1$, we define the form

$$\mathcal{A}_\varepsilon[\tilde v] \quad := \quad -\big\langle L^h_{1,\varepsilon}(\tilde v)\,,\;\tilde v\big\rangle \tag{3.49a}$$

$$= \quad \int_0^1 \big[\varepsilon^2\tilde v^2_{xx} + W''(u^h)\tilde v^2_x\big]\,dx \tag{3.49b}$$

where we performed integration by parts and recall that $L^h_{1,\varepsilon}$ stands for the linearized Cahn-Hilliard operator $\mathrm{A}_{1,\varepsilon}$ at u^h (see (3.9), (3.18)), and is given by

$$L^h_{1,\varepsilon}(\tilde{v}) = -\varepsilon^2 \tilde{v}_{xxxx} + \big(W''(u^h)\,\tilde{v}_x\big)_x,$$

associated with the BVP for the integrated Cahn-Hilliard equation

$$\text{(ICH)} \qquad \begin{cases} \tilde{u}_t = -\varepsilon^2 \tilde{u}_{xxxx} + \big(W'(\tilde{u}_x)\big)_x, & 0 < x < 1, \\ \tilde{u}(0,\,t) = 0, \qquad \tilde{u}(1,\,t) \ = M, & \\ \tilde{u}_{xx}(0,\,t) = \tilde{u}_{xx}(1,\,t) \ = 0. & \end{cases}$$

The definition of $\mathcal{A}_\varepsilon$ is motivated by Lemma 4.2 of [6]. There, this Lemma, combined with the estimates on the growth of $|\xi_i|$ in terms of B_ε (see [6, (84)]), together with the estimates on the growth of $\mathcal{A}_\varepsilon$ (see [6, (96)-(98)]) obtained by the equations of motion, led to the characterization of the "slow channel":

$$\Gamma := \Big\{\tilde{u}(x) : \ \tilde{u} = \tilde{u}^\xi + \tilde{v}, \ \ \mathcal{A}_\varepsilon[\tilde{v}] \le c\varepsilon^{-5}\alpha^2(r)\Big\}, \tag{3.50}$$

for the solutions of the integrated Cahn-Hilliard near N-layered equillibria. This, stands as a special case of our problem, for $\delta := 1$, $\mu(\varepsilon) := 0$.

In particular, according to [6, Lemma 4.2], there is a $\rho_0 > 0$ such that if $0 < \rho < \rho_0$ and $h \in \Omega_\rho$, then for any $\tilde{v} \in C^2$ with $\tilde{v} = 0$ at $x = 0, 1$ and $\langle \tilde{v}, E_j\rangle = 0$, $j = 1, \cdots, N-1$, there exists a constant C independent of ε and $\tilde{v}$ such that

$$\varepsilon^2 B_\varepsilon[\tilde{v}] \le C\mathcal{A}_\varepsilon[\tilde{v}]. \tag{3.51}$$

Let us point out that the forms $\mathcal{A}_\varepsilon[\tilde{v}], B_\varepsilon[\tilde{v}]$ as defined here in §3 are the forms associated with the 4^{th} order Cahn-Hilliard operator and they are defined by Bates-Xun [6, (76)].

Considering our problem, we define the slow channel for (IACH) by (cf. (3.50))

$$\Gamma_\rho := \Big\{\tilde{u}(x) : \ \tilde{u} = \tilde{u}^\xi + \tilde{v}, \ \ \mathcal{A}_\varepsilon[\tilde{v}] \le c\,\gamma(\varepsilon)\,\alpha^2(r)\Big\}, \tag{3.52}$$

with

$$\gamma(\varepsilon) := \mu^2(\varepsilon)\delta^{-1}(\varepsilon)\varepsilon^{-1} \ + \ \delta(\varepsilon)\varepsilon^{-5} \ + \ \varepsilon^{-2} \ + \ \mu^2(\varepsilon)\,\delta^{-2}(\varepsilon)\,\varepsilon. \tag{3.53}$$

It is clear that $\gamma(\varepsilon) \gg 1$, and in view of (3.30),

$$\gamma(\varepsilon) \ = \ \mathcal{O}\big(\delta(\varepsilon)\varepsilon^{-5} \ + \ \varepsilon^{-2}\big). \tag{3.54}$$

The next Main Theorem establishes attractiveness, and the slow evolution of states within the channel (3.52); cf. [6, Theorem B] for an analogous result in the Cahn-Hilliard case.

Theorem 3.2. *Let $\tilde{u}(x,t) = \tilde{u}^{\xi(t)}(x) + \tilde{v}(x,t)$ be an orbit of* (IACH) *starting outside but near the slow channel Γ_ρ in the sense that $\tilde{v}(\cdot, 0)$ satisfies condition* (3.29)*. Then $B_\varepsilon[\tilde{v}]$ will decrease exponentially until $\tilde{u}$ enters the channel and will remain in the channel following the approximate manifold $\mathcal{M}$ with speed $\mathcal{O}(e^{-c/r})$, thus staying in the channel for an exponentially long time. It can leave Γ_ρ only through the ends of the channel i.e at a time that $(h_j - h_{j-1})$ is reduced to $\frac{\varepsilon}{\rho}$ for some j.*

Proof. Applying (3.51) into (3.34) we immediately get

$$
\begin{aligned}
|\dot{\xi}_i| \le & C\delta(\varepsilon)\Big(\varepsilon^{-2}\alpha(r) + \varepsilon^{-6}\beta(r)\mathcal{A}_\varepsilon^{1/2}[\tilde{v}] + \varepsilon^{-4}\mathcal{A}_\varepsilon[\tilde{v}]\Big) \\
& + C\mu(\varepsilon)\Big(\alpha(r) + \varepsilon^{-2}\mathcal{A}_\varepsilon^{1/2}[\tilde{v}] + \varepsilon^{-4}\mathcal{A}_\varepsilon[\tilde{v}]\Big).
\end{aligned} \tag{3.55}
$$

In view of (3.55), our aim is to establish estimates on the growth of $\mathcal{A}_\varepsilon[\tilde{v}(\cdot,t)]$.

Let us set

$$
\begin{aligned}
\mathcal{I}_\varepsilon[\tilde{v}] &:= \frac{1}{2}\frac{d}{dt}\mathcal{A}_\varepsilon[\tilde{v}] \\
&= \frac{1}{2}\frac{d}{dt}\big\langle -L^h_{1,\varepsilon}(\tilde{v}),\, \tilde{v}\big\rangle \\
&= \Big\langle -\frac{1}{2}\frac{\partial}{\partial t}L^h_{1,\varepsilon}(\tilde{v}),\, \tilde{v}\Big\rangle - \frac{1}{2}\big\langle L^h_{1,\varepsilon}(\tilde{v}),\, \tilde{v}_t\big\rangle.
\end{aligned} \tag{3.56}
$$

In order to write $\mathcal{I}_\varepsilon[\tilde{v}]$ in a more convenient form, we first observe that

$$
\frac{\partial}{\partial t}L^h_{1,\varepsilon}(\tilde{v}) = L^h_{1,\varepsilon}(\tilde{v}_t) + \Big(\big(W''(u^\xi)\big)_t\,\tilde{v}_x\Big)_x. \tag{3.57}
$$

Moreover, using integrations by parts (i.e. symmetry of the integrated linearized CH operator $L^h_{1,\varepsilon}$), we obtain

$$
\big\langle L^h_{1,\varepsilon}(\tilde{v}_t),\, \tilde{v}\big\rangle = \big\langle \tilde{v}_t,\, L^h_{1,\varepsilon}(\tilde{v})\big\rangle, \tag{3.58}
$$

where the boundary terms vanish due to the zero boundary values of $\tilde{v}, \tilde{v}_{xx}$.

Therefore, by (3.56), (3.57), (3.58) we get

$$
\frac{1}{2}\frac{d}{dt}\mathcal{A}_\varepsilon[\tilde{v}] = -\big\langle L^h_{1,\varepsilon}(\tilde{v}), \tilde{v}_t\big\rangle - \frac{1}{2}\Big\langle \Big(\big(W''(u^\xi)\big)_t\,\tilde{v}_x\Big)_x,\, \tilde{v}\Big\rangle. \tag{3.59}
$$

Regarding the second term in (3.59), we integrate by parts and use Lemma 2.7, to derive as in [6, (93)]

$$
\begin{aligned}
\Big|\Big\langle \big(\big(W''(u^\xi)\big)_t\,\tilde{v}_x\big)_x,\, \tilde{v}\Big\rangle\Big| &= \Big|\Big\langle \big(W''(u^\xi)\big)_t\,\tilde{v}_x,\, \tilde{v}_x\Big\rangle\Big| \\
&\le \|\tilde{v}_x\|^2_{L^\infty}\,\|W'''(u^\xi)\|_{L^\infty}\sum_{j=1}^{N-1}\|u^\xi_j\|_{L^1}\,|\dot{\xi}_j| \\
&\le C\,\varepsilon^{-1}\,B_\varepsilon[\tilde{v}]\sum_{j=1}^{N-1}|\dot{\xi}_j| \\
&\le C\,\varepsilon^{-1}\big(B^2_\varepsilon[\tilde{v}] + \max_j|\dot{\xi}_j|^2\big)
\end{aligned} \tag{3.60}
$$

where we used Propositions 4.4, 4.5, 4.6, for the boundedness of the L^1-norm of u^ξ_j.

