MALLIAVIN CALCULUS FOR THE STOCHASTIC CAHN-HILLIARD/ALLEN-CAHN EQUATION WITH UNBOUNDED NOISE DIFFUSION

DIMITRA C. ANTONOPOULOU, DIMITRIS FARAZAKIS, AND GEORGIA KARALI

Abstract. The stochastic partial differential equation analyzed in this work, is motivated by a simplified mesoscopic physical model for phase separation. It describes pattern formation due to adsorption and desorption mechanisms involved in surface processes, in the presence of a stochastic driving force. This equation is a combination of Cahn-Hilliard and Allen-Cahn type operators with a multiplicative, white, space-time noise of unbounded diffusion. We apply Malliavin calculus, in order to investigate the existence of a density for the stochastic solution u. In dimension one, according to the regularity result in [5], u admits continuous paths a.s. Using this property, and inspired by a method proposed in [8], we construct a modified approximating sequence for u, which properly treats the new second order Allen-Cahn operator. Under a localization argument, we prove that the Malliavin derivative of u exists locally, and that the law of u is absolutely continuous, establishing thus that a density exists.

Keywords: stochastic partial differential equations, reaction-diffusion equations, phase transitions, Malliavin calculus.

1. Introduction

1.1. **The Stochastic Model.** We consider the following stochastic partial differential equation which is given as a combination of Cahn-Hilliard and Allen-Cahn type equations, perturbed by a multiplicative space-time noise $\dot{W}$ with a non-linear diffusion coefficient σ

$$u_t = -\varrho\Delta\Big(\Delta u - f(u)\Big) + \Big(\Delta u - f(u)\Big) + \sigma(u)\dot{W}, \quad t > 0,\ x \in \mathcal{D}, \tag{1.1}$$

where $\mathcal{D} \subset \mathbb{R}^d$, for $d = 1, 2, 3$, is a bounded spatial domain. Here, $f(u) = u^3 - u$ is the derivative of a double equal-well potential. The constant $\varrho > 0$ is a positive bifurcation parameter referring to an attractive potential for the related physical model, while the noise $\dot{W} = \dot{W}(x, t)$ is a space-time white noise in the sense of Walsh, [18], given as the formal derivative of a Wiener process. More specifically, $dW := W(dx, ds)$ is a d-dimensional space-time white noise, induced by the one-dimensional $(d+1)$-parameter Wiener process W defined as $W := \big\{W(x,t) :\ t \in [0, T],\ x \in \mathcal{D}\big\}$. The noise diffusion $\sigma(u)$ has a sub-linear growth of the form

$$|\sigma(u)| \leq C(1 + |u|^q),$$

for some $C > 0$ and any $q \in (0, \frac{1}{3})$.

The initial and boundary value problem for this equation, satisfies the initial condition

$$u(x, 0) = u_0(x) \text{ in } \mathcal{D},$$

and the next homogeneous Neumann boundary conditions

$$\frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0 \text{ on } \partial\mathcal{D} \times [0, T). \tag{1.2}$$

The Cahn-Hilliard equation was initially proposed as a simple model for the description of the phase separation of a binary alloy, being in a non-equilibrium state, [10]. Cook in [11], extended the deterministic

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1991 *Mathematics Subject Classification.* 35K55, 35K40, 60H30, 60H15.

partial differential equation to a stochastic one by introducing thermal fluctuations in the form of an additive noise. There exist some interesting results in the relevant literature on existence and uniqueness of solution for the stochastic problem, as for example in [8, 12], where the i.b.v.p. was posed on cubic domains, and rectangles, or on Lipschitz domains of more general topography, [4]. In [7, 12, 8, 9], the authors considered the version of an odd polynomial nonlinearity for the potential. Moreover, in [3], the one-dimensional stochastic Cahn-Hilliard equation has been approximated by a manifold of solutions and the dynamics of the stochastic motion of the fronts were described. In [7], the effect of noise on evolving interfaces during the initial stage of phase separation was analyzed, while in [6], the singular limit of the generalized Cahn-Hilliard equation has been rigorously derived by means of the Hilbert expansion method, imitating the behavior of a stochastic model. The sharp interface limit of the Cahn-Hilliard equation with additive noise has been examined in [2]; in this case, depending on the noise strength, the chemical potential satisfies on the limit a deterministic or a stochastic Hele-Shaw problem of Stefan type. Funaki studied the interface motion and applied a singular perturbation analysis for the Allen-Cahn equation with mild noise, when the initial data are close to an instanton, [14, 13]. In the presence of a non-local integral term the Allen-Cahn equation exhibits the mass conservation property; for the dynamics of the mass conserving stochastic Allen-Cahn equation, we refer to the results presented in [1].

In the deterministic setting, Karali and Katsoulakis, in [15], introduced a simplified mean field type model written as a combination of Cahn-Hilliard and Allen-Cahn type equations, in order to study the effect of diffusion and adsorption/desorption in the context of surface processes. Antonopoulou, Karali and Millet in [5], by inserting a noise term additive in the equation and stemming from the free energy and thermal fluctuations, derived the stochastic non-linear equation version of the aforementioned model. There in, the authors described the physical motivation of such a stochastic forcing. In addition, they investigated the existence and regularity of solution for the stochastic Cahn-Hilliard/Allen-Cahn equation with unbounded noise diffusion, when posed in dimensions $d = 1, 2, 3$.

Our aim in this work, is to study the existence of a density for the stochastic solution. The dimensions of the problem in spatial coordinates are expected to play a crucial role. Note that in dimensions $d = 1$ the stochastic solution has continuous paths a.s., while in higher dimensions existence of maximal solutions has been established, [5].

1.2. **The Malliavin derivative.** Let $(\Omega, \mathcal{F}, P)$ be a probability space, where Ω is a sample space, $\mathcal{F}$ is a σ-algebra consisting of subsets of Ω and P a probability measure $P : \mathcal{F} \to [0, 1]$, and consider a random variable $F : \Omega \to \mathbb{R}$. The sample space Ω consists of all the possible outcomes ω (simple events) of a random experiment. The Malliavin derivative measures the rate of change of F as a function of $\omega \in \Omega$ and implements the idea of differentiating F with respect to the 'chance parameter' ω, [17]. When Ω has a topological structure, the derivative operator is induced by a directional Fréchet derivative of F along a certain direction ω_0 in Ω, of the form

$$\frac{d}{d\varepsilon}F(\omega + \varepsilon\omega_0)|_{\varepsilon=0},$$

[17]. The function F can be a stochastic process as for example the solution of a stochastic pde (such as u in (1.1)). In our case, $\mathcal{F}$ is the σ-algebra generated by the Wiener process $W := \big\{W(x, t) : \ t \in [0, T], \ x \in \mathcal{D}\big\}$, and the relevant topological structure is this of the Hilbert space $L^2([0, T] \times \mathcal{D})$.

1.3. **Main Results.** We investigate if u, the solution of (1.1), as a random variable, has a density; an affirmative answer is given by proving that the law of u is absolutely continuous.

Here, we follow the strategy proposed by Cardon-Weber in [8], and approximate u by a sequence u_n for which we prove existence of Malliavin derivative; we then check that a certain norm of this derivative is almost surely strictly positive. Strict positivity establishes the absolute continuity of the sequence u_n and on the limit, as $n \to \infty$, the same result follows for u, cf. Subsections 3.1, 3.2.

We use carefully some important definitions and results from the theory of Malliavin Calculus, presented by Nualart in [17].

More precisely, in dimension $d=1$, we show that the stochastic solution is locally differentiable in the sense of Malliavin calculus. Under some non-degeneracy condition on the noise diffusion coefficient, we prove that the law of the solution is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$.

Cardon-Weber in [8] studied the stochastic Cahn-Hilliard equation with bounded noise diffusion. In our case we consider a more general problem; this of the stochastic Cahn-Hilliard/Allen-Cahn equation with unbounded noise diffusion, for which when $d=1$ in [5], the authors established existence of a continuous solution a.s. This equation contains a new second order nonlinear operator, fact that arises the use of a new spde, quite different than this proposed in [8], which defines a proper approximating sequence u_n. Additionally, we treat efficiently the existing growth of the unbounded diffusion, by proving estimates in expectation in the stronger $L^\infty(\mathcal{D})$-norm, in various places, involving u_n and its Malliavin derivative.

The novelty of this paper is the proof of Theorem 1.1 (i.e., Theorem 2.9 and Theorem 3.4), for the equation (1.1), which consists a stochastic pde with a white space-time noise and unbounded noise diffusion. This is an important contribution to the literature of stochastic equations stemming from physical problems, such as phase separation in the presence of randomness. Our result is set in the very active area of research on well posedness (existence and regularity) of solutions of spdes. Moreover, these solutions are random variables depending not only on space and time but also on the parameter $\omega \in \Omega$. Hence, by proving that a density exists for u, we integrate significantly the theoretical analysis of this stochastic model.

In particular, we prove the next Main Theorem.

Theorem 1.1. *Let u be the solution of the stochastic Cahn-Hilliard/Allen-Cahn equation* (1.1)*, with the Neumann b.c.* (1.2) *in dimension $d=1$, for $\mathcal{D}:=(0,\pi)$, with smooth initial condition u_0.*

Let the noise diffusion σ satisfy:

(1) *σ has a sublinear growth uniformly for any $x\in\mathbb{R}$ of the form*

$$|\sigma(x)| \leq C(1+|x|^q), \tag{1.3}$$

for $C>0$, and $q\in(0,\frac{1}{3})$,

(2) *σ is Lipschitz on $\mathbb{R}$, i.e., there exists K:*

$$|\sigma(x)-\sigma(y)| \leq K|x-y|, \ \ \forall\, x,\ y\in\mathbb{R}, \tag{1.4}$$

(3) *σ is continuously differentiable on $\mathbb{R}$ (i.e., $\exists\, \sigma'$, and σ, σ' are continuous), and since σ' exists, due to* (1.4) *it follows that*

$$|\sigma'(x)| \leq K, \ \ \forall\, x\in\mathbb{R}. \tag{1.5}$$

Then the derivative of u in the Malliavin sense exists locally (cf. Theorem 2.9).

Moreover, if, in addition, σ is non-degenerate, i.e., there exists $c_0>0$ such that

$$|\sigma(x)| \geq c_0 > 0, \ \ \forall\, x\in\mathbb{R}, \tag{1.6}$$

then the law of u is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$ (cf. Theorem 3.4).

Remark 1.2. *The above theorem is also valid in the more general case of*

$$u_t = -\varrho\Delta\Big(\Delta u - f(u)\Big) + \tilde{q}\Big(\Delta u - f(u)\Big) + \sigma(u)\dot{W}, \quad t>0,\ x\in\mathcal{D}, \tag{1.7}$$

for $\varrho>0$ and $\tilde{q}\geq 0$, cf. Section 4 of [5] *for the relevant discussion for the existence and regularity of solution for this more general problem, and the observations for the Green's function. In our case, when establishing existence of a density, all our results hold true for* (1.7) *also.*

Thus, for $\varrho=1$, $\tilde{q}:=0$, the Main Theorem 1.1 (existence of Malliavin derivative locally and of a density for u) is valid for the one-dimensional stochastic Cahn-Hilliard equation with unbounded noise diffusion and non-smooth in space and in time space-time noise.

The structure of the rest of this paper is as follows: Section 2 presents some basic definitions from Malliavin calculus such as the definitions of the spaces of random variables $D^{1,2}$, $L^{1,2}$, and their local versions $D^{1,2}_{\rm loc}$, $L^{1,2}_{\rm loc}$. Moreover, due to the fact that u is a.s. continuous, we are able to approximate efficiently the solution u by some u_n defined through an spde, for which we prove existence of the Malliavin derivative; u_n is proven to be a localization in the Malliavin sense of u, which yields finally the existence of the Malliavin derivative of u locally. In details, u is written in the integral representation given by (2.2). This representation motivates the piece-wise approximation u_n definition as the solution of the spde (2.6). Lemma 2.7 establishes existence and uniqueness of u_n, and provides a useful bound in expectation. We then prove that the Malliavin derivative of u_n is well defined, and that $u_n \in D^{1,2}$, cf. Proposition 2.8; a direct consequence is the Main Theorem 2.9, i.e., that u belongs to $L^{1,2}_{\rm loc} \subseteq D^{1,2}_{\rm loc}$.

In Section 3, we prove the absolute continuity of the approximations u_n which again through a localization argument (see Remark 3.1) yields the existence of a density for the stochastic solution u. More specifically, we present first the very technical Lemma 3.2, where the growth of the unbounded noise diffusion σ is crucial. In the sequel, under the additional assumption (1.6) (non-degenerating σ), we establish, in Theorem 3.3, the absolute continuity of u_n, and thus, the existence of a density for u (Main Theorem 3.4).

For the rest of this paper, we consider $d = 1$, $\mathcal{D} := (0, \pi)$, a smooth u_0, and the assumptions (1), (2), (3), of the statement of Theorem 1.1, for the diffusion σ; for simplicity, we set in (1.1) $\varrho := 1$. The additional assumption (1.6) for a non-degenerate σ appears only in the statements (and proofs) of Theorems 3.3, 3.4.

2. Malliavin calculus

2.1. Basic definitions.

Definition 2.1. *Following the notation of* [8]*, we denote by $D^{1,2}$ the set of random variables v such that the Malliavin derivative (in space and time) $D_{y,s}v(x,t)$ exists, for any $y \in \mathcal{D}$ and any $s \geq 0$ and any $(x,t) \in \mathcal{D} \times [0,T]$, and satisfies*

$$\|v\|_{D^{1,2}} := \Big(\mathbf{E}(|v|^2) + \mathbf{E}(\|D_{\cdot,\cdot}v\|^2_{L^2([0,T]\times\mathcal{D})})\Big)^{1/2} < \infty, \tag{2.1}$$

for

$$\|D_{\cdot,\cdot}v(x,t)\|_{L^2([0,T]\times\mathcal{D})} := \Big(\int_0^T \int_{\mathcal{D}} |D_{y,s}v(x,t)|^2 dyds\Big)^{1/2}.$$

Indeed, according to [17]*, p. 27 (where the definition of $D^{1,p}$, $p \geq 1$, is given), $D^{1,2}$ is a Hilbert space and consists the closure of the class of smooth random variables v in the norm*

$$\|v\|_{D^{1,2}} := \Big(\mathbf{E}(|v|^2) + \mathbf{E}(\|D_{\cdot,\cdot}v\|^2_H)\Big)^{1/2},$$

where $\|\cdot\|_H$ is the norm induced by the inner product $<\cdot,\cdot>_H$ and the norm $\|\cdot\|_{D^{1,2}}$ is induced by the inner product

$$< f, g >:= \mathbf{E}(fg) + \mathbf{E}(< D_{\cdot,\cdot}f, D_{\cdot,\cdot}g >_H),$$

where, in our case, $H := L^2([0,T] \times \mathcal{D})$ and $<\cdot,\cdot>_H$ is the usual L^2 inner product on $[0,T] \times \mathcal{D}$.

Definition 2.2. *The set $L^{1,2}$ is defined as the class of all stochastic processes $v = v(x,t) \in L^2(\Omega\times[0,T]\times\mathcal{D})$, i.e.,*

$$\|v\|_{L^2(\Omega\times[0,T]\times\mathcal{D})} := \Big(\mathbf{E}\Big(\int_0^T \int_{\mathcal{D}} |v(x,t)|^2 dxdt\Big)\Big)^{1/2} < \infty,$$

such that $v \in D^{1,2}$ and satisfy

$$\mathbf{E}\Big(\int_0^T \int_{\mathcal{D}} \int_0^T \int_{\mathcal{D}} |D_{y,s}v(x,t)|^2 dydsdxdt\Big) < \infty,$$

cf. [17]*, p. 42 (to avoid any confusion, we point out that the notation T used by Nualart at p. 42, in our case corresponds to $[0,T] \times \mathcal{D}$), and* [8].

Definition 2.3. *According to* [17], *p. 49, for* $L := L^{1,2}$ *(a class of stochastic processes),* $(L^{1,2})_{\rm loc} =: L^{1,2}_{\rm loc}$ *is defined as the set of random variables* v*:* $\exists$ *a sequence* $\{(\Omega_n, v_n),\ n \geq 1\} \subset \mathcal{F} \times L$ (here $L := L^{1,2}$) *such that*

(1) $\Omega_n \uparrow \Omega$ *a.s.,*
(2) $v = v_n$ *a.s. on* Ω_n.

Also, for $L := D^{1,2}$ *(a class of random variables),* $(D^{1,2})_{\rm loc} =: D^{1,2}_{\rm loc}$ *is defined as the set of random variables* v*:* $\exists$ *a sequence* $\{(\Omega_n, v_n),\ n \geq 1\} \subset \mathcal{F} \times L$ (here $L := D^{1,2}$) *such that*

(1) $\Omega_n \uparrow \Omega$ *a.s.,*
(2) $v = v_n$ *a.s. on* Ω_n.

