EXISTENCE OF MAXIMAL SOLUTIONS FOR THE FINANCIAL STOCHASTIC STEFAN PROBLEM OF A VOLATILE ASSET WITH SPREAD

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Abstract. In this work, we consider the outer Stefan problem for the short-time prediction of the spread of a volatile asset traded in a financial market. The stochastic equation for the evolution of the density of sell and buy orders is the Heat Equation with a space-time white noise, posed in a moving boundary domain with velocity given by the Stefan condition. This condition determines the dynamics of the spread, and the solid phase $[s^-(t), s^+(t)]$ defines the bid-ask spread area wherein the transactions vanish. We introduce a reflection measure and prove existence and uniqueness of maximal solutions up to stopping times in which the spread $s^+(t) - s^-(t)$ stays a.s. non-negative and bounded. For this, we define an approximation scheme, and use some of the estimates of [16] for the Green's function and the associated to the reflection measure obstacle problem. Analogous results are obtained for the equation without reflection corresponding to a signed density.

Keywords: Phase field models, Stefan problem, volatility, limit order books, spreads.
AMS subject classification: 35K55, 35K40, 60H30, 60H15, 91G80, 91B70.

1. Introduction

1.1. **The one-dimensional Stochastic Stefan problem with a solid phase.** The parabolic Stochastic Stefan problem with a solid phase $\overline{S(t)} := [s^-(t), s^+(t)] \subset \mathbb{R}$ is defined by

$$\begin{cases} \partial_t w = \alpha \Delta w + \sigma(\text{dist}(x, \partial S))\dot{W}_s, \quad x \in \mathbb{R} - \overline{S(t)} \ \text{('liquid' phase)}, \quad t > 0, \\ w = 0, \quad x \in \overline{S(t)} \qquad \text{('solid' phase)}, \\ V := -\nabla w|_{\partial S} \qquad \text{(Stefan condition)}, \\ \partial S(0) = \{s^-(0), s^+(0)\} = \text{given}. \end{cases} \tag{1.1}$$

Here, $w = w(x, t)$ is a density, $\alpha > 0$ is a positive constant liquidity coefficient, σ is a function of

$$\text{dist}(x, \partial S(t)) = \min\{|x - s^+(t)|, |x - s^-(t)|\} \tag{1.2}$$

the distance of $x \notin \overline{S(t)}$ from the solid phase boundary $\partial S(t) = \{s^-(t), s^+(t)\}$, and

$$\dot{W}_s(x, t) := \dot{W}(x - s^+(t), t) \ \text{if} \ x \geq s^+(t), \quad \dot{W}_s(x, t) := \dot{W}(-x + s^-(t), t) \ \text{if} \ x \leq s^-(t), \tag{1.3}$$

where $\dot{W}(\pm x \mp s^\pm(t), t)$ is the non smooth in space and in time space-time white noise. The initial condition $w(x, 0)$ is considered given for all $x \in \mathbb{R}$. The parabolic equation of the Stefan problem is the stochastic Heat equation with space-time white noise.

The moving boundary of (1.1) is the union for all $t \geq 0$ of the curves $x = s^+(t)$, $x = s^-(t)$ enclosing the solid phase $\overline{S(t)}$ with midpoint $s(t) := (s^-(t) + s^+(t))/2$ and length $s^+(t) - s^-(t)$

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defining the spread. If $x \in \overline{S(t)}$, then the solution $w(x,t)$ vanishes. The Stefan condition

$$V(s^+(t),t) := \partial_t s^+(t) = -(\nabla w)^+(s^+(t),t),$$
$$V(s^-(t),t) := \partial_t s^-(t) = -(\nabla w)^-(s^-(t),t), \tag{1.4}$$

describes the change of liquidity for stochastic Stefan problems in financial models for the evolution of buy and sell orders (see Section 1.3), where V is the velocity at the boundary points of $\overline{S(t)}$, and $(\nabla\cdot)^{\pm}$ denotes the derivative from the right $(x > s^+)$ and left $(x < s^-)$. The solid phase dynamics are given by

$$\partial_t s^+(t) - \partial_t s^-(t) = -(\nabla w)^+(s^+(t),t) + (\nabla w)^-(s^-(t),t)). \tag{1.5}$$

The gradients are taken along the 'outer' normal vector, i.e., the direction is towards the liquid phase, so here in $d = 1$ they coincide to the right and left derivatives. Models with a.s. non-negative density w, when for example a reflection measure is introduced to the stochastic equation, due to the fact that $w = 0$ at $x = s^{\pm}$ will result in an a.s. decreasing spread $s^+(t)-s^-(t)$. More specifically $(\nabla w)^+(s^+(t),t) \geq 0$ and $(\nabla w)^-(s^-(t),t) \leq 0$ and thus by (1.5) $\partial_t(s^+(t) - s^-(t)) \leq 0$ for all $t \geq 0$ a.s. In contrast, when a signed density is considered the spread is not monotone.

Motivated by the analysis of [25, 26, 6] in higher dimensions, we define the bounded and time independent space domain $\Omega = (a,b)$ by

$$\Omega = \Omega_{\text{Liq}}(t) \cup [s^-(t), s^+(t)], \tag{1.6}$$

for a liquid phase $\Omega_{\text{Liq}} \subset \Omega$ where

$$\Omega_{\text{Liq}} := \{x \in \Omega : \ 0 \leq -x + s^-, \ x - s^+ \leq \lambda\}, \tag{1.7}$$

for $\lambda = b - a$ a positive constant relatively much larger than the initial spread $s^+(0) - s^-(0)$. The density $w(x,t)$ will be observed for x in Ω. As $\lambda \to \infty$ the liquid phase becomes infinite as in (1.1) and Ω will correspond to $\mathbb{R}$. The problem is one-dimensional and the liquid phase consists of two separate bounded linear segments. This enables the splitting of the Stefan problem equation in two equations posed for $x \in \Omega_{\text{Liq}}$ on $x \geq s^+$ and on $x \leq s^-$ where we shall apply the change of variables

$$y = x - s^+(t) \ \text{ if } x \geq s^+(t), \ \ y = -x + s^-(t) \ \text{ if } x \leq s^-(t), \tag{1.8}$$

and thus

$$y_t = -\partial_t s^+(t) \ \text{ if } x \geq s^+(t), \ \ \partial_t s^-(t) \ \text{ if } x \leq s^-(t). \tag{1.9}$$

As we shall see the equation is transformed due to the Stefan condition into two independent ones posed each on the fixed space domain $\mathcal{D} := (0, \lambda)$ with Dirichlet b.c. The value $y = 0$ occurs when x is $s^{\pm}$, while $y = \lambda$ when the spread is zero and $s^+ = s^-$ hits the boundary of Ω. These equations are of the general form

$$v_t(y,t) = \alpha\Delta v(y,t) \mp \nabla v(0^+,t)\nabla v(y,t) \pm \sigma(y)\dot{W}(y,t) + \dot{\eta}(y,t), \ \ y \in \mathcal{D}, \ \ t \geq 0, \tag{1.10}$$

for η a reflection measure keeping v a.s. non-negative, while $\eta = 0$ will correspond to the unreflected problem and a signed v.

We also note that when a system is considered in place of the Stefan problem (1.1) with different liquidity coefficients α_1, α_2 on the diffusion term Δw, the same equation of the above general form (1.10) will appear after the change of variables for $\alpha = \alpha_1, \alpha_2$; there, the volatility in each equation will depend on the distance $|x-s^+(t)|$ and $|x-s^-(t)|$ respectively. Hambly, and Kalsi proved in [16] existence and uniqueness of stochastic solutions for such two phases Stefan systems with reflection, when the solid phase has zero spread, i.e., when $s^+(t) = s^-(t) = s(t)$ $\forall t \geq 0$. Considering 2-phases

1-dimensional stochastic Stefan systems for the evolution of sell and buy orders without spread we refer also to [12, 31], and to the more recent results of [24, 17].

We prove existence of unique maximal solutions (v, η) for the stochastic equation (1.10) for the stopping time $\sup_{M>0} \tau_M$ where

$$\tau_M := \inf\Big\{T \geq 0 : \sup_{r\in(0,T)} |\nabla v(0^+, r)| \geq M\Big\}, \tag{1.11}$$

up to which $|\nabla v(0^+, r)| = \nabla v(0^+, r)$ stays a.s. bounded. In the case of the unreflected problem, η is just replaced by zero and the absolute value is kept. In order to return to the initial variables and to the moving boundary problem, the stopping time will be further reduced so that the spread stays a.s. non-negative and the spread area in the domain Ω. These restrictions will be induced by the Stefan condition and the resulting solid phase dynamics (1.5), the initial spread $s^+(0) - s^-(0)$, and the magnitude of λ.

Deterministic parabolic Stefan problems have been so far extensively studied when describing the phenomenon of phase separation of alloys. In [25], the deterministic version of (1.1) was introduced in higher dimensions in the physical context of the LSW theory for the Ostwald ripening of alloys; there, a first order approximation was established for the dynamics of the radii of spherical moving boundaries in dimensions $d = 3$. In [1, 3, 2], the authors considered the quasi-static problem and obtained second order approximations by taking into account the variable in general geometry of the solid phase. We also refer to [6] for the analysis of the parabolic Stefan problem of [25] in the presence of kinetic undercooling and additive forcing.

In [5], the authors derived the rigorous financial interpretation of the parabolic Stefan stochastic model, which applies for the density w of trading of a portfolio of assets with spread when $d \geq 2$; a quasi-static version thereof approximates the parabolic one when the diffusion tends to infinity as in the case of very large trading. In contrast to the deterministic Stefan problem where a spherical initial solid phase or the interval $[s^-(0), s^+(0)]$ in dimension $d = 1$ are static solutions, in the stochastic case the boundary changes as time evolves due to the random perturbation in the spde; see for example the numerical simulations in [5] when $d = 3$.

Remark 1.1. *In dimensions $d = 1$ as here, and for σ constant, the initial and boundary value problem with Dirichlet or Neumann b.c. for the stochastic Heat equation with space-time white noise admits a unique solution, see for example in* [15] *for the Neumann problem. Here of course we analyze a different moving boundary problem. However, the stochastic Heat equation when the noise is white in space in spatial (integer) dimensions higher than or equal to 2 does not admit function-valued solutions and is known to only exist as a distribution cf. for example in* [30] *and the discussion of* [18] *on the issues arising when $d \geq 2$; we refer also to the very interesting results of* [18] *where the stochastic Heat equation is posed on a class of self similar sets with spectral dimension in $[1, 2)$, corresponding to a fractional spatial dimension where the authors proved that function-valued space-time continuous solutions exist. This indicates strongly the use of a white in time but smooth in space noise when the Stefan problem is considered in dimensions $d \geq 2$; existence and regularity of solutions for such a problem is a work in progress.*

1.2. **Main Results.** Let us focus on the novel aspects of the present paper in comparison with previous work and in particular with the important references [16, 17]. These aspects concern the introduction of spread in the solid phase in the context of the financial model which is motivated by the standard outer Stefan problem statement in the physical application, and the mathematical proof of local existence up to a stopping time by using an spde of reference posed on a fixed domain.

- Our analysis covers 3 versions of the Stefan problem.

(1) Let $w_1, w_2 \geq 0$ be the densities corresponding to the two liquid phases separated by the solid phase. When $x > s^+(t)$ then only sell orders are executed ($w_2 = 0$), while when $x < s^-(t)$ then only buy orders are executed ($w_1 = 0$). Moreover at $x = s^+(t)$ $w_1 = 0$ and at $x = s^-(t)$ $w_2 = 0$. The signed density $w = w_1 - w_2$ is given by

$$w(x,t) = w_1(x,t) \text{ if } x > s^+(t), \;\; w(x,t) = -w_2(x,t) \text{ if } x < s^-(t), \;\; w(x,t) = 0 \text{ otherwise.} \tag{1.12}$$

We introduce in (1.1) the additive term $\dot{\eta}_s$ defined by

$$\dot{\eta}_s(x,t) := \dot{\eta}_1(x - s^+(t), t) \text{ if } x \geq s^+(t), \;\; \dot{\eta}_s(x,t) := -\dot{\eta}_2(-x + s^-(t), t) \text{ if } x \leq s^-(t), \tag{1.13}$$

where η_1, η_2 are reflection measures so that $w_1, w_2 \geq 0$.

(2) We consider the reflected problem where $w \geq 0$. The Stefan condition due to the non-negativity of w which vanishes at $x = s^{\pm}$ yields an a.s. decreasing spread. The reflection additive term on (1.1) is of the form

$$\dot{\eta}_s(x,t) := \dot{\eta}_1(x - s^+(t), t) \text{ if } x \geq s^+(t), \;\; \dot{\eta}_s(x,t) := \dot{\eta}_2(-x + s^-(t), t) \text{ if } x \leq s^-(t), \tag{1.14}$$

where η_1, η_2 are reflection measures keeping $w \geq 0$ for any $x \in \Omega_{\rm Liq}$.