We next have to estimate the first term in (3.59).

Recall the equation of motion (3.27)

$$
\tilde{v}_t = \delta(\varepsilon)\Big[\mathrm{A}_{1,\varepsilon}(\tilde{u}^\xi) + L^h_{1,\varepsilon}(\tilde{v}) + \big(f^\xi\,\tilde{v}_x^2\big)_x\Big] + \mu(\varepsilon)\Big[\mathrm{A}_{2,\varepsilon}(\tilde{u}^\xi) + L^h_{2,\varepsilon}(\tilde{v}) + \int_0^x f^\xi\,\tilde{v}_x^2\,dy\Big] - \sum_{j=1}^{N-1}\tilde{u}^\xi_j\,\dot{\xi}_j,
$$

with

$$f^{\xi}(x) := \int_0^1 (1-\tau)\, W'''(\tilde{u}^{\xi}_x + \tau \tilde{v}_x)\, d\tau,$$

given in (3.24).

We may write the first term in (3.59) as follows,

$$-\big\langle L^h_{1,\varepsilon}(\tilde{v})\,,\, \tilde{v}_t \big\rangle = -\delta(\varepsilon)\|L^h_{1,\varepsilon}(\tilde{v})\|^2 \;-\; \overbrace{\Big\langle L^h_{1,\varepsilon}(\tilde{v})\,,\, \delta(\varepsilon)\big[\mathrm{A}_{1,\varepsilon}(\tilde{u}^{\xi}) \,+\, \big(f^{\xi}\,\tilde{v}_x^2\big)_x\big] \,-\, \sum_{j=1}^{N-1} \tilde{u}^{\xi}_j\, \dot{\xi}_j \Big\rangle}^{=:I_{0,\varepsilon}[\tilde{v}],\text{ can be estimated in terms of } B_\varepsilon[\tilde{v}],\, \|L^h_{1,\varepsilon}(\tilde{v})\| \text{ by [6, (88)-(90)]}}$$

$$-\,\mu(\varepsilon)\,\underbrace{\Big\langle L^h_{1,\varepsilon}(\tilde{v})\,,\, \mathrm{A}_{2,\varepsilon}(\tilde{u}^{\xi})\Big\rangle}_{I_{1,\varepsilon}[\tilde{v}]} - \mu(\varepsilon)\,\underbrace{\Big\langle L^h_{1,\varepsilon}(\tilde{v})\,,\, L^h_{2,\varepsilon}(\tilde{v})\Big\rangle}_{I_{2,\varepsilon}[\tilde{v}]} - \mu(\varepsilon)\,\underbrace{\Big\langle L^h_{1,\varepsilon}(\tilde{v})\,,\, \int_0^x f^{\xi}\,\tilde{v}_x^2\, dy\Big\rangle}_{I_{3,\varepsilon}[\tilde{v}]}$$

$$= -\delta(\varepsilon)\, \|L^h_{1,\varepsilon}(\tilde{v})\|^2 \,-\, I_{0,\varepsilon}[\tilde{v}] \,-\, \mu(\varepsilon)\, I_{1,\varepsilon}[\tilde{v}] \,-\, \mu(\varepsilon)\, I_{2,\varepsilon}[\tilde{v}] \,-\, \mu(\varepsilon)\, I_{3,\varepsilon}[\tilde{v}]. \tag{3.61}$$

Arguing as in [6, (88)-(90)] and applying [6, (101)], the term $I_{0,\varepsilon}[\tilde{v}]$ is estimated by

$$\Big|I_{0,\varepsilon}[\tilde{v}]\Big| \le \frac{\delta(\varepsilon)}{4}\|L^h_{1,\varepsilon}(\tilde{v})\|^2 \,+\, C\left(\varepsilon^{-1}\max_j |\dot{\xi}_j|^2 \,+\, \delta(\varepsilon)\big(\varepsilon^{-2}\alpha^2(r) \,+\, \varepsilon^{-4} B^2_\varepsilon[\tilde{v}]\big)\right). \tag{3.62}$$

Let us estimate the terms $I_{1,\varepsilon}[\tilde{v}], I_{2,\varepsilon}[\tilde{v}], I_{3,\varepsilon}[\tilde{v}]$ which are the ones coming from the AC part. Regarding the term $I_{1,\varepsilon}[\tilde{v}]$ in (3.61), we have the estimate

$$\begin{aligned} I_{1,\varepsilon}[\tilde{v}] &= \Big\langle L^h_{1,\varepsilon}(\tilde{v})\,,\, \mathrm{A}_{2,\varepsilon}(\tilde{u}^{\xi})\Big\rangle \\ &\le \|L^h_{1,\varepsilon}(\tilde{v})\|\, \|\mathrm{A}_{2,\varepsilon}(\tilde{u}^{\xi})\| \\ &\le \frac{\delta(\varepsilon)}{12\,\mu(\varepsilon)}\|L^h_{1,\varepsilon}(\tilde{v})\|^2 \,+\, 3\,\delta^{-1}(\varepsilon)\,\mu(\varepsilon)\, \|\mathrm{A}_{2,\varepsilon}(\tilde{u}^{\xi})\|^2 \\ &\le \frac{\delta(\varepsilon)}{12\,\mu(\varepsilon)}\|L^h_{1,\varepsilon}(\tilde{v})\|^2 \,+\, C\,\delta^{-1}(\varepsilon)\,\mu(\varepsilon)\,\varepsilon\,\alpha^2(r). \end{aligned} \tag{3.63}$$

In the last inequality we used (2.16), (4.43), and (3.38).

As for the term $I_{2,\varepsilon}[\tilde{v}]$ in (3.61), we first easily get

$$\begin{aligned} I_{2,\varepsilon}[\tilde{v}] &= \Big\langle L^h_{1,\varepsilon}(\tilde{v})\,,\, L^h_{2,\varepsilon}(\tilde{v})\Big\rangle \\ &\le \|L^h_{1,\varepsilon}(\tilde{v})\|\, \|L^h_{2,\varepsilon}(\tilde{v})\| \\ &\le \frac{\delta(\varepsilon)}{12\,\mu(\varepsilon)}\|L^h_{1,\varepsilon}(\tilde{v})\|^2 \,+\, 3\,\delta^{-1}(\varepsilon)\,\mu(\varepsilon)\, \|L^h_{2,\varepsilon}(\tilde{v})\|^2, \end{aligned} \tag{3.64}$$

and then, as $u^h = \mathcal{O}(1)$, we have $|W''(u^h)| = \mathcal{O}(1)$.

By the definitions (3.18), (3.19) of the linearized AC operator $L^h_{2,\varepsilon}$ together with (3.33), we get

$$\begin{aligned} [L^h_{2,\varepsilon}(\tilde{v})]^2 &:= \Big[\varepsilon^2 \tilde{v}_{xx} - \int_0^x W''\big(u^{h(t)}(y)\big)\, \tilde{v}_x(y,t)\, dy\Big]^2 \\ &\le 2\varepsilon^4 \tilde{v}^2_{xx} \,+\, C\|\tilde{v}_x\|^2_{L^\infty} \end{aligned}$$

$$\leq \quad 2\varepsilon^4 \tilde{v}_{xx}^2 \;+\; C\,\varepsilon^{-1}\,B_\varepsilon[\tilde{v}] \leq C\,\varepsilon^{-1}\,B_\varepsilon[\tilde{v}]. \tag{3.65}$$

So, (3.64) yields

$$I_{2,\varepsilon}[\tilde{v}] \leq \frac{\delta(\varepsilon)}{12\,\mu(\varepsilon)} \|L^h_{1,\varepsilon}(\tilde{v})\|^2 \;+\; C\,\delta^{-1}(\varepsilon)\,\mu(\varepsilon)\,\varepsilon^{-1}\,B_\varepsilon[\tilde{v}]. \tag{3.66}$$

For the term $I_{3,\varepsilon}[\tilde{v}]$ in (3.61), we have

$$\begin{aligned} I_{3,\varepsilon}[\tilde{v}] \quad &= \quad \Big\langle L^h_{1,\varepsilon}(\tilde{v})\,,\, \int_0^x f^\xi\,\tilde{v}_x^2\,dy \Big\rangle \\ &\leq \quad \|L^h_{1,\varepsilon}(\tilde{v})\|\;\Big\|\int_0^x f^\xi\,\tilde{v}_x^2\,dy\Big\| \\ &\leq \quad \frac{\delta(\varepsilon)}{12\,\mu(\varepsilon)}\|L^h_{1,\varepsilon}(\tilde{v})\|^2 \;+\; 3\,\delta^{-1}(\varepsilon)\,\mu(\varepsilon)\,\Big\|\int_0^x f^\xi\,\tilde{v}_x^2\,dy\Big\|^2, \end{aligned} \tag{3.67}$$

where we recall (3.24)

$$f^\xi(x) := \int_0^1 (1-\tau)\;W'''(\tilde{u}^\xi_x + \tau\tilde{v}_x)\;d\tau.$$