Here, $\mathcal{F}$ *is the* σ*-algebra, while* $\Omega_n \uparrow \Omega$ *a.s., is equivalent to* $\Omega_1 \subseteq \Omega_2 \subseteq \Omega_3 \subseteq \cdots \subseteq \Omega$*, such that*

$$\lim_{n\to\infty} P(\Omega_n) = P(\Omega) = 1.$$

Remark 2.4. *If* $v \in D^{1,2}_{\rm loc}$ *and* (Ω_n, v_n) *localizes* v *in* $D^{1,2}$ *in the aforementioned way (cf. the previous definition), then the Malliavin derivative* $D_{y,s}v$ *is defined without ambiguity by* $D_{y,s}v = D_{y,s}v_n$ *on* Ω_n *for* $n \geq 1$ *(i.e.,* $D_{y,s}v$ *is well defined by localization in the space* $D^{1,2}_{\rm loc}$*), cf.* [17]*, p. 49.*

2.2. **Localization of u in $L^{1,2}$.** Our aim is to prove that the stochastic solution u of (1.1) belongs to the space $L^{1,2}_{\rm loc}$, (observe that $L^{1,2}_{\rm loc} \subseteq D^{1,2}_{\rm loc}$).

Remark 2.5. *Note, that* $L^{1,2}$ *is a subset of* $D^{1,2}$*, consisting of more regular random variables in* $L^2(\Omega \times [0,T] \times \mathcal{D})$*, with Malliavin derivative bounded in* $L^2(\Omega \times ([0,T] \times \mathcal{D})^2)$*. Hence, a constructed localization* (Ω_n, u_n) *of* u *in* $L^{1,2}$ *is also a localization in* $D^{1,2}$ *and thus, will define well the Malliavin derivative of* u *through the Malliavin derivative of* u_n *(see Remark 2.4). Moreover, the previous construction, will establish local regularity of the solution* u *of* (1.1) *in the sense of Malliavin calculus.*

The solution u of the stochastic equation (1.1) is written in integral representation as

$$\begin{aligned} u(x,t) = &\int_{\mathcal{D}} u_0(y) G(x,y,t)dy \\ &+ \int_0^t \int_{\mathcal{D}} [\Delta G(x,y,t-s) - G(x,y,t-s)] f(u(y,s))dyds \\ &+ \int_0^t \int_{\mathcal{D}} G(x,y,t-s)\sigma(u(y,s))W(dy,ds), \end{aligned} \tag{2.2}$$

for,

$$G(x,y,t) := \sum_{k=0}^{\infty} e^{-(\lambda_k^2+\lambda_k)t}\alpha_k(x)\alpha_k(y), \tag{2.3}$$

where λ_k are the eigenvalues of the negative Neumann Laplacian with Neumann b.c. posed on $\mathcal{D}$, and $\{\alpha_k\}_{k\in\mathbb{N}}$ a corresponding eigenfunction orthonormal basis of $L^2(\mathcal{D})$; see [5] for more details on (2.2) and the definition of Green's function G.

2.2.1. *Piece-wise approximation of the stochastic solution.* We shall construct a 'piecewise' approximation $u_n \in L^{1,2}$ of u.

Let $H_n : \mathbb{R}^+ \to \mathbb{R}^+$ be a $\mathcal{C}^1$ cut-off function satisfying

$$|H_n| \leq 1, \text{ and } |H_n'| \leq 2,$$

for any $n > 0$, with

$$H_n(x) := \begin{cases} 1 & \text{if } |x| < n \\ 0 & \text{if } |x| \geq n+1. \end{cases} \tag{2.4}$$

We set

$$f_n(x) := H_n(|x|)f(x), \tag{2.5}$$

obviously f_n is a $\mathcal{C}^1$ function and its derivative is bounded, [18]; this bound depends on n and consists a Lipschitz coefficient for f_n.

We define

$$\Omega_n := \Big\{\omega \in \Omega : \sup_{t\in[0,T]} \sup_{x\in\mathcal{D}} |u(x,t;\omega)| < n\Big\}.$$

Obviously, it holds that

$$\Omega_1 \subseteq \Omega_2 \subseteq \cdots \subseteq \Omega.$$

Let

$$\begin{aligned} u_n(x,t) :=& \int_{\mathcal{D}} u_0(y)G(x,y,t)dy \\ &+ \int_0^t \int_{\mathcal{D}} [\Delta G(x,y,t-s) - G(x,y,t-s)]f_n(u_n(y,s))dyds \\ &+ \int_0^t \int_{\mathcal{D}} G(x,y,t-s)\sigma(u_n(y,s))W(dy,ds). \end{aligned} \tag{2.6}$$

We shall prove existence and uniqueness of solution u_n of (2.6), and we shall establish that u_n belongs in the space $L^{1,2}$; this will yield that the solution u is in the space $L^{1,2}_{\text{loc}}$.

We assume that the initial condition u_0 is smooth; according to [5], in dimensions $d = 1$, due to the stated at the introduction assumptions for σ, in particular the Lipschitz property and the growth of order $q < \frac{1}{3}$ (in [5], σ is just Lipschitz with sublinear growth of order $q < \frac{1}{3}$ and not assumed also continuously differentiable or non-degenerate), the solution u of (1.1) exists and is a.s. continuous.

Remark 2.6. *In* [5]*, the authors proved, for $d = 1$, global existence of an a.s. continuous solution u for* (1.1)*, when the diffusion coefficient satisfies a sub-linear growth condition of order q bounded by $\frac{1}{3}$, where $\frac{1}{3}$ is the inverse of the polynomial order of the nonlinear function used in* (1.1)*, i.e. of*

$$f(u) = u^3 - u.$$

We need a.s. continuity of u in order to establish our arguments, and this is the main reason why our Main Result is restricted in dimensions $d = 1$. More precisely, the a.s. continuity of u yields, cf. also in [8]

$$P(\Omega_n) \to 1 \ \text{ as } n \to \infty,$$

which is needed for the definition of the localization of u.

The rest of this paragraph, will be devoted to the proof of the next, quite technical lemma, which establishes the existence of the piece-wise approximation u_n, and provides a useful bound in expectation.

Lemma 2.7. *The problem (2.6) has a unique solution u_n, in dimensions $d = 1$.*

Moreover, u_n satisfies for any $p \geq 2$

$$\sup_{t\in[0,T]} \mathbf{E}(\|u_n(\cdot,t)\|^p_{L^\infty(\mathcal{D})}) < \infty. \tag{2.7}$$

Proof. The basic idea is the construction of a Cauchy sequence, through a Picard iteration scheme, which converges, at a certain norm, to the solution u_n of (2.6).

For given n, we define

$$u_{n,0}(x,t) := G_t u_0(x),$$

and for any integer $k \geq 0$, we consider the following Picard iteration scheme, which is motivated by (2.6),

$$\begin{aligned} u_{n,k+1}(x,t) = &\int_{\mathcal{D}} u_0(y)G(x,y,t)dy \\ &+ \int_0^t \int_{\mathcal{D}} [\Delta G(x,y,t-s) - G(x,y,t-s)] f_n(u_{n,k}(y,s))dyds \\ &+ \int_0^t \int_{\mathcal{D}} G(x,y,t-s)\sigma(u_{n,k}(y,s))W(dy,ds). \end{aligned} \tag{2.8}$$

Relation (2.8) yields for any $k \geq 1$,

$$\begin{aligned} u_{n,k+1}(x,t) - u_{n,k}(x,t) = &\int_0^t \int_{\mathcal{D}} [\Delta G(x,y,t-s) - G(x,y,t-s)]\Big(f_n(u_{n,k}(y,s)) - f_n(u_{n,k-1}(y,s))\Big)dyds \\ &+ \int_0^t \int_{\mathcal{D}} G(x,y,t-s)\Big(\sigma(u_{n,k}(y,s)) - \sigma(u_{n,k-1}(y,s))\Big)W(dy,ds). \end{aligned} \tag{2.9}$$

Hence, we obtain

$$\begin{aligned} |u_{n,k+1}(x,t) - u_{n,k}(x,t)| \leq &\int_0^t \int_{\mathcal{D}} |\Delta G(x,y,t-s)||f_n(u_{n,k}(y,s)) - f_n(u_{n,k-1}(y,s))|dyds \\ &+ \int_0^t \int_{\mathcal{D}} |G(x,y,t-s)||f_n(u_{n,k}(y,s)) - f_n(u_{n,k-1}(y,s))|dyds \\ &+ \Big| \int_0^t \int_{\mathcal{D}} G(x,y,t-s)(\sigma(u_{n,k}(y,s)) - \sigma(u_{n,k-1}(y,s)))W(dy,ds)\Big|. \end{aligned}$$

Thus, taking p powers for $p \geq 2$, and then supremum for any $x \in \mathcal{D}$ and supremum in time for the stochastic integral, and then expectation, we get

$$\begin{aligned} \mathbf{E}(\sup_{x\in\mathcal{D}}|u_{n,k+1}(x,t) - u_{n,k}(x,t)|^p) &= \mathbf{E}(\|u_{n,k+1}(\cdot,t) - u_{n,k}(\cdot,t)\|^p_{L^\infty(\mathcal{D})}) \\ \leq &c\mathbf{E}\bigg(\sup_{x\in\mathcal{D}}\Big(\int_0^t \int_{\mathcal{D}} |\Delta G(x,y,t-s)||f_n(u_{n,k}(y,s)) - f_n(u_{n,k-1}(y,s))|dyds\Big)^p\bigg) \\ &+ c\mathbf{E}\bigg(\sup_{x\in\mathcal{D}}\Big(\int_0^t \int_{\mathcal{D}} |G(x,y,t-s)||f_n(u_{n,k}(y,s)) - f_n(u_{n,k-1}(y,s))|dyds\Big)^p\bigg) \\ &+ c\mathbf{E}\bigg(\sup_{\tau\in[0,t]}\sup_{x\in\mathcal{D}}\Big|\int_0^\tau \int_{\mathcal{D}} G(x,y,\tau-s)[\sigma(u_{n,k}(y,s)) - \sigma(u_{n,k-1}(y,s))]W(dy,ds)\Big|^p\bigg). \end{aligned}$$

The function f_n is Lipschitz and so,

$$\begin{aligned} \mathbf{E}(\|u_{n,k+1}(\cdot,t) - &u_{n,k}(\cdot,t)\|^p_{L^\infty(\mathcal{D})}) \\ \leq &c\mathbf{E}\bigg(\sup_{x\in\mathcal{D}}\Big(\int_0^t \int_{\mathcal{D}} |\Delta G(x,y,t-s)||u_{n,k}(y,s) - u_{n,k-1}(y,s)|dyds\Big)^p\bigg) \\ &+ c\mathbf{E}\bigg(\sup_{x\in\mathcal{D}}\Big(\int_0^t \int_{\mathcal{D}} |G(x,y,t-s)||u_{n,k}(y,s) - u_{n,k-1}(y,s)|dyds\Big)^p\bigg) \\ &+ c\mathbf{E}\bigg(\sup_{\tau\in[0,t]}\sup_{x\in\mathcal{D}}\Big|\int_0^\tau \int_{\mathcal{D}} G(x,y,\tau-s)[\sigma(u_{n,k}(y,s)) - \sigma(u_{n,k-1}(y,s))]W(dy,ds)\Big|^p\bigg). \end{aligned}$$

Burkholder-Davis-Gundy inequality applied to the stochastic term of the previous inequality gives

$$\begin{aligned}
&\mathbf{E}\Big(\|u_{n,k+1}(\cdot,t)-u_{n,k}(\cdot,t)\|^p_{L^\infty(\mathcal{D})}\Big)\\
&\qquad\le c\mathbf{E}\Big(\sup_{x\in\mathcal{D}}\Big(\int_0^t\int_{\mathcal{D}}|\Delta G(x,y,t-s)||u_{n,k}(y,s)-u_{n,k-1}(y,s)|dyds\Big)^p\Big)\\
&\qquad\quad+c\mathbf{E}\Big(\sup_{x\in\mathcal{D}}\Big(\int_0^t\int_{\mathcal{D}}|G(x,y,t-s)||u_{n,k}(y,s)-u_{n,k-1}(y,s)|dyds\Big)^p\Big)\\
&\qquad\quad+c\mathbf{E}\Big(\sup_{\tau\in[0,t]}\sup_{x\in\mathcal{D}}\Big(\int_0^\tau\int_{\mathcal{D}}|G(x,y,\tau-s)|^2|\sigma(u_{n,k}(y,s))-\sigma(u_{n,k-1}(y,s))|^2dyds\Big)^{p/2}\Big)\\
&\qquad\le c\mathbf{E}\Big(\sup_{x\in\mathcal{D}}\Big(\int_0^t\int_{\mathcal{D}}|\Delta G(x,y,t-s)||u_{n,k}(y,s)-u_{n,k-1}(y,s)|dyds\Big)^p\Big)\\
&\qquad\quad+c\mathbf{E}\Big(\sup_{x\in\mathcal{D}}\Big(\int_0^t\int_{\mathcal{D}}|G(x,y,t-s)||u_{n,k}(y,s)-u_{n,k-1}(y,s)|dyds\Big)^p\Big)\\
&\qquad\quad+c\mathbf{E}\Big(\sup_{\tau\in[0,t]}\sup_{x\in\mathcal{D}}\Big(\int_0^\tau\int_{\mathcal{D}}|G(x,y,\tau-s)|^2|u_{n,k}(y,s)-u_{n,k-1}(y,s)|^2dyds\Big)^{p/2}\Big),
\end{aligned}\tag{2.10}$$

where for the last inequality we used that the diffusion coefficient σ is Lipschitz, uniformly for any n.

Thus, from (2.10), we derived

$$\mathbf{E}\Big(\|u_{n,k+1}(\cdot,t)-u_{n,k}(\cdot,t)\|^p_{L^\infty(\mathcal{D})}\Big)\le Q_1(t)+Q_2(t)+Q_3(t),\tag{2.11}$$

where

$$\begin{aligned}
Q_1(t)&:=c\mathbf{E}\Big(\Big\|\int_0^t\int_{\mathcal{D}}|\Delta G(\cdot,y,t-s)||u_{n,k}(y,s)-u_{n,k-1}(y,s)|dyds\Big\|^p_{L^\infty(\mathcal{D})}\Big),\\
Q_2(t)&:=c\mathbf{E}\Big(\Big\|\int_0^t\int_{\mathcal{D}}|G(\cdot,y,t-s)||u_{n,k}(y,s)-u_{n,k-1}(y,s)|dyds\Big\|^p_{L^\infty(\mathcal{D})}\Big),\\
Q_3(t)&:=c\mathbf{E}\Big(\sup_{\tau\in[0,t]}\Big\|\int_0^\tau\int_{\mathcal{D}}|G(\cdot,y,\tau-s)|^2|u_{n,k}(y,s)-u_{n,k-1}(y,s)|^2dyds\Big\|^{p/2}_{L^\infty(\mathcal{D})}\Big).
\end{aligned}$$

In the sequel, we shall estimate the terms involving the Green's function G by using Lemma 1.6 of [8] for $\rho=q:=\infty$, $r:=1$ (which holds true when H and υ are replaced by their absolute values, cf. the proof of lemma presented in [8]).

The statement of Lemma 1.6 of [8] involves general parameters denoted, in [8], by ρ,q,r. Throughout our manuscript, whenever we use this Lemma, we assign from the start specific values to these parameters, which are proper for our proofs; here, we use this Lemma for $\rho:=\infty$, $q:=\infty$, $r:=1$.