(3) The unreflected problem is analyzed with a signed density w where as in (1) the spread is non-monotone.

- In all the above cases we derive a system of independent spdes of the form (1.10) for $v = v_1$, $v = v_2$. Then s^+, s^- are specified through integration of the Stefan condition. For stopping times wherein $s^- \leq s^+$ and $(s^-, s^+) \subseteq \Omega$ by applying the change of variables (1.8), $v_1 \to w|_{x \geq s^+}$, $v_2 \to -w|_{x \leq s^-}$ in (1) or $w|_{x \leq s^-}$ in (2) and (3), we return to the initial Stefan problem.

  The suggested transformation is efficient for representing the stochastic equation of the Stefan problem as a system of independent spdes posed on the fixed domain $\mathcal{D} = (0, \lambda)$, of the same general form. Such a transformation is natural when local existence of Stefan problems is considered in higher dimensions, cf. for example in [25, 26]. Additionally, for the reflected equations, we impose the non-negativity of $v_{1,2}$ by proving existence of the measures $\eta_{1,2}$ on the fixed domain which then define the additive reflection term in the initial equation. Our approach on transforming first the one-dimensional problem to an spde of reference and then establishing maximal solutions to the initial one by using the Stefan condition for the stopping times is also applicable for various other one-dimensional versions with financial interest being analyzed for example in [16, 17, 24] without spread. We note that we arrive to a system of decoupled spdes of similar form posed on the same fixed bounded interval. An analogous transformation was used for example in [24] for a quite different Stefan problem posed on the whole real line and the resulting system of the so-called centered spdes consisted of two equations posed on the positive and negative semi-axis respectively.
- Variables of the form $y = -x + s(t)$ when $x \leq s$, $y = x - s(t)$ when $x \geq s$ for the zero spread model where s is the sell/buy price, are used in [16, 24]. In [16] the problem is not transformed, a weak solution formulation for proper test functions compactly supported in $[0, 1]$ induces somehow a relevant (not splited) system posed on the fixed domain $(0, 1)$ that seems to facilitate the authors proof of maximal solutions.
- Our model permits zero spread as a special case and in this sense stands as an extension of the results of [16, 17, 24].
- The function σ and the noise depend on the distance of x from the spread area boundary and not on the position of x. This yields, since the velocity is given by the standard Stefan condition of (1.1), to spdes in the y variables where $s^{\pm}$, that belong to the initial problem unknowns, are absent.

In Section 2, we present analytically the change of variables $y = x - s^+$, $y = -x + s^-$ for $x \in \Omega_{\text{Liq}}$, use the Stefan condition and derive per case the Stefan problems as systems of two independent spdes of the form (1.10) for $v = v_1, v_2$, cf. (2.10), (2.15), (2.18). Section 3 is devoted to the existence of unique weak maximal solutions (v, η) of the Dirichlet problem on $\mathcal{D}$ for (1.10) with reflection, and then of maximal solutions to the initial variables with stopping times restricted by the Stefan condition, the non-negativity of spread and the boundedness of the liquid phase. In detail, we write the spde in an integral form using the Green's function of the negative Dirichlet Laplacian and construct an approximate scheme for the truncated problem. In Theorem 3.1 we prove existence and uniqueness a.s. for the approximate solutions, and on the limit existence and uniqueness of the truncated solution. For this, we use some of the Green's estimates of [16] and a proper Banach space introduced therein. The reflection measure η is associated to the obstacle problem estimated in [16]. In Theorem 3.2, using the consistency of the truncated solutions we prove existence of a unique maximal solution (v, η) a.s. in the maximal time interval $[0, \sup_{M>0} \tau_M)$ for τ_M given by (1.11). Given the maximal solution (v, η), for $v = v_{1,2} \geq 0$, $\eta = \eta_{1,2}$, in Theorem 3.3 we prove existence of unique maximal solutions (w_1, η_1), (w_2, η_2) to the reflected Stefan problem (2.3)-(2.19)-(2.20) corresponding to (1), and of $w|_{x \geq s^+} = w_1 \geq 0$, $w|_{x \leq s^-} = -w_2 \leq 0$, in the maximal interval $\mathcal{I}_1 := [0, \hat{\tau})$ for $\hat{\tau} := \min\{\sup_{M>0} \tau_{1M}, \tau_{1s}, \tau_1^*\}$, with $\tau_{1M}, \tau_{1s}, \tau_1^*$ given by (3.36), (3.38), (3.39) for which the spread exists and stays a.s. non-negative for any $t \in \mathcal{I}_1$. An analogous result for the case (2) is proven in Theorem 3.4 but in a different maximal interval $\mathcal{I}_2 := [0, \hat{\tau})$ for $\hat{\tau} := \min\{\sup_{M>0} \tau_{1M}, \tau_{2s}\}$, with τ_{1M}, τ_{2s} given by (3.36), (3.40). There, the decreasing property of the spread is used.

In Section 4 we consider the Stefan problem without reflection, i.e., (3), and the Dirichlet problem on $\mathcal{D}$ for the spde (4.1) that v satisfies. Theorem 4.1 establishes existence and uniqueness a.s. of the truncated equation, and Theorem 4.2 existence of a unique maximal solution v in $[0, \sup_{M>0} \tau_M)$ for τ_M as in (1.11). Then the existence and uniqueness of maximal solution in the initial variables is proven in Theorem 4.3 for the resulting stopping time.

Remark 1.2. *This work was motivated by the analysis of the problem without spread of Hambly and Kalsi in* [16] *from which we heavily used the results derived therein for the Heat equation Obstacle problem and some of the new estimates of the Green's function. These stand as an important contribution in the mathematical theory of evolutionary stochastic pdes since they can be used directly as main tools when existence and uniqueness is proven for spdes of second or fourth order with reflection; we refer respectively to the Green's estimates when the Laplacian or the bi-Laplacian operator is involved and to the Obstacle problem when a reflection measure is introduced for the positivity of stochastic solutions. In contrast to* [16]*, we shall not consider the more general case where* σ *depends also on the solution where the shape of the order book and the transactions density is expected to influence investors decisions about order placement; it is left to the interested reader such a case where the growth and Lipschitz conditions proposed in* [16] *seem sufficient. The dependence of* σ *on the distance from the moving boundary is present in our problem.*

We consider the Stefan problem with a solid phase modeling the financial spread. We state the weak formulation on the transformed problem which is posed on the fixed domain $(0, \lambda)$ *for a generic* $\lambda > 0$ *and not only for* $\lambda := 1$*; this would permit in future work, as in the physical problem's literature, rigorous mean-field arguments on long range asymptotics and spread dynamics as* $\lambda \to \infty$*. An increasing domain of length* $\lambda \to \infty$ *would be essential for the analysis of the Stefan problem posed on the whole real line, as done by rescaling in* [25, 26, 6] *for the deterministic problem on infinite domains. This manuscript, considers three different Stefan problems proven well posed,* *one of which (Case 2) justifies the assumption of zero spread used so far in the literature, which is that under positivity of solutions the Stefan condition makes the spread decreasing and so, it makes sense to assume zero spread in later times. Moreover, we address the unreflected problem (Case 3) and Case 1 where (even with reflection for Case 1) the solution changes sign. We also note that we carefully introduced an implicit-explicit discrete scheme for the approximate solution which is quite different than this used in* [16].

1.3. **Financial Motivation.** The limit orders are instructions for trading of a portion of an asset, [22], based on information from the limit order book. The lowest sell order $s^+(t)$ defined as ask price, is the minimum price at which the investor is willing to receive, and $s^-(t)$ is the highest buy order or bid price which is the maximum price at which the investor is willing to pay. Orders come in two types - market and limit orders. If a market order arrives it is executed at the best available price. For a limit order, if it is a buy order, it joins the order book provided it is below the best ask, otherwise it becomes a market order and is executed. The case of sell orders is analogous. An order is executed if the price set (spot price) lies outside the spread interval $[s^-(t), s^+(t)]$, if not it is sorted in the order book list and not traded, see for example in [13, 21, 27].

In view of the Stefan problem (1.1), the solution $w(x,t)$ may model the density of the sell and buy orders of a stochastically volatile liquid asset traded at a price x at time $t > 0$. The solid phase $\overline{S(t)}$ stands as the spread area with length $s^+(t) - s^-(t)$ defining the spread at time t. The asset price x has been transformed through a logarithmic scale and in general can take negative and positive values; in particular the interval $[e^{s^-(t)}, e^{s^+(t)}]$ is the financial spread area and its length the financial spread with midpoint defining the so-called mid price. The exponential function is monotone and in fact increasing, which permits the definition of a solid phase $[s^-(t), s^+(t)]$ corresponding to the zero density area for the transformed problem. We refer to [5] for some simulations including the logarithmic transformation, the computed spread in the logarithmic scale and the resulting financial spread. If $x \in \overline{S(t)}$, then there is no volume of orders to sell or buy at price x, hence the asset cannot be traded, and so the density $w(x,t)$ of sell and buy orders is zero. If $x \notin \overline{S(t)}$ then the order is performed. The parameter $\alpha > 0$ estimates the total liquidity index of the market and measures the diffusion strength of sell and buy orders; it can be approximated, in small time periods, by the total volume of orders divided by an average spread, [5]. The stochastic term $\sigma\dot{W}_s$ defines the volatility, it captures the random arrival and cancellation of orders and gives rise to spreads and prices that vary. In the absence of volatility, i.e., when $\sigma := 0$, the solid phase is static, since one sole interval defining the initial solid phase (one ball in dimensions higher than 1) is a static solution (solution of the elliptic static problem), see in [25, 26, 6, 5]. We also note that the Gibbs Thomson condition on the moving boundary $\partial S(t)$ which is present in dimensions $d \geq 2$, [25], involving the mean curvature, and the constant value of $w = w_0$ in the solid phase are both replaced by the condition $w = w_0 := 0$ in $\overline{S(t)}$.

The spread is usually seen as a measure of liquidity, which is one type of risk of investment, see for example in [4]. Highly traded assets tend to have very small spreads, while a relatively large spread indicates a higher risk. An order is a commitment from the traders, a buyer or a seller, to buy or sell respectively at an appropriate price at a given time $t > 0$, for which the profit of the trade is maximized for both sides, [13], also called limit price. The total volume of active limit orders in a financial market at a given time is stored in the asset's limit order book. The limit orders remain active until they achieve the expected 'closing' price, and target to better future prices for maximizing profits, [22], they are low risk commitments since the price of execution is predetermined reducing thus the odds of significant failure. However, the process is time consuming and the order may never be executed. The spread and the density of transactions reflect asset's liquidity. Considering the financial spreads in relation with market's liquidity, we refer to [7] where the German power market liquidity was studied, we also refer to [28] for a statistical analysis of the fluctuations of the average spread where the relation of spread with shares volume and volatility was examined, or to [20] for a stochastic equation model estimating the liquidity risk. In [10], the authors analyzed how transaction costs affect the spreads while in case of zero cost then the market price should act as a Wiener process; see also in [23] for the liquidity risk with respect to the transaction costs and market manipulation under a Brownian motion problem formulation, or in [9, 29, 19, 8], and in [14] for various empirical approaches on spread's forecast.

2. The Stefan problems

2.1. **Assumptions.** As it is common in the literature, we shall consider the initial Stefan problem (1.1) on a static very large bounded domain $\Omega := (a, b)$, with $\lambda := b - a >> 0$, instead of $\mathbb{R}$, which is occupied by the moving liquid and solid phases. The static outer boundary of the Stefan problem will be $\partial\Omega := \{a, b\}$, while the moving inner boundary is $\{s^-(t), s^+(t)\}$ for $t > 0$.

The initial solid phase should be in Ω and far away from the outer boundary $\partial\Omega := \{a, b\}$. Therefore, we consider initial data satisfying

$$s^{\pm}(0) \in \Omega, \quad s^-(0) \le s^+(0),$$

and so

$$0 \le s^+(0) - s^-(0) << b - a := \lambda.$$

We shall observe evolution as long as the solid phase exists, it stays in Ω (and thus it is not touching or crossing the outer boundary $\partial\Omega$), i.e., for times $t > 0$ such that

$$0 \le s^+(t) - s^-(t) < \lambda.$$

Since Ω is bounded, we impose a Dirichlet condition $w(a, t) = w(b, t) = 0$ at the outer boundary $\partial\Omega = \{a, b\}$.

Finally, we shall assume that the initial condition is smooth and compactly supported in the liquid phase, in particular we assume that $w_0 \in C_c^\infty([a, s^-(0)]) \cap C_c^\infty([s^+(0), b])$ which yields the weaker condition $w_0 \in C_c^\infty([a, b])$.