By (3.29), (3.33) and the fact that $u^h = \mathcal{O}(1)$, we have (cf. (2.95))

$$\tilde{u}^\xi_x + \tau\tilde{v}_x = u^h + \tau v = \mathcal{O}(1),$$

and thus the integrand $|W'''(\tilde{u}^\xi_x + \tau\tilde{v}_x)|$ in the definition (3.24) of f^ξ is uniformly bounded. So,

$$\Big\|\int_0^x f^\xi\,\tilde{v}_x^2\,dy\Big\| \leq C\,B_\varepsilon[\tilde{v}], \tag{3.68}$$

and therefore, (3.67) yields

$$I_{3,\varepsilon}[\tilde{v}] \leq \frac{\delta(\varepsilon)}{12\,\mu(\varepsilon)}\|L^h_{1,\varepsilon}(\tilde{v})\|^2 \;+\; C\,\delta^{-1}(\varepsilon)\,\mu(\varepsilon)\,B^2_\varepsilon[\tilde{v}]. \tag{3.69}$$

Gathering (3.59)-(3.63), (3.66), (3.69), we get

$$\begin{aligned} \frac{d}{dt}\mathcal{A}_\varepsilon[\tilde{v}] \;+\; \frac{\delta(\varepsilon)}{2}\|L^h_{1,\varepsilon}(\tilde{v})\|^2 \;\leq\; C\,\Big[&\varepsilon^{-1}\max_j|\dot{\xi}_j|^2 \;+\; \Big(\delta(\varepsilon)\,\varepsilon^{-2} \;+\; \mu^2(\varepsilon)\,\delta^{-1}(\varepsilon)\,\varepsilon\Big)\alpha^2(r) \\ &+\; \Big(\big(\varepsilon^{-1} + \delta(\varepsilon)\,\varepsilon^{-4} \;+\; \mu^2(\varepsilon)\,\delta^{-1}(\varepsilon)\big)\,B_\varepsilon[\tilde{v}] \;+\; \mu^2(\varepsilon)\,\delta^{-1}(\varepsilon)\,\varepsilon^{-1}\Big)B_\varepsilon[\tilde{v}]\Big]. \end{aligned} \tag{3.70}$$

In the above, we apply the estimate (3.34) for $\max_j|\dot{\xi}_j|$ into the first term in the RHS of (3.70) to get

$$\begin{aligned} \frac{d}{dt}\mathcal{A}_\varepsilon[\tilde{v}] \;+\; \frac{\delta(\varepsilon)}{2}\|L^h_{1,\varepsilon}(\tilde{v})\|^2 \;\leq\; C\,\Big[&\Big(\mu^2(\varepsilon)\varepsilon^{-1} \;+\; \delta^2(\varepsilon)\varepsilon^{-5} \;+\; \delta(\varepsilon)\,\varepsilon^{-2} \;+\; \mu^2(\varepsilon)\,\delta^{-1}(\varepsilon)\,\varepsilon\Big)\,\alpha^2(r) \\ &+\; \Big(\delta^2(\varepsilon)\,\varepsilon^{-11}\,\beta^2(r) \;+\; \mu^2(\varepsilon)\,\delta^{-1}(\varepsilon)\,\varepsilon^{-1} \;+\; \mu^2(\varepsilon)\,\varepsilon^{-3} \\ &+\; \big(\delta^2(\varepsilon)\,\varepsilon^{-5} + \mu(\varepsilon)^2\varepsilon^{-5} + \varepsilon^{-1} + \;\delta(\varepsilon)\,\varepsilon^{-4} \;+\; \mu^2(\varepsilon)\,\delta^{-1}(\varepsilon)\big)\,B_\varepsilon[\tilde{v}]\Big)B_\varepsilon[\tilde{v}]\Big]. \end{aligned} \tag{3.71}$$

Let us now note that by [6, Lemma 3.2] we have the spectral estimate

$$0<\Lambda\le\lambda_N\le\frac{\mathcal{A}_\varepsilon[\tilde v]}{\|\tilde v\|^2}, \tag{3.72}$$

where Λ is a constant independent of ε and ξ, and λ_N denotes the $N^{\rm th}$ eigenvalue of $L^h_{1,\varepsilon}$,

$$\text{(EVP)}\qquad \begin{cases} L^h_{1,\varepsilon}(\phi):=-\varepsilon^2\phi'''' + \big(W''(u^\xi)\phi'\big)' \;=\; \lambda(\varepsilon,\xi)\,\phi, & 0<x<1,\\ \phi(0)=\phi(1)=0, & \\ \phi''(0)=\phi''(1)=0. & \end{cases}$$

From (3.72) we obtain

$$\big|\mathcal{A}_\varepsilon[\tilde v]\big| := \;|-\langle L^h_{1,\varepsilon}(\tilde v)\,,\,\tilde v\rangle|\;\le\;\|L^h_{1,\varepsilon}(\tilde v)\|\,\|\tilde v\|\overset{(3.72)}{\le}\frac{1}{\sqrt{\Lambda}}\cdot\|L^h_{1,\varepsilon}(\tilde v)\|\cdot\big|\mathcal{A}_\varepsilon[\tilde v]\big|^{1/2},$$

therefore

$$\mathcal{A}_\varepsilon[\tilde v]\;\le\;\frac{1}{\Lambda}\,\|L^h_{1,\varepsilon}(\tilde v)\|^2. \tag{3.73}$$

Combining (3.51) with (3.73), we get

$$B_\varepsilon[\tilde v]\;\le\;\frac{1}{\Lambda}\,\varepsilon^{-2}\,\|L^h_{1,\varepsilon}(\tilde v)\|^2. \tag{3.74}$$

Applying (3.74) into (3.71) we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{A}_\varepsilon[\tilde v]\;+\;\frac{\delta(\varepsilon)}{2}\|L^h_{1,\varepsilon}(\tilde v)\|^2\;\le\;C\Big[&\Big(\mu^2(\varepsilon)\varepsilon^{-1}\;+\;\delta^2(\varepsilon)\varepsilon^{-5}\;+\;\delta(\varepsilon)\,\varepsilon^{-2}\;+\;\mu^2(\varepsilon)\,\delta^{-1}(\varepsilon)\,\varepsilon\Big)\,\alpha^2(r)\\ &+\;\Big(\delta^2(\varepsilon)\,\varepsilon^{-13}\,\beta^2(r)\;+\;\mu^2(\varepsilon)\,\delta^{-1}(\varepsilon)\,\varepsilon^{-3}\;+\;\mu^2(\varepsilon)\,\varepsilon^{-5}\\ &+\;\big(\delta^2(\varepsilon)\,\varepsilon^{-7}+\mu(\varepsilon)^2\varepsilon^{-7}+\varepsilon^{-3}+\;\delta(\varepsilon)\,\varepsilon^{-6}\;+\;\mu^2(\varepsilon)\,\delta^{-1}(\varepsilon)\,\varepsilon^{-2}\big)\,B_\varepsilon[\tilde v]\Big)\,\|L^h_{1,\varepsilon}(\tilde v)\|^2\Big]\end{aligned} \tag{3.75}$$

and taking into account (4.3), (3.29) and (3.73), we arrive at (cf. [6, (96)] for an analogous argument)

$$\frac{d}{dt}\mathcal{A}_\varepsilon[\tilde v(t)]+\frac{\Lambda\,\delta(\varepsilon)}{3}\mathcal{A}_\varepsilon[\tilde v(t)]\;\le\;C\,\Big(\mu^2(\varepsilon)\varepsilon^{-1}+\delta^2(\varepsilon)\varepsilon^{-5}+\delta(\varepsilon)\,\varepsilon^{-2}+\mu^2(\varepsilon)\,\delta^{-1}(\varepsilon)\,\varepsilon\Big)\,\alpha^2(r) \tag{3.76}$$

with the light abuse of notation $\mathcal{A}_\varepsilon[\tilde v(t)]$ in place of $\mathcal{A}_\varepsilon[\tilde v(\cdot,t)]$.

Integrating (3.76) we get

$$\begin{aligned}\mathcal{A}_\varepsilon[\tilde v(t)]\;&\le\;\mathcal{A}_\varepsilon[\tilde v(0)]\,e^{-C_\delta t}\;+\;C\,\gamma(\varepsilon)\,\alpha^2(r)\big(1-e^{-C_\delta t}\big)\\ &\le\;\max\big\{\mathcal{A}_\varepsilon[\tilde v(0)]\;,\;C\,\gamma(\varepsilon)\,\alpha^2(r)\big\},\end{aligned} \tag{3.77}$$

where $C_\delta:=\frac{\Lambda\,\delta(\varepsilon)}{3}$, C is a positive constant independent of $\varepsilon,\tilde v,$ and the coefficient $\gamma(\varepsilon)$ is given in (3.53). We see by (3.77) that the solution $\tilde u$ evolves exponentially towards the slow channel (3.52).