For estimating the term $Q_1(t)$, we choose the inequality (1.12) of [8], p. 781, for

$$H(x,y,t-s):=|\Delta G(x,y,t-s)|,\quad \upsilon(y,s):=|u_{n,k}(y,s)-u_{n,k-1}(y,s)|,\quad \rho=q=\infty,\quad r=1.$$

Thus, we have for $p>2$

$$
\begin{aligned}
Q_1(t) \leq & c\mathbf{E}\Big(\Big|\int_0^t \frac{1}{(t-s)^{(d+2)/4-d/4}}\|u_{n,k}(\cdot,s)-u_{n,k-1}(\cdot,s)\|_{L^\infty(\mathcal{D})}ds\Big|^p\Big)\\
\leq & c\mathbf{E}\Big(\Big|\int_0^t \frac{1}{(t-s)^{\frac{1}{2}}}\|u_{n,k}(\cdot,s)-u_{n,k-1}(\cdot,s)\|_{L^\infty(\mathcal{D})}ds\Big|^p\Big)\\
\leq & c\mathbf{E}\Big(\Big(\int_0^t \frac{1}{(t-s)^{q_1\frac{1}{2}}}ds\Big)^{p/q_1}\Big(\int_0^t \|u_{n,k}(\cdot,s)-u_{n,k-1}(\cdot,s)\|^p_{L^\infty(\mathcal{D})}ds\Big)^{p/p}\Big)\\
= & c\mathbf{E}\Big(\Big(\int_0^t \frac{1}{(t-s)^{q_1\frac{1}{2}}}ds\Big)^{p/q_1}\int_0^t \|u_{n,k}(\cdot,s)-u_{n,k-1}(\cdot,s)\|^p_{L^\infty(\mathcal{D})}ds\Big)\\
\leq & c\mathbf{E}\Big(\int_0^t \|u_{n,k}(\cdot,s)-u_{n,k-1}(\cdot,s)\|^p_{L^\infty(\mathcal{D})}ds\Big),
\end{aligned}
\tag{2.12}
$$

where we used Hölder inequality for $q_1:=p/(p-1)$, i.e., $1/p+1/q_1=1/p+(p-1)/p=1$, and the fact that $0<\frac{q_1}{2}<1$ or equivalently $0<\frac{p}{p-1}<2$, which is true for any $p>2$, and thus

$$
\Big(\int_0^t \frac{1}{(t-s)^{q_1\frac{1}{2}}}ds\Big)^{p/q_1}<c.
$$

For the term $Q_2(t)$ we choose the inequality (1.11) of [8], p. 781, for

$$
H(x,y,t-s)=|G(x,y,t-s)|,\quad \upsilon(y,s)=|u_{n,k}(y,s)-u_{n,k-1}(y,s)|,\quad \rho=q=\infty,\quad r=1.
$$

Then we get

$$
\begin{aligned}
Q_2(t) \leq & c\mathbf{E}\Big(\Big|\int_0^t \frac{1}{(t-s)^{(\frac{d}{4})(1-1)}}\|u_{n,k}(\cdot,s)-u_{n,k-1}(\cdot,s)\|_{L^\infty(\mathcal{D})}ds\Big|^p\Big)\\
= & c\mathbf{E}\Big(\Big|\int_0^t \|u_{n,k}(\cdot,s)-u_{n,k-1}(\cdot,s)\|_{L^\infty(\mathcal{D})}ds\Big|^p\Big)\\
\leq & c\mathbf{E}\Big(\Big(\int_0^t 1^{q_2}ds\Big)^{p/q_2}\Big(\int_0^t \|u_{n,k}(\cdot,s)-u_{n,k-1}(\cdot,s)\|^p_{L^\infty(\mathcal{D})}ds\Big)^{p/p}\Big)\\
\leq & c\mathbf{E}\Big(\int_0^t \|u_{n,k}(\cdot,s)-u_{n,k-1}(\cdot,s)\|^p_{L^\infty(\mathcal{D})}ds\Big),
\end{aligned}
\tag{2.13}
$$

where we used Hölder inequality for $q_2:=p/(p-1)$.

For the term $Q_3(t)$ we choose the inequality (1.13) of [8], p. 781, for

$$
H(x,y,t-s)=G^2(x,y,t-s),\quad \upsilon(y,s)=|u_{n,k}(y,s)-u_{n,k-1}(y,s)|^2,\quad \rho=q=\infty,\quad r=1,
$$

and we obtain

$$
\begin{aligned}
Q_3(t) \leq & c\mathbf{E}\Big(\sup_{\tau\in[0,t]}\Big|\int_0^\tau \frac{1}{(\tau-s)^{\frac{d}{2}-\frac{d}{4}}}\||u_{n,k}(\cdot,s)-u_{n,k-1}(\cdot,s)|^2\|_{L^\infty(\mathcal{D})}ds\Big|^{p/2}\Big)\\
= & c\mathbf{E}\Big(\sup_{\tau\in[0,t]}\Big|\int_0^\tau \frac{1}{(\tau-s)^{\frac{d}{4}}}\|u_{n,k}(\cdot,s)-u_{n,k-1}(\cdot,s)\|^2_{L^\infty(\mathcal{D})}ds\Big|^{p/2}\Big)\\
\leq & c\mathbf{E}\Big(\sup_{\tau\in[0,t]}\Big[\Big(\int_0^\tau \frac{1}{(\tau-s)^{q_3\frac{d}{4}}}ds\Big)^{\frac{p}{2q_3}}\Big[\int_0^\tau \|u_{n,k}(\cdot,s)-u_{n,k-1}(\cdot,s)\|^{2p/2}_{L^\infty(\mathcal{D})}ds\Big]^{\frac{p/2}{p/2}}\Big]\Big)\\
\leq & c\mathbf{E}\Big(\sup_{\tau\in[0,t]}\Big[\Big(\int_0^\tau \frac{1}{(\tau-s)^{q_3\frac{d}{4}}}ds\Big)^{\frac{p}{2q_3}}\Big]\sup_{\tau\in[0,t]}\Big[\int_0^\tau \|u_{n,k}(\cdot,s)-u_{n,k-1}(\cdot,s)\|^p_{L^\infty(\mathcal{D})}ds\Big]\Big)\\
\leq & c\mathbf{E}\Big(\int_0^t \|u_{n,k}(\cdot,s)-u_{n,k-1}(\cdot,s)\|^p_{L^\infty(\mathcal{D})}ds\Big),
\end{aligned}
\tag{2.14}
$$

where we used Hölder inequality for q_3 such that $1/q_3+1/(p/2)=(p-2)/p+2/p=1$, i.e., for $q_3:=p/(p-2)$ which gives $0<q_3\frac{d}{4}=\frac{p}{p-2}\frac{d}{4}<1$ true for $p>\frac{8}{4-d}$ and thus,

$$\Big(\int_0^\tau \frac{1}{(\tau-s)^{q_3\frac{d}{4}}}ds\Big)^{\frac{p}{2q_3}}<\infty.$$

Replacing (2.12), (2.13) and (2.14) to (2.11), we obtain for any $p>\max\{2,8/(4-d)\}$ and any integer $k\geq 1$

$$\begin{aligned}\mathbf{E}(\|u_{n,k+1}(\cdot,t)-u_{n,k}(\cdot,t)\|^p_{L^\infty(\mathcal{D})})&\leq c\mathbf{E}\bigg(\int_0^t\|u_{n,k}(\cdot,s)-u_{n,k-1}(\cdot,s)\|^p_{L^\infty(\mathcal{D})}ds\bigg)\\&\leq c\int_0^t\mathbf{E}(\|u_{n,k}(\cdot,s_k)-u_{n,k-1}(\cdot,s_k)\|^p_{L^\infty(\mathcal{D})})ds_k,\end{aligned}\tag{2.15}$$

where we used Fubini's Theorem.

Inequality (2.15) applied for the term $\mathbf{E}(\|u_{n,k}(\cdot,s_k)-u_{n,k-1}(\cdot,s_k)\|^p_{L^\infty(\mathcal{D})})$ gives

$$\mathbf{E}(\|u_{n,k+1}(\cdot,t)-u_{n,k}(\cdot,t)\|^p_{L^\infty(\mathcal{D})})\leq c^2\int_0^t\int_0^{s_k}\mathbf{E}(\|u_{n,k-1}(\cdot,s_{k-1})-u_{n,k-2}(\cdot,s_{k-1})\|^p_{L^\infty(\mathcal{D})})ds_{k-1}ds_k,$$

i.e., we get

$$\begin{aligned}&\mathbf{E}(\|u_{n,k+1}(\cdot,t)-u_{n,k}(\cdot,t)\|^p_{L^\infty(\mathcal{D})})\leq c^2\int_0^t\int_0^{s_k}\mathbf{E}(\|u_{n,k-1}(\cdot,s_{k-1})-u_{n,k-2}(\cdot,s_{k-1})\|^p_{L^\infty(\mathcal{D})})ds_{k-1}ds_k\\&\quad\leq c^3\int_0^t\int_0^{s_k}\int_0^{s_{k-1}}\mathbf{E}(\|u_{n,k-2}(\cdot,s_{k-2})-u_{n,k-3}(\cdot,s_{k-2})\|^p_{L^\infty(\mathcal{D})})ds_{k-2}ds_{k-1}ds_k\leq\cdots\\&\quad\leq c^k\int_0^t\int_0^{s_k}\int_0^{s_{k-1}}\cdots\int_0^{s_2}\mathbf{E}(\|u_{n,1}(\cdot,s_1)-u_{n,0}(\cdot,s_1)\|^p_{L^\infty(\mathcal{D})})d_{s_1}\cdots ds_{k-2}ds_{k-1}ds_k\\&\quad\leq c^k\int_0^t\int_0^{s_k}\int_0^{s_{k-1}}\cdots\int_0^{s_2}1d_{s_1}\cdots ds_{k-2}ds_{k-1}ds_k\sup_{t\in[0,T]}\mathbf{E}(\|u_{n,1}(\cdot,t)-u_{n,0}(\cdot,t)\|^p_{L^\infty(\mathcal{D})})\leq c\frac{c^{2k}}{k!},\end{aligned}\tag{2.16}$$

for any $t\in[0,T]$, where we applied the next calculation

$$\begin{aligned}&\int_0^t\int_0^{s_k}\int_0^{s_{k-1}}\cdots\int_0^{s_2}1d_{s_1}\cdots ds_{k-2}ds_{k-1}ds_k=\int_0^t\int_0^{s_k}\int_0^{s_{k-1}}\cdots\int_0^{s_3}s_2d_{s_2}\cdots ds_{k-2}ds_{k-1}ds_k\\&=\int_0^t\int_0^{s_k}\int_0^{s_{k-1}}\cdots\int_0^{s_4}\frac{s_3^2}{2}d_{s_3}\cdots ds_{k-2}ds_{k-1}ds_k=\int_0^t\int_0^{s_k}\int_0^{s_{k-1}}\cdots\int_0^{s_5}\frac{s_4^3}{2\cdot 3}d_{s_4}\cdots ds_{k-2}ds_{k-1}ds_k\\&=\cdots=\mathcal{O}\Big(\frac{c^k}{k!}\Big).\end{aligned}$$

We also used the fact that for the first step ($k:=0$), we have easily

$$\sup_{t\in[0,T]}\mathbf{E}(\|u_{n,1}(\cdot,t)-u_{n,0}(\cdot,t)\|^p_{L^\infty(\mathcal{D})})\leq c\sup_{t\in[0,T]}\mathbf{E}(\|u_{n,1}(\cdot,t)\|^p_{L^\infty(\mathcal{D})})+c\sup_{t\in[0,T]}\mathbf{E}(\|u_{n,0}(\cdot,t)\|^p_{L^\infty(\mathcal{D})})<\infty,\tag{2.17}$$

since $u_{n,0}$ is deterministic and $u_{n,1}$ is given by the Picard scheme involving $f_n(u_{n,0})$ and $\sigma(u_{n,0})$ at the right-hand side, for f_n and σ Lipschitz, and u_0 smooth. In details, by Picard scheme, we have

$$\begin{aligned}|u_{n,1}(x,t)| \leq &\int_0^t |G(x,y,t)||u_0(y)|dy + \int_0^t\int_{\mathcal{D}} |\Delta G(x,y,t-s)||f_n(u_{n,0}(y,s))|dyds\\ &+\int_0^t\int_{\mathcal{D}} |G(x,y,t-s)||f_n(u_{n,0}(y,s))|dyds\\ &+\Big|\int_0^t\int_{\mathcal{D}} G(x,y,t-s)(\sigma(u_{n,0}(y,s))W(dy,ds)\Big|.\end{aligned}$$

Thus, taking p powers then supremum on $x\in\mathcal{D}$ and then expectation, exactly as before, using the Green's function estimates, Burkholder-Davis-Gunty inequality and then Hölder's inequality, we arrive at

$$\begin{aligned}\sup_{t\in[0,T]}\mathbf{E}(\|u_{n,1}(\cdot,t)\|^p_{L^\infty(\mathcal{D})}) \leq &\sup_{t\in[0,T]}\mathbf{E}\Big(\Big\|\int_0^t |G(\cdot,y,t)||u_0(y)|dy\Big\|^p_{L^\infty(\mathcal{D})}\Big)\\ &+c\mathbf{E}\bigg(\int_0^t 1ds\bigg) \leq c+c<\infty.\end{aligned}$$

So, (2.17) is valid and indeed (2.16) holds true.

Taking now supremum in t at (2.16) we obtain

$$\sup_{t\in[0,T]}\mathbf{E}(\|u_{n,k+1}(\cdot,t)-u_{n,k}(\cdot,t)\|^p_{L^\infty(\mathcal{D})}) \leq c\frac{c^{2k}}{k!},$$

and by summation, we get

$$\sum_{k=0}^\infty \sup_{t\in[0,T]}\mathbf{E}(\|u_{n,k+1}(\cdot,t)-u_{n,k}(\cdot,t)\|^p_{L^\infty(\mathcal{D})}) \leq c\sum_{k=0}^\infty\frac{c^{2k}}{k!} = c\exp(c^2)<\infty, \tag{2.18}$$

for any $p>\max\{2,\frac{8}{4-d}\}=\frac{8}{4-d}$.

Therefore, it follows that, for n fixed, the limit $\lim\limits_{k\to\infty} u_{n,k}$, in the $L^p(\Omega)$ norm, exists for any $(x,t)\in\mathcal{D}\times[0,T]$. Indeed, we have for any $(x,t)\in\mathcal{D}\times[0,T]$

$$\begin{aligned}\mathbf{E}(|u_{n,k+1}(x,t)-u_{n,k}(x,t)|^p) \leq &\mathbf{E}(\|u_{n,k+1}(\cdot,t)-u_{n,k}(\cdot,t)\|^p_{L^\infty(\mathcal{D})})\\ \leq &\sup_{t\in[0,T]}\mathbf{E}(\|u_{n,k+1}(\cdot,t)-u_{n,k}(\cdot,t)\|^p_{L^\infty(\mathcal{D})}) \to 0 \text{ as } k\to\infty.\end{aligned} \tag{2.19}$$

So, for n fixed, the sequence $u_{n,k}$ is Cauchy in $L^p(\Omega)$, and convergent as $k\to\infty$ to some u_n in this norm, i.e.,

$$\exists\, u_n:\ \lim_{k\to\infty}\mathbf{E}\Big(|u_{n,k}(\cdot,t)-u_n(\cdot,t)|^p\Big)=0.$$

Moreover, we observe that $u_{n,k}$, for n fixed, is also Cauchy in the norm $L^p(\infty,\Omega)$ defined by

$$\|v(\cdot,t)\|_{L^p(\infty,\Omega)} := \Big(\mathbf{E}(\|v(\cdot,t)\|^p_{L^\infty(\mathcal{D})})\Big)^{1/p},$$

and convergent in this norm, i.e.,

$$\exists\, \tilde{u_n}:\ \lim_{k\to\infty}\mathbf{E}\Big(\|u_{n,k}(\cdot,t)-\tilde{u_n}(\cdot,t)\|^p_{L^\infty(\mathcal{D})}\Big)=0.$$

Obviously, since

$$\|u_{n,k}(\cdot,t)-\tilde{u_n}(\cdot,t)\|_{L^p(\Omega)} \leq \|u_{n,k}(\cdot,t)-\tilde{u_n}(\cdot,t)\|_{L^p(\infty,\Omega)},$$

from uniqueness of limits, we have $u_n=\tilde{u_n}$, and thus

$$\mathbf{E}(\|u_n(\cdot,t)-u_{n,k}(\cdot,t)\|^p_{L^\infty(\mathcal{D})}) \to 0, \ \text{ as } k\to\infty,$$

and so

$$\mathbf{E}(\|u_n(\cdot,t)-u_{n,k}(\cdot,t)\|^p_{L^\infty(\mathcal{D})})<\infty, \tag{2.20}$$

for any k.

We then have, using (2.18) and (2.20), for any t

$$\begin{aligned}\mathbf{E}(\|u_n(\cdot,t)\|^p_{L^\infty(\mathcal{D})}) \leq & \mathbf{E}(\|u_n(\cdot,t)-u_{n,k}(\cdot,t)\|^p_{L^\infty(\mathcal{D})})+c\sum_{j=k}^{\infty}\mathbf{E}(\|u_{n,j+1}(\cdot,t)-u_{n,j}(\cdot,t)\|^p_{L^\infty(\mathcal{D})})\\ \leq & \mathbf{E}(\|u_n(\cdot,t)-u_{n,k}(\cdot,t)\|^p_{L^\infty(\mathcal{D})})+c\sum_{j=k}^{\infty}\sup_{t\in[0,T]}\mathbf{E}(\|u_{n,j+1}(\cdot,t)-u_{n,j}(\cdot,t)\|^p_{L^\infty(\mathcal{D})})\\ \leq & \mathbf{E}(\|u_n(\cdot,t)-u_{n,k}(\cdot,t)\|^p_{L^\infty(\mathcal{D})})+c\leq c.\end{aligned}$$

Hence, by taking supremum over all $t\in[0,T]$, we obtain for any $p>\max\{2,8/(4-d)\}=\frac{8}{3}$, in dimensions $d=1$,

$$\sup_{t\in[0,T]}\mathbf{E}(\|u_n(\cdot,t)\|^p_{L^\infty(\mathcal{D})})<\infty. \tag{2.21}$$

Note that for power $\hat{p}$ such that $2\leq\hat{p}\leq\frac{8}{3}$, we use Hölder's inequality for the expectation as follows. Observe that $2\hat{p}>\frac{8}{3}$, and take

$$\mathbf{E}(\|u_n(\cdot,t)\|^{\hat{p}}_{L^\infty(\mathcal{D})})\leq c\mathbf{E}(\|u_n(\cdot,t)\|^{2\hat{p}}_{L^\infty(\mathcal{D})})^{1/2}$$

Thus, we get

$$\sup_{t\in[0,T]}\mathbf{E}(\|u_n(\cdot,t)\|^{\hat{p}}_{L^\infty(\mathcal{D})})\leq c\sup_{t\in[0,T]}\mathbf{E}(\|u_n(\cdot,t)\|^{2\hat{p}}_{L^\infty(\mathcal{D})})^{1/2}<\infty,$$

by (2.21), since $2\hat{p}>\frac{8}{3}$. So, we have finally for any $p\geq 2$, in dimensions $d=1$

$$\sup_{t\in[0,T]}\mathbf{E}(\|u_n(\cdot,t)\|^p_{L^\infty(\mathcal{D})})<\infty. \tag{2.22}$$

Through the scheme (2.8), by a standard argument, where we take limits in the $L^p(\Omega)$ norm, and use the fact that f_n and σ are uniformly continuous since Lipschitz, we have

$$\begin{aligned}\lim_{k\to\infty}u_{n,k}(x,t)=&\int_{\mathcal{D}}u_0(y)G(x,y,t)dy\\ &+\int_0^t\int_{\mathcal{D}}[\Delta G(x,y,t-s)-G(x,y,t-s)]f_n(\lim_{k\to\infty}u_{n,k}(y,s))dyds\\ &+\int_0^t\int_{\mathcal{D}}G(x,y,t-s)\sigma(\lim_{k\to\infty}u_{n,k}(y,s))W(dy,ds).\end{aligned} \tag{2.23}$$

Note that for the stochastic term, since

$$\|u_n(\cdot,t)-u_{n,k}(\cdot,t)\|_{L^p(\Omega)}\to 0,\ \text{ as } k\to\infty,$$

we can easily prove that

$$\Big\|\int_0^t\int_{\mathcal{D}}G(x,y,t-s)[\sigma(u_n(y,s))-\sigma(u_{n,k}(y,s))]W(dy,ds)\Big\|_{L^p(\Omega)}\to 0,\ \text{ as } k\to\infty,$$

by using Burkholder-Davis-Gundy inequality as before, the Lipschitz property (or uniform continuity of σ), Hölder inequality and the estimates of G.

