MOTION OF A DROPLET FOR THE MASS-CONSERVING STOCHASTIC ALLEN-CAHN EQUATION

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ABSTRACT. We study the stochastic mass-conserving Allen-Cahn equation posed on a bounded domain of $\mathbb{R}^2$ with additive spatially smooth space-time noise. This equation associated with a small positive parameter ε describes the stochastic motion of a small almost semicircular droplet attached to domain's boundary and moving towards a point of locally maximum curvature. We apply Itô calculus to derive the stochastic dynamics of the droplet by utilizing the approximately invariant manifold introduced by Alikakos, Chen and Fusco [2] for the deterministic problem. In the stochastic case depending on the scaling, the motion is driven by the change in the curvature of the boundary and the stochastic forcing. Moreover, under the assumption of a sufficiently small noise strength, we establish stochastic stability of a neighborhood of the manifold of droplets in L^2 and H^1, which means that with overwhelming probability the solution stays close to the manifold for very long time-scales.

Keywords: Stochastic Allen-Cahn, mass conservation, droplet's motion, additive noise, invariant manifold, stochastic dynamics, stochastic stability, Itô calculus.

1. INTRODUCTION

1.1. **The problem.** We consider the IBVP for the mass conserving Allen-Cahn equation posed on a two-dimensional bounded smooth domain Ω and introduce an additive spatially smooth and white in time space-time noise $\dot{V}$

$$
\partial_t \phi^{\hat{\varepsilon}}(y,t) = \hat{\varepsilon}^2 \Delta_y \phi^{\hat{\varepsilon}}(y,t) - f(\phi^{\hat{\varepsilon}}(y,t)) + \frac{1}{|\Omega|}\int_\Omega f(\phi^{\hat{\varepsilon}}(y,t))dy + \dot{V}(y,t), \quad y \in \Omega, \quad t > 0,
$$

(1.1)
$$
\partial_n \phi^{\hat{\varepsilon}}(y,t) = 0, \quad y \in \partial\Omega, \quad t > 0,
$$
$$
\phi^{\hat{\varepsilon}}(y,0) = \phi_0^{\hat{\varepsilon}}(y), \quad y \in \Omega.
$$

Here, $\hat{\varepsilon}$ is a small positive parameter, $\Omega \subset \mathbb{R}^2$ of area $|\Omega|$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$, and ∂_n is the exterior normal derivative to $\partial\Omega$. The function f is the derivative of a double-well potential, which we denote by F. We assume that f is smooth, $f(\pm 1) = 0 < f'(\pm 1)$ and f has exactly one other zero that lies in $(-1, 1)$. The standard example is $f(u) = u^3 - u$, which we will assume for simplicity in the whole presentation, although the result holds for more general nonlinearities.

The deterministic problem, i.e., when $\dot{V} = 0$, was first studied by Rubinstein and Sternberg, [26], then by Alikakos, Chen and Fusco, [2], and later by Bates and Jin in [5]. In [2], the authors analyzed the problem's long-time dynamics and established existence of stable sets of solutions corresponding to the motion of a small, almost semicircular interface (droplet) intersecting the boundary of the domain and moving towards

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a point of locally maximal curvature. In [5], the authors established the existence of a global invariant manifold of droplet states using the approximation given in [2].

The Allen-Cahn equation, also called Model A in the theory of dynamics of critical phenomena (cf. [20]), describes the evolution of the concentration $\phi^{\hat\varepsilon}$ of one species of a two-phase mixture, for example a binary alloy, occupying a bounded domain Ω. The small positive parameter $\hat\varepsilon$ represents the surface tension associated with interfacial regions that are generated during phase separation, cf. [3]. The double-well potential F favors layered functions that take values close to its minima ± 1. The zero level sets of such a function are called interfaces and the values close to ± 1 are called states. Usually they are assigned almost uniformly away from the interface .

Due to mass conservation, a phase separation begins either by spinodal decomposition, or as in our case, as the mass is very asymmetric by nucleation. For the case of Cahn-Hilliard equation see [9, 15].