In view of (3.77), the estimate (3.55) yields

$$|\dot\xi_i|\le C\Big[\delta(\varepsilon)\,\varepsilon^{-6}\beta(r)\Big(\mathcal{A}^{1/2}_\varepsilon[\tilde v(0)]\;+\;\gamma^{1/2}(\varepsilon)\,\alpha(r)\Big)\;+\;\big(\delta(\varepsilon)\;+\;\mu(\varepsilon)\varepsilon^2\big)\varepsilon^{-2}\alpha(r)+\mu(\varepsilon)\,\varepsilon^{-4}\mathcal{A}_\varepsilon[\tilde v(0)] \tag{3.78}$$

$$+\,\delta(\varepsilon)\,\varepsilon^{-4}\Big(\mathcal{A}_\varepsilon[\tilde v(0)]\,+\,\gamma(\varepsilon)\,\alpha^2(r)\Big)\,+\,\mu(\varepsilon)\,\varepsilon^{-2}\Big(\mathcal{A}_\varepsilon^{1/2}[\tilde v(0)]\,+\,\gamma^{1/2}(\varepsilon)\,\alpha(r)\Big)\Big],$$

and in the slow channel (3.52) we have

$$\mathcal{A}_\varepsilon[\tilde v(0)] \le c\,\gamma(\varepsilon)\,\alpha^2(r), \tag{3.79}$$

so (3.78) on its turn gives

$$|\dot\xi_i| \le C\ \max\Big\{\,\big(\delta(\varepsilon)\,+\,\mu(\varepsilon)\varepsilon^2\big)\,\varepsilon^{-2}\,,\ \delta(\varepsilon)\varepsilon^{-4}\gamma(\varepsilon)\alpha(r)\,,\ \mu(\varepsilon)\varepsilon^{-2}\gamma^{1/2}(\varepsilon)\Big\}\ \alpha(r), \tag{3.80}$$

where $\alpha(r)$ is exponentially small in ε (see the detailed definition in Appendix and the estimate (4.2)).

So, provided that (3.29) is satisfied, and if

$$\big(\delta(\varepsilon)\,+\,\mu(\varepsilon)\varepsilon^2\big)\,\varepsilon^{-2}\ll\alpha^{-1}\,,\qquad \delta(\varepsilon)\varepsilon^{-4}\gamma(\varepsilon)\ll\alpha^{-2}\qquad\text{and}\qquad \mu(\varepsilon)\,\varepsilon^{-2}\,\gamma^{1/2}(\varepsilon)\ll\alpha^{-1}, \tag{3.81}$$

by (3.80) we have

$$|\dot\xi_i| = \mathcal{O}(e^{-c/r}),$$

and the solution $\tilde u$ stay in the channel for an exponentially long time.

Note that any $\delta(\varepsilon)$, $\mu(\varepsilon)$ of polynomial or negative polynomial order in ε satisfy (3.81). □

4. Appendix

In §4.2-§4.2 we prove various estimates for the non mass-conserving manifold approximation used throughout this paper, and collect together existing results thereof from the work of Carr and Pego, [11]. Some of the estimates have been also proven in [4] and then used in their integrated version for the mass-conserving case. Then in §4.3 we derive certain a priori energy estimates for establishing the well-posedness of the mass-conserving problem considered in §3.

4.1. **Estimates for the stationary Dirichlet problem (2.7).** As it is clear from the definition (2.10), many of our subsequent estimates involving u^h, rest upon certain properties of the stationary states ϕ of (1.1), namely the solutions of the Dirichlet problem (2.7). In this section we record these properties and for their proof we refer to [11].

Since $\phi_\varepsilon(0,\ell,\pm1)$ depends on ε and ℓ only through the ratio $\mathtt{r}=\varepsilon/\ell$, we may define

$$\alpha_\pm(\mathtt{r}) := F\big(\phi_\varepsilon(0,\ell,\pm1)\big),\qquad \beta_\pm(\mathtt{r}) := 1\mp\phi_\varepsilon(0,\ell,\pm1). \tag{4.1}$$

In what follows, C will denote a positive constant not necessarily the same at each occurrence and we stress that C is independent of ε, x, h_j's, j's.

Proposition 4.1 ([11, Proposition 3.4])**.** *There exists $r_0>0$ such that if $0<r<r_0$, then*

$$\alpha_\pm(r) \;=\; \frac{1}{2}K_\pm^2A_\pm^2\exp\Big(\frac{-A_\pm}{r}\Big)\left[1+\mathcal{O}\left(r^{-1}\exp\big(\frac{-A_\pm}{2r}\big)\right)\right], \tag{4.2}$$

$$\beta_\pm(r) \;=\; K_\pm\exp\Big(\frac{-A_\pm}{2r}\Big)\left[1+\mathcal{O}\left(r^{-1}\exp\big(\frac{-A_\pm}{2r}\big)\right)\right], \tag{4.3}$$

where

$$A_\pm \;:=\; f'(\pm1)>0, \tag{4.4}$$

$$K_\pm \;:=\; 2\,\exp\left[\int_0^1\Big(\frac{A}{\sqrt{2F(\pm t)}}-\frac{1}{1-t}\Big)\,dt\right], \tag{4.5}$$

with $A := \min\{A_+, A_-\}$, *and the asymptotic formulas* (4.2), (4.3) *also hold when they are differentiated a finite number of times, e.g.*

$$\alpha'_+(r) = -A_+ \, r^{-2} \, \alpha_+(r) \, \left[1 + \mathcal{O}\left(r^{-1} \exp(\frac{-A_+}{2r})\right)\right]. \tag{4.6}$$

Proposition 4.2 ([11, Lemma 7.4])**.** *Let* $0 < r < r_0$. *Then there exist constants* $C_1 \in (0,1)$ *and* $C_2 > 0$ *such that, for* $|x \pm \frac{\ell}{2}| \le \varepsilon$,

$$|\phi(x, \ell, \pm 1)| \le C_1, \tag{4.7a}$$

$$F(\phi(x, \ell, \pm 1)) \ge C_2. \tag{4.7b}$$

Proposition 4.3 ([11, Lemma 7.5])**.** *For* $|x| \le 2\varepsilon$, *we have*

$$|\phi(x, \ell, \pm 1) - (-1)^j| \le C\beta(r), \tag{4.8a}$$

$$|\phi_x(x, \ell, \pm 1)| \le C\varepsilon^{-1}\beta(r). \tag{4.8b}$$

Proposition 4.4 ([11, Lemma 7.7])**.** *We have*

$$\int_{-\ell/2}^{\ell/2} |\phi_x| \, dx \le 2, \qquad \int_{-\ell/2}^{\ell/2} |\phi_{xx}| \, dx \le C\varepsilon^{-1}, \qquad \int_{-\ell/2}^{\ell/2} |\phi_{xx}|^2 \, dx \le C\varepsilon^{-3}. \tag{4.9}$$

Beside the above estimates for ϕ and its derivatives with respect to x, we will also need estimates on the derivatives $\phi_\ell(x, \ell, \pm 1) := \frac{\partial}{\partial \ell}\phi(x, \ell, \pm 1)$.

Proposition 4.5 ([11, Lemma 7.8])**.** *For* $x \in [-\ell, \ell]$,

$$\phi_\ell\big(x, \ell, \pm 1\big) \; = \; -\frac{1}{2} \, \mathrm{sgn}(x) \, \phi_x\big(x, \ell, \pm 1\big) \; + \; \mathrm{w}\big(x, \ell, \pm 1\big), \tag{4.10}$$

where, for $x \neq 0$,

$$\mathrm{w}\big(x, \ell, \pm 1\big) \; = \; \varepsilon^{-1} \, \ell^{-2} \, \alpha'_\pm(r) \, \phi_x(|x|, \ell, \pm 1) \int_{\ell/2}^{|x|} \phi_x\big(s, \ell, \pm 1\big)^{-2} \, ds, \tag{4.11}$$

and

$$\mathrm{w}\big(0, \ell, \pm 1\big) \; = \; \frac{-\varepsilon^{-1} \, \ell^{-2} \, \alpha'_\pm(r)}{\phi_{xx}(0, \ell, \pm 1)}. \tag{4.12}$$

Proposition 4.6 ([11, Lemma 7.9])**.** *Let* w *be defined in Proposition 4.5. There exists* $r_0 > 0$ *such that if* $0 < r < r_0$, *then*

$$|\mathrm{w}\big(x, \ell, \pm 1\big)| \le C\varepsilon^{-1}\beta_\pm(r), \qquad \textit{for} \quad x \in \big[-\frac{\ell}{2} - \varepsilon, \, \frac{\ell}{2} + \varepsilon\big], \tag{4.13}$$

$$|\mathrm{w}\big(x, \ell, \pm 1\big)| \le C\varepsilon^{-1}\alpha_\pm(r), \qquad \textit{for} \quad \big|x \pm \frac{\ell}{2}\big| < \varepsilon. \tag{4.14}$$

Lemma 4.7 ([11, Lemma 7.10])**.** *For* $x \in [-\frac{\ell}{2} - \varepsilon, \, \frac{\ell}{2} + \varepsilon]$, $x \neq 0$,

$$\big|\mathrm{w}_x\big(x, \ell, \pm 1\big)\big| \; = \; \left|\phi_{\ell x}\big(x, \ell, \pm 1\big) + \frac{1}{2} \, \mathrm{sgn}(x) \, \phi_{xx}\big(x, \ell, \pm 1\big)\right| \; \le \; C \, \varepsilon^{-2} \, r^{-1} \, \beta_\pm(r). \tag{4.15}$$