So, since $u_n=\lim\limits_{k\to\infty}u_{n,k}$, in the $L^p(\Omega)$ norm, we derive that u_n satisfies the stochastic pde (2.6); as we shall prove in the sequel, (2.6) is uniquely solvable (due to the fact that f_n, σ are Lipschitz in $\mathbb{R}$). Moreover, $u=u_n$ on Ω_n a.s. (see also in [8], for the analogous argument for the stochastic Cahn-Hilliard case, where the same cut-off function was used).

We proceed by establishing uniqueness of solution for the problem (2.6).

Let us suppose that ω_n is a solution of (2.6). Then since u_n is a solution also, by using (2.6) for ω_n and u_n respectively, and subtracting, we get

$$\begin{aligned}u_n(x,t)-\omega_n(x,t)=&\int_0^t\int_{\mathcal{D}}[\Delta G(x,y,t-s)-G(x,y,t-s)]\Big(f_n(u_n(y,s))-f_n(\omega_n(y,s))\Big)dyds\\&+\int_0^t\int_{\mathcal{D}}G(x,y,t-s)\Big(\sigma(u_n(y,s))-\sigma(\omega_n(y,s))\Big)W(dy,ds).\end{aligned}$$

Hence, we obtain

$$\begin{aligned}|u_n(x,t)-\omega_n(x,t)|\leq&\int_0^t\int_{\mathcal{D}}|\Delta G(x,y,t-s)||f_n(u_n(y,s))-f_n(u_n(y,s))|dyds\\&+\int_0^t\int_{\mathcal{D}}|G(x,y,t-s)||f_n(u_n(y,s))-f_n(\omega_n(y,s))|dyds\\&+\Big|\int_0^t\int_{\mathcal{D}}G(x,y,t-s)(\sigma(u_n(y,s))-\sigma(\omega_n(y,s)))W(dy,ds)\Big|.\end{aligned}$$

We take p powers for $p\geq 2$, and proceed as we did for deriving (2.10), i.e., we take supremum in space, expectations at both sides, use that f_n and σ are Lipschitz, and apply the Burkholder-Davis-Gundy inequality to the stochastic term. This yields

$$\begin{aligned}&\mathbf{E}\Big(\|u_n(\cdot,t)-\omega_n(\cdot,t)\|^p_{L^\infty(\mathcal{D})}\Big)\\&\leq c\mathbf{E}\Big(\sup_{x\in\mathcal{D}}\Big(\int_0^t\int_{\mathcal{D}}|\Delta G(x,y,t-s)||u_n(y,s)-\omega_n(y,s)|dyds\Big)^p\Big)\\&\quad+c\mathbf{E}\Big(\sup_{x\in\mathcal{D}}\Big(\int_0^t\int_{\mathcal{D}}|G(x,y,t-s)||u_n(y,s)-\omega_n(y,s)|dyds\Big)^p\Big)\\&\quad+c\mathbf{E}\Big(\sup_{\tau\in[0,t]}\sup_{x\in\mathcal{D}}\Big(\int_0^\tau\int_{\mathcal{D}}|G(x,y,\tau-s)|^2|u_n(y,s)-\omega_n(y,s)|^2dyds\Big)^{p/2}\Big).\end{aligned}\tag{2.24}$$

Observe that the previous inequality is the same as (2.10), where the differences $u_{n,k+1}-u_{n,k}$, $u_{n,k}-u_{n,k-1}$ are replaced by $u_n-\omega_n$. Thus, a direct result is the analogous of (2.15), i.e., for any $p>\max\{2,8/(4-d)\}=\frac{8}{3}$

$$\begin{aligned}\mathbf{E}(\|u_n(\cdot,t)-\omega_n(\cdot,t)\|^p_{L^\infty(\mathcal{D})})&\leq c\mathbf{E}\Big(\int_0^t\|u_n(\cdot,s)-\omega_n(\cdot,s)\|^p_{L^\infty(\mathcal{D})}ds\Big)\\&\leq c\Big(\int_0^t\mathbf{E}(\|u_n(\cdot,s)-\omega_n(\cdot,s)\|^p_{L^\infty(\mathcal{D})})ds\Big),\end{aligned}\tag{2.25}$$

where again we used Fubini's Theorem.

Hence, by applying Gronwall's Lemma to the previous inequality for the term $\mathbf{E}(\|u_n(\cdot,t)-\omega_n(\cdot,t)\|^p_{L^\infty(\mathcal{D})})$, we obtain

$$\mathbf{E}(\|u_n(\cdot,t)-\omega_n(\cdot,t)\|^p_{L^\infty(\mathcal{D})})\leq 0,$$

for any $t\in[0,T]$. So for any t in $[0,T]$,

$$\mathbf{E}(\|u_n(\cdot,t)-\omega_n(\cdot,t)\|^p_{L^\infty(\mathcal{D})})=0.$$

This yields that $u_n(x,t)=\omega_n(x,t)$ almost surely in Ω and in Ω_n (since $\Omega_n\subset\Omega$ and thus $\|v\|_{L^p(\Omega_n)}\leq\|v\|_{L^p(\Omega)}$), for any t, x, i.e., for $\hat{\Omega}:=\Omega$, or Ω_n

$$P\Big(w\in\hat{\Omega}:\ u_n(x,t;w)=\omega_n(x,t;w)\Big)=1,\ \text{ for any }\ t\in[0,T],\ \text{ and any }\ x\in\mathcal{D},$$

and so by definition u_n, ω_n are equivalent in Ω and in Ω_n.

We shall use now the fact that when two processes are equivalent in a set and a.s. continuous in the same set, then they are indistinguishable in this set.

The solution u of the stochastic Cahn-Hilliard/Allen-Cahn equation (1.1) is almost surely continuous in space and time, in dimensions $d = 1$, cf. [5], and the approximations u_n, ω_n of u satisfy the equation (1.1) a.s. in Ω_n (since $f_n(u_n) = f(u_n)$ and $f_n(\omega_n) = f(\omega_n)$ in Ω_n a.s.). So, the equivalent processes u_n, ω_n are almost surely continuous in Ω_n also and thus indistinguishable in Ω_n (having the same paths), i.e.,

$$P\Big(w \in \Omega_n : \ u_n(x,t;w) = \omega_n(x,t;w), \text{ for any } (x,t) \in \mathcal{D} \times [0,T]\Big) = 1. \tag{2.26}$$

Since u_n, ω_n are indistinguishable on Ω_n then we have uniqueness of solution of (2.6) with uniquely defined paths a.s. on Ω_n.

Thus, u_n is well defined by (2.6), and suitable for localizing u. □

2.2.2. *The Malliavin derivative of u_n.* We proceed by proving that the derivative of the approximation u_n in the Malliavin sense, is well defined as the solution of an spde. In addition, we establish the regularity of u_n in $D^{1,2}$ and $L^{1,2}$; this is accomplished at the next proposition.

Proposition 2.8. *Let $u_n(x,t)$ be the solution of* (2.6)*, then:*

(1) *u_n belongs to the space $D^{1,2}$.*

(2) *The Malliavin derivative of u_n satisfies for any $s \le t$, uniquely, the spde of the form*

(2.27)

$$\begin{aligned}D_{y,s}u_n(x,t) := D_{y,s}(u_n(x,t)) = &\int_s^t \int_{\mathcal{D}} [\Delta G(x,z,t-\tau) - G(x,z,t-\tau)]\tilde{\mathcal{G}}_2(n)(z,\tau) D_{y,s}(u_n(z,\tau)) dz d\tau \\ &+ G(x,y,t-s)\sigma(u_n(y,s)) \\ &+ \int_s^t \int_{\mathcal{D}} G(x,z,t-\tau)\tilde{\mathcal{G}}_1(n)(z,\tau) D_{y,s}(u_n(z,\tau)) W(dz,d\tau),\end{aligned}$$

while

$$D_{y,s}u_n(x,t) = 0 \quad \text{for any } s > t.$$

Here, $\tilde{\mathcal{G}}_1(n)(z,\tau)$, $\tilde{\mathcal{G}}_2(n)(z,\tau)$ are bounded, and satisfy

$$D_{y,s}(\sigma(u_n(z,\tau))) = \tilde{\mathcal{G}}_1(n)(z,\tau) D_{y,s}(u_n(z,\tau)),$$

$$D_{y,s}(f_n(u_n(z,\tau))) = \tilde{\mathcal{G}}_2(n)(z,\tau) D_{y,s}(u_n(z,\tau)).$$

(3) *u_n belongs to $L^{1,2}$.*

Proof. First, we will prove that the Cauchy sequence $\{u_{n,k}\}_{k\in\mathbb{N}}$ (as we described in Lemma (2.7)) belongs to the space $D^{1,2}$ for all $(x,t) \in \mathcal{D} \times [0,T]$, by using induction and the Picard iteration scheme.

For $k = 0$, the function $u_{n,0}$ is deterministic with Malliavin derivative $Du_{n,0} = 0$. Thus $u_{n,0} \in D^{1,2}$.

We proceed with induction.

We suppose for $k \ge 0$ that for any $i \le k$, $u_{n,i} \in D^{1,2}$ for every $(x,t) \in \mathcal{D} \times [0,T]$, and that

$$\sup_{t\in[0,T]} \sup_{i\le k} \mathbf{E}\Big(\int_0^t \int_{\mathcal{D}} \|D_{y,s}u_{n,i}(\cdot,t)\|^2_{L^\infty(\mathcal{D})} dy ds\Big) < \infty.$$

We shall prove that for any $i \le k+1$, $u_{n,i} \in D^{1,2}$ for every $(x,t) \in \mathcal{D} \times [0,T]$ also (i.e., $u_{n,k+1} \in D^{1,2}$ for every $(x,t) \in \mathcal{D} \times [0,T]$), and

$$\sup_{t\in[0,T]} \sup_{i\le k+1} \mathbf{E}\Big(\int_0^t \int_{\mathcal{D}} \|D_{y,s}u_{n,i}(\cdot,t)\|^2_{L^\infty(\mathcal{D})} dy ds\Big) < \infty,$$

also (the bounds being independent of k). Note that the integral for $s \in [0,t]$ coincides with the integral for $s \in [0,T]$, since the Malliavin derivative involved is zero for any $s > t$. This will result that

$$\begin{aligned} &\forall\, k \quad \exists \;\; u_{n,k} \in D^{1,2} \;\; \forall\, (x,t) \in \mathcal{D} \times [0,T], \;\; \text{and} \\ &\sup_{t\in[0,T]} \sup_{k} \mathbf{E}\Big(\int_0^T \int_{\mathcal{D}} \|D_{y,s} u_{n,k}(\cdot,t)\|^2_{L^\infty(\mathcal{D})} dy ds \Big) < \infty. \end{aligned} \tag{2.28}$$

We apply the Malliavin derivative to (2.8), and get, since it is a linear operator

$$\begin{aligned} D_{y,s}(u_{n,k+1}(x,t)) =: D_{y,s}u_{n,k+1}(x,t) = {}& D_{y,s}\Big[\int_{\mathcal{D}} u_0(y) G(x,z,t) dz \Big] \\ &+ D_{y,s}\Big[\int_0^t \int_{\mathcal{D}} [\Delta G(x,z,t-\tau) - G(x,z,t-\tau)] f_n(u_{n,k}(z,\tau)) dz d\tau \Big] \\ &+ D_{y,s}\Big[\int_0^t \int_{\mathcal{D}} G(x,z,t-\tau) \sigma(u_{n,k}(z,\tau)) W(dz,d\tau) \Big] \\ ={}& 0 + \int_0^t \int_{\mathcal{D}} D_{y,s}\Big([\Delta G(x,z,t-\tau) - G(x,z,t-\tau)] f_n(u_{n,k}(z,\tau)) \Big) dz d\tau \\ &+ G(x,y,t-s) \sigma(u_{n,k}(y,s)) \\ &+ \int_0^t \int_{\mathcal{D}} D_{y,s}\Big(G(x,z,t-\tau) \sigma(u_{n,k}(z,\tau)) \Big) W(dz,d\tau) \\ ={}& \int_0^t \int_{\mathcal{D}} D_{y,s}\Big(\Delta G(x,z,t-\tau) - G(x,z,t-\tau) \Big) f_n(u_{n,k}(z,\tau)) dz d\tau \\ &+ \int_0^t \int_{\mathcal{D}} [\Delta G(x,z,t-\tau) - G(x,z,t-\tau)] D_{y,s}\Big(f_n(u_{n,k}(z,\tau)) \Big) dz d\tau \\ &+ G(x,y,t-s) \sigma(u_{n,k}(y,s)) \\ &+ \int_0^t \int_{\mathcal{D}} D_{y,s}\Big(G(x,z,t-\tau) \Big) \sigma(u_{n,k}(z,\tau)) W(dz,d\tau) \\ &+ \int_0^t \int_{\mathcal{D}} G(x,z,t-\tau) D_{y,s}\Big(\sigma(u_{n,k}(z,\tau)) \Big) W(dz,d\tau) \\ ={}& 0 + \int_0^t \int_{\mathcal{D}} [\Delta G(x,z,t-\tau) - G(x,z,t-\tau)] D_{y,s}\Big(f_n(u_{n,k}(z,\tau)) \Big) dz d\tau \\ &+ G(x,y,t-s) \sigma(u_{n,k}(y,s)) \\ &+ 0 + \int_0^t \int_{\mathcal{D}} G(x,z,t-\tau) D_{y,s}\Big(\sigma(u_{n,k}(z,\tau)) \Big) W(dz,d\tau) \\ ={}& \int_s^t \int_{\mathcal{D}} [\Delta G(x,z,t-\tau) - G(x,z,t-\tau)] D_{y,s}\Big(f_n(u_{n,k}(z,\tau)) \Big) dz d\tau \\ &+ G(x,y,t-s) \sigma(u_{n,k}(y,s)) \\ &+ \int_s^t \int_{\mathcal{D}} G(x,z,t-\tau) D_{y,s}\Big(\sigma(u_{n,k}(z,\tau)) \Big) W(dz,d\tau), \end{aligned} \tag{2.29}$$

where we used also that the Malliavin derivative is zero when applied to the deterministic terms G, ΔG (since no change is observed on $\omega \in \Omega$, they are constant as functions of $\omega \in \Omega$). Moreover, since the Malliavin derivative is zero for any $\tau < s$, this resulted to integrals on $\tau \geq s$.

Here, we note that $D_{y,s}(u_{n,k+1}(x,t))$ is a function of $y,\ s,\ x,\ t$. In this work, the notation $D_{y,s}f(x,t)$, for a general function f, is used to denote $D_{y,s}(f(x,t))$.