2.2. **Change of variables.** We consider Ω given by (1.6), Ω_{Liq} by (1.7), and y defined by (1.8) for any $x \in \Omega_{\text{Liq}} \cup \{s^-, s^+\}$. Let $\tilde{w}_1(x, t)$ be defined in $\{x \in \Omega_{\text{Liq}} \cup \{s^-, s^+\} : \ x \ge s^+\}$ and $\tilde{w}_2(x, t)$ be defined in $\{x \in \Omega_{\text{Liq}} \cup \{s^-, s^+\} : \ x \le s^-\}$ and set for $y := x - s^+$

$$\tilde{w}_1(x, t) := \tilde{v}_1(y, t) \ \ \forall\, x \in \Omega_{\text{Liq}} \cup \{s^-(t), s^+(t)\} : \ x \ge s^+(t),$$

while for $y := -x + s^-$

$$\tilde{w}_2(x, t) := \tilde{v}_2(y, t) \ \ \forall\, x \in \Omega_{\text{Liq}} \cup \{s^-(t), s^+(t)\} : \ x \le s^-(t).$$

If $x \ge s^+(t)$ we get

$$\begin{aligned}
&\tilde{w}_1(x, t) = \tilde{v}_1(x - s^+(t), t) = \tilde{v}_1(y, t), \quad y = x - s^+(t), \ \ y_x = 1, \ \ y_t = -\partial_t s^+(t) \\
&(\tilde{w}_1)_t(x, t) = (\tilde{v}_1)_y(y, t) y_t(y, t) + (\tilde{v}_1)_t(y, t) = -\partial_t s^+(t)(\tilde{v}_1)_y(y, t) + (\tilde{v}_1)_t(y, t), \\
&(\tilde{w}_1)_x(x, t) = (\tilde{v}_1)_y(y, t) y_x = +(\tilde{v}_1)_y(y, t), \\
&(\tilde{w}_1)_{xx}(x, t) = (\tilde{v}_1)_{yy}(y, t)(y_x(y, t))^2 = (\tilde{v}_1)_{yy}(y, t),
\end{aligned} \tag{2.1}$$

and if $x \le s^-(t)$

$$
\begin{aligned}
&\tilde{w}_2(x,t) = \tilde{v}_2(-x+s^-(t),t) = \tilde{v}_2(y,t), \;\; y=-x+s^-(t), \;\; y_x=-1, \;\; y_t=\partial_t s^-(t)\\
&(\tilde{w}_2)_t(x,t) = (\tilde{v}_2)_y(y,t)y_t(y,t) + (\tilde{v}_2)_t(y,t) = \partial_t s^-(t)(\tilde{v}_2)_y(y,t) + (\tilde{v}_2)_t(y,t),\\
&(\tilde{w}_2)_x(x,t) = (\tilde{v}_2)_y(y,t)y_x = -(\tilde{v}_2)_y(y,t),\\
&(\tilde{w}_2)_{xx}(x,t) = (\tilde{v}_2)_{yy}(y,t)(y_x(y,t))^2 = (\tilde{v}_2)_{yy}(y,t).
\end{aligned}
\tag{2.2}
$$

2.3. **Case 1.** Let for any $x \in \Omega$ the signed density w be given by

$$
w(x,t) = w_1(x,t) - w_2(x,t) = \begin{cases} w_1(x,t) & \text{if } x > s^+(t),\\ -w_2(x,t) & \text{if } x < s^-(t),\\ 0 & \text{otherwise,}\end{cases}
$$

for w_1, w_2 the densities of orders corresponding to the two liquid phases. We then have $w(x,t)|_{x\ge s^+} = w_1(x,t)$, $w(x,t)|_{x\le s^-} = -w_2(x,t)$.

The equation (1.1) by introducing the additive term $\dot{\eta}_s$ given by (1.13) takes in $\Omega_{\rm Liq}$ the form

$$
\partial_t w = \alpha\Delta w + \sigma(\text{dist}(x,\partial S))\dot{W}_s(x,t) + \dot{\eta}_s(x,t), \qquad x\in\Omega_{\rm Liq}, \;\; t>0,
$$

or equivalently for $x \in \Omega_{\rm Liq}$

$$
\begin{aligned}
\partial_t w_1 =& \alpha\Delta w_1 + \sigma(x-s^+(t))\dot{W}(x-s^+(t),t) + \dot{\eta}_1(x-s^+(t),t), \qquad x>s^+(t), \;\; t>0,\\
\partial_t w_2 =& \alpha\Delta w_2 - \sigma(-x+s^-(t))\dot{W}(-x+s^-(t),t) + \dot{\eta}_2(-x+s^-(t),t), \;\; x<s^-(t), \;\; t>0,
\end{aligned}
\tag{2.3}
$$

while $w(x,t) = w_1(x,t) = w_2(x,t) = 0$, $\forall\, x \in [s^-(t), s^+(t)]$, $\forall\, t>0$, and $w_1(x,0) = w(x,0)$ for any $x \ge s^+(0)$, $w_2(x,0) = -w(x,0)$ for any $x \le s^-(0)$. We shall assume that $w_1(x,0), w_2(x,0) \ge 0$. The reflection measures η_1, η_2, if they exist, will keep $w_1, w_2 \ge 0$ for all t a.s. Using the Stefan condition (1.4), we obtain

$$
\begin{aligned}
V(s^+(t),t) =& \partial_t s^+(t) = -(\nabla w)^+(s^+(t),t) = -(\nabla w_1)^+(s^+(t),t),\\
V(s^-(t),t) =& \partial_t s^-(t) = -(\nabla w)^-(s^-(t),t) = (\nabla w_2)^-(s^-(t),t),
\end{aligned}
\tag{2.4}
$$

and so the spread dynamics are given by

$$
\partial_t s^+(t) - \partial_t s^-(t) = -(\nabla w_1)^+(s^+(t),t) - (\nabla w_2)^-(s^-(t),t)). \tag{2.5}
$$

We apply the change of variables

$$
w_1(x,t) = v_1(y,t) \quad \text{for} \quad y = x - s^+ \tag{2.6}
$$

and so

$$
(\nabla w_1)^+(s^+,t) = \nabla v_1(0^+,t), \tag{2.7}
$$

and

$$
w_2(x,t) = v_2(y,t) \quad \text{for} \quad y = -x + s^- \tag{2.8}
$$

and so

$$
(\nabla w_2)^-(s^-,t) = -\nabla v_2(0^+,t). \tag{2.9}
$$

We use (2.1), (2.2), and (2.4) which yields that

$$
\partial_t s^+(t) = -(\nabla w_1)^+(s^+(t),t) = -\nabla v_1(0^+,t),
$$

and that

$$
\partial_t s^-(t) = (\nabla w_2)^-(s^-(t),t) = -\nabla v_2(0^+,t),
$$

and derive the system of two independent initial and boundary value problems

(2.10)

$$
\begin{aligned}
\partial_t v_1(y,t) =&\alpha\Delta v_1(y,t) + \partial_t s^+(t)\nabla v_1(y,t) + \sigma(y)\dot{W}(y,t) + \dot{\eta}_1(y,t)\\
=&\alpha\Delta v_1(y,t) - \nabla v_1(0^+,t)\nabla v_1(y,t) + \sigma(y)\dot{W}(y,t) + \dot{\eta}_1(y,t),\ \ y\in(0,\lambda)=:\mathcal{D},\ \ t>0,\\
&v_1(0,t)=v_1(\lambda,t)=0,\ \ t>0,\ \ v_1(y,0)=w(y+s^+(0),0)\ge 0,\ \ y\in\mathcal{D},
\end{aligned}
$$

and

$$
\begin{aligned}
\partial_t v_2(y,t) =&\alpha\Delta v_2(y,t) - \partial_t s^-(t)\nabla v_2(y,t) - \sigma(y)\dot{W}(y,t) + \dot{\eta}_2(y,t)\\
=&\alpha\Delta v_2(y,t) + \nabla v_2(0^+,t)\nabla v_2(y,t) - \sigma(y)\dot{W}(y,t) + \dot{\eta}_2(y,t),\ \ y\in(0,\lambda)=:\mathcal{D},\ \ t>0,\\
&v_2(0,t)=v_2(\lambda,t)=0,\ \ t>0,\ \ v_2(y,0)=-w(s^-(0)-y,0)\ge 0,\ \ y\in\mathcal{D}.
\end{aligned}
$$

In the above, we used the Dirichlet b.c. $v_1(\lambda,t)=v_2(\lambda,t)=0$.

By using (2.5), the spread evolution is given by

$$\partial_t(s^+(t)-s^-(t)) = -\nabla v_1(0^+,t)+\nabla v_2(0^+,t). \tag{2.11}$$

2.4. **Case 2.** Let for any $x\in\Omega$ the density w, and define

$$w_1(x,t):=w(x,t)\ \ x\ge s^+,\ \ w_2(x,t)=w(x,t)\ \ x\le s^-,$$

and so

$$w(x,t)=\begin{cases} w_1(x,t) & \text{if } x>s^+(t),\\ w_2(x,t) & \text{if } x<s^-(t),\\ 0 & \text{otherwise.}\end{cases}$$

In this case, we have $w(x,t)|_{x\ge s^+}=w_1(x,t)$, $w(x,t)|_{x\le s^-}=w_2(x,t)$.

We introduce in (1.1) the additive term $\dot{\eta}_s(x,t)$ given by (1.14), for η_1, η_2 the reflection measures keeping $w_1,w_2\ge 0$ and thus $w\ge 0$. The equation on $\Omega_{\rm Liq}$ takes the form

$$\partial_t w = \alpha\Delta w + \sigma(\mathrm{dist}(x,\partial S))\dot{W}_s(x,t) + \dot{\eta}_s(x,t), \qquad x\in\Omega_{\rm Liq},\ \ t>0,$$

or equivalently for $x\in\Omega_{\rm Liq}$

$$
\begin{aligned}
\partial_t w_1 =&\alpha\Delta w_1 + \sigma(x-s^+(t))\dot{W}(x-s^+(t),t) + \dot{\eta}_1(x-s^+(t),t), \qquad x>s^+(t),\ \ t>0,\\
\partial_t w_2 =&\alpha\Delta w_2 + \sigma(-x+s^-(t))\dot{W}(-x+s^-(t),t) + \dot{\eta}_2(-x+s^-(t),t),\ \ x<s^-(t),\ \ t>0,
\end{aligned}
\tag{2.12}
$$

while $w(x,t)=w_1(x,t)=w_2(x,t)=0,\ \ \forall\, x\in[s^-(t),s^+(t)],\ \ \forall\, t>0$, and $w_1(x,0)=w(x,0)$ for any $x\ge s^+(0)$, $w_2(x,0)=w(x,0)$ for any $x\le s^-(0)$. We shall assume that $w_1(x,0),w_2(x,0)\ge 0$. The reflection measures η_1, η_2, if they exist, will keep $w_1,w_2,w\ge 0$ for all t a.s. Using the Stefan condition (1.4), we obtain

$$
\begin{aligned}
V(s^+(t),t)=&\partial_t s^+(t) = -(\nabla w)^+(s^+(t),t) = -(\nabla w_1)^+(s^+(t),t),\\
V(s^-(t),t)=&\partial_t s^-(t) = -(\nabla w)^-(s^-(t),t) = -(\nabla w_2)^-(s^-(t),t),
\end{aligned}
\tag{2.13}
$$

and so the spread dynamics are given by

$$\partial_t s^+(t) - \partial_t s^-(t) = -(\nabla w_1)^+(s^+(t),t) + (\nabla w_2)^-(s^-(t),t)). \tag{2.14}$$

We apply again the change of variables (2.6), (2.8) resulting to (2.7) and (2.9) respectively. Using then (2.1), (2.2), and (2.4) which yields for that case that

$$\partial_t s^+(t) = -(\nabla w_1)^+(s^+(t),t) = -\nabla v_1(0^+,t),$$

and that

$$\partial_t s^-(t) = -(\nabla w_2)^-(s^-(t),t) = \nabla v_2(0^+,t),$$

we arrive at the next system of two independent initial and boundary value problems

(2.15)
$$
\begin{aligned}
\partial_t v_1(y,t) =& \alpha\Delta v_1(y,t) + \partial_t s^+(t)\nabla v_1(y,t) + \sigma(y)\dot{W}(y,t) + \dot{\eta}_1(y,t)\\
=& \alpha\Delta v_1(y,t) - \nabla v_1(0^+,t)\nabla v_1(y,t) + \sigma(y)\dot{W}(y,t) + \dot{\eta}_1(y,t), \;\; y\in(0,\lambda) =: \mathcal{D}, \;\; t>0,\\
& v_1(0,t) = v_1(\lambda,t) = 0, \;\; t>0, \;\; v_1(y,0) = w(y+s^+(0),0) \geq 0, \;\; y\in\mathcal{D},\\
& \text{and}\\
\partial_t v_2(y,t) =& \alpha\Delta v_2(y,t) - \partial_t s^-(t)\nabla v_2(y,t) + \sigma(y)\dot{W}(y,t) + \dot{\eta}_2(y,t)\\
=& \alpha\Delta v_2(y,t) - \nabla v_2(0^+,t)\nabla v_2(y,t) + \sigma(y)\dot{W}(y,t) + \dot{\eta}_2(y,t), \;\; y\in(0,\lambda) =: \mathcal{D}, \;\; t>0,\\
& v_2(0,t) = v_2(\lambda,t) = 0, \;\; t>0, \;\; v_2(y,0) = w(s^-(0)-y,0) \geq 0, \;\; y\in\mathcal{D}.
\end{aligned}
$$

By using (2.5), the spread evolution is given by

$$
\partial_t(s^+(t) - s^-(t)) = -\nabla v_1(0^+,t) - \nabla v_2(0^+,t). \tag{2.16}
$$

As we already mentioned, if the reflection measures exist and keep $v_1, v_2 \geq 0$ and since $v_1 = v_2 = 0$ at $y = 0$, then the spread is decreasing.