One may show that w is C^2 on $[0, \ell]$ and satisfies (see [11, (7.19)])

$$\varepsilon^2 \mathrm{w}_{xx} = f'(\phi)\mathrm{w}, \tag{4.16}$$

which together with (4.13) yields

$$\big|\mathrm{w}_{xx}\big(x, \ell, \pm 1\big)\big| \le C \, \varepsilon^{-3} \, \beta_\pm(r). \tag{4.17}$$

4.2. **Estimates on the states u^h.** For $j = 1, 2, \dots, N+1$, and the ℓ_j that are given in (2.9), we set

$$r_j := \frac{\varepsilon}{\ell_j}, \tag{4.18}$$

and

$$\alpha^j := \begin{cases} \alpha_+(r_j), & \text{for } j \text{ even}, \\ \alpha_-(r_j), & \text{for } j \text{ odd}, \end{cases} \quad \text{and} \quad \beta^j := \begin{cases} \beta_+(r_j), & \text{for } j \text{ even}, \\ \beta_-(r_j), & \text{for } j \text{ odd}. \end{cases} \tag{4.19}$$

We also set

$$r := \max_{1\le j\le N+1} r_j = \frac{\varepsilon}{\min_j \ell_j}, \tag{4.20}$$

and

$$\alpha(r) := \max_{1\le j\le N+1} \alpha^j, \qquad \text{and} \qquad \beta(r) := \max_{1\le j\le N+1} \beta^j. \tag{4.21}$$

From the first estimate in (4.9) and the zero boundary values in (2.7) we deduce that $|\phi| \le 2$ on $[-\frac{\ell}{2}, \frac{\ell}{2}]$. Therefore, for each $j = 1, \dots, N+1$,

$$|\phi^j| \le 2 \qquad \text{on} \quad \left[m_j - \tfrac{\ell_j}{2},\, m_j + \tfrac{\ell_j}{2}\right], \tag{4.22}$$

and as a consequence of the definition (2.10), u^h is uniformly bounded on $[0,1]$, thus $f(u^h)$ and $f'(u^h)$ are uniformly bounded too.

Similarly, from the second estimate in (4.9) we get that

$$|\phi^j_x| \le C\varepsilon^{-1} \qquad \text{on} \quad \left[m_j - \tfrac{\ell_j}{2},\, m_j + \tfrac{\ell_j}{2}\right]. \tag{4.23}$$

By (2.15) and (4.22) we get

$$|\phi^j_{xx}| \le C\varepsilon^{-2} \qquad \text{on} \quad \left[m_j - \tfrac{\ell_j}{2},\, m_j + \tfrac{\ell_j}{2}\right], \tag{4.24}$$

and a differentiation of (2.15) together with (4.22), (4.23) yields

$$|\phi^j_{xxx}| \le C\varepsilon^{-3}, \qquad \text{on} \quad \left[m_j - \tfrac{\ell_j}{2},\, m_j + \tfrac{\ell_j}{2}\right], \tag{4.25}$$

and in general, we may see that, see also in [4]

$$\left|\partial^n_x \phi^j\right| \le C\varepsilon^{-n}, \qquad \text{on} \quad \left[m_j - \tfrac{\ell_j}{2},\, m_j + \tfrac{\ell_j}{2}\right]. \tag{4.26}$$

By Proposition 4.3, we have

$$|\phi^j(x) - (-1)^j| \;\le\; C\beta(r), \tag{4.27a}$$
$$|\phi^j_x(x)| \;\le\; C\varepsilon^{-1}\beta(r), \qquad \text{for} \quad |x - m_j| < 2\varepsilon. \tag{4.27b}$$

As a consequence of (4.27a), we have

$$|f(\phi^j)| \;=\; |f(\phi^j) - f((-1)^j)| \le C\beta(r), \qquad \text{for} \quad |x - m_j| < 2\varepsilon, \tag{4.28}$$

which on its turn together with (2.15) implies

$$|\phi^j_{xx}| \le C\varepsilon^{-2}\beta(r), \qquad \text{for} \quad |x - m_j| < 2\varepsilon. \tag{4.29}$$

Proposition 4.8 ([11, Lemma 8.2])**.** *Let $r_0 > 0$ be sufficiently small. There exist constants C_1, C_2 such that if we assume that $\varepsilon/\ell_j < r_0$ and $\varepsilon/\ell_{j+1} < r_0$ for $j \in \{1, 2, \dots, N\}$, then*

$$\left|\phi^j(x) - \phi^{j+1}(x)\right| \leq C_1 \left|a^j - a^{j+1}\right|, \tag{4.30a}$$

$$\left|\phi^j_x(x) - \phi^{j+1}_x(x)\right| \leq C_2\, \varepsilon^{-1} \left|a^j - a^{j+1}\right|, \qquad \textit{for} \quad |x - h_j| < \varepsilon. \tag{4.30b}$$

Moreover, by (2.15), mean value theorem and (4.30a) we have, with some θ_x between $\phi^j(x)$ and $\phi^{j+1}(x)$,

$$\begin{aligned} |\phi^j_{xx}(x) - \phi^{j+1}_{xx}(x)| &= \varepsilon^{-2} |f(\phi^j(x)) - f(\phi^{j+1}(x))| \\ &= \varepsilon^{-2} |f'(\theta_x)|\, |\phi^{j+1} - \phi^j| \\ &\leq C\varepsilon^{-2} \left|a^j - a^{j+1}\right|, \qquad \text{for} \quad |x - h_j| < \varepsilon. \end{aligned} \tag{4.31}$$

By differentiating (2.15), we may proceed recursively to get, for $n = 1, 2, 3, \cdots$,

$$|\partial^n_x \phi^j - \partial^n_x \phi^{j+1}| \leq C\varepsilon^{-n} \left|a^j - a^{j+1}\right|, \qquad \text{for} \quad |x - h_j| < \varepsilon. \tag{4.32}$$

Considering the smooth cut-off function χ^j let us notice that, for $n = 1, 2, 3, \cdots$,

$$\left|\frac{d^n}{dx^n} \chi^j\right| \leq C\varepsilon^{-n}. \tag{4.33}$$

By (2.13), (4.23) and (4.33) we have

$$|u^h_x| \leq C\varepsilon^{-1}, \qquad \text{on} \quad [0, 1], \tag{4.34}$$

and by (2.14), (4.24), (4.30), (4.33) we easily get

$$|u^h_{xx}| \leq C\varepsilon^{-2}, \qquad \text{on} \quad [0, 1]. \tag{4.35}$$

Differentiating (2.14) we immediately get

$$u^h_{xxx} = \begin{cases} \phi^j_{xxx} & , m_j \leq x \leq h_j - \varepsilon, \\ \chi^j_{xxx} \left(\phi^{j+1} - \phi^j\right) + 3\chi^j_{xx} \left(\phi^{j+1}_x - \phi^j_x\right) + 3\chi^j_x \left(\phi^{j+1}_{xx} - \phi^j_{xx}\right) & \\ \qquad + (1 - \chi^j)\phi^j_{xxx} + \chi^j \phi^{j+1}_{xxx} & , |x - h_j| < \varepsilon, \\ \phi^{j+1}_{xxx} & , h_j + \varepsilon \leq x \leq m_{j+1}. \end{cases} \tag{4.36}$$

By (4.25), (4.32), (4.33), (4.36), we easily obtain

$$|u^h_{xxx}| \leq C\varepsilon^{-3}, \qquad \text{on} \quad [0, 1]. \tag{4.37}$$

Also for a smooth function $\mathcal{F} = \mathcal{F}(s)$, $s \in [0, 1]$, it is straightforward to show that the remainder $R(\chi)$ of the linear Lagrange interpolation of $\mathcal{F}$ at $s = 0$ and $s = 1$,

$$R(x) := \mathcal{F}(x) - (1 - x)\mathcal{F}(0) - x\mathcal{F}(1), \qquad x \in [0, 1], \tag{4.38}$$

is given by

$$R(x) = (1 - x) \int_0^\chi s\mathcal{F}''(s)\, ds + x \int_x^1 (1 - s)\mathcal{F}''(s)\, ds. \tag{4.39}$$

We now use (2.13) and employ (4.38)-(4.39) for the function

$$\mathcal{F}(s) := f((1 - s)\phi^j + s\phi^{j+1}), \tag{4.40}$$

to get

$$\mathscr{L}^b(u^h) = \varepsilon^2\chi^j_{xx}\left(\phi^{j+1} - \phi^j\right) \;+\; 2\varepsilon^2\chi^j_x\left(\phi^{j+1}_x - \phi^j_x\right) \;+\; R^j, \qquad \text{for} \quad |x - h_j| < \varepsilon, \tag{4.41}$$

where

$$R^j = \left(\phi^{j+1} - \phi^j\right)^2\left[(1-\chi^j)\int_0^{\chi^j} sf''\big(\theta(s)\big)\,ds \;+\; \chi^j\int_{\chi^j}^1 (1-s)f''\big(\theta(s)\big)\,ds\right], \tag{4.42}$$

and $\theta(s) := (1-s)\phi^j + s\phi^{j+1}$.

We then combine (4.41)-(4.42) with (4.30), (4.33) to conclude that

$$|\mathscr{L}^b(u^h)| \le C\alpha(r), \qquad \text{for} \quad |x - h_j| < \varepsilon; \tag{4.43}$$

cf. [11, Theorem 3.5].