If the states are separated, the total perimeter of interfaces decreases in time, [16, 17, 11]. For the one-dimensional case see also [12, 21, 22, 27, 24]. For the two-dimensional single layer problem, Chen and Kowalczyk in [13] proved that in the limit $\hat\varepsilon \to 0^+$ this layer becomes a circular arc interface intersecting the boundary orthogonally and encloses a point on the boundary where the curvature has a local maximum. Alikakos, Chen and Fusco in [2] restricted the analysis to a single connected interface (curve) of shape close to a small semicircular-arc intersecting the outer boundary $\partial\Omega$. This so called droplet maintains a semicircular shape for economizing the perimeter and therefore, in [2] its evolution was fully described in terms of the motion of its center along the outer boundary.

In the absence of the non-local term, the multi-dimensional stochastic Allen-Cahn equation driven by a multiplicative noise non-smooth in time and smooth in space was considered in [25]; the authors therein prove the tightness of solutions for the sharp interface limit problem. We refer also to the results in [19], where a mollified additive white space-time noise was introduced and the limiting behavior of solution was investigated for very rough noise. Considering the one-dimensional Allen-Cahn equation with an additive space-time white noise, in [28], the author proved exponential convergence towards a curve of minimizers of the energy.

In this paper, as in [2], we consider a single small droplet and so, the average concentration $m \in (-1, 1)$ is assumed to satisfy

$$m = 1 - \frac{\pi\delta^2}{|\Omega|},$$

for some $0 < \delta \ll 1$ while the parameter $\hat\varepsilon$ is sufficiently small such that $0 < \hat\varepsilon \ll \delta^3$.

When $\dot V := 0$, if $z(\hat\xi_0)$ is a point of $\partial\Omega$ where the curvature has a strict extremum, then there exists a unique equilibrium $\phi(y)$ of (1.1) with zero level set close to the circle of radius δ centered at this point. Moreover, for layered initial data whose interface is close to the semicircle centered at $z(\hat\xi_0)$ of radius δ, the solution of (1.1) is layered also, with interface close to a semicircle of the same radius centered at some point $z(\hat\xi)$ of $\partial\Omega$, [2].

The mass conservation constraint

$$\frac{1}{|\Omega|}\int_\Omega \phi^{\hat\varepsilon}(y,t)dy = m \quad \text{for any} \quad t \geq 0$$

holds if and only if

$$\begin{aligned} 0 = \partial_t\Big(\int_\Omega \phi^{\hat\varepsilon}(y,t)dy\Big) &= \int_\Omega \phi_t^{\hat\varepsilon}(y,t)dy \\ &= \int_\Omega \hat\varepsilon^2 \Delta_y \phi^{\hat\varepsilon}(y,t)dy - \int_\Omega f(\phi^{\hat\varepsilon}(y,t))dy + \int_\Omega \frac{1}{|\Omega|}\int_\Omega f(\phi^{\hat\varepsilon}(y,t))dydy + \int_\Omega \dot V(y,t)dy \\ &= \int_\Omega \dot V(y,t)dy. \end{aligned}$$

For the calculation above we integrated (1.1) in space and used the Neumann boundary conditions. Hence, mass conservation holds for the stochastic problem only if the spatially smooth additive noise satisfies

$$
(1.2) \qquad \int_\Omega \dot{V}(y,t)dy = 0 \quad \text{for any} \quad t \geq 0.
$$

This means that in a Fourier series expansion, there is no noise on the constant mode. Otherwise the average mass would behave like a Brownian motion, and would not stay close to 1.

Following [2], in order to fix the size of the droplet, we introduce in (1.1) the following change of variables:

$$
(1.3) \qquad y = \delta x, \quad \hat{\varepsilon} = \varepsilon\delta, \quad u^\varepsilon(x,t) = \phi^{\hat{\varepsilon}}(y,t), \quad \dot{W}(x,t) = \dot{V}(y,t), \quad \Omega_\delta = \delta^{-1}\Omega := \{x \in \mathbb{R}^2 : \delta x \in \Omega\},
$$

and obtain the equivalent problem

$$
(1.4)
\begin{aligned}
&\partial_t u^\varepsilon(x,t) = \varepsilon^2 \Delta u^\varepsilon(x,t) - f(u^\varepsilon(x,t)) + \frac