2.5. **Case 3.** As in Case 2, we consider for any $x \in \Omega$ the signed density w, and define

$$
w_1(x,t) := w(x,t) \;\; x \geq s^+, \;\; w_2(x,t) = w(x,t) \;\; x \leq s^-.
$$

We do not require here $w \geq 0$ and so we consider the unreflected equation (1.1) posed in $\Omega_{\text{Liq}}, \;\; t > 0$, or equivalently for $x \in \Omega_{\text{Liq}}$

$$
\begin{aligned}
\partial_t w_1 =& \alpha\Delta w_1 + \sigma(x - s^+(t))\dot{W}(x - s^+(t),t), \qquad\quad x > s^+(t), \;\; t>0,\\
\partial_t w_2 =& \alpha\Delta w_2 + \sigma(-x + s^-(t))\dot{W}(-x + s^-(t),t), \;\; x < s^-(t), \;\; t>0,
\end{aligned} \tag{2.17}
$$

while $w(x,t) = w_1(x,t) = w_2(x,t) = 0, \;\; \forall\, x \in [s^-(t), s^+(t)], \;\; \forall\, t > 0$, and $w_1(x,0) = w(x,0)$ for any $x \geq s^+(0)$, $w_2(x,0) = w(x,0)$ for any $x \leq s^-(0)$. Using the Stefan condition (1.4), we obtain (2.13) again for the velocity and the spread dynamics are given by (2.14). We apply the change of variables $w_1(x,t) = v_1(y,t)$ for $y = x - s^+$, and $w_2(x,t) = v_2(y,t)$ for $y = -x + s^-$ to obtain as in Case 2 the system of two independent initial and boundary value problems

$$
\begin{aligned}
\partial_t v_1(y,t) =& \alpha\Delta v_1(y,t) - \nabla v_1(0^+,t)\nabla v_1(y,t) + \sigma(y)\dot{W}(y,t), \;\; y\in(0,\lambda) =: \mathcal{D}, \;\; t>0,\\
& v_1(0,t) = v_1(\lambda,t) = 0, \;\; t>0, \;\; v_1(y,0) = w(y+s^+(0),0), \;\; y\in\mathcal{D},\\
& \text{and}\\
\partial_t v_2(y,t) =& \alpha\Delta v_2(y,t) - \nabla v_2(0^+,t)\nabla v_2(y,t) + \sigma(y)\dot{W}(y,t), \;\; y\in(0,\lambda) =: \mathcal{D}, \;\; t>0,\\
& v_2(0,t) = v_2(\lambda,t) = 0, \;\; t>0, \;\; v_2(y,0) = w(s^-(0)-y,0), \;\; y\in\mathcal{D}.
\end{aligned} \tag{2.18}
$$

The spread evolution is given as in Case 2 by (2.16), but since v_1, v_2 may change sign even if $v_1 = v_2 = 0$ at $y = 0$, the spread is not monotone.

We shall assume that coefficient σ of the noise is a sufficiently smooth function; its minimal regularity will be specified in the sequel. The random measure $W(dy,ds)$ is defined as the 1-dimensional space-time white noise induced by the 2-dimensional Wiener process $W := \{W(y,t) : t \in [0,T], \; y \in (0,\lambda)\}$ which generates, for any $t \geq 0$, the filtration $\mathcal{F}_t := \sigma(W(y,s) : \; s \leq t, \; y \in (0,\lambda))$, where the notation σ here denotes the σ-algebra.

When reflection measures are considered, i.e., for the Cases 1,2, each problem's unknowns for $t \in [0,T]$ is a pair (v,η) where the reflection measure η is defined to satisfy

$$\text{for all measurable functions } \psi : \overline{\mathcal{D}} \times (0,T) \to [0,\infty)$$
$$\int_0^t \int_{\mathcal{D}} \psi(y,s)\eta(dy,ds) \text{ is } \mathcal{F}_t - \text{measurable}, \tag{2.19}$$

and the constraint

$$\int_0^T \int_{\mathcal{D}} v(y,s)\eta(dy,ds) = 0. \tag{2.20}$$

Remark 2.1. *In all the above cases, given the solutions v_1, v_2 for $t \in [0,T]$, $s^+(t)$, $s^-(t)$ and the spread $s^+(t)-s^-(t)$ are derived by direct formulae after integration of the Stefan condition in $[0,t]$.*

Remark 2.2. *We observe that the transformed spdes of Cases 1,2,3 are of the general form (2.2), i.e.,*

$$v_t(y,t) = \alpha\Delta v(y,t) \mp \nabla v(0^+,t)\nabla v(y,t) \pm \sigma(y)\dot{W}(y,t) + \dot{\eta}(y,t),$$

posed on $\mathcal{D} := (0,\lambda)$ for $t \in [0,T]$, with Dirichlet b.c. $v(x,t) = 0$ at $\partial\mathcal{D}$, and $v(y,0)$ given, for $v \geq 0$ when η not the zero measure, and signed v when $\eta \equiv 0$.

Remark 2.3. *Given $v_{1,2}$ for any $y \in \mathcal{D}$ and any t in $[0,T]$, then the Stefan condition will determine after integration $s^{\pm}(t)$ in $(0,T]$. Let $x \in \Omega_{\text{Liq}}$ then for any given $t \in [0,T]$ and any $x \geq s^+(t)$ since $y = x-s^+(t)$, $w(x,t)$ will be defined by $v_1(x-s^+(t),t)$, while for any $x \leq s^-(t)$ since $y = -x+s^-(t)$, $w(x,t)$ will be defined by $-v_2(-x+s^-(t),t)$ for case (1) or by $v_2(-x+s^-(t),t)$ for Cases 2,3.*

Remark 2.4. *Evolution for $v_{1,2}$ will be observed as long as $a \leq s^- \leq s^+ \leq b$, while $a \leq x \leq b$. In particular, consider $y = \lambda = b-a$. Then if $x \geq s^+$ then $y = b-a = x-s^+ \leq b-s^+$ will yield $-a \leq -s^+$ i.e., $s^+ \leq a$ and thus $s^+ = a$ and $s^- = s^+ = a$ and $x = b$ which is the case when the the spread is zero and hits the boundary at b and $v_1(\lambda,t) = 0$. If $x \leq s^-$ then $y = b-a = -x+s^- \leq -a+s^-$ will yield $b \leq s^-$ and thus $s^- = b$ and $s^+ = s^- = b$ and $x = a$ which is the case when the the spread is zero and hits the boundary at a and $v_2(\lambda,t) = 0$. When $y = 0$ then either $x = s^-$ and $v_2(0,t) = 0$ or $x = s^+$ and $v_1(0,t) = 0$. For all $x \in (s^-,s^+)$ the density $w(x,t)$ will be set to 0. The initial values of $v_{1,2}$ are well defined through the initial value $w(x,0)$ which is given for all $x \in \mathbb{R}$. We assume that $w(x,0)$ is compactly supported in $\overline{\Omega}$ to obtain a compatibility condition to $v_{1,2}(\lambda,t) = 0$ at $t = 0$.*

We will analyze in detail in the sequel how the restrictions of a non-negative spread and spread area in the domain Ω, i.e., $a < s^- \leq s^+ < b$, reduce the stopping time up to which maximal solutions $w_{1,2}$ exist.

3. Existence of maximal solutions with reflection

In what follows we shall present the analytical proof of existence of unique maximal solutions (v,η) for the initial and boundary value problem for

$$v_t(y,t) = \alpha\Delta v(y,t) - \nabla v(0^+,t)\nabla v(y,t) + \sigma(y)\dot{W}(y,t) + \dot{\eta}(y,t), \tag{3.1}$$

posed for any y in $\mathcal{D} = (0,\lambda)$ for $t \in [0,T]$ with Dirichlet b.c., with $v(y,0) \geq 0$ given, and η a reflection measure satisfying (2.19) and (2.20) keeping v non-negative. As $\alpha > 0$ the proof for the 2d i.b.v. problem of (2.10) of Case 1 is completely analogous, while the results for Case 3 (unreflected problem) will be derived at a next section by setting $\eta \equiv 0$. We will keep the absolute values on $\nabla v(0^+,t)$ appearing in the following proofs (even if non-negative in (3.1)) so that the results are applicable for these cases directly.

3.1. **Weak formulation.** Let us define an $L^2(\mathcal{D})$ basis of eigenfunctions $w_n := \sin\Big(\frac{n\pi}{\lambda}x\Big)$, $n = 0, 1, 2, \cdots$, corresponding to the eigenvalues μ_n, $n = 0, 1, \cdots$ of $-\Delta u = \mu u$, $u(0) = 0$, $u(\lambda) = 0$, where $\mu_n := \frac{n^2\pi^2}{\lambda^2}$, $n = 0, 1, 2, \cdots$ The associate Green's function for the negative of the Dirichlet Laplacian can then be given by $\frac{2}{\lambda}\sum_{n=0}^{\infty} e^{-\mu_n t} w_n(x)w_n(y)$, see [11], so that the Green's function corresponding to $-\alpha\Delta$ with Dirichlet b.c. is given by $G(x, y, t) = \frac{2}{\lambda}\sum_{n=0}^{\infty} e^{-\alpha\mu_n t} w_n(x)w_n(y)$.

We say that v is a weak (analytic) solution of (3.1) if it satisfies for all $\phi = \phi(y)$ in $C^2(\overline{\mathcal{D}})$ with $\phi(0) = \phi(\lambda) = 0$, the following weak formulation

$$
\begin{aligned}
\int_{\mathcal{D}} \Big(v(y,t) - v_0(y)\Big)\phi(y)dy = &\int_0^t \int_{\mathcal{D}} \Big(\alpha\Delta\phi(y)v(y,s) + \nabla\phi(y)\nabla v(0^+, s)v(y,s)\Big)dyds \\
&+ \int_0^t \int_{\mathcal{D}} \phi(y)\sigma(y)W(dy,ds) + \int_0^t \int_{\mathcal{D}} \phi(y)\eta(dy,ds) \quad \text{for all } t \in (0,T).
\end{aligned}
\tag{3.2}
$$

The solution of (3.1) admits for any $y \in \mathcal{D}$, $t \in [0,T]$, the next integral representation

$$
\begin{aligned}
v(y,t) = &\int_{\mathcal{D}} v_0(z)G(y,z,t)dz \\
&+ \int_0^t \int_{\mathcal{D}} \nabla v(0^+, s)\nabla G(y,z,t-s)v(z,s)dzds \\
&+ \int_0^t \int_{\mathcal{D}} G(y,z,t-s)\sigma(z)W(dz,ds) + \int_0^t \int_{\mathcal{D}} G(y,z,t-s)\eta(dz,ds),
\end{aligned}
\tag{3.3}
$$

and η satisfies (2.19), (2.20).

3.2. **Main Theorems.** Let the Banach space $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$

$$
\mathcal{B} := \Big\{ f \in C(\overline{\mathcal{D}}) : \ \exists f'(0), \ f(0) = f(\lambda) = 0 \Big\},
$$

with the norm $\|\cdot\|_{\mathcal{B}} : \mathcal{B} \to \mathbb{R}^+$, defined by

$$
\|f\|_{\mathcal{B}} := \sup_{y \in \mathcal{D}} \Big| \frac{f(y)}{y} \Big|.
$$

Let $M > 0$ fixed, we define in the Banach space $\mathcal{B}$, as in [16], the operator $\mathcal{T}_M : \mathcal{B} \to \mathcal{B}$ given for any $y \in \overline{\mathcal{D}}$ and $u \in \mathcal{B}$ by

$$
\mathcal{T}_M(u)(y,\cdot) = \begin{cases} y \min\left\{ \frac{u(y,\cdot)}{y}, M \right\} & y \neq 0, \\ 0 & y = 0. \end{cases}
\tag{3.4}
$$

We also define as $L^p(\mathbf{\Omega}, C[0,T]; \mathcal{B})$, the space of functions $f : \overline{\mathcal{D}} \times [0,T] \to \mathbb{R}$ such that $f(\cdot,t) \in \mathcal{B}$ $\forall\, t \in [0,T]$, $f(y,\cdot) \in C[0,T]$ $\forall\, y \in \overline{\mathcal{D}}$, and satisfy for any $t \in [0,T]$ that

$$
\|f(\cdot,t)\|_{L^p(\mathbf{\Omega};\mathcal{B})} := \Big(E(\|f(\cdot,t)\|^p_{\mathcal{B}})\Big)^{1/p} < \infty,
$$

for $\|\cdot\|_{\mathcal{B}}$ the norm of the Banach space $\mathcal{B}$.