At this point let us recall (2.16), i.e. $\mathscr{L}^b(u^h) = 0$, for $|x-h_j| \ge \varepsilon$, which together with boundary values (2.19) and (4.43) show that u^h "almost" satisfy the steady-state problem (2.2).

Remark 4.9. To show that

$$u^h_j \sim -u^h_x, \qquad \text{as} \quad r \to 0, \qquad \text{uniformly on} \quad I_j := [m_j,\, m_{j+1}] \tag{4.44}$$

first notice that only ϕ^j and ϕ^{j+1} depend on h_j, so the support of u^h_j is contained in $[h_{j-1}-\varepsilon,\, h_{j+1}+\varepsilon]$.

Applying (4.10) for the translate ϕ^j of ϕ,

$$\phi^j(x) := \phi\big(x - \tfrac{h_{j-1}+h_j}{2},\, h_j - h_{j-1},\, (-1)^j\big), \qquad x \in [h_{j-1} - \varepsilon,\, h_j + \varepsilon],$$

we have, for $x \in [h_{j-1} - \varepsilon,\, h_j + \varepsilon]$,

$$\begin{aligned} \frac{\partial}{\partial h_j}\phi^j &= \phi^j_x\,\frac{\partial}{\partial h_j}\Big(x - \tfrac{h_{j-1}+h_j}{2}\Big) + \phi^j_\ell\,\frac{\partial}{\partial h_j}\big(h_j - h_{j-1}\big) \\ &= -\frac{1}{2}\phi^j_x - \frac{1}{2}\,\mathrm{sgn}(x - m_j)\,\phi^j_x + \mathrm{w}^j, \end{aligned} \tag{4.45}$$

therefore

$$\frac{\partial}{\partial h_j}\phi^j \;=\; -\phi^j_x \,+\, \mathrm{w}^j, \qquad \text{in} \quad I_j := [m_j,\, h_j + \varepsilon], \tag{4.46}$$

since $\mathrm{sgn}(x - m_j) > 0$.

Similarly, we obtain

$$\frac{\partial}{\partial h_j}\phi^{j+1} \;=\; -\phi^{j+1}_x \,-\, \mathrm{w}^{j+1}, \qquad \text{for} \quad x \in [h_j,\, h_{j+1}], \tag{4.47}$$

and thus

$$\frac{\partial}{\partial h_j}\big(\phi^{j+1} - \phi^j\big) \;=\; \phi^j_x \,-\, \phi^{j+1}_x \,-\, \mathrm{w}^j \,-\, \mathrm{w}^{j+1}, \qquad \text{in} \quad I_j. \tag{4.48}$$

So recalling that $u^h = \big(1-\chi^j\big)\,\phi^j + \chi^j\,\phi^{j+1}$, and noticing that $\chi^j_x = -\chi^j_{h_j}$, it is straightforward to see that

$$\begin{aligned}
\frac{\partial}{\partial h_j}u^h &= \chi^j_x\big(\phi^j - \phi^{j+1}\big) + \chi^j\frac{\partial}{\partial h_j}\big(\phi^{j+1}-\phi^j\big) + \frac{\partial}{\partial h_j}\phi^j\\
&\overset{(4.48)}{=\!=} \chi^j_x\big(\phi^j - \phi^{j+1}\big) + \chi^j\big[\phi^j_x - \phi^{j+1}_x - \mathrm{w}^j - \mathrm{w}^{j+1}\big] - \phi^j_x + \mathrm{w}^j\\
&= \chi^j_x\big(\phi^j - \phi^{j+1}\big) + \chi^j\big[\phi^j_x - \phi^{j+1}_x\big] - \chi^j\big[\mathrm{w}^j + \mathrm{w}^{j+1}\big] - \phi^j_x + \mathrm{w}^j\\
&\overset{(2.13)}{=\!=} -u^h_x + \big(1-\chi^j\big)\mathrm{w}^j - \chi^j\mathrm{w}^{j+1} \qquad \text{for} \quad |x-h_j|<\varepsilon. \qquad (4.49)
\end{aligned}$$

For the translate ϕ^{j+1} of ϕ,

$$\phi^{j+1}(x) := \phi\big(x - \tfrac{h_j+h_{j+1}}{2},\, h_{j+1}-h_j;\, (-1)^{j+1}\big) \qquad x\in[h_j-\varepsilon,\, h_{j+1}+\varepsilon]$$

we have, by (4.10),

$$\begin{aligned}
\frac{\partial}{\partial h_j}\phi^{j+1} &= \phi^{j+1}_x\frac{\partial}{\partial h_j}\Big(x-\tfrac{h_j+h_{j+1}}{2}\Big) + \phi^{j+1}_\ell\frac{\partial}{\partial h_j}\big(h_{j+1}-h_j\big)\\
&= -\frac{1}{2}\phi^{j+1}_x + \frac{1}{2}\,\mathrm{sgn}(x-m_{j+1})\,\phi^{j+1}_x - \mathrm{w}^{j+1} = -\,\mathrm{w}^{j+1}, \qquad \text{in} \quad [m_{j+1},\, h_{j+1}+\varepsilon], \qquad (4.50)
\end{aligned}$$

since $\mathrm{sgn}(x-m_{j+1})>0$.

Recall that

$$u^h = \big(1-\chi^{j+1}\big)\,\phi^{j+1} + \chi^{j+1}\,\phi^{j+2}, \qquad x\in[m_{j+1},\, m_{j+2}],$$

so using (4.50) and noticing that $\chi^{j+1}_{h_j} = 0 = \phi^{j+2}_{h_j}$, it is straightforward to see that

$$\frac{\partial}{\partial h_j}u^h = \big(1-\chi^{j+1}\big)\frac{\partial}{\partial h_j}\phi^{j+1} \overset{(4.50)}{=\!=} -\big(1-\chi^{j+1}\big)\mathrm{w}^{j+1}, \qquad x\in[m_{j+1},\, h_{j+1}+\varepsilon]. \qquad (4.51)$$

Analogously, taking into account that $u^h = \big(1-\chi^{j-1}\big)\,\phi^{j-1} + \chi^{j-1}\,\phi^j$ for $x\in[m_{j-1},\, m_j]$, using (4.45) and noticing that $\chi^{j-1}_{h_j} = 0 = \phi^{j-1}_{h_j}$, we obtain that

$$\frac{\partial}{\partial h_j}u^h = \chi^{j-1}\frac{\partial}{\partial h_j}\phi^j = \chi^{j-1}\mathrm{w}^j, \qquad x\in[h_{j-1}-\varepsilon,\, m_j]. \qquad (4.52)$$

Gathering (4.49), (4.51), (4.52) we have that

$$\frac{\partial}{\partial h_j}u^h = \begin{cases}
\chi^{j-1}\mathrm{w}^j, & \text{for} \quad h_{j-1}-\varepsilon \le x \le m_j,\\
-u^h_x + \mathrm{w}^j, & \text{for} \quad m_j \le x \le h_j-\varepsilon,\\
-u^h_x + \big(1-\chi^j\big)\mathrm{w}^j - \chi^j\mathrm{w}^{j+1}, & \text{for} \quad |x-h_j|<\varepsilon,\\
-u^h_x - \mathrm{w}^{j+1}, & \text{for} \quad h_j+\varepsilon \le x \le m_{j+1},\\
-\,(1-\chi^{j+1})\mathrm{w}^{j+1}, & \text{for} \quad m_{j+1} \le x \le h_{j+1}+\varepsilon.
\end{cases} \qquad (4.53)$$

Then (4.44) follows from (4.53) combined with (4.13)-(4.14). See also in [4], for some analogous results.

4.3. **A priori estimates for the problem (1.1)-(BC1)-(BC2)-(MC).** For the well-posedness of the initial and boundary value problem we may argue as in [15, §2]; next we derive the estimates needed in our case where we have replaced the b.c. at $x = 1$ with the mass conservation condition and added the Allen-Cahn lower order term in the pde.