We now use Proposition 1.2.4 of [17], cf. also in [8], in dimensions $m = 1$ (following the Nualart's book notation, since $u_{n,k}(x,t) \in \mathbb{R}^m$, $m = 1$) with the norm used for the Lipschitz condition being the absolute value. More specifically, since $u_{n,k}$ belongs to $D^{1,2}$ (true by the induction hypothesis) and σ is Lipschitz uniformly on any x in $\mathbb{R}$ with K_σ its Lipschitz coefficient, then $\sigma(u_{n,k})$ belongs to $D^{1,2}$ also, and there exists a random variable $\mathcal{G}_1 = \mathcal{G}_1(n,k)$ such that

$$D_{y,s}\Big(\sigma(u_{n,k}(x,t))\Big) = \mathcal{G}_1(n,k)(x,t) D_{y,s} u_{n,k}(x,t), \tag{2.30}$$

with $\mathcal{G}_1$ bounded (in the absolute value norm) by K_σ, uniformly for any x, t, i.e.,

$$|\mathcal{G}_1(n,k)(x,t)| \leq K_\sigma, \quad \forall\, x \in \mathcal{D},\ \forall t \in [0,T].$$

Since K_σ is independent of n, k, we have finally

$$\sup_{n,k,(x,t)\in\mathcal{D}\times[0,T]} |\mathcal{G}_1(n,k)(x,t)| \leq K_\sigma. \tag{2.31}$$

The same argument can be applied for f_n in place of σ, since f_n is also Lipschitz uniformly on $\mathbb{R}$. Indeed, there exists a random variable $\mathcal{G}_2 = \mathcal{G}_2(n,k)$ such that

$$D_{y,s}\Big(f_n(u_{n,k}(x,t))\Big) = \mathcal{G}_2(n,k)(x,t) D_{y,s} u_{n,k}(x,t), \tag{2.32}$$

and

$$\sup_{k,(x,t)\in\mathcal{D}\times[0,T]} |\mathcal{G}_2(n,k)(x,t)| \leq K_{f_n}, \tag{2.33}$$

for K_{f_n} a positive constant, depending on n through f_n.

Therefore, (2.30) and (2.32), together with (2.29), give finally for any $s \leq t$

$$\begin{aligned} D_{y,s}u_{n,k+1}(x,t) = &\int_s^t \int_{\mathcal{D}} [\Delta G(x,z,t-\tau) - G(x,z,t-\tau)] \mathcal{G}_2(n,k)(z,\tau) D_{y,s}u_{n,k}(z,\tau)\,dz d\tau \\ &+ G(x,y,t-s)\sigma(u_{n,k}(y,s)) \\ &+ \int_s^t \int_{\mathcal{D}} G(x,z,t-\tau)\mathcal{G}_1(n,k)(z,\tau) D_{y,s}u_{n,k}(z,\tau) W(dz,d\tau), \end{aligned} \tag{2.34}$$

while for $s > t$

$$D_{y,s}u_{n,k+1}(x,t) = 0.$$

Taking absolute value at both sides of (2.34), and then p powers for $p \geq 2$, we get

$$\begin{aligned} |D_{y,s}u_{n,k+1}(x,t)|^p \leq &c|G(x,y,t-s)\sigma(u_{n,k}(y,s))|^p \\ &+ c\Big|\int_s^t \int_{\mathcal{D}} [\Delta G(x,z,t-\tau) - G(x,z,t-\tau)] \mathcal{G}_2(n,k)(z,\tau) D_{y,s}u_{n,k}(z,\tau)\,dz d\tau\Big|^p \\ &+ c\Big|\int_s^t \int_{\mathcal{D}} G(x,z,t-\tau)\mathcal{G}_1(n,k)(z,\tau) D_{y,s}u_{n,k}(z,\tau) W(dz,d\tau)\Big|^p, \end{aligned}$$

which gives by (2.33)

$$\begin{aligned} \|D_{y,s}u_{n,k+1}(\cdot,t)\|^p_{L^\infty(\mathcal{D})} \leq &c\|G(\cdot,y,t-s)\sigma(u_{n,k}(y,s))\|^p_{L^\infty(\mathcal{D})} \\ &+ cK_{f_n}\Big\|\int_s^t \int_{\mathcal{D}} |\Delta G(\cdot,z,t-\tau) - G(\cdot,z,t-\tau)||D_{y,s}u_{n,k}(z,\tau)|\,dz d\tau\Big\|^p_{L^\infty(\mathcal{D})} \\ &+ c\Big\|\int_s^t \int_{\mathcal{D}} G(\cdot,z,t-\tau)\mathcal{G}_1(n,k)(z,\tau) D_{y,s}u_{n,k}(z,\tau) W(dz,d\tau)\Big\|^p_{L^\infty(\mathcal{D})}. \end{aligned}$$

We integrate for $y \in \mathcal{D},\ s \in [0,t]$ and then take expectation, to derive

$$
\begin{aligned}
&\mathbf{E}\Big(\int_0^t\int_{\mathcal{D}}\|D_{y,s}u_{n,k+1}(\cdot,t)\|^p_{L^\infty(\mathcal{D})}dyds\Big) \leq c\mathbf{E}\Big(\int_0^t\int_{\mathcal{D}}\|G(\cdot,y,t-s)\sigma(u_{n,k}(y,s))\|^p_{L^\infty(\mathcal{D})}dyds\Big)\\
&\quad + cK_{f_n}\mathbf{E}\Big(\int_0^t\int_{\mathcal{D}}\Big\|\int_s^t\int_{\mathcal{D}}|\Delta G(\cdot,z,t-\tau)-G(\cdot,z,t-\tau)||D_{y,s}u_{n,k}(z,\tau)|dzd\tau\Big\|^p_{L^\infty(\mathcal{D})}dyds\Big)\\
&\quad + c\mathbf{E}\Big(\int_0^t\int_{\mathcal{D}}\Big\|\int_s^t\int_{\mathcal{D}}G(\cdot,z,t-\tau)\mathcal{G}_1(n,k)(z,\tau)D_{y,s}u_{n,k}(z,\tau)W(dz,d\tau)\Big\|^p_{L^\infty(\mathcal{D})}dyds\Big)\\
&:= M_1(t;k)+M_2(t;k)+M_3(t;k).
\end{aligned}
\tag{2.35}
$$

We shall estimate the terms $M_i(t;k)$ for $i=1,2,3$.

Considering the term $M_1(t;k)$, we have

$$
\begin{aligned}
M_1(t;k) =& c\mathbf{E}\Big(\int_0^t\int_{\mathcal{D}}\|G(\cdot,y,t-s)\sigma(u_{n,k}(y,s))\|^p_{L^\infty(\mathcal{D})}dyds\Big)\\
\leq& c\mathbf{E}\Big(\int_0^t\int_{\mathcal{D}}\|G(\cdot,y,t-s)\|^{2p}_{L^\infty(\mathcal{D})}dyds\Big)+c\mathbf{E}\Big(\int_0^t\int_{\mathcal{D}}|\sigma(u_{n,k}(y,s))|^{2p}dyds\Big)\\
\leq& c\mathbf{E}\Big(\int_0^t\int_{\mathcal{D}}\|G(\cdot,y,t-s)\|^{2p}_{L^\infty(\mathcal{D})}dyds\Big)+c\mathbf{E}\Big(\int_0^t\int_{\mathcal{D}}c(1+|u_{n,k}(y,s)|^{2pq})dyds\Big)\\
\leq& c\mathbf{E}\Big(\int_0^t\int_{\mathcal{D}}\|G(\cdot,y,t-s)\|^{2p}_{L^\infty(\mathcal{D})}dyds\Big)+c+c\mathbf{E}\Big(\int_0^t c\|u_{n,k}(\cdot,s)\|^{2pq}_{L^\infty(\mathcal{D})}ds\Big)\\
\leq& c+c\mathbf{E}\Big(\int_0^t\int_{\mathcal{D}}\|G(\cdot,y,t-s)\|^{2p}_{L^\infty(\mathcal{D})}dyds\Big)+c\int_0^t\mathbf{E}\Big(\|u_{n,k}(\cdot,s)\|^{2pq}_{L^\infty(\mathcal{D})}\Big)ds,
\end{aligned}
\tag{2.36}
$$

where we used the growth of the unbounded noise diffusion, for $q\in(0,1/3)$, and Fubini's Theorem.

We use the next estimate (1.6) of [8], to get

$$
\int_0^t\int_{\mathcal{D}}\|G(\cdot,y,t-s)\|^{2p}_{L^\infty(\mathcal{D})}dyds \leq C\int_0^t|t-s|^{-2pd/4+d/4}ds<\infty,
\tag{2.37}
$$

for $-2pd/4+d/4=(-2p+1)/4>-1$ (since $d=1$) i.e., for $(2\leq)p<5/2$.

Also since $2pq\leq 2p<5$ in dimensions $d=1$, using (2.21) and (2.20), we obtain for any $t\in[0,T]$

$$
\begin{aligned}
\mathbf{E}(\|u_{n,k}(\cdot,t)\|^{2pq}_{L^\infty(\mathcal{D})}) \leq& c+c\mathbf{E}(\|u_{n,k}(\cdot,t)-u_n(\cdot,t)\|^5_{L^\infty(\mathcal{D})})+c\mathbf{E}(\|u_n(\cdot,t)\|^5_{L^\infty(\mathcal{D})})\\
\leq& c+c\sup_{t\in[0,T]}\mathbf{E}(\|u_n(\cdot,t)\|^5_{L^\infty(\mathcal{D})})\leq c+c\leq c.
\end{aligned}
\tag{2.38}
$$

Using (2.37), (2.38) in (2.36), yields for $2\leq p<5/2$

$$
\sup_{k,t\in[0,T]}M_1(t;k)<\infty.
\tag{2.39}
$$

Considering the term $M_2(t;k)$, we choose the inequality (1.12) of [8], p. 781, for

$$
H(x,y,t-\tau):=|\Delta G(x,z,t-\tau)|,\quad \upsilon(z,\tau):=|D_{y,s}u_{n,k}(z,\tau)|,\quad \rho=q\ (\text{of [8] notation})\ =\infty,\quad r=1.
$$

As in (2.12), we have

$$
\Big\|\int_s^t\int_{\mathcal{D}}|\Delta G(\cdot,z,t-\tau)||D_{y,s}u_{n,k}(z,\tau)|dzd\tau\Big\|^p_{L^\infty(\mathcal{D})}\leq c\int_0^t\|D_{y,s}u_{n,k}(\cdot,\tau)\|^p_{L^\infty(\mathcal{D})}d\tau.
\tag{2.40}
$$

Using the inequality (1.11) of [8], p. 781, for

$$
H(x,z,t-\tau)=|G(x,z,t-\tau)|,\quad \upsilon(z,\tau):=|D_{y,s}u_{n,k}(z,\tau)|,\quad \rho=q=\infty,\quad r=1,
$$

we get

$$\Big\|\int_s^t\int_{\mathcal{D}}\|G(\cdot,z,t-\tau)||D_{y,s}u_{n,k}(z,\tau)|dzd\tau\Big\|^p_{L^\infty(\mathcal{D})}\leq c\int_0^t\|D_{y,s}u_{n,k}(\cdot,\tau)\|^p_{L^\infty(\mathcal{D})}d\tau. \tag{2.41}$$

Relations (2.40), (2.41) yield

$$\begin{aligned}
M_2(t;k)=&cK_{f_n}\mathbf{E}\Big(\int_0^t\int_{\mathcal{D}}\Big\|\int_s^t\int_{\mathcal{D}}|\Delta G(\cdot,z,t-\tau)-G(\cdot,z,t-\tau)||D_{y,s}u_{n,k}(z,\tau)|dzd\tau\Big\|^p_{L^\infty(\mathcal{D})}dyds\Big)\\
\leq&cK_{f_n}\mathbf{E}\Big(\int_0^t\int_{\mathcal{D}}\int_0^t\|D_{y,s}u_{n,k}(\cdot,\tau)\|^p_{L^\infty(\mathcal{D})}d\tau dyds\Big)\\
\leq&cK_{f_n}\int_0^t\mathbf{E}\Big(\int_0^t\int_{\mathcal{D}}\|D_{y,s}u_{n,k}(\cdot,\tau)\|^p_{L^\infty(\mathcal{D})}dyds\Big)d\tau\\
=&cK_{f_n}\int_0^t\mathbf{E}\Big(\int_0^\tau\int_{\mathcal{D}}\|D_{y,s}u_{n,k}(\cdot,\tau)\|^p_{L^\infty(\mathcal{D})}dyds\Big)d\tau,
\end{aligned}\tag{2.42}$$

where we used Fubini's Theorem; the integral for s is taken finally in $[0,\tau]$ since for $s>\tau$ the Malliavin derivative satisfies $D_{y,s}u_{n,k}(x,\tau)=0$, for any x.

For the term $M_3(t;k)$, we have, using Fubini's Theorem and Burkholder-Davis-Gundy inequality

$$\begin{aligned}
M_3(k;t)=&c\mathbf{E}\Big(\int_0^t\int_{\mathcal{D}}\Big\|\int_s^t\int_{\mathcal{D}}G(\cdot,z,t-\tau)\mathcal{G}_1(n,k)(z,\tau)D_{y,s}u_{n,k}(z,\tau)W(dz,d\tau)\Big\|^p_{L^\infty(\mathcal{D})}dyds\Big)\\
\leq&c\int_0^t\int_{\mathcal{D}}\mathbf{E}\Big(\Big\|\int_s^t\int_{\mathcal{D}}G(\cdot,z,t-\tau)\mathcal{G}_1(n,k)(z,\tau)D_{y,s}u_{n,k}(z,\tau)W(dz,d\tau)\Big\|^p_{L^\infty(\mathcal{D})}\Big)dyds\\
\leq&c\int_0^t\int_{\mathcal{D}}\mathbf{E}\Big(\sup_{r\in[0,t]}\sup_{x\in\mathcal{D}}\Big|\int_0^r\int_{\mathcal{D}}G(x,z,t-\tau)\mathcal{G}_1(n,k)(z,\tau)D_{y,s}u_{n,k}(z,\tau)W(dz,d\tau)\Big|^p\Big)dyds\\
\leq&c\int_0^t\int_{\mathcal{D}}\mathbf{E}\Big(\sup_{r\in[0,t]}\sup_{x\in\mathcal{D}}\Big|\int_0^r\int_{\mathcal{D}}|G(x,z,t-\tau)|^2|\mathcal{G}_1(n,k)(z,\tau)|^2|D_{y,s}u_{n,k}(z,\tau)|^2dzd\tau\Big|^{p/2}\Big)dyds\\
\leq&c\int_0^t\int_{\mathcal{D}}\mathbf{E}\Big(\sup_{r\in[0,t]}\sup_{x\in\mathcal{D}}\Big(\int_0^r\int_{\mathcal{D}}|G(x,z,t-\tau)|^2|D_{y,s}u_{n,k}(z,\tau)|^2dzd\tau\Big)^{p/2}\Big)dyds\\
=&c\int_0^t\int_{\mathcal{D}}\mathbf{E}\Big(\sup_{r\in[0,t]}\Big\|\int_0^r\int_{\mathcal{D}}|G(\cdot,z,t-\tau)|^2|D_{y,s}u_{n,k}(z,\tau)|^2dzd\tau\Big\|^{p/2}_{L^\infty(\mathcal{D})}\Big)dyds,
\end{aligned}\tag{2.43}$$

where we also used the relation (2.31).

As in (2.14), we choose the inequality (1.13) of [8], p. 781, for

$$H(x,y,t-s)=G^2(x,z,t-\tau),\quad v(z,\tau)=|D_{y,s}u_{n,k}(z,\tau)|^2,\quad \rho=q\text{ (following [8] notation)}=\infty,\quad r=1,$$

and we obtain

$$
\begin{aligned}
&\mathbf{E}\Big(\sup_{r\in[0,t]}\Big\|\int_0^r\int_{\mathcal{D}}|G(\cdot,z,t-\tau)|^2|D_{y,s}u_{n,k}(z,\tau)|^2dzd\tau\Big\|^{p/2}_{L^\infty(\mathcal{D})}\Big)\\
&\le\mathbf{E}\Big(\sup_{r\in[0,t]}\Big\|\int_0^r\|D_{y,s}u_{n,k}(\cdot,\tau)\|^2_{L^\infty(\mathcal{D})}\int_{\mathcal{D}}|G(\cdot,z,t-\tau)|^2dzd\tau\Big\|^{p/2}_{L^\infty(\mathcal{D})}\Big)\\
&\le\mathbf{E}\Big(\Big(\int_0^t\|D_{y,s}u_{n,k}(\cdot,\tau)\|^2_{L^\infty(\mathcal{D})}(t-\tau)^{-2d/4+d/4}d\tau\Big)^{p/2}\Big)\\
&=\mathbf{E}\Big(\Big(\int_0^t\|D_{y,s}u_{n,k}(\cdot,\tau)\|^2_{L^\infty(\mathcal{D})}(t-\tau)^{-d/4}d\tau\Big)^{p/2}\Big)\\
&=\mathbf{E}\Big(\Big(\int_0^t\|D_{y,s}u_{n,k}(\cdot,\tau)\|^2_{L^\infty(\mathcal{D})}(t-\tau)^{-d/4}d\tau\Big)\Big),
\end{aligned}
\tag{2.44}
$$

where we took $p=2$. We use now estimate (2.44) to (2.43), and arrive at

$$
\begin{aligned}
M_3(k;t)\le&c\int_0^t\int_{\mathcal{D}}\mathbf{E}\Big(\sup_{r\in[0,t]}\Big\|\int_0^r\int_{\mathcal{D}}|G(\cdot,z,t-\tau)|^2|D_{y,s}u_{n,k}(z,\tau)|^2dzd\tau\Big\|_{L^\infty(\mathcal{D})}\Big)dyds\\
\le&c\int_0^t\int_{\mathcal{D}}\mathbf{E}\Big(\int_0^t(t-\tau)^{-d/4}\|D_{y,s}u_{n,k}(\cdot,\tau)\|^2_{L^\infty(\mathcal{D})}d\tau\Big)dyds\\
=&c\mathbf{E}\Big(\int_0^t\int_{\mathcal{D}}\int_0^t(t-\tau)^{-d/4}\|D_{y,s}u_{n,k}(\cdot,\tau)\|^2_{L^\infty(\mathcal{D})}d\tau dyds\Big)\\
=&c\mathbf{E}\Big(\int_0^t\int_0^t\int_{\mathcal{D}}(t-\tau)^{-d/4}\|D_{y,s}u_{n,k}(\cdot,\tau)\|^2_{L^\infty(\mathcal{D})}dydsd\tau\Big)\\
=&c\mathbf{E}\Big(\int_0^t\int_0^\tau\int_{\mathcal{D}}(t-\tau)^{-d/4}\|D_{y,s}u_{n,k}(\cdot,\tau)\|^2_{L^\infty(\mathcal{D})}dydsd\tau\Big)\\
=&c\mathbf{E}\Big(\int_0^t(t-\tau)^{-d/4}\int_0^\tau\int_{\mathcal{D}}\|D_{y,s}u_{n,k}(\cdot,\tau)\|^2_{L^\infty(\mathcal{D})}dydsd\tau\Big)\\
=&c\int_0^t(t-\tau)^{-d/4}\mathbf{E}\Big(\int_0^\tau\int_{\mathcal{D}}\|D_{y,s}u_{n,k}(\cdot,\tau)\|^2_{L^\infty(\mathcal{D})}dyds\Big)d\tau,
\end{aligned}
\tag{2.45}
$$

since for $s>\tau$, $D_{y,s}u_{n,k}(x,\tau)=0$, for any x.