We consider a truncated problem through the action of the operator $\mathcal{T}_M$ on the gradient terms of the spde (3.1) for which we prove the next existence-uniqueness theorem.

Theorem 3.1. *Let the noise diffusion σ satisfy*

$$\sigma \in C(\overline{\mathcal{D}}), \;\; \sigma(0) = \sigma(\lambda) = 0, \;\; \exists\, \sigma'(0). \tag{3.5}$$

Let also a deterministic $M > 0$ fixed, and some $p \geq p_0 > 8$, and let $v_0 \in C_c^\infty(\overline{\mathcal{D}})$ be the initial condition of (3.1)*. Then there exists a unique weak solution (v^M, η^M) with $v^M \in L^p(\mathbf{\Omega}, C[0,T]; \mathcal{B})$, depending on M, to the truncated problem*

$$\begin{aligned} & v_t^M(y,t) = \alpha \Delta v^M(y,t) - \nabla(\mathcal{T}_M(v^M))(0^+,t)\nabla(\mathcal{T}_M(v^M))(y,t) \\ & \qquad\qquad + \sigma(y)\dot{W}(y,t) + \dot{\eta}^M(y,t), \;\; t \in (0,T], \;\; y \in \mathcal{D}, \\ & v^M(y,0) := v_0(y), \;\; y \in \mathcal{D}, \\ & v^M(0,t) = v^M(\lambda,t) = 0, \;\; t \in (0,T], \end{aligned} \tag{3.6}$$

where $T := T_M > 0$ such that

$$\sup_{r \in (0,T)} |\nabla(\mathcal{T}_M(v^M))(0^+,r)|^p < \infty \;\; a.s. \tag{3.7}$$

More specifically, for any $t \in (0,T)$, v^M satisfies the weak formulation

$$\begin{aligned} v^M(y,t) = & \int_{\mathcal{D}} v_0(z) G(y,z,t) dz \\ & + \int_0^t \int_{\mathcal{D}} \nabla(\mathcal{T}_M(v^M))(0^+,s) \nabla G(y,z,t-s) \mathcal{T}_M(v^M)(z,s) dz ds \\ & + \int_0^t \int_{\mathcal{D}} G(y,z,t-s)\sigma(z) W(dz,ds) \\ & + \int_0^t \int_{\mathcal{D}} G(y,z,t-s)\eta^M(dz,ds), \end{aligned} \tag{3.8}$$

for $v^M(y,0) := v_0(y)$, and η^M satisfies (2.19) *and* (2.20)*, i.e.,*

$$\begin{aligned} & \textit{for all measurable functions } \psi : \; \overline{\mathcal{D}} \times (0,T) \to [0,\infty) \\ & \int_0^t \int_{\mathcal{D}} \psi(y,s)\eta^M(dy,ds) \;\; \textit{is } \mathcal{F}_t - \textit{measurable}, \end{aligned} \tag{3.9}$$

and the constraint

$$\int_0^T \int_{\mathcal{D}} v^M(y,s)\eta^M(dy,ds) = 0. \tag{3.10}$$

Proof. Note that for a given bounded M an upper bound for the supremum in (3.7) should be a bounded constant $C_2(M,p)$ depending on M and p.

The weak formulation (3.8) is equivalent to the so-called mild solution formulation for the stochastic equation.

The operator $\mathcal{T}_M : \mathcal{B} \to \mathcal{B}$ is well defined, [16], and thus, for any u in the space $\mathcal{B}$, $\mathcal{T}_M(u)$ returns in $\mathcal{B}$, and so $\mathcal{T}_M(u) \in C(\overline{\mathcal{D}})$ and vanishes at the boundary of $\mathcal{D}$, while the gradient $\nabla(\mathcal{T}_M(u))$ at $x = 0$ exists.

Motivated by the integral representation (3.8), we define through an iteration scheme the approximation v_n^M of v^M as the solution of the approximate problem

$$\begin{aligned} v_n^M(y,t) &= \int_{\mathcal{D}} v_0(z)G(y,z,t)dz \\ &+ \int_0^t \int_{\mathcal{D}} \nabla(\mathcal{T}_M(v^M))(0^+,s)\nabla G(y,z,t-s)\mathcal{T}_M(v_{n-1}^M)(z,s)dzds \\ &+ \int_0^t \int_{\mathcal{D}} G(y,z,t-s)\sigma(z)W(dz,ds) \\ &+ \int_0^t \int_{\mathcal{D}} G(y,z,t-s)\eta_n^M(dz,ds), \quad n := 1,2\cdots \end{aligned} \tag{3.11}$$

for $v_0^M(y,t) := v_0(y)$, and η_n^M, which approximates η^M, satisfying (2.19) and (2.20), i.e.,

$$\begin{aligned} &\text{for all measurable functions } \psi : \overline{\mathcal{D}} \times (0,T) \to [0,\infty) \\ &\int_0^t \int_{\mathcal{D}} \psi(y,s)\eta_n^M(dy,ds) \ \text{ is } \ \mathcal{F}_t - \text{measurable}, \end{aligned} \tag{3.12}$$

and the constraint

$$\int_0^T \int_{\mathcal{D}} v_n^M(y,s)\eta_n^M(dy,ds) = 0. \tag{3.13}$$

In order to keep v_n^M non-negative, and having in mind the integral property (3.13), we will absorb the reflection term η_n^M in the scheme (3.11), by splitting v_n^M as follows

$$v_n^M(y,t) = u_n(y,t) + \mathbb{O}_n(y,t), \tag{3.14}$$

where $\mathbb{O}_n(y,t)$ solves in the weak sense the Heat Equation Obstacle problem for any $y \in \mathcal{D}$, $t \in [0,T]$

$$\begin{aligned} &\partial_t \mathbb{O}_n(y,t) = \alpha\Delta\mathbb{O}_n(y,t) + \tilde{\eta}_n(dy,dt), \quad u_n + \mathbb{O}_n \geq 0 (\Leftrightarrow \mathbb{O}_n \geq -u_n), \\ &\mathbb{O}_n(0,t) = \mathbb{O}_n(\lambda,t) = 0, \quad \mathbb{O}_n(y,0) = 0, \\ &\int_0^T \int_{\mathcal{D}} (u_n(y,s) + \mathbb{O}_n(y,s))\tilde{\eta}_n(dy,ds) = 0. \end{aligned} \tag{3.15}$$

Note that the above problem has a unique weak solution $(\mathbb{O}_n, \tilde{\eta}_n)$ as long as u_n exists and is smooth, see in [16] and the references therein. We observe that $\mathbb{O}_n(y,0) = 0$ yields that $u_n(y,0) = v_n^M(y,0) = v_0(y)$.

We define $\eta_n^M := \tilde{\eta}_n$, and as we shall see it satisfies (3.11) when v_n^M satisfies (3.14). Indeed, we replace $v_n^M = u_n(y,t) + \mathbb{O}_n(y,t)$ at the left-hand side of (3.11) and obtain for $\eta_n^M := \tilde{\eta}_n$

$$\begin{aligned} u_n(y,t) + \mathbb{O}_n(y,t) &= \int_{\mathcal{D}} v_0(z) G(y,z,t) dz \\ &+ \int_0^t \int_{\mathcal{D}} \nabla(\mathcal{T}_M(v^M))(0^+,s) \nabla G(y,z,t-s) \mathcal{T}_M(v_{n-1}^M)(z,s) dz ds \\ &+ \int_0^t \int_{\mathcal{D}} G(y,z,t-s) \sigma(z) W(dz,ds) \\ &+ \int_0^t \int_{\mathcal{D}} G(y,z,t-s) \eta_n^M(dz,ds), \\ &= \int_{\mathcal{D}} v_0(z) G(y,z,t) dz \\ &+ \int_0^t \int_{\mathcal{D}} \nabla(\mathcal{T}_M(v^M))(0^+,s) \nabla G(y,z,t-s) \mathcal{T}_M(v_{n-1}^M)(z,s) dz ds \\ &+ \int_0^t \int_{\mathcal{D}} G(y,z,t-s) \sigma(z) W(dz,ds) \\ &+ \int_0^t \int_{\mathcal{D}} G(y,z,t-s) \tilde{\eta}_n(dz,ds) \quad n := 1,2\cdots \end{aligned} \tag{3.16}$$

Since $\mathbb{O}_n$ solves in the weak sense (3.15), and $\mathbb{O}_n(y,0) = 0$, then using the same Green's function G for the integral representation of $\mathbb{O}_n$, we see that the last term of (3.16) coincides with $\mathbb{O}_n(y,t)$, so we obtain

$$\begin{aligned} u_n(y,t) = &\int_{\mathcal{D}} v_0(z) G(y,z,t) dz \\ &+ \int_0^t \int_{\mathcal{D}} \nabla(\mathcal{T}_M(v^M))(0^+,s) \nabla G(y,z,t-s) \mathcal{T}_M(v_{n-1}^M)(z,s) dz ds \\ &+ \int_0^t \int_{\mathcal{D}} G(y,z,t-s) \sigma(z) W(dz,ds), \quad n := 1,2\cdots \end{aligned} \tag{3.17}$$

We split now v^M by

$$v^M(y,t) = u(y,t) + \mathbb{O}(y,t), \tag{3.18}$$

and set $\eta^M := \tilde{\eta}$, where $(\mathbb{O}(y,t), \tilde{\eta}(y,t))$ solves in the weak sense the Heat Equation Obstacle problem for any $y \in \mathcal{D}$, $t \in [0,T]$

$$\begin{aligned} &\partial_t \mathbb{O}(y,t) = \alpha \Delta \mathbb{O}(y,t) + \tilde{\eta}(dy,dt), \quad u + \mathbb{O} \geq 0 (\Leftrightarrow \mathbb{O} \geq -u), \\ &\mathbb{O}(0,t) = \mathbb{O}(\lambda,t) = 0, \quad \mathbb{O}(y,0) = 0, \\ &\int_0^T \int_{\mathcal{D}} (u(y,s) + \mathbb{O}(y,s)) \tilde{\eta}(dy,ds) = 0. \end{aligned} \tag{3.19}$$

We observe that $\mathbb{O}(y,0)=0$ yields that $u(y,0)=v^M(y,0)=v_0(y)$, and as we argued for the derivation of (3.17), we obtain that u satisfies

$$\begin{aligned} u(y,t)=&\int_{\mathcal{D}} v_0(z)G(y,z,t)dz \\ &+\int_0^t\int_{\mathcal{D}}\nabla(\mathcal{T}_M(v^M))(0^+,s)\nabla G(y,z,t-s)\mathcal{T}_M(v^M)(z,s)dzds \\ &+\int_0^t\int_{\mathcal{D}} G(y,z,t-s)\sigma(z)W(dz,ds). \end{aligned} \tag{3.20}$$

Using (3.17) for u_n, u_{n-1}, by subtraction, we get for $n=2,3,\cdots$

$$\begin{aligned} u_n(y,t)-u_{n-1}(y,t)=&\int_0^t\int_{\mathcal{D}}\Big[\nabla(\mathcal{T}_M(v^M))(0^+,s)\mathcal{T}_M(v^M_{n-1})(z,s) \\ &-\nabla(\mathcal{T}_M(v^M))(0^+,s)\mathcal{T}_M(v^M_{n-2})(z,s)\Big]\nabla G(y,z,t-s)dzds \\ =&\int_0^t\int_{\mathcal{D}}\Big[\nabla(\mathcal{T}_M(v^M))(0^+,s)z\min\Big\{\frac{v^M_{n-1}(z,s)}{z},M\Big\} \\ &-\nabla(\mathcal{T}_M(v^M))(0^+,s)z\min\Big\{\frac{v^M_{n-2}(z,s)}{z},M\Big\}\Big]\nabla G(y,z,t-s)dzds. \end{aligned} \tag{3.21}$$