Local in time existence may be proved by fixed-point theory, applying a Picard-type iteration scheme. In order to prove global existence, i.e. existence on $[0, T]$ for any $T > 0$, we need to derive certain a priori uniform estimates on u. To this aim, first notice that by (MC), (ACH) and (BC1)-(BC2) we have

$$\begin{aligned} 0 &= \frac{d}{dt}\int_0^1 u(x,t)\,dx = \int_0^1 u_t(x,t)\,dx \\ &= -\delta(\varepsilon)\int_0^1 \big(\varepsilon^2 u_{xx} - W'(u)\big)_{xx}\,dx + \mu(\varepsilon)\int_0^1 \big(\varepsilon^2 u_{xx} - W'(u)\big)\,dx \\ &= -\delta(\varepsilon)\varepsilon^2 u_{xxx}(1,t) - \mu(\varepsilon)\int_0^1 W'(u)\,dx, \end{aligned}$$

so we have

$$\mu(\varepsilon)\int_0^1 W'(u)\,dx = -\delta(\varepsilon)\varepsilon^2 u_{xxx}(1,t). \tag{4.54}$$

Also, as in the proof of Lemma 2.7, we can see that for differentiable v and any positive ϵ_1,

$$v^2(1,t) \le \|v\|_\infty^2 \le 2\epsilon_1\|v_x\|^2 + \frac{2}{\epsilon_1}\|v\|^2. \tag{4.55}$$

For the special case of

$$W(u) = \frac{1}{4}(u^2-1)^2, \qquad \text{thus} \qquad W'(u) = u^3 - u, \tag{4.56}$$

we see that,

$$W'(u) \le c_1 W(u) + c_2, \qquad \forall u \in \mathbb{R},$$

for some positive constants c_1, c_2 independent of u, and so

$$\int_0^1 |W'(u)|\,dx \le c_1\int_0^1 W(u)\,dx + c_2. \tag{4.57}$$

Growth estimate for the energy: We set

$$E(t) := \int_0^1 \frac{\varepsilon^2}{2}u_x^2 + W(u)\,dx, \tag{4.58}$$

and we have

$$\begin{aligned} \frac{d}{dt}E(t) &= \int_0^1 \varepsilon^2 u_x\,(u_t)_x + W'(u)\,u_t\,dx \\ &= -\int_0^1 \big(\varepsilon^2 u_{xx} - W'(u)\big)\,u_t\,dx, \end{aligned}$$

where we integrated by parts the first term and applied (BC1). Then, by the pde (ACH) and integrations by parts combined with (BC1)-(BC2) we get

$$\begin{aligned}\frac{d}{dt}E(t) &= \int_0^1 \big(\varepsilon^2 u_{xx} - W'(u)\big)\Big[\delta(\varepsilon)\big(\varepsilon^2 u_{xx} - W'(u)\big)_{xx} - \mu(\varepsilon)\big(\varepsilon^2 u_{xx} - W'(u)\big)\Big]\,dx\\
&= -\,\delta(\varepsilon)\int_0^1 \Big[\big(\varepsilon^2 u_{xx} - W'(u)\big)_x\Big]^2\,dx - \mu(\varepsilon)\int_0^1 \big[\varepsilon^2 u_{xx} - W'(u)\big]^2\,dx\\
&\quad + \delta(\varepsilon)\,\varepsilon^2 u_{xxx}(1,t)\Big[\varepsilon^2 u_{xx}(1,t) - W'\big(u(1,t)\big)\Big]\\
&\overset{(4.54)}{=} -\,\delta(\varepsilon)\,\big\|\big(\varepsilon^2 u_{xx} - W'(u)\big)_x\big\|^2 - \mu(\varepsilon)\,\big\|\varepsilon^2 u_{xx} - W'(u)\big\|^2\\
&\quad - \mu(\varepsilon)\Big[\varepsilon^2 u_{xx}(1,t) - W'\big(u(1,t)\big)\Big]\int_0^1 W'(u)\,dx\\
&\le -\,\delta(\varepsilon)\,\big\|\big(\varepsilon^2 u_{xx} - W'(u)\big)_x\big\|^2 - \mu(\varepsilon)\,\big\|\varepsilon^2 u_{xx} - W'(u)\big\|^2\\
&\quad + \frac{\mu(\varepsilon)}{4\epsilon}\Big[\varepsilon^2 u_{xx}(1,t) - W'\big(u(1,t)\big)\Big]^2 + \epsilon\,\mu(\varepsilon)\left(\int_0^1 W'(u)\,dx\right)^2. \qquad (4.59)\end{aligned}$$

By (BC1) we have

$$\int_0^1 W'(u)\,dx = \int_0^1 \big(W'(u) - \varepsilon^2 u_{xx}\big)\,dx \le \big\|W'(u) - \varepsilon^2 u_{xx}\big\|, \qquad (4.60)$$

and by (4.55) for $\upsilon = W'(u) - \varepsilon^2 u_{xx}$ therein, we have

$$\frac{\mu(\varepsilon)}{4\epsilon}\upsilon^2(1,t) \le \frac{\mu(\varepsilon)\,\epsilon_1}{2\epsilon}\|\upsilon_x\|^2 + \frac{\mu(\varepsilon)}{2\epsilon\,\epsilon_1}\|\upsilon\|^2, \qquad (4.61)$$

so, choosing ϵ, ϵ_1, so that

$$\frac{\mu(\varepsilon)\,\epsilon_1}{2\epsilon} \le \delta(\varepsilon) \qquad \text{and} \qquad 0 \le 1 - \epsilon - \frac{1}{2\epsilon\,\epsilon_1}, \qquad (4.62)$$

we substitute (4.60), (4.61) into (4.59) to get that

$$\frac{d}{dt}E(t) \le 0.$$

Therefore, $E(t) \le E(0)$ that is

$$\frac{\varepsilon^2}{2}\|u_x\|^2 + \int_0^1 W(u)\,dx \le E_0 := \int_0^1 \frac{\varepsilon^2}{2}(u_0)_x^2 + W\big(u_0\big)\,dx, \qquad (4.63)$$

with $u_0(x) := u(x,0)$, and so

$$\frac{\varepsilon^2}{2}\|u_x\|^2 \le E_0, \qquad (4.64)$$

and

$$\int_0^1 W(u)\,dx \le E_0. \qquad (4.65)$$

Furthermore, integrating (4.59) we get

$$\int_0^t \big\|\varepsilon^2 u_{xx} - W'(u)\big\|_{H^1}^2\,d\tau \le C, \qquad 0 \le t \le T, \qquad (4.66)$$

for some constant C depending on u_0, T and $\delta(\varepsilon), \mu(\varepsilon), \varepsilon$.

Remark: A trivial calculus shows that the weakest condition (4.62) for $\delta(\varepsilon), \mu(\varepsilon)$ is attained by choosing $\epsilon_1/\epsilon = 27/8$ for $\epsilon = 2/3$, and so $\frac{27}{16}\,\mu(\varepsilon) \le \delta(\varepsilon)$. Let us also emphasize that the condition $c\,\mu(\varepsilon) \le \delta(\varepsilon)$ for some $c > 0$, is weaker than the assumption (3.30) for establishing the slow evolution within the channel (3.52) (Theorem 3.2); e.g. take $\delta(\varepsilon), \mu(\varepsilon)$ such that $\varepsilon^{-3/2}\mu(\varepsilon) \ll \delta(\varepsilon) = \mathcal{O}(\varepsilon^8)$.

Growth estimate for $\|u\|^2$: Multiply (ACH) by u, then integrate with respect to x and apply (BC1)-(BC2) to get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u\|^2 \;+\; \delta(\varepsilon)\,\varepsilon^2\,\|u_{xx}\|^2 \;+\; \mu(\varepsilon)\,\varepsilon^2\,\|u_x\|^2 \\
&\quad = \; -\,\delta(\varepsilon)\,\varepsilon^2\,u_{xxx}(1,t)\,u(1,t) \;-\; \delta(\varepsilon)\int_0^1 \big(W'(u)\big)_x\,u_x\,dx \;-\; \mu(\varepsilon)\int_0^1 W'(u)\,u\,dx \\
&\quad = \; -\,\delta(\varepsilon)\,\varepsilon^2\,u_{xxx}(1,t)\,u(1,t) \;-\; \delta(\varepsilon)\int_0^1 W''(u)\,u_x^2\,dx \;-\; \mu(\varepsilon)\int_0^1 W'(u)\,u\,dx \\
&\quad \overset{W''\ge -1}{\le} \; -\,\delta(\varepsilon)\,\varepsilon^2\,u_{xxx}(1,t)\,u(1,t) \;+\; \delta(\varepsilon)\int_0^1 u_x^2\,dx \;-\; \mu(\varepsilon)\int_0^1 W'(u)\,u\,dx \\
&\quad \overset{(4.54)}{=} \; \delta(\varepsilon)\,\|u_x\|^2 \;+\; \mu(\varepsilon)\left(u(1,t)\int_0^1 W'(u)\,dx \;-\; \int_0^1 W'(u)\,u\,dx\right) \\
&\quad \le \delta(\varepsilon)\,\|u_x\|^2 \;+\; \mu(\varepsilon)\,\Big(\|u\|_\infty^2 \;+\; \big\|W'(u)\big\|_1^2\Big) \\
&\quad \overset{(4.55)}{\le} \; \delta(\varepsilon)\,\|u_x\|^2 \;+\; \mu(\varepsilon)\,\Big(2\,\|u_x\|^2 \;+\; 2\,\|u\|^2 \;+\; \big\|W'(u)\big\|_1^2\Big).
\end{aligned}
\tag{4.67}
$$

Regarding the term $\|W'(u)\|_1$ in (4.67), we combine (4.57) and (4.65) to see that

$$
\big\|W'(u)\big\|_1 \le c_1\,\|W(u)\|_1 + c_2 \le C. \tag{4.68}
$$

In view of (4.64) and (4.68), the estimate (4.67) yields

$$
\frac{1}{2}\frac{d}{dt}\|u\|^2 \;+\; \delta(\varepsilon)\,\varepsilon^2\,\|u_{xx}\|^2 \;+\; \mu(\varepsilon)\,\varepsilon^2\,\|u_x\|^2 \le C_1\|u\|^2 \;+\; C_2, \tag{4.69}
$$

the constants C_1, C_2 depending only on u_0 and $\delta(\varepsilon), \mu(\varepsilon)$.