Thus, choosing $p=2$ on (2.35), and using the estimates (2.39), (2.42) and (2.45), we finally proved since $d=1$

$$
\begin{aligned}
\mathbf{E}\Big(\int_0^t\int_{\mathcal{D}}\|D_{y,s}u_{n,k+1}(\cdot,t)\|^2_{L^\infty(\mathcal{D})}dyds\Big)\le&C_0+cK_{f_n}\int_0^t\mathbf{E}\Big(\int_0^\tau\int_{\mathcal{D}}\|D_{y,s}u_{n,k}(\cdot,\tau)\|^2_{L^\infty(\mathcal{D})}dyds\Big)d\tau\\
&+c\int_0^t(t-\tau)^{-1/4}\mathbf{E}\Big(\int_0^\tau\int_{\mathcal{D}}\|D_{y,s}u_{n,k}(\cdot,\tau)\|^2_{L^\infty(\mathcal{D})}dyds\Big)d\tau,
\end{aligned}
\tag{2.46}
$$

for $C_0,c>0$ constants independent of $k,\ t$.

We take supremum on $i \leq k$ (the above inequality is true for any such i, from the first induction hypothesis: $D_{y,s}u_{n,i} \in \mathbf{D}_{1,2}$ for any $i \leq k$), and get

$$
\begin{aligned}
&\sup_{i\leq k}\mathbf{E}\Big(\int_0^t\int_{\mathcal{D}}\|D_{y,s}u_{n,i+1}(\cdot,t)\|^2_{L^\infty(\mathcal{D})}dyds\Big) \leq C_0 + cK_{f_n}\int_0^t\sup_{i\leq k}\mathbf{E}\Big(\int_0^\tau\int_{\mathcal{D}}\|D_{y,s}u_{n,i}(\cdot,\tau)\|^2_{L^\infty(\mathcal{D})}dyds\Big)d\tau\\
&\quad + c\int_0^t(t-\tau)^{-1/4}\sup_{i\leq k}\mathbf{E}\Bigg(\int_0^\tau\int_{\mathcal{D}}\|D_{y,s}u_{n,i}(\cdot,\tau)\|^2_{L^\infty(\mathcal{D})}dyds\Bigg)d\tau\\
\leq& c + C_0 + cK_{f_n}\int_0^t\sup_{i\leq k}\mathbf{E}\Big(\int_0^\tau\int_{\mathcal{D}}\|D_{y,s}u_{n,i+1}(\cdot,\tau)\|^2_{L^\infty(\mathcal{D})}dyds\Big)d\tau\\
&\quad + c\int_0^t(t-\tau)^{-1/4}\sup_{i\leq k}\mathbf{E}\Bigg(\int_0^\tau\int_{\mathcal{D}}\|D_{y,s}u_{n,i+1}(\cdot,\tau)\|^2_{L^\infty(\mathcal{D})}dyds\Bigg)d\tau,
\end{aligned}
$$

which gives for

$$
A_{n,k+1}(t) := \sup_{i\leq k+1}\mathbf{E}\Big(\int_0^t\int_{\mathcal{D}}\|D_{y,s}u_{n,i}(\cdot,t)\|^2_{L^\infty(\mathcal{D})}dyds\Big),
$$

$$
A_{n,k+1}(t) \leq c + C_0 + cK_{f_n}\int_0^t A_{n,k+1}(\tau)d\tau + c\int_0^t(t-\tau)^{-1/4}A_{n,k+1}(\tau)d\tau. \tag{2.47}
$$

From (2.47) and since $A_{n,k+1} \geq 0$, we obtain

$$
\begin{aligned}
\int_0^t(t-\tau)^{-1/4}A_{n,k+1}(\tau)d\tau \leq& (c+C_0)\int_0^t(t-\tau)^{-1/4}d\tau + cK_{f_n}\int_0^t(t-\tau)^{-1/4}\int_0^\tau A_{n,k+1}(s)dsd\tau\\
&+ c\int_0^t(t-\tau)^{-1/4}\int_0^\tau(\tau-s)^{-1/4}A_{n,k+1}(s)dsd\tau\\
\leq& c + cK_{f_n}\int_0^t(t-\tau)^{-1/4}\int_0^t A_{n,k+1}(s)dsd\tau\\
&+ c\int_0^t(t-\tau)^{-1/4}\int_0^t(\tau-s)^{-1/4}A_{n,k+1}(s)dsd\tau\\
=& c + cK_{f_n}\int_0^t(t-\tau)^{-1/4}d\tau\int_0^t A_{n,k+1}(s)ds\\
&+ c\int_0^t\int_0^t(t-\tau)^{-1/4}(\tau-s)^{-1/4}A_{n,k+1}(s)dsd\tau\\
=& c + cK_{f_n}\int_0^t(t-\tau)^{-1/4}d\tau\int_0^t A_{n,k+1}(s)ds\\
&+ c\int_0^t\Big[\int_0^t(t-\tau)^{-1/4}(\tau-s)^{-1/4}d\tau\Big]A_{n,k+1}(s)ds\\
\leq& c + c\int_0^t A_{n,k+1}(s)ds,
\end{aligned} \tag{2.48}
$$

where we used that

$$
\int_0^t(t-\tau)^{-1/4}d\tau < \infty,
$$

and

$$
\int_0^t(t-\tau)^{-1/4}(\tau-s)^{-1/4}d\tau \leq \Big[\int_0^t(t-\tau)^{-1/2}d\tau\Big]^{1/2}\Big[\int_0^t(\tau-s)^{-1/2}d\tau\Big]^{1/2} < \infty.
$$

So, by using (2.48) in (2.47), yields

$$A_{n,k+1}(t) \leq c + c\int_0^t A_{n,k+1}(\tau)d\tau, \tag{2.49}$$

and by Gronwall's Lemma, we get

$$\sup_{i\leq k+1} \mathbf{E}\Big(\int_0^t \int_{\mathcal{D}} \|D_{y,s}u_{n,i}(\cdot,t)\|^2_{L^\infty(\mathcal{D})}dyds\Big) = A_{n,k+1}(t) \leq c = c(n),$$

which gives

$$\sup_{t\in[0,T]}\sup_{i\leq k+1} \mathbf{E}\Big(\int_0^T \int_{\mathcal{D}} \|D_{y,s}u_{n,i}(\cdot,t)\|^2_{L^\infty(\mathcal{D})}dyds\Big) = \sup_{t\in[0,T]}\sup_{i\leq k+1} \mathbf{E}\Big(\int_0^t \int_{\mathcal{D}} \|D_{y,s}u_{n,i}(\cdot,t)\|^2_{L^\infty(\mathcal{D})}dyds\Big) < \infty. \tag{2.50}$$

Here, we used that $D_{y,s}u_{n,i}(x,t) = 0$ for any $s > t$ and thus the integration is for $s \in [0,T]$, while we note that the bound is independent of k. So, we have, by (2.50), that

$$\begin{aligned}
\|u_{n,k+1}(x,t)\|_{D^{1,2}} :=& \Big(\mathbf{E}(|u_{n,k+1}(x,t)|^2) + \mathbf{E}\Big(\int_0^T \int_{\mathcal{D}} |D_{y,s}u_{n,k+1}(x,t)|^2 dyds\Big)\Big)^{1/2}\\
\leq& c + c\Big[\sup_{t\in[0,T]} \mathbf{E}\Big(\int_0^T \int_{\mathcal{D}} \|D_{y,s}u_{n,k+1}(\cdot,t)\|^2_{L^\infty(\mathcal{D})}dyds\Big)\Big]^{1/2}\\
\leq& c + c\Big[\sup_{t\in[0,T]}\sup_{i\leq k+1} \mathbf{E}\Big(\int_0^T \int_{\mathcal{D}} \|D_{y,s}u_{n,i}(\cdot,t)\|^2_{L^\infty(\mathcal{D})}dyds\Big)\Big]^{1/2} < \infty,
\end{aligned}$$

uniformly for any k; here, since $2 < 5$, we used the same argument of proving (2.38), but for 2 in place of $2pq$ (i.e., $\mathbf{E}(|u_{n,k+1}(x,t)|^2) < \infty$, the bound again independent of k). This yields that

$$\exists\ D_{y,s}u_{n,k+1}(x,t)\ \in D^{1,2}\ \ \forall\ (x,t) \in \mathcal{D}\times[0,T]. \tag{2.51}$$

Relations (2.50), (2.51) complete the induction, and establish (2.28).

As proved, for $p \geq 2$

$$\|u_n(\cdot,t) - u_{n,k}(\cdot,t)\|_{L^p(\Omega)} \to 0 \ \text{ as } \ k\to\infty,$$

and so,

$$u_{n,k}(\cdot,t) \to u_n(\cdot,t) \ \text{ as } \ k\to\infty \ \text{ in the } L^2(\Omega) \text{ norm,} \tag{2.52}$$

while as we also proved

$$u_{n,k} \in D^{1,2}\ \ \forall\ k. \tag{2.53}$$

Moreover for

$$\|D_{\cdot,\cdot}u_{n,k}(x,t)\|_H := \Big[\int_0^T \int_{\mathcal{D}} |D_{y,s}u_{n,k}(x,t)|^2 dyds\Big]^{1/2},$$

it holds that

$$\sup_k \mathbf{E}(\|D_{\cdot,\cdot}u_{n,k}(x,t)\|^2_H) < \infty, \tag{2.54}$$

since by (2.28)

$$\begin{aligned}
\sup_k \mathbf{E}(\|D_{\cdot,\cdot}u_{n,k}(x,t)\|^2_H) =& \sup_k \mathbf{E}\Big(\int_0^T \int_{\mathcal{D}} |D_{y,s}u_{n,k}(x,t)|^2 dyds\Big)\\
&\leq \sup_{t\in[0,T]}\sup_k \mathbf{E}\Big(\int_0^T \int_{\mathcal{D}} \|D_{y,s}u_{n,k}(\cdot,t)\|^2_{L^\infty(\mathcal{D})}dyds\Big) < \infty.
\end{aligned}$$

Using Lemma 1.2.3 of [17], due to (2.53), (2.52), (2.54), we have the first result of this proposition, i.e., that

$$u_n(x,t) \in D^{1,2}, \tag{2.55}$$

and

$$D_{y,s}u_{n,k}(x,t) \to D_{y,s}u_n(x,t),$$

in the weak topology of $L^2(\Omega;H) := L^2(\Omega \times ([0,T]\times\mathcal{D}))$, where

$$\|v\|_{L^2(\Omega;H)} := \Big[\mathbf{E}\Big(\int_0^T\int_{\mathcal{D}}|v(y,s)|^2dyds\Big)\Big]^{1/2}.$$

(Observe that for (x,t) fixed, $D_{y,s}u_{n,k}(x,t) = v(y,s)$ for some v.)

We remind that $D_{y,s}(u_{n,k+1}(x,t))$ was defined through (2.29). We shall show that $D_{y,s}u_n(x,t)$ satisfies uniquely (2.27).

Taking Malliavin derivatives in both sides of spde (2.6) (see the analogous calculus and arguments for $D_{y,s}u_{n,k+1}$ given by (2.34)), we obtain that for any $s \leq t$

$$\begin{aligned}
D_{y,s}u_n(x,t) = &\int_s^t\int_{\mathcal{D}}[\Delta G(x,z,t-\tau) - G(x,z,t-\tau)]\tilde{\mathcal{G}}_2(n)(z,\tau)D_{y,s}u_{n,k}(z,\tau)dzd\tau\\
&+ G(x,y,t-s)\sigma(u_n(y,s))\\
&+ \int_s^t\int_{\mathcal{D}}G(x,z,t-\tau)\tilde{\mathcal{G}}_1(n)(z,\tau)D_{y,s}u_n(z,\tau)W(dz,d\tau),
\end{aligned}$$

i.e., (2.27) is satisfied, while for $s > t$

$$D_{y,s}u_n(x,t) = 0.$$

Here, $\tilde{\mathcal{G}}_1(n)(z,\tau)$, $\tilde{\mathcal{G}}_2(n)(z,\tau)$ are bounded, and satisfy

$$\begin{aligned}
D_{y,s}(\sigma(u_n(z,\tau))) &= \tilde{\mathcal{G}}_1(n)(z,\tau)D_{y,s}(u_n(z,\tau)),\\
D_{y,s}(f_n(u_n(z,\tau))) &= \tilde{\mathcal{G}}_2(n)(z,\tau)D_{y,s}(u_n(z,\tau)).
\end{aligned}$$

Indeed, by Proposition 1.2.4 of [17] (as we already used for $u_{n,k+1}$), since u_n belongs to $D^{1,2}$ and σ is Lipschitz uniformly on any x in $\mathbb{R}$ with K_σ its Lipschitz coefficient, then $\sigma(u_n)$ belongs to $D^{1,2}$ also, and there exists a random variable $\tilde{\mathcal{G}}_1 = \tilde{\mathcal{G}}_1(n)$ such that

$$D_{y,s}\Big(\sigma(u_n(x,t))\Big) = \tilde{\mathcal{G}}_1(n)(x,t)D_{y,s}u_n(x,t), \tag{2.56}$$

with $\tilde{\mathcal{G}}_1$ bounded (in the absolute value norm) by K_σ, uniformly for any x, t, i.e.,

$$|\tilde{\mathcal{G}}_1(n)(x,t)| \leq K_\sigma, \ \ \forall\, x\in\mathcal{D},\ \forall t\in[0,T].$$

Taking f_n in place of σ, the same argument - since f_n is also Lipschitz uniformly on $\mathbb{R}$ - yields

$$D_{y,s}\Big(f_n(u_n(x,t))\Big) = \tilde{\mathcal{G}}_2(n)(x,t)D_{y,s}u_n(x,t), \tag{2.57}$$

and

$$|\tilde{\mathcal{G}}_2(n)(x,t)| \leq \hat{K}_{f_n},$$

for $\hat{K}_{f_n}$ a positive constant, depending on n through f_n.

Remind that σ is continuously differentiable and Lipschitz.

We note that as stated in the proof of Proposition 1.2.4 in [17], since f_n is continuously differentiable, then

$$\mathcal{G}_2(n,k)(x,t) = f_n'(u_{n,k}(x,t)), \quad \tilde{\mathcal{G}}_2(n)(x,t) = f_n'(u_n(x,t)),$$

while for the same reason

$$\mathcal{G}_1(n,k)(x,t) = \sigma'(u_{n,k}(x,t)), \quad \tilde{\mathcal{G}}_1(n)(x,t) = \sigma'(u_n(x,t)).$$

We need only to show uniqueness of solution of (2.27); note that from uniqueness of the Malliavin derivative, $\tilde{\mathcal{G}}_1$, $\tilde{\mathcal{G}}_2$ are uniquely determined. So, if $\hat{D}_{y,s}u_n(x,t)$ is another solution of (2.27), then through linearity of (2.27) on $D_{y,s}u_n(x,t)$ or on $\hat{D}_{y,s}u_n(x,t)$, we get, applying the same arguments, the analogous result as this for (2.49). More specifically, for

$$B_n(t) := \mathbf{E}\Big(\int_0^t\int_{\mathcal{D}}\|D_{y,s}u_n(\cdot,t)-\hat{D}_{y,s}u_n(\cdot,t)\|^2_{L^\infty(\mathcal{D})}dyds\Big),$$

we can analogously derive,

$$B_n(t) \leq 0 + c\int_0^t B_n(\tau)d\tau, \tag{2.58}$$

and by Gronwall's Lemma we get that $B_n(t)=0$ for any t, i.e.,

$$\mathbf{E}\Big(\int_0^t\int_{\mathcal{D}}\|D_{y,s}u_n(\cdot,t)-\hat{D}_{y,s}u_n(\cdot,t)\|^2_{L^\infty(\mathcal{D})}dyds\Big)=0,\ \ \forall\, t\in[0,T],$$

which yields finally uniqueness of solution of (2.27).