In the above, we apply $\|\cdot\|_{\mathcal{B}}$-norm at both sides and then take p-powers for some $p>0$, and then $\sup\limits_{t\in(0,T)}$, and then expectation, to obtain for $n=2,3,\cdots$

$$\begin{aligned} &E\Big(\sup_{t\in(0,T)}\|u_n(\cdot,t)-u_{n-1}(\cdot,t)\|^p_{\mathcal{B}}\Big) \\ =&E\Big(\sup_{t\in(0,T)}\Big\|\int_0^t\int_{\mathcal{D}}\Big[\nabla(\mathcal{T}_M(v^M))(0^+,s)z\min\Big\{\frac{v^M_{n-1}(z,s)}{z},M\Big\} \\ &-\nabla(\mathcal{T}_M(v^M))(0^+,s)z\min\Big\{\frac{v^M_{n-2}(z,s)}{z},M\Big\}\Big]\nabla G(y,z,t-s)dzds\Big\|^p_{\mathcal{B}}\Big). \end{aligned} \tag{3.22}$$

In [16], various useful bounds were proven in the norm $\|\cdot\|_{\mathcal{B}}$ for the heat kernel G defined explicitly by a different series representation than the standard trigonometric series (bounds holding obviously true for αt in place of the time variable t, and $\mathcal{D}=(0,\lambda)$ in place of $(0,1)$ there). In particular, we use the estimate of Proposition 4.4. therein, to derive directly for some constant $c=c(T,p)>0$, that

$$E(\sup_{t\in(0,T)}\|J\|^p_{\mathcal{B}})\leq c(T,p)\int_0^T E(\sup_{\tau\in(0,s)}\|f(\cdot,\tau)\|^p_{\mathcal{B}})ds,$$

for

$$J(y,t):=\int_0^t\int_{\mathcal{D}} f(z,s)\nabla G(y,z,t-s)dzds,$$

and

$$\begin{aligned} f(z,s):=&\nabla(\mathcal{T}_M(v^M))(0^+,s)z\min\Big\{\frac{v^M_{n-1}(z,s)}{z},M\Big\} \\ &-\nabla(\mathcal{T}_M(v^M))(0^+,s)z\min\Big\{\frac{v^M_{n-2}(z,s)}{z},M\Big\}. \end{aligned}$$

Using the above in (3.22) yields for $n = 2, 3, \cdots$

$$
\begin{aligned}
&E\Big(\sup_{t\in(0,T)} \|u_n(\cdot,t) - u_{n-1}(\cdot,t)\|^p_{\mathcal{B}}\Big) \\
&\leq cC(T,p)\int_0^T E\Big(\sup_{\tau\in(0,s)}\sup_{z\in\mathcal{D}}\Big|\nabla(\mathcal{T}_M(v^M))(0^+,\tau)\min\Big\{\frac{v^M_{n-1}(z,\tau)}{z}, M\Big\} \\
&\qquad\qquad - \nabla(\mathcal{T}_M(v^M))(0^+,\tau)\min\Big\{\frac{v^M_{n-2}(z,\tau)}{z}, M\Big\}\Big|^p\Big)ds \qquad (3.23)\\
&= cC(T,p)\int_0^T E\Big(\sup_{\tau\in(0,s)}\sup_{z\in\mathcal{D}}\Big|\nabla(\mathcal{T}_M(v^M))(0^+,\tau)\Big|^p \\
&\qquad\qquad \Big|\min\Big\{\frac{v^M_{n-1}(z,\tau)}{z}, M\Big\} - \min\Big\{\frac{v^M_{n-2}(z,\tau)}{z}, M\Big\}\Big|^p\Big)ds.
\end{aligned}
$$

Observing that $a \leq M$ and $b \leq M$ yields $\min\{a, M\} - \min\{b, M\} = a - b$, while $a \geq M$ and $b \geq M$ yields $\min\{a, M\} - \min\{b, M\} = M - M = 0 \leq |a - b|$, while $a \leq M$ and $b \geq M$ yields $\min\{a, M\} - \min\{b, M\} = a - M \leq 0 \leq |a - b|$, we have

$$|\min\{a, M\} - \min\{b, M\}| \leq |a - b|,$$

and so, we obtain by (3.23) for $n = 2, 3, \cdots$

$$
\begin{aligned}
&E\Big(\sup_{t\in(0,T)} \|u_n(\cdot,t) - u_{n-1}(\cdot,t)\|^p_{\mathcal{B}}\Big) \\
&\leq cC(T,p)\int_0^T E\Big(\sup_{\tau\in(0,s)}\sup_{z\in\mathcal{D}}\Big|\nabla(\mathcal{T}_M(v^M))(0^+,\tau)\Big|^p \\
&\qquad\qquad \Big|\min\Big\{\frac{v^M_{n-1}(z,\tau)}{z}, M\Big\} - \min\Big\{\frac{v^M_{n-2}(z,\tau)}{z}, M\Big\}\Big|^p\Big)ds \qquad (3.24)\\
&\leq cC(T,p)\int_0^T E\Big(\sup_{\tau\in(0,s)}|\nabla(\mathcal{T}_M(v^M))(0^+,\tau)|^p \sup_{\tau\in(0,s)}\sup_{z\in\mathcal{D}}\Big|\frac{v^M_{n-1}(z,\tau)}{z} - \frac{v^M_{n-2}(z,\tau)}{z}\Big|^p\Big)ds \\
&= cC(T,p)\int_0^T E\Big(\sup_{\tau\in(0,s)}|(\nabla(\mathcal{T}_M(v^M))(0^+,\tau)|^p \sup_{\tau\in(0,s)}\|v^M_{n-1}(\cdot,\tau) - v^M_{n-2}(\cdot,\tau)\|^p_{\mathcal{B}}\Big)ds.
\end{aligned}
$$

But by (3.14), it holds that

$$
\begin{aligned}
\sup_{\tau\in(0,s)}\sup_{z\in\mathcal{D}}\Big|\frac{v^M_{n-1}(z,\tau)}{z} - \frac{v^M_{n-2}(z,\tau)}{z}\Big|^p &\leq c(p)\sup_{\tau\in(0,s)}\sup_{z\in\mathcal{D}}\Big|\frac{u_{n-1}(z,\tau)}{z} - \frac{u_{n-2}(z,\tau)}{z}\Big|^p \\
&\quad + c(p)\sup_{\tau\in(0,s)}\sup_{z\in\mathcal{D}}\Big|\frac{\mathbb{O}_{n-1}(z,\tau)}{z} - \frac{\mathbb{O}_{n-2}(z,\tau)}{z}\Big|^p \qquad (3.25)\\
&\leq c(p)\sup_{\tau\in(0,s)}\sup_{z\in\mathcal{D}}\Big|\frac{u_{n-1}(z,\tau)}{z} - \frac{u_{n-2}(z,\tau)}{z}\Big|^p,
\end{aligned}
$$

where for the last inequality we used the stability bound in $\sup_{\tau\in(0,s)}\sup_{z\in\mathcal{D}}$ of the obstacle problem solutions by the obstacle, cf. [16] in the proof of Theorem 3.2. So, for $s = T$, we obtain by bounding the last term of (3.24) by (3.25), and for $n = 2, 3, \cdots$

(3.26)
$$\begin{aligned}
c(p)E\Big(\sup_{t\in(0,T)}\|v_n^M(\cdot,t)-v_{n-1}^M(\cdot,t)\|_{\mathcal{B}}^p\Big) &\le E\Big(\sup_{t\in(0,T)}\|u_n(\cdot,t)-u_{n-1}(\cdot,t)\|_{\mathcal{B}}^p\Big)\\
&\le c(p)2^pC(T,p)\int_0^T E\Big(\sup_{\tau\in(0,s)}|\nabla(\mathcal{T}_M(v^M))(0^+,\tau)|^p\sup_{\tau\in(0,s)}\|v_{n-1}^M(\cdot,\tau)-v_{n-2}^M(\cdot,\tau)\|_{\mathcal{B}}^p\Big)ds\\
&\le C(T,p)\int_0^T E\Big(\sup_{\tau\in(0,s)}|\nabla(\mathcal{T}_M(v^M))(0^+,\tau)|^p\sup_{\tau\in(0,s)}\|u_{n-1}(\cdot,\tau)-u_{n-2}(\cdot,\tau)\|_{\mathcal{B}}^p\Big)ds.
\end{aligned}$$

We apply the same argumentation as for deriving the above inequality, on (3.17) now. By using that $v_0 \in C_c^\infty(\overline{\mathcal{D}})$, and the estimate of Proposition 4.3 from [16], as $p > 8 > 2$, we obtain for $n = 1, 2, \cdots$

$$\begin{aligned}
c(p)E\Big(\sup_{t\in(0,T)}\|v_n^M(\cdot,t)\|_{\mathcal{B}}^p\Big) &\le E\Big(\sup_{t\in(0,T)}\|u_n(\cdot,t)\|_{\mathcal{B}}^p\Big)\\
&\le C(T,p)\int_0^T E\Big(\sup_{\tau\in(0,s)}|\nabla(\mathcal{T}_M(v^M))(0^+,\tau)|^p\sup_{\tau\in(0,s)}\|v_{n-1}^M(\cdot,\tau)\|_{\mathcal{B}}^p\Big)ds\\
&\quad + CT\sup_{\tau\in(0,T)}\|\sigma(\cdot,\tau)\|_{\mathcal{B}}^p + C\\
&\le C(T,p)\int_0^T E\Big(\sup_{\tau\in(0,s)}|\nabla(\mathcal{T}_M(v^M))(0^+,\tau)|^p\sup_{\tau\in(0,s)}\|u_{n-1}(\cdot,\tau)\|_{\mathcal{B}}^p\Big)ds\\
&\quad + CT\sup_{\tau\in(0,T)}\|\sigma(\cdot,\tau)\|_{\mathcal{B}}^p + C.
\end{aligned}\tag{3.27}$$

Here, we used (3.14) and the bound of the solution $\mathbb{O}_n$ by the obstacle (comparing with the zero solution) for the first and the third inequality. Moreover, since $p > 8$ we applied the estimate of Proposition 4.5 in [16] to bound the noise term. In the above, note that since σ satisfies (3.5), i.e., $\sigma \in C(\overline{\mathcal{D}})$, $\sigma(0) = \sigma(\lambda) = 0$, and that it exists $\sigma'(0)$, then $\|\sigma\|_{\mathcal{B}}$ is well defined. This assumption models a zero volatility at the boundary of $\mathcal{D}$ in accordance to the Dirichlet b.c. for the density of the transactions v, i.e., the solution of (3.1), that vanishes on $\partial\mathcal{D}$.

Thus, by (3.27), since σ satisfies (3.5), v_n^M, u_n stay in $L^p(\mathbf{\Omega}, C[0,T];\mathcal{B})$ if $v_0^M(y,t) := v_0(y) \in C_c^\infty(\overline{\mathcal{D}})$ and $T = T_M$ such that (3.7) holds true, i.e.,

$$\sup_{r\in(0,T)}|\nabla(\mathcal{T}_M(v^M))(0^+,r)|^p \le C_2(M,p) < \infty \ \ a.s.$$

Furthermore, by (3.26), we get that v_n^M, u_n are Cauchy in $L^p(\mathbf{\Omega}, C[0,T];\mathcal{B})$ while they also stay in $L^p(\mathbf{\Omega}, C[0,T];\mathcal{B})$, for $T = T_M$. So, by the completeness of the Banach space $\mathcal{B}$ in this norm, both they converge in $L^p(\mathbf{\Omega}, C[0,T];\mathcal{B})$ to some unique $\hat{v}^M, \hat{u} \in L^p(\mathbf{\Omega}, C[0,T];\mathcal{B})$ as $n \to \infty$. In details, by using (3.26), we obtain

$$
\begin{aligned}
&E\Big(\sup_{t\in(0,T)}\|v_n^M(\cdot,t)-v_{n-1}^M(\cdot,t)\|_{\mathcal{B}}^p\Big)\leq CE\Big(\sup_{t\in(0,T)}\|u_n(\cdot,t)-u_{n-1}(\cdot,t)\|_{\mathcal{B}}^p\Big)\\
&\leq \tilde{c}\int_0^T E\Big(\sup_{\tau\in(0,s)}|\nabla(\mathcal{T}_M(v^M))(0^+,\tau)|^p\sup_{\tau\in(0,s)}\|v_{n-1}^M(\cdot,\tau)-v_{n-2}^M(\cdot,\tau)\|_{\mathcal{B}}^p\Big)ds\\
&\leq \tilde{c}\int_0^T E\Big(\sup_{\tau\in(0,s)}\|v_{n-1}^M(\cdot,\tau)-v_{n-2}^M(\cdot,\tau)\|_{\mathcal{B}}^p\Big)ds\\
&\leq c^{n-1}\int_0^T\int_0^{s_{n-1}}\int_0^{s_{n-2}}\cdots\int_0^{s_2}1d_{s_1}\cdots d_{s_{n-2}}d_{s_{n-1}}E\Big(\sup_{\tau\in(0,T)}\|v_1^M(\cdot,\tau)-v_0^M(\cdot,\tau)\|_{\mathcal{B}}^p\Big)\\
&\leq C\frac{c^{n-1}}{(n-1)!}\to 0\quad as\;\; n\to\infty,
\end{aligned}
\tag{3.28}
$$

where we used that $v_1^M\in L^p(\mathbf{\Omega},C[0,T];\mathcal{B})$ since v_n^M stays in $L^p(\mathbf{\Omega},C[0,T];\mathcal{B})$ for all n if $v_0^M(y,t):=v_0(y)\in C_c^\infty(\overline{\mathcal{D}})$. Therefore, v_n^M, u_n are Cauchy in $L^p(\mathbf{\Omega},C[0,T];\mathcal{B})$.