In particular, (4.69) implies

$$
\frac{1}{2}\frac{d}{dt}\|u\|^2 \le C_1\|u\|^2 \;+\; C_2,
$$

and integrating this inequality we get

$$
\|u(\cdot,t)\|^2 \le e^{2C_1 t}\|u_0\|^2 + C_2(e^{2C_1 t} - 1)/C_1, \tag{4.70}
$$

and so

$$
\|u(\cdot,t)\|^2 \le c_1\|u_0\|^2 + c_2, \qquad 0 \le t \le T,
$$

with $c_1 = e^{2C_1 T}$, $c_2 = C_2(e^{2C_1 T} - 1)/C_1$, that is

$$
\|u\| \le C, \qquad 0 \le t \le T, \tag{4.71}
$$

with a constant C depending only on u_0, T and $\delta(\varepsilon), \mu(\varepsilon)$.

By (4.64) and (4.71) we get that

$$\|u(\cdot,t)\|_\infty \le C, \qquad 0 \le t \le T. \tag{4.72}$$

Now we return to (4.69), ignore the positive term $\|u_{xx}\|$, then integrate and employ (4.70) to get

$$\|u(\cdot,T)\|^2 \;+\; 2\,\mu(\varepsilon)\,\varepsilon^2 \int_0^t \|u_x\|^2\,d\tau \le \|u_0\|^2\,e^{2C_1t} \;+\; C_3\big(e^{2C_1t}-1\big),$$

therefore

$$\int_0^t \|u_x\|^2\,d\tau \le C, \qquad 0 \le t \le T, \tag{4.73}$$

for some positive constant C depending only on u_0, T and $\delta(\varepsilon), \mu(\varepsilon), \varepsilon$.

Returning once more to (4.69), we get as above that

$$\int_0^t \|u_{xx}\|^2\,d\tau \le C, \qquad 0 \le t \le T, \tag{4.74}$$

as well.

For improving the regularity of the weak solution to be a classical one we may use a bootstrap argument; see e.g. (2-20)-(2-25) of [15]. Let us also remark that in view of (3.6), the H^k-regularity of u implies the H^{k+1}-regularity for the solution $\tilde{u}$ of the integrated problem.

Uniqueness: Let u, υ be solutions of the problem (ACH)-(BC1)-(BC2)-(MC) and consider the difference $\mathrm{v} = u - \upsilon$. In view of (ACH), we have

$$\mathrm{v}_t = -\delta(\varepsilon)\,\varepsilon^2\mathrm{v}_{xxxx} \;+\; \delta(\varepsilon)\,\big(W'(u)\;-\;W'(\upsilon)\big)_{xx} \;+\; \mu(\varepsilon)\,\varepsilon^2\mathrm{v}_{xx} \;-\; \mu(\varepsilon)\,\big(W'(u)\;-\;W'(\upsilon)\big), \tag{4.75}$$

the (BC1)-(BC2) yield the boundary conditions

$$\mathrm{v}_x(0,\,t) = \mathrm{v}_x(1,\,t) \;= 0, \tag{4.76}$$

$$\mathrm{v}_{xxx}(0,\,t) \;= 0, \tag{4.77}$$

and (MC) implies

$$\int_0^1 \mathrm{v}(x,t)\,dx = 0, \qquad t > 0. \tag{4.78}$$

Multiply the pde (4.75) by v, then integrate with respect to x and apply (4.76)-(4.77) to get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\mathrm{v}\|^2 \;+\; \delta(\varepsilon)\,\varepsilon^2\,\|\mathrm{v}_{xx}\|^2 \;+\; \mu(\varepsilon)\,\varepsilon^2\,\|\mathrm{v}_x\|^2 \;&=\; -\delta(\varepsilon)\,\varepsilon^2\,\mathrm{v}_{xxx}(1,t)\,\mathrm{v}(1,t)\\ &\;+\; \delta(\varepsilon)\int_0^1 \big(W'(u)\;-\;W'(\upsilon)\big)\,\mathrm{v}_{xx}\,dx \;-\; \mu(\varepsilon)\int_0^1 \big(W'(u)\;-\;W'(\upsilon)\big)\,\mathrm{v}\,dx.\end{aligned} \tag{4.79}$$

Let us next estimate the terms in the RHS of (4.79). To this aim, we set $K_T := \sup\big\{\|u(\cdot,t)\|_\infty, \|\upsilon(\cdot,t)\|_\infty : 0 \le t \le T\big\}$ and $L = \max\big\{|W''(w)| : |w| \le K_T\big\}$. In view of (4.72) we have $K_T < \infty$, and L depends on u, υ, W, T, but it is independent of t.

We then have that

$$\big|W'\big(\upsilon(x,t)\big) \;-\; W'\big(u(x,t)\big)\big| \le L\,|\upsilon(x,t) - u(x,t)|, \qquad 0 \le t \le T. \tag{4.80}$$

Regarding the first term in the RHS of (4.79), we clearly have

$$\mathrm{v}(1,t) = \int_y^1 \mathrm{v}_x\,dx \;+\; \mathrm{v}(y,t) \le \int_0^1 |\mathrm{v}_x|\,dx \;+\; \mathrm{v}(y,t),$$

and integrate this inequality with respect to y, to get, by virtue of (4.78),

$$\mathrm{v}(1,t) \leq \int_0^1 |\mathrm{v}_x|\,dx + \int_0^1 \mathrm{v}(x,t)\,dx \;=\; \|\mathrm{v}_x(\cdot,t)\|_1 . \tag{4.81}$$

Moreover, by (4.54) and (4.80) we get

$$\delta(\varepsilon)\,\varepsilon^2\,\mathrm{v}_{xxx}(1,t) = \mu(\varepsilon)\int_0^1 \big(W'(\upsilon) - W'(u)\big)\,dx \;\leq\; L\,\mu(\varepsilon)\int_0^1 |\upsilon-u|\,dx = L\,\mu(\varepsilon)\,\|\mathrm{v}\|_1 . \tag{4.82}$$

Consequently, by (4.81)-(4.82) we obtain that

$$\begin{aligned} \delta(\varepsilon)\,\varepsilon^2\,\mathrm{v}_{xxx}(1,t)\,\mathrm{v}(1,t) &\leq \frac{L^2\,\mu(\varepsilon)}{4\epsilon}\,\|\mathrm{v}\|_1^2 + \epsilon\,\mu(\varepsilon)\,\|\mathrm{v}_x\|_1^2 \\ &\leq \frac{L^2\,\mu(\varepsilon)}{4\epsilon}\,\|\mathrm{v}\|^2 + \epsilon\,\mu(\varepsilon)\,\|\mathrm{v}_x\|^2 , \end{aligned} \tag{4.83}$$

for an arbitrary positive $\epsilon < 1$.

As for the second term in the RHS of (4.79), again we use (4.80) to see that

$$\begin{aligned} \delta(\varepsilon)\int_0^1 \big(W'(u) - W'(\upsilon)\big)\,\mathrm{v}_{xx}\,dx &\leq \delta(\varepsilon)\,L\int_0^1 \big|u-\upsilon\big|\,|\mathrm{v}_{xx}|\,dx \\ &\leq \frac{L^2\,\delta(\varepsilon)}{4\epsilon}\,\|\mathrm{v}\|^2 + \epsilon\,\delta(\varepsilon)\,\|\mathrm{v}_{xx}\|^2 , \end{aligned} \tag{4.84}$$

and for the last term in (4.79), estimate (4.80) yields the bound

$$\mu(\varepsilon)\int_0^1 \big(W'(u) - W'(\upsilon)\big)\,\mathrm{v}\,dx \;\leq\; \mu(\varepsilon)\,L\int_0^1 \big|u-\upsilon\big|\,|\mathrm{v}|\,dx \;=\; \mu(\varepsilon)\,L\,\|\mathrm{v}\|^2 . \tag{4.85}$$

We apply (4.83), (4.84), (4.85) into (4.79) to obtain

$$\frac{1}{2}\frac{d}{dt}\|\mathrm{v}\|^2 + \delta(\varepsilon)\,(\varepsilon^2-\epsilon)\,\|\mathrm{v}_{xx}\|^2 + \mu(\varepsilon)\,(\varepsilon^2-\epsilon)\,\|\mathrm{v}_x\|^2 \leq \left(\frac{L^2\,\mu(\varepsilon)}{4\epsilon} + \frac{L^2\,\delta(\varepsilon)}{4\epsilon} + \mu(\varepsilon)\,L\right)\|\mathrm{v}\|^2 . \tag{4.86}$$

Therefore

$$\frac{d}{dt}\|\mathrm{v}\|^2 \leq c\,\|\mathrm{v}\|^2 , \qquad 0 \leq t \leq T, \tag{4.87}$$

for some constant c depending on $\delta(\varepsilon), \mu(\varepsilon), T$, but independent of t and integrating with respect to t, we obtain

$$\|\mathrm{v}(\cdot,t)\|^2 \leq e^{ct}\,\|\mathrm{v}(\cdot,0)\|^2 \;=\; 0, \qquad 0 \leq t \leq T,$$

that is $u \equiv \upsilon$, so the solution of (ACH)-(BC1)-(BC2)-(MC) is unique.

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