For (x,t) given, we derive that

$$\begin{aligned}
&\mathbf{E}\Big(\int_0^T\int_{\mathcal{D}}\int_0^T\int_{\mathcal{D}}|D_{y,s}u_n(x,t)|^2dydsdxdt\Big)=\int_0^T\int_{\mathcal{D}}\mathbf{E}\Big(\int_0^T\int_{\mathcal{D}}|D_{y,s}u_n(x,t)|^2dyds\Big)dxdt\\
&\leq c\int_0^T\int_{\mathcal{D}}\mathbf{E}\Big(\int_0^T\int_{\mathcal{D}}|D_{y,s}u_n(x,t)-D_{y,s}u_{n,k}(x,t)|^2dyds\Big)dxdt\\
&\quad+c\int_0^T\int_{\mathcal{D}}\mathbf{E}\Big(\int_0^T\int_{\mathcal{D}}|D_{y,s}u_{n,k}(x,t)|^2dyds\Big)dxdt\\
&\leq c\int_0^T\int_{\mathcal{D}}\mathbf{E}\Big(\int_0^T\int_{\mathcal{D}}|D_{y,s}u_n(x,t)-D_{y,s}u_{n,k}(\cdot,t)|^2dyds\Big)dxdt\\
&\quad+c\int_0^T\int_{\mathcal{D}}\mathbf{E}\Big(\int_0^T\int_{\mathcal{D}}\|D_{y,s}u_{n,k}(\cdot,t)\|^2_{L^\infty(\mathcal{D})}dyds\Big)dxdt<\infty,
\end{aligned}\tag{2.59}$$

since, by (2.28)

$$\mathbf{E}\Big(\int_0^T\int_{\mathcal{D}}\|D_{y,s}u_{n,k}(\cdot,t)\|^2_{L^\infty(\mathcal{D})}dyds\Big)<\infty,$$

and due to

$$\mathbf{E}\Big(\int_0^T\int_{\mathcal{D}}|D_{y,s}u_n(x,t)-D_{y,s}u_{n,k}(\cdot,t)|^2dyds\Big)<\infty. \tag{2.60}$$

In the previous argument we applied Fubini's Theorem. Moreover (2.60) holds true since $D_{y,s}u_{n,k}\to D_{y,s}u_n$ as $k\to\infty$ in $L^2(\Omega)$; this $L^2(\Omega)$ convergence can be easily established analogously to the way that the $L^2(\Omega)$ convergence of $u_{n,k}$ was established, i.e, we subtract the relation (2.34) - which defines the sequence of Malliavin derivatives $D_{y,s}u_{n,k}$, and (2.27) - which is uniquely solvable for $D_{y,s}u_n$, and derive after straight forward calculations, and since f_n', σ' are continuous, the $L^2(\Omega)$ convergence of the sequence of derivatives.

Also, by the estimate (2.7) of Lemma 2.7, we have

$$\begin{aligned}
\mathbf{E}\Big(\int_0^T\int_{\mathcal{D}}|u_n(x,t)|^2dxdt\Big)&\leq\int_0^T\mathbf{E}\Big(\int_{\mathcal{D}}|u_n(x,t)|^2dx\Big)dt\\
&\leq\int_0^T\mathbf{E}\Big(\|u_n(\cdot,t)\|^2_{L^\infty(\mathcal{D})}\Big)dt\\
&\leq\sup_{t\in[0,T]}\mathbf{E}\Big(\|u_n(\cdot,t)\|^2_{L^\infty(\mathcal{D})}\Big)<\infty.
\end{aligned}\tag{2.61}$$

Relations (2.59) and (2.61), by definition, yield the final regularity result of this proposition, i.e.,

$$u_n(x,t)\in L^{1,2}. \tag{2.62}$$

□

The next Main Theorem is a direct consequence of the previous arguments.

Theorem 2.9. *Let u be the solution of the stochastic Cahn-Hilliard/Allen-Cahn equation* (1.1)*, in dimension $d = 1$, with smooth initial data u_0. Moreover, let σ satisfy for any $x \in \mathbb{R}$* (1.3)*, i.e.,*

$$|\sigma(x)| \leq C(1 + |x|^q),$$

for some $C > 0$ and any $q \in (0, \frac{1}{3})$, and the Lipschitz property on $\mathbb{R}$ (1.4)*, and also let σ be continuously differentiable on $\mathbb{R}$. Then the solution u of* (1.1) *belongs to $L^{1,2}_{\text{loc}} \subseteq D^{1,2}_{\text{loc}}$.*

Proof. Indeed, since we constructed a localization of u, by (Ω_n, u_n), $n \in \mathbb{N}$, with u_n proven to be in $L^{1,2} \subseteq D^{1,2}$. □

Remark 2.10. *As already stated, the Malliavin derivative $D_{y,s}u$ is defined well by the Malliavin derivatives of the restrictions $u|_{\Omega_n}$ on Ω_n:*

$$D_{y,s}u := D_{y,s}u_n, \quad \text{on } \Omega_n.$$

3. Existence of a density for u

In order to establish existence of a density for the solution u of (1.1), we prove first the absolute continuity of the approximation u_n.

3.1. **Absolute continuity of u_n.** Our aim is to prove that for $t > 0$ and for $x \in [0, \pi]$

$$\|D_{\cdot,\cdot}u_n(x,t)\|_H^2 = \int_0^t \int_{\mathcal{D}} |D_{y,s}u_n(x,t)|^2 dyds > 0, \tag{3.1}$$

with probability $P = 1$.

Remark 3.1. *If we prove the above, then $\|D_{\cdot,\cdot}u_n(x,t)\|_H > 0$ almost surely, while we have proved that $u_n \in D^{1,2} \subseteq D^{1,2}_{\text{loc}} \subseteq D^{1,1}_{\text{loc}}$ (obviously applying Hölder's inequality on the formula of $\|\cdot\|_{D^{1,1}}$-norm where, cf. p. 27 of* [17]*, $\|v\|_{D^{1,1}} := \mathbf{E}(|v|) + \mathbf{E}(\|D_{\cdot,\cdot}\|_H)$, we see that $D^{1,2} \subseteq D^{1,1}$) and thus, according to Theorem 2.1.3 of* [17] *p. 98, u_n is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$ (see also the analogous argument used in* [8]*).*

Applying the same argument of Theorem 2.1.3 of [17]*, for u this time, since by Theorem 2.9 $u \in D^{1,2}_{\text{loc}}$, in order to prove absolute continuity for u, we need to prove that*

$$\|D_{\cdot,\cdot}u\|_H > 0, \quad \text{almost surely.}$$

More specifically, the aforementioned Theorem 2.1.3 states: Let F be a random variable of the space $D^{1,1}_{\text{loc}}$, and suppose that $\|DF\|_H > 0$ a.s. Then the law of F is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$.

In our case, we defined the space-time Malliavin derivative operator $D := D_{y,s}$ and $H := L^2([0,T] \times \mathcal{D})$, while we apply this theorem for u_n, u, for which we have shown that $u_n \in D^{1,2} \subseteq D^{1,2}_{\text{loc}} \subseteq D^{1,1}_{\text{loc}}$, $u \in D^{1,2}_{\text{loc}} \subseteq D^{1,1}_{\text{loc}}$.

We prove now why the validity of (3.1) *is sufficient for establishing the absolute continuity of u.*

Let $\omega \in A := \cup_{k=1}^n \Omega_k = \Omega_n \subseteq \Omega$, then $u(x,t;\omega) = u_n(x,t;\omega)$ a.s., and thus, for

$$B := \{\omega \in \Omega : \|D_{\cdot,\cdot}u(x,t;\omega)\|_H > 0\} \supseteq C := \{\omega \in A : \|D_{\cdot,\cdot}u(x,t;\omega)\|_H > 0\},$$

$$D_n := \{\omega \in A = \Omega_n : \|D_{\cdot,\cdot}u_n(x,t;\omega)\|_H > 0\},$$

we have $P(C) = P(D_n)$ for any n. Set

$$Z := \{\omega \in \Omega : \|D_{\cdot,\cdot}u_n(x,t;\omega)\|_H > 0\}.$$

So, if (3.1) *is valid, then* $P(Z) = 1$*, which gives* $P(Z^c) = 0$*. But, observe that*

$$D_n^c = \{\omega \in A = \Omega_n : \|D_{\cdot,\cdot}u_n(x,t;\omega)\|_H \leq 0\} \subseteq \{\omega \in \Omega : \|D_{\cdot,\cdot}u_n(x,t;\omega)\|_H \leq 0\} = Z^c,$$

so,

$$P(D_n^c) \leq P(Z^c) = 0,$$

i.e., $P(D_n^c) = 0$ *and so* $P(D_n) = 1$*. Thus, we have*

$$1 \geq P(B) \geq P(C) = P(D_n) = 1,$$

which yields $P(B) = 1$*. Hence, indeed* $\|D_{\cdot,\cdot}u\|_H > 0$*, almost surely, and as already argued, the law of* u *is absolutely continuous with respect to the Lebesgue measure on* $\mathbb{R}$*.*

In the sequel, we shall present two very important and difficult estimates that are derived after treating carefully the growth of the unbounded noise diffusion σ.

Lemma 3.2. *Under the assumptions of Theorem 2.9, the next estimates hold true*

$$\sup_{t\in[\hat{s}-\varepsilon,\hat{s}]} \mathbf{E}\Big(\int_{\hat{s}-\varepsilon}^{\hat{s}}\int_{\mathcal{D}} \|D_{y,s}u_n(\cdot,t)\|^2_{L^\infty(\mathcal{D})}dyds\Big) < C(n)\varepsilon^{2/3}, \tag{3.2}$$

and

$$\sup_{t\in[\varepsilon,T]} \mathbf{E}\Big(\int_{t-\varepsilon}^{t}\int_{\mathcal{D}} \|D_{y,s}u_n(\cdot,t) - G(\cdot,y,t-s)\sigma(u_n(y,s))\|^2_{L^\infty(\mathcal{D})}dyds\Big) < C(n)\varepsilon^{17/12}, \tag{3.3}$$

for any $\hat{s} \geq 0$*, where* $\varepsilon < \min\{1,\hat{s}\}$*, and* $C(n) > 0$ *is a constant independent of* t*,* ε*.*

Proof. Using the spde (2.27) for the Malliavin derivative of u_n, we proceed as when using equation (2.34) (when we estimated the Malliavin derivative of $u_{n,k}$) but integrating now on (a,t) for $t \geq a \geq 0$, instead of $(0,t)$. At the end, we will use our result for $a := 0$ and for $a := \hat{s} - \varepsilon$.

More specifically, for $p \geq 2$, we get

$$\begin{aligned}|D_{y,s}u_n(x,t)|^p \leq& c|G(x,y,t-s)\sigma(u_n(y,s))|^p\\ &+ c\Big|\int_s^t\int_{\mathcal{D}}[\Delta G(x,z,t-\tau) - G(x,z,t-\tau)]\tilde{\mathcal{G}}_2(n)(z,\tau)D_{y,s}u_n(z,\tau)dzd\tau\Big|^p\\ &+ c\Big|\int_s^t\int_{\mathcal{D}}G(x,z,t-\tau)\tilde{\mathcal{G}}_1(n)(z,\tau)D_{y,s}u_n(z,\tau)W(dz,d\tau)\Big|^p,\end{aligned}$$

which yields by the boundedness of $\tilde{\mathcal{G}}_2$

$$\begin{aligned}\|D_{y,s}u_n(\cdot,t)\|^p_{L^\infty(\mathcal{D})} \leq& c\|G(\cdot,y,t-s)\sigma(u_n(y,s))\|^p_{L^\infty(\mathcal{D})}\\ &+ c\Big\|\int_s^t\int_{\mathcal{D}}|\Delta G(\cdot,z,t-\tau) - G(\cdot,z,t-\tau)||D_{y,s}u_n(z,\tau)|dzd\tau\Big\|^p_{L^\infty(\mathcal{D})}\\ &+ c\Big\|\int_s^t\int_{\mathcal{D}}G(\cdot,z,t-\tau)\tilde{\mathcal{G}}_1(n)(z,\tau)D_{y,s}u_n(z,\tau)W(dz,d\tau)\Big\|^p_{L^\infty(\mathcal{D})}.\end{aligned}$$

We integrate the previous for $y \in \mathcal{D},\ s \in [a,t]$ and then take expectation, to derive

$$\begin{aligned}\mathbf{E}\Big(\int_a^t\int_{\mathcal{D}}\|D_{y,s}u_n(\cdot,t)\|^p_{L^\infty(\mathcal{D})}dyds\Big) &\leq c\mathbf{E}\Big(\int_a^t\int_{\mathcal{D}}\|G(\cdot,y,t-s)\sigma(u_n(y,s))\|^p_{L^\infty(\mathcal{D})}dyds\Big)\\ &+ c\mathbf{E}\Big(\int_a^t\int_{\mathcal{D}}\Big\|\int_s^t\int_{\mathcal{D}}|\Delta G(\cdot,z,t-\tau) - G(\cdot,z,t-\tau)||D_{y,s}u_n(z,\tau)|dzd\tau\Big\|^p_{L^\infty(\mathcal{D})}dyds\Big)\\ &+ c\mathbf{E}\Big(\int_a^t\int_{\mathcal{D}}\Big\|\int_s^t\int_{\mathcal{D}}G(\cdot,z,t-\tau)\tilde{\mathcal{G}}_1(n)(z,\tau)D_{y,s}u_n(z,\tau)W(dz,d\tau)\Big\|^p_{L^\infty(\mathcal{D})}dyds\Big)\\ &:= E_1(t) + E_2(t) + E_3(t).\end{aligned} \tag{3.4}$$

We set $p=2$. We shall estimate the terms $E_i(t)$ for $i=1,2,3$ when $p=2$.

We have for $1/\alpha+1/\beta=1$

$$\begin{aligned}
E_1(t) =&c\mathbf{E}\Big(\int_a^t\int_{\mathcal{D}}\|G(\cdot,y,t-s)\sigma(u_n(y,s))\|^p_{L^\infty(\mathcal{D})}dyds\Big)\\
\leq& c\mathbf{E}\Big(\int_a^t\int_{\mathcal{D}}\|G(\cdot,y,t-s)\|^{p\alpha}_{L^\infty(\mathcal{D})}dyds\Big)+c\mathbf{E}\Big(\int_a^t\int_{\mathcal{D}}|\sigma(u_n(y,s))|^{p\beta}dyds\Big)\\
\leq& c\mathbf{E}\Big(\int_a^t\int_{\mathcal{D}}\|G(\cdot,y,t-s)\|^{p\alpha}_{L^\infty(\mathcal{D})}dyds\Big)+c\mathbf{E}\Big(\int_a^t\int_{\mathcal{D}}c(1+|u_n(y,s)|^{p\beta q})dyds\Big)\\
\leq& c\mathbf{E}\Big(\int_a^t\int_{\mathcal{D}}\|G(\cdot,y,t-s)\|^{p\alpha}_{L^\infty(\mathcal{D})}dyds\Big)+c(t-a)+c\mathbf{E}\Big(\int_{t-\varepsilon}^t c\|u_n(\cdot,s)\|^{p\beta q}_{L^\infty(\mathcal{D})}ds\Big)\\
\leq& c(t-a)+\mathbf{E}\Big(\int_a^t\int_{\mathcal{D}}\|G(\cdot,y,t-s)\|^{p\alpha}_{L^\infty(\mathcal{D})}dyds\Big)+c\int_a^t\mathbf{E}\Big(\|u_n(\cdot,s)\|^{p\beta q}_{L^\infty(\mathcal{D})}\Big)ds,
\end{aligned}\tag{3.5}$$

where we used the growth of the unbounded noise diffusion, for $q\in(0,1/3)$, and Fubini's Theorem. We shall use $\alpha=7/6$, $\beta=7$, and $p=2$.