Moreover, we also obtain, as in (3.27)

$$
\begin{aligned}
&E\Big(\sup_{t\in(0,T)}\|v^M(\cdot,t)\|_{\mathcal{B}}^p\Big)\leq CE\Big(\sup_{t\in(0,T)}\|u(\cdot,t)\|_{\mathcal{B}}^p\Big)\\
&\leq C\int_0^T E\Big(\sup_{\tau\in(0,s)}|\nabla(\mathcal{T}_M(v^M))(0^+,\tau)|^p\sup_{\tau\in(0,s)}\|v^M(\cdot,\tau)\|_{\mathcal{B}}^p\Big)ds\\
&\quad+CT\sup_{\tau\in(0,T)}\|\sigma(\cdot,\tau)\|_{\mathcal{B}}^p+C\\
&\leq C\int_0^T E\Big(\sup_{\tau\in(0,s)}|\nabla(\mathcal{T}_M(v^M))(0^+,\tau)|^p\sup_{\tau\in(0,s)}\|u(\cdot,\tau)\|_{\mathcal{B}}^p\Big)ds\\
&\quad+CT\sup_{\tau\in(0,T)}\|\sigma(\cdot,\tau)\|_{\mathcal{B}}^p+C.
\end{aligned}
\tag{3.29}
$$

Therefore, by Gronwall's inequality on (3.29), and using (3.7), (3.5), we arrive at

$$
E\Big(\sup_{t\in(0,T)}\|v^M(\cdot,t)\|_{\mathcal{B}}^p\Big)+E\Big(\sup_{t\in(0,T)}\|u(\cdot,t)\|_{\mathcal{B}}^p\Big)\leq C.
\tag{3.30}
$$

Thus, we get that $v^M,u\in L^p(\mathbf{\Omega},C[0,T];\mathcal{B})$. By subtraction of the integral representations (3.17), (3.20), we obtain (as previously when deriving the 4th inequality of (3.26)), that u_n,u satisfy for $n=1,2,\cdots$

$$
\begin{aligned}
&E\Big(\sup_{t\in(0,T)}\|u_n(\cdot,t)-u(\cdot,t)\|_{\mathcal{B}}^p\Big)\\
&\leq C\int_0^T E\Big(\sup_{\tau\in(0,s)}|\nabla(\mathcal{T}_M(v^M))(0^+,\tau)|^p\sup_{\tau\in(0,s)}\|u_{n-1}(\cdot,\tau)-u(\cdot,\tau)\|_{\mathcal{B}}^p\Big)ds\\
&\leq\frac{C^{n-1}}{(n-1)!}E\Big(\sup_{t\in(0,T)}\|u_1(\cdot,t)-u(\cdot,t)\|_{\mathcal{B}}^p\Big)\to 0\;\; as\;\; n\to\infty.
\end{aligned}
\tag{3.31}
$$

Here, we used once again (3.5), (3.7), the argument for the last bound of (3.28), together with the fact that, as proven, $u_1,u\in L^p(\mathbf{\Omega},C[0,T];\mathcal{B})$. Therefore, for $T=T_M$, $u_n\to u$ as $n\to\infty$ in $L^p(\mathbf{\Omega},C[0,T];\mathcal{B})$. By uniqueness of the limits, we get that $\hat{u}=u$ a.s.

Since u exists uniquely, then the solution $(\mathbb{O},\tilde{\eta})$ of (3.19) exists uniquely. But it holds that

$$\begin{aligned}\sup_{\tau\in(0,s)}\sup_{z\in\mathcal{D}}\Big|\frac{v_n^M(z,\tau)}{z}-\frac{v^M(z,\tau)}{z}\Big|^p \leq & c\sup_{\tau\in(0,s)}\sup_{z\in\mathcal{D}}\Big|\frac{u_n(z,\tau)}{z}-\frac{u(z,\tau)}{z}\Big|^p\\ &+c\sup_{\tau\in(0,s)}\sup_{z\in\mathcal{D}}\Big|\frac{\mathbb{O}_n(z,\tau)}{z}-\frac{\mathbb{O}(z,\tau)}{z}\Big|^p\\ \leq & c\sup_{\tau\in(0,s)}\sup_{z\in\mathcal{D}}\Big|\frac{u_n(z,\tau)}{z}-\frac{u(z,\tau)}{z}\Big|^p,\end{aligned}$$

where again we used the stability bound in $\sup\limits_{\tau\in(0,s)}\sup\limits_{z\in\mathcal{D}}$ of the obstacle problem solutions by the obstacle, see in [16] in the proof of Theorem 3.2. So, for $T=T_M$, by taking expectation, since, as we have shown, $u_n\to u$ in $L^p(\mathbf{\Omega},C[0,T];\mathcal{B})$, we obtain that $v_n^M\to v^M$ as $n\to\infty$ in $L^p(\mathbf{\Omega},C[0,T];\mathcal{B})$. By uniqueness of the limits, we get that $\hat{v}^M=v^M$ a.s.

So, for all $T=T_M>0$ such that (4.3) holds true, we derive the following:

(1) v^M,u exist and belong in $L^p(\mathbf{\Omega},C[0,T];\mathcal{B})$.
(2) Since u exists, we may define $\eta^M:=\tilde{\eta}$ a.s. for $\tilde{\eta}$ the second term of the solution $(\mathbb{O},\tilde{\eta})$ of (3.19).
(3) The pair (v^M,η^M) exists, $v^M\in L^p(\mathbf{\Omega},C[0,T];\mathcal{B})$, and is the weak solution of the truncated problem (3.6) in $(0,T)$, where v^M, η^M satisfy the weak formulation (3.8) for $v^M(y,0):=v_0(y)$, where η^M satisfies (3.9) and (3.10).
(4) The pair (v^M,η^M) is unique. Indeed, uniqueness of the limit of v_n^M showed that v^M is unique. Uniqueness of η^M follows by the uniqueness of $\tilde{\eta}$ of the obstacle problem (3.19) since as we have shown u exists uniquely as the limit of u_n in $L^p(\mathbf{\Omega},C[0,T];\mathcal{B})$.

Therefore, by taking limits on the iteration scheme, there exists a unique solution (v^M,η^M) of the weak formulation (3.8) with $v^M\in L^p(\mathbf{\Omega},C[0,T];\mathcal{B})$ and with η^M satisfying (3.9) and (3.10), which completes the proof. □

We return to the M-independent problem (3.1), and we shall prove that it admits a unique maximal solution by concatenation of the solution of the M-truncated problem (3.6)-(3.9)-(3.10). This is established by the next Main Theorem.

Theorem 3.2. *Let the noise diffusion σ satisfy* (3.5), *and $v_0\in C_c^\infty(\overline{\mathcal{D}})$, and $p\geq p_0>8$. Then, there exists a unique weak maximal solution (v,η) to the problem* (3.1)-(2.19)-(2.20) *in the maximal interval $[0,\sup\limits_{M>0}\tau_M)$, where*

$$\begin{aligned}\tau_M:=&\inf\Big\{T\geq 0:\ \sup_{r\in(0,T)}|\nabla v(0^+,r)|\geq M\Big\}\\ =&\inf\Big\{T\geq 0:\ \sup_{r\in(0,T)}\nabla v(0^+,r)\geq M\Big\}.\end{aligned}\tag{3.32}$$

Proof. We note that for the reflected problem since $v\geq 0$ a.s. and $v(0,t)=0$, then $\nabla v(0^+,t)\geq 0$ a.s. for any t. As we mentioned, we continue to keep the absolute value on $\nabla v(0^+,t)$ in this proof also in order to present a more general result applicable to the 2d i.b.v. problem of (2.10), and to the problem without reflection of next section.

Let v^M as in Theorem 3.1. We observe first that by the operator definition (3.4), and since $v^M(0,t)=0$, we have

$$\sup_{r\in(0,T)}|\nabla(\mathcal{T}_M(v^M))(0^+,r)|^p \le \min\Big\{\sup_{r\in(0,T)}|\nabla v^M(0^+,r)|^p, M^p\Big\} \le M^p, \tag{3.33}$$

and so by (3.7), Theorem 3.1 holds also for any $T=T(M)$ such that

$$\min\Big\{\sup_{r\in(0,T)}|\nabla v^M(0^+,r)|, M\Big\} \le M < \infty \;\; a.s. \tag{3.34}$$

We fix $M>0$, and consider arbitrary $\tilde{M}>0$ such that $\tilde{M}\le M$. We define the (random) stopping time $\tilde{\tau}_M$ as follows

$$\begin{aligned}\tilde{\tau}_M :=& \inf\Big\{T\ge 0:\; \min\Big\{\sup_{r\in(0,T)}|\nabla v^M(0^+,r)|, M\Big\} \ge \tilde{M}\Big\}\\ =& \inf\Big\{T\ge 0:\; \sup_{r\in(0,T)}|\nabla v^M(0^+,r)| \ge \tilde{M}\Big\}.\end{aligned} \tag{3.35}$$

Let $\tau := \min\{\tilde{\tau}_M, \tilde{\tau}_{\tilde{M}}\}$, and an arbitrary deterministic $t>0$. Then for any $s\le\min\{t,\tau\}$, the term $\nabla\mathcal{T}_M(v^M)(0^+,s)$ would be equal $\nabla v^M(0^+,s)$ since

$$\nabla v^M(0^+,s) \le |\nabla v^M(0^+,s)| \le \tilde{M} \le M,$$

and the term $\nabla\mathcal{T}_{\tilde{M}}(v^{\tilde{M}})(0^+,s)$ would be equal to $\nabla v^{\tilde{M}}(0^+,s)$ since

$$\nabla v^{\tilde{M}}(0^+,s) \le |\nabla v^{\tilde{M}}(0^+,s)| \le \tilde{M}.$$

Thus, the weak solution (v^M,η^M) of the M-dependent problem (3.6)-(3.9)-(3.10) solves weakly the M-independent problem (3.1)-(2.19)-(2.20) or equivalently the problem (3.6)-(3.9)-(3.10) is not depending on M as $\nabla\mathcal{T}_M(v^M)(0^+,s) = \nabla v^M(0^+,s)$ there, and the weak solution $(v^{\tilde{M}},\eta^{\tilde{M}})$ of the $\tilde{M}$-dependent problem (3.6)-(3.9)-(3.10) solves weakly the $\tilde{M}$-independent same problem (3.1)-(2.19)-(2.20) or equivalently the problem (3.6)-(3.9)-(3.10) is not depending on $M:=\tilde{M}$ now, as $\nabla\mathcal{T}_{\tilde{M}}(v^{\tilde{M}})(0^+,s) = \nabla v^{\tilde{M}}(0^+,s)$ there (since they share the same initial condition). So, by the uniqueness of weak solutions they coincide for any $s\le\min\{t,\tau\}$, i.e.,

$$v^M(\cdot,s)=v^{\tilde{M}}(\cdot,s), \quad \eta^M(\cdot,s)=\eta^{\tilde{M}}(\cdot,s) \quad \forall\, s\le\min\{t,\tau\}.$$

Since t is a deterministic arbitrary constant, the above yields that

$$v^M(\cdot,s)=v^{\tilde{M}}(\cdot,s), \quad \eta^M(\cdot,s)=\eta^{\tilde{M}}(\cdot,s) \;\; \forall\, s\le\tau \;\; a.s.$$

So, the weak solutions of the M-truncated problem (3.6)-(3.9)-(3.10) are consistent, and we can proceed to concatenation.

Let us define the stochastic process (v,η) such that for all $M>0$ it coincides with the weak solution (v^M,η^M) of the M-truncated problem (3.6)-(3.9)-(3.10) until the stopping time

$$\begin{aligned}\tau_M =& \inf\Big\{T\ge 0:\; \sup_{r\in(0,T)}|\nabla v^M(0^+,r)| \ge M\Big\}\\ =& \inf\Big\{T\ge 0:\; \sup_{r\in(0,T)}|\nabla v(0^+,r)| \ge M\Big\}.\end{aligned}$$

By its definition, $(v(\cdot,s),\eta(\cdot,s))$ is a weak solution of the M-independent problem (3.1)-(2.19)-(2.20), for any $s\in[0,\sup_{M>0}\tau_M)$, and τ_M is a localising sequence. Then, $(v(\cdot,s),\eta(\cdot,s))$ is a maximal weak solution of (3.1)-(2.19)-(2.20), since

$$\lim_{t\to\left(\sup\limits_{M>0}\tau_M\right)^-}\sup_{r\in(0,t)}|\nabla v^M(0^+,r)|=\infty \ \ a.s.$$

Uniqueness of the maximal weak solution $(v(\cdot,s),\eta(\cdot,s))$ for $s\in[0,\sup\limits_{M>0}\tau_M)$, follows from the consistency of the solution of the M-truncated problem with which by its definition coincides. □

Let us now consider Case 1. The above analysis is valid for both i.b.v. problems of (2.10), and due to Theorem 3.2, and under its assumptions there exist unique weak maximal solutions (v_1,η_1), (v_2,η_2) satisfying (2.19)-(2.20) in the maximal interval $[0,\sup\limits_{M>0}\tau_{1M})$, where

$$\begin{aligned}\tau_{1M}&:=\inf\Big\{T\geq 0:\ \sup_{r\in(0,T)}(|\nabla v_1(0^+,r)|+|\nabla v_2(0^+,r)|)\geq M\Big\}\\&=\inf\Big\{T\geq 0:\ \sup_{r\in(0,T)}(\nabla v_1(0^+,r)+\nabla v_2(0^+,r))\geq M\Big\}.\end{aligned}\tag{3.36}$$

Recall that $\Omega=(a,b)$, $\lambda=b-a$, and $a\leq s^-(0)\leq s^+(0)\leq b$. We need $a\leq s^-(t)\leq s^+(t)\leq b$ in order to return to the initial variables. This will restrict the stopping time. By using the Stefan condition (2.4) we obtain

$$\partial_t s^-(t)=-\nabla v_2(0^+,t)\leq 0,\ \ \partial_t s^+(t)=-\nabla v_1(0^+,t)\leq 0,$$

and so

$$s^-(t)\leq s^-(0)\leq b,\ \ s^+(t)\leq s^+(0)\leq b,$$

so we need $a\leq s^-(t)\leq s^+(t)$ which yields

$$a\leq s^-(0)-\int_0^t\nabla v_2(0^+,s)ds\leq s^+(0)-\int_0^t\nabla v_1(0^+,s)ds.\tag{3.37}$$

We define the stopping time

$$\tau_{1s}:=\inf\Big\{T>0:\ \sup_{r\in(0,T)}|\nabla v_1(0^+,r)-\nabla v_2(0^+,r)|\geq(T)^{-1}(s^+(0)-s^-(0))\Big\},\tag{3.38}$$

to keep the spread non-negative, since if

$$\sup_{r\in(0,T)}|\nabla v_1(0^+,r)-\nabla v_2(0^+,r)|<(T)^{-1}(s^+(0)-s^-(0))$$

then the second inequality of (3.37) holds and thus the spread is non-negative for all $t\in(0,T)$, and

$$\tau_1^*:=\inf\Big\{T>0:\ \sup_{r\in(0,T)}\nabla v_2(0^+,r)\geq(T)^{-1}(s^-(0)-a)\Big\},\tag{3.39}$$

to keep the spread area in $\mathcal{D}$, since if

$$\sup_{r\in(0,T)}\nabla v_2(0^+,r)<(T)^{-1}(s^-(0)-a)$$

then the first inequality of (3.37) holds and thus the spread area is in $\mathcal{D}$ for all $t\in(0,T)$.

So, the next theorem follows.

Theorem 3.3. *Under the assumptions of Theorem 3.2, and if the initial spread satisfies* $\lambda > s^+(0) - s^-(0) \geq 0$*, then there exist unique weak maximal solutions* (w_1, η_1)*,* (w_2, η_2) *to the reflected Stefan problem* (2.3)-(2.19)-(2.20)*, and* $w|_{x \geq s^+} = w_1$*,* $w|_{x \leq s^-} = -w_2$*, in the maximal interval* $\mathcal{I}_1 := [0, \hat{\tau})$ *for* $\hat{\tau} := \min\{\sup_{M>0} \tau_{1M}, \tau_{1s}, \tau_1^*\}$*, with* $\tau_{1M}, \tau_{1s}, \tau_1^*$ *given by* (3.36)*,* (3.38)*,* (3.39) *for which the spread* $s^+(t) - s^-(t)$ *defined by the Stefan condition* (2.4) *exists and stays a.s. non-negative for any* $t \in \mathcal{I}_1$*.*

We consider now Case 2. Due to Theorem 3.2, and under its assumptions there exist unique weak maximal solutions (v_1, η_1), (v_2, η_2) satisfying (2.19)-(2.20) in the maximal interval $[0, \sup_{M>0} \tau_{1M})$ for τ_{1M} given by (3.36). We need $a \leq s^-(t) \leq s^+(t) \leq b$ in order to return to the initial variables. By using the Stefan condition (2.13) we obtain

$$\partial_t s^-(t) = \nabla v_2(0^+, t) \geq 0, \ \ \partial_t s^+(t) = -\nabla v_1(0^+, t) \leq 0,$$

and so

$$a \leq s^-(0) \leq s^-(t), \ \ s^+(t) \leq s^+(0) \leq b,$$

so we need $s^-(t) \leq s^+(t)$ which yields

$$s^-(0) + \int_0^t \nabla v_2(0^+, s) ds \leq s^+(0) - \int_0^t \nabla v_1(0^+, s) ds.$$

We define the stopping time

$$\tau_{2s} := \inf\Big\{T > 0 : \sup_{r \in (0,T)} (\nabla v_1(0^+, r) + \nabla v_2(0^+, r)) \geq (T)^{-1}(s^+(0) - s^-(0))\Big\}, \tag{3.40}$$

to keep the spread non-negative, while the spread area stays in $\mathcal{D}$ as the spread is decreasing.

So, the next theorem holds.

Theorem 3.4. *Under the assumptions of Theorem 3.2, and if the initial spread satisfies* $\lambda > s^+(0) - s^-(0) \geq 0$*, then there exist unique weak maximal solutions* (w_1, η_1)*,* (w_2, η_2) *to the reflected Stefan problem* (2.12)-(2.19)-(2.20)*, and* $w|_{x \geq s^+} = w_1$*,* $w|_{x \leq s^-} = w_2$*, in the maximal interval* $\mathcal{I}_2 := [0, \hat{\tau})$ *for* $\hat{\tau} := \min\{\sup_{M>0} \tau_{1M}, \tau_{2s}\}$*, with* τ_{1M}, τ_{2s} *given by* (3.36)*,* (3.40)*, for which the spread* $s^+(t) - s^-(t)$ *defined by the Stefan condition* (2.13) *exists and stays a.s. non-negative for any* $t \in \mathcal{I}_2$*.*

4. The problem without reflection

4.1. **Existence of maximal solutions.** We shall consider the unreflected initial and boundary value problem for

$$v_t(y, t) = \alpha \Delta v(y, t) - \nabla v(0^+, t) \nabla v(y, t) + \sigma(y) \dot{W}(y, t), \tag{4.1}$$

posed for any y in $\mathcal{D} = (0, \lambda)$ for $t \in [0, T]$ with Dirichlet b.c., with $v(y, 0)$ given.

In the proofs of the previous section we replace the reflection measure by 0 and keep as presented the absolute value on the changing in general sign $\nabla v(0^+, t)$ (as v may take negative values), and we derive the next results.

Theorem 4.1. *Let the noise diffusion* σ *satisfy the condition* (3.5)*,* $M > 0$ *fixed,* $p \geq p_0 > 8$*, and let* $v_0 \in C_c^\infty(\overline{\mathcal{D}})$ *be the initial condition of* (4.1)*. Then there exists a unique weak solution*

$v^M \in L^p(\mathbf{\Omega}, C[0,T]; \mathcal{B})$ *to the truncated problem*

$$\begin{aligned} &v_t^M(y,t) = \alpha\Delta v^M(y,t) - \nabla(\mathcal{T}_M(v^M))(0^+,t)\nabla(\mathcal{T}_M(v^M))(y,t) \\ &\qquad\qquad + \sigma(y)\dot{W}(y,t), \ \ t\in(0,T], \ \ y\in\mathcal{D}, \\ &v^M(y,0) := v_0(y), \ \ y\in\mathcal{D}, \\ &v^M(0,t) = v^M(\lambda,t) = 0, \ \ t\in(0,T], \end{aligned} \tag{4.2}$$

where $T := T_M > 0$ *such that*

$$\sup_{r\in(0,T)} |\nabla(\mathcal{T}_M(v^M))(0^+,r)|^p < \infty \ \ a.s., \tag{4.3}$$

where for any $t\in(0,T)$, v^M *satisfies the weak formulation*

$$\begin{aligned} v^M(y,t) = &\int_{\mathcal{D}} v_0(z)G(y,z,t)dz \\ &+ \int_0^t\int_{\mathcal{D}} \nabla(\mathcal{T}_M(v^M))(0^+,s)\nabla G(y,z,t-s)\mathcal{T}_M(v^M)(z,s)dzds \\ &+ \int_0^t\int_{\mathcal{D}} G(y,z,t-s)\sigma(z)W(dz,ds), \end{aligned} \tag{4.4}$$

for $v^M(y,0) := v_0(y)$.

Theorem 4.2. *Let the noise diffusion* σ *satisfy* (3.5), *and* $v_0 \in C_c^\infty(\overline{\mathcal{D}})$, *and* $p \geq p_0 > 8$. *Then, there exists a unique weak maximal solution* v *to the problem* (4.1) *in the maximal interval* $[0, \sup\limits_{M>0}\tilde{\tau}_M)$, *where*

$$\tilde{\tau}_M := \inf\Big\{T\geq 0 : \sup_{r\in(0,T)}|\nabla v(0^+,r)| \geq M\Big\}. \tag{4.5}$$

We consider now Case 3. Due to Theorem 4.2, and under its assumptions there exist unique weak maximal solutions (v_1,η_1), (v_2,η_2) satisfying (2.19)-(2.20) in the maximal interval $[0, \sup\limits_{M>0}\tau_{3M})$ for τ_{3M} given by

$$\tau_{3M} := \inf\Big\{T\geq 0 : \sup_{r\in(0,T)}\big(|\nabla v_1(0^+,r)| + |\nabla v_2(0^+,r)|\big) \geq M\Big\}. \tag{4.6}$$

We need $a \leq s^-(t) \leq s^+(t) \leq b$ in order to return to the initial variables. By using the Stefan condition (2.13) we obtain

$$\partial_t s^-(t) = \nabla v_2(0^+,t), \ \ \partial_t s^+(t) = -\nabla v_1(0^+,t),$$

and we need

$$a \leq s^-(0) + \int_0^t \nabla v_2(0^+,s)ds \leq s^+(0) - \int_0^t \nabla v_1(0^+,s)ds \leq b.$$

We define the stopping time

$$\tau_{3s} := \inf\Big\{T>0 : \sup_{r\in(0,T)}|\nabla v_1(0^+,r) + \nabla v_2(0^+,r))| \geq (T)^{-1}(s^+(0)-s^-(0))\Big\}, \tag{4.7}$$

to keep the spread non-negative, and

$$\tau_3^* := \inf\Big\{T>0 : \sup_{r\in(0,T)}|\nabla v_2(0^+,r)| \geq (T)^{-1}(s^-(0)-a)\Big\}, \tag{4.8}$$

$$\tau_3^{**} := \inf\Big\{T > 0 : \sup_{r\in(0,T)} |\nabla v_1(0^+, r)| \geq (T)^{-1}(b - s^+(0))\Big\}\Big\}, \tag{4.9}$$

to keep the spread area in $\mathcal{D}$.

So, the next theorem holds.

Theorem 4.3. *Under the assumptions of Theorem 4.2, and if the initial spread satisfies* $\lambda > s^+(0) - s^-(0) \geq 0$*, then there exist unique weak maximal solutions* w_1*,* w_2 *to the unreflected Stefan problem* (2.17)*, and* $w|_{x\geq s^+} = w_1$*,* $w|_{x\leq s^-} = w_2$*, in the maximal interval* $\mathcal{I}_3 := [0, \hat{\tau})$ *for* $\hat{\tau} := \min\{\sup_{M>0} \tau_{3M}, \tau_{3s}, \tau_3^*, \tau_3^{**}\}$*, with* $\tau_{3M}, \tau_{3s}, \tau_3^*, \tau_3^{**}$ *given by* (4.6)*,* (4.7)*,* (4.8)*,* (4.9) *for which the spread* $s^+(t) - s^-(t)$ *defined by the Stefan condition* (2.13) *exists and stays a.s. non-negative for any* $t \in \mathcal{I}_3$*.*

Acknowledgment

The authors would like to thank the anonymous referees for their valuable comments and suggestions. The research work was supported by the Hellenic Foundation for Research and Innovation (H.F.R.I.) under the First Call for H.F.R.I. Research Projects to support Faculty members and Researchers. (Project Number: HFRI-FM17-45).

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