By (1.6) of [8], we have for $d=1$, $p=2$

$$\begin{aligned}
\int_a^t\int_{\mathcal{D}}\|G(\cdot,y,t-s)\|^{p\alpha}_{L^\infty(\mathcal{D})}dyds \leq& C\int_a^t|t-s|^{-p\alpha d/4+d/4}ds=C\int_a^t|t-s|^{-2(7/6)(1/4)+1/4}ds\\
=&C\int_a^t|t-s|^{-1/3}ds\leq C(t-a)^{2/3}.
\end{aligned}\tag{3.6}$$

Also since $p\beta q=2\cdot 7\cdot q<2\cdot 7\cdot 1/3=14/3(<5)$, in dimensions $d=1$, using (2.7), we obtain for any $s\in[0,T]$ and thus for any $s\in[a,t]$

$$\mathbf{E}(\|u_n(\cdot,s)\|^{p\beta q}_{L^\infty(\mathcal{D})})\leq c.\tag{3.7}$$

Using (3.6), (3.7) in (3.5), yields

$$\begin{aligned}
E_1(t)\leq& c(t-a)+c\mathbf{E}\Big(\int_a^t\int_{\mathcal{D}}\|G(\cdot,y,t-s)\|^{p\alpha}_{L^\infty(\mathcal{D})}dyds\Big)+c\int_a^t\mathbf{E}\Big(\|u_n(\cdot,s)\|^{p\beta q}_{L^\infty(\mathcal{D})}\Big)ds\\
\leq& c(t-a)+c(t-a)^{2/3}+c(t-a)\leq c(t-a)+c(t-a)^{2/3},
\end{aligned}\tag{3.8}$$

uniformly for all t, and thus for $p=2$

$$E_1(t)\leq c(t-a)+c(t-a)^{2/3}.\tag{3.9}$$

Considering the term $E_2(t)$, by (1.12) of [8], we have, as in deriving (2.42), but observing that $s\leq\tau\leq t$ and $a\leq s\leq t$, which yields $s\in[a,\tau]$ when changing the order of integration

$$\begin{aligned}
E_2(t)=&c\mathbf{E}\Big(\int_a^t\int_{\mathcal{D}}\Big\|\int_s^t\int_{\mathcal{D}}|\Delta G(\cdot,z,t-\tau)-G(\cdot,z,t-\tau)||D_{y,s}u_n(z,\tau)|dzd\tau\Big\|^p_{L^\infty(\mathcal{D})}dyds\Big)\\
\leq& c\int_a^t\mathbf{E}\Big(\int_a^\tau\int_{\mathcal{D}}\|D_{y,s}u_n(\cdot,\tau)\|^p_{L^\infty(\mathcal{D})}dyds\Big)d\tau.
\end{aligned}\tag{3.10}$$

For the term $E_3(t)$, Fubini's Theorem and Burkholder-Davis-Gundy inequality, together with the boundedness of $\tilde{\mathcal{G}}_1$, yields as in (2.43)

$$\begin{aligned}
E_3(t)=&c\mathbf{E}\Big(\int_a^t\int_{\mathcal{D}}\Big\|\int_s^t\int_{\mathcal{D}}G(\cdot,z,t-\tau)\tilde{\mathcal{G}}_1(n)(z,\tau)D_{y,s}u_n(z,\tau)W(dz,d\tau)\Big\|^p_{L^\infty(\mathcal{D})}dyds\Big)\\
\leq& c\int_a^t\int_{\mathcal{D}}\mathbf{E}\Big(\Big\|\int_a^\tau\int_{\mathcal{D}}|G(\cdot,z,t-\tau)|^2|D_{y,s}u_n(z,\tau)|^2dzd\tau\Big\|^{p/2}_{L^\infty(\mathcal{D})}\Big)dyds.
\end{aligned}\tag{3.11}$$

As in (2.45), we derive since $d = 1$ and $p = 2$

$$E_3(t) \leq c \int_a^t (t-\tau)^{-1/4} \mathbf{E}\Big(\int_a^\tau \int_{\mathcal{D}} \|D_{y,s} u_{n,k}(\cdot,\tau)\|^2_{L^\infty(\mathcal{D})} dy ds \Big) d\tau. \tag{3.12}$$

Hence, by the estimates (3.9), (3.10) and (3.12), we get for (3.4)

$$\begin{aligned} \mathbf{E}\Big(\int_a^t \int_{\mathcal{D}} \|D_{y,s} u_n(\cdot,t)\|^2_{L^\infty(\mathcal{D})} dy ds \Big) \leq & c(t-a) + (t-a)^{2/3} \\ & + c \int_a^t \mathbf{E}\Big(\int_a^\tau \int_{\mathcal{D}} \|D_{y,s} u_n(\cdot,\tau)\|^2_{L^\infty(\mathcal{D})} dy ds \Big) d\tau \\ & + c \int_a^t (t-\tau)^{-1/4} \mathbf{E}\Big(\int_a^\tau \int_{\mathcal{D}} \|D_{y,s} u_n(\cdot,\tau)\|^2_{L^\infty(\mathcal{D})} dy ds \Big) d\tau, \end{aligned} \tag{3.13}$$

for $c > 0$ constants independent of t.

Define

$$L_n(t) := \mathbf{E}\Big(\int_a^t \int_{\mathcal{D}} \|D_{y,s} u_n(\cdot,t)\|^2_{L^\infty(\mathcal{D})} dy ds \Big),$$

then (3.13) is written as

$$L_n(t) \leq c(t-a) + c(t-a)^{2/3} + c \int_a^t L_n(\tau) d\tau + c \int_a^t (t-\tau)^{-1/4} L_n(\tau) d\tau. \tag{3.14}$$

This yields

$$\begin{aligned} c \int_a^t (t-\tau)^{-1/4} L_n(\tau) d\tau \leq & c(t-a)^{1+1-1/4} + c(t-a)^{2/3+1-1/4} \\ & + c(t-a)^{1-1/4} \int_a^t L_n(s) ds \\ & + c \int_a^t [\int_a^t (t-\tau)^{-1/4} (\tau-s)^{-1/4} d\tau] L_n(s) ds \\ \leq & c(t-a)^{1+1-1/4} + c(t-a)^{2/3+1-1/4} \\ & + c(t-a)^{1-1/4} \int_a^t L_n(s) ds + c(t-a)^{1/4} \int_a^t L_n(s) ds \\ \leq & c(t-a)^{7/4} + c(t-a)^{17/12} + c \int_a^t L_n(s) ds. \end{aligned}$$

Thus (3.14) becomes

$$L_n(t) \leq C_0(t,a) + c \int_a^t L_n(\tau) d\tau, \tag{3.15}$$

for

$$C_0(t,a) := c(t-a) + c(t-a)^{2/3} + c(t-a)^{7/4} + c(t-a)^{17/12}.$$

By (3.15), we get

$$L_n(t) \leq C_0(t,a), \tag{3.16}$$

which yields,

$$\begin{aligned} \mathbf{E}\Big(\int_a^t \int_{\mathcal{D}} \|D_{y,s} u_n(\cdot,t)\|^2_{L^\infty(\mathcal{D})} dy ds \Big) &= L_n(t) \\ &\leq c(t-a) + c(t-a)^{2/3} + c(t-a)^{7/4} + c(t-a)^{17/12}, \end{aligned} \tag{3.17}$$

uniformly for any t.

In the above, $c = C(n) > 0$ is independent of t, but generally may depend on n. Since $D_{y,s}u_n(\cdot, t) = 0$ when $s > t$, then for any $\hat{s} \geq t$, by using (3.17), we have

$$\begin{aligned}\mathbf{E}\Big(\int_a^{\hat{s}}\int_{\mathcal{D}}\|D_{y,s}u_n(\cdot,t)\|^2_{L^\infty(\mathcal{D})}dyds\Big) =&\mathbf{E}\Big(\int_a^t\int_{\mathcal{D}}\|D_{y,s}u_n(\cdot,t)\|^2_{L^\infty(\mathcal{D})}dyds\Big)\\ \leq& c(t-a)+c(t-a)^{2/3}+c(t-a)^{7/4}+c(t-a)^{17/12}\\ \leq& c(\hat{s}-a)+c(\hat{s}-a)^{2/3}+c(\hat{s}-a)^{7/4}+c(\hat{s}-a)^{17/12}.\end{aligned}\tag{3.18}$$

So, choosing in the above $a := \hat{s} - \varepsilon \leq t$ (we need $a \leq t$), we have for any $\hat{s} \geq t \geq \hat{s} - \varepsilon$,

$$\mathbf{E}\Big(\int_{\hat{s}-\varepsilon}^{\hat{s}}\int_{\mathcal{D}}\|D_{y,s}u_n(\cdot,t)\|^2_{L^\infty(\mathcal{D})}dyds\Big) \leq c\varepsilon^{2/3},\tag{3.19}$$

for $\varepsilon < 1$. Taking supremum on any such $t \in [\hat{s} - \varepsilon, \hat{s}]$, we have the result, i.e., (3.2).

Moreover, we have

$$\begin{aligned}\mathbf{E}\Big(\int_{t-\varepsilon}^t\int_{\mathcal{D}}\|D_{y,s}u_n(\cdot,t)-G(\cdot,y,t-s)\sigma(u_n(y,s))\|^2_{L^\infty(\mathcal{D})}dyds\Big) &\leq cE_2(t)+cE_3(t)\\ &\leq c\varepsilon^{2/3}\varepsilon+c\varepsilon^{2/3}\varepsilon^{1-1/4}\\ &\leq c\varepsilon^{17/12},\end{aligned}\tag{3.20}$$

where we used (3.10) and (3.12) for $a = t - \varepsilon$ and the estimate (3.2). So, the estimate (3.3) is established. □

We are now ready to prove the next important theorem, which will yield by localization the second result of the Main Theorem 1.1 of this paper. Here, we need a non-degenerating extra assumption for the diffusion σ.

Theorem 3.3. *Under the assumptions of Theorem 2.9, if additionally, σ satisfies (1.6), i.e*

$$|\sigma(x)| \geq c_0 > 0,$$

for any $x \in \mathbb{R}$, then the law of the solution $u_n(x,t)$ of (2.6) when $t > 0$ and $x \in (0,\pi)$, is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$.

Proof. Relation (2.27) yields

$$\begin{aligned}|D_{y,s}u_n(x,t)|^2 =&\Big|\int_s^t\int_{\mathcal{D}}[\Delta G(x,z,t-\tau)-G(x,z,t-\tau)]\tilde{\mathcal{G}}_2(n)(z,\tau)D_{y,s}(u_n(z,\tau))dzd\tau\\ &+G(x,y,t-s)\sigma(u_n(y,s))\\ &+\int_s^t\int_{\mathcal{D}}G(x,z,t-\tau)\tilde{\mathcal{G}}_1(n)(z,\tau)D_{y,s}(u_n(z,\tau))W(dz,d\tau)\Big|^2\\ \geq&\frac{1}{2}A-B,\end{aligned}\tag{3.21}$$

for

$$A(x,y,s,t) := G(x,y,t-s)^2\sigma(u_n(y,s))^2,$$

and

$$\begin{aligned}B(x,y,s,t) :=&\Big|\int_s^t\int_{\mathcal{D}}[\Delta G(x,z,t-\tau)-G(x,z,t-\tau)]\tilde{\mathcal{G}}_2(n)(z,\tau)D_{y,s}(u_n(z,\tau))dzd\tau\\ &+\int_s^t\int_{\mathcal{D}}G(x,z,t-\tau)\tilde{\mathcal{G}}_1(n)(z,\tau)D_{y,s}(u_n(z,\tau))W(dz,d\tau)\Big|^2,\end{aligned}$$

where we used that $(2^{-1/2}a + 2^{1/2}b)^2 \geq 0$ which yields $(a+b)^2 \geq \frac{1}{2}a^2 - b^2$.

So, we have

$$\int_0^t\int_{\mathcal{D}}|D_{y,s}u_n(x,t)|^2dyds \geq \int_{t-\varepsilon}^t\int_{\mathcal{D}}|D_{y,s}u_n(x,t)|^2dyds \geq \frac{1}{2}\int_{t-\varepsilon}^t\int_{\mathcal{D}}Adyds - \int_{t-\varepsilon}^t\int_{\mathcal{D}}Bdyds. \tag{3.22}$$

We will give an upper bound in expectation for the term

$$\int_{t-\varepsilon}^t\int_{\mathcal{D}}Bdyds.$$

For this, we shall use Lemma 3.2 (relation (3.3)), which yields

$$\mathbf{E}\Big(\int_{t-\varepsilon}^t\int_{\mathcal{D}}B(x,y,s,t)dyds\Big) \leq \mathbf{E}\Big(\int_{t-\varepsilon}^t\int_{\mathcal{D}}\|B(\cdot,y,s,t)\|^2_{L^\infty(\mathcal{D})}dyds\Big) \leq C(n)\varepsilon^{17/12}. \tag{3.23}$$

We now provide a lower bound for $\int_{t-\varepsilon}^t\int_{\mathcal{D}}Adyds$. The non-degeneracy condition of the diffusion and using the spectrum in $[0,\pi]$ of the negative Neumann Laplacian, yields

$$\begin{aligned}\int_{t-\varepsilon}^t\int_{\mathcal{D}}Adyds =& \int_{t-\varepsilon}^t\int_{\mathcal{D}}G(x,y,t-s)^2\sigma(u_n(y,s))^2dyds\\ \geq& c\int_{t-\varepsilon}^t\int_{\mathcal{D}}G(x,y,t-s)^2dyds = c\int_{t-\varepsilon}^t\Big[\sum_{k=0}^\infty a_k^2(x)e^{-2(\lambda_k^2+\lambda_k)(t-s)}\Big]ds\\ \geq& c\int_{t-\varepsilon}^t\Big[\sum_{k=0}^\infty a_k^2(x)e^{-4\lambda_k^2(t-s)}\Big]ds = c\sum_{k=1}^\infty a_k^2(x)\frac{1}{4\lambda_k^2}[1-e^{-4\lambda_k^2\varepsilon}]+C\varepsilon\\ =& c\frac{1}{2}\sum_{k=1}^\infty a_k^2(x)\frac{1}{2\lambda_k^2}[1-e^{-2\lambda_k^2(2\varepsilon)}]+C\varepsilon \geq C(2\varepsilon)^{1-d/4} = C\varepsilon^{3/4},\end{aligned} \tag{3.24}$$

where we used the orthonormal $L^2(\mathcal{D})$ eigenfunctions basis $\{a_k\}$ for $k=0,1,2,\cdots$, in dimensions $d=1$, the fact that $\lambda_k = k^2$, and that $(t-s)\geq 0$ and that $-(\lambda_k^2+\lambda_k) = -(k^4+k^2) \geq -2\lambda_k^2 = -2k^4$, and the estimate (3.25) of [8]. Thus, we have proven that

$$\int_{t-\varepsilon}^t\int_{\mathcal{D}}Adyds \geq C_0\varepsilon^{3/4}. \tag{3.25}$$

Using the estimates (3.23), (3.25), we arrive at

$$\begin{aligned}P\Big(\int_0^T\int_{\mathcal{D}}|D_{y,s}u_n(x,t)|^2dyds > 0\Big) \geq& P\Big(\frac{1}{2}\int_{t-\varepsilon}^t\int_{\mathcal{D}}A(x,y,s,t)dyds - \int_{t-\varepsilon}^t\int_{\mathcal{D}}B(x,y,s,t)dyds > 0\Big)\\ \geq& P\Big(\int_{t-\varepsilon}^t\int_{\mathcal{D}}B(x,y,s,t)dyds < \frac{C_0}{2}\varepsilon^{3/4}\Big)\\ \geq& 1 - c\mathbf{E}\Big(\int_{t-\varepsilon}^t\int_{\mathcal{D}}B(x,y,s,t)dyds\Big)\varepsilon^{-3/4}\\ \geq& 1 - c\varepsilon^{17/12}\varepsilon^{-3/4} = 1 - c\varepsilon^{2/3} \to 1, \text{ as } \varepsilon \to 0,\end{aligned} \tag{3.26}$$

where we applied Markov's inequality.

This yields the result.

We note that the proof of this theorem was influenced by the very interesting arguments of Cardon-Weber in [8], for an analogous result, where the stochastic Cahn-Hilliard equation with bounded noise diffusion was considered. However, we used in a direct way the property of σ, i.e., that $|\sigma(x)| \geq c_0 > 0$ for any $x \in \mathbb{R}$. $\square$

3.2. **Absolute continuity of the stochastic solution u.** The next theorem establishes the second result of Main Theorem 1.1, this of the existence of a density for u.

Theorem 3.4. *Under the assumptions of Theorem 2.9, if additionally, σ satisfies* (1.6)*, i.e.,*

$$|\sigma(x)| \geq c_0 > 0,$$

for any $x \in \mathbb{R}$, then the law of the solution u of (1.1) *is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$.*

Proof. This is a direct result of Theorem 3.3 through localization, see the arguments of Remark 3.1. $\square$

Acknowledgments

The authors would like to thank the anonymous referee for the valuable comments provided.

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Dimitra Antonopoulou, Department of Mathematics, University of Chester, Thornton Science Park, CH2 4NU, Chester, UK, and, Institute of Applied and Computational Mathematics, FORTH, GR–711 10 Heraklion, Greece.
E-mail address: d.antonopoulou@chester.ac.uk

Dimitris Farazakis, Department of Mathematics and Applied Mathematics, University of Crete, GR–714 09 Heraklion, Greece.
E-mail address: D.Farazakis@outlook.com

Georgia Karali, Department of Mathematics and Applied Mathematics, University of Crete, GR–714 09 Heraklion, Greece, and, Institute of Applied and Computational Mathematics, FORTH, GR–711 10 Heraklion, Greece.
E-mail address: gkarali@tem.uoc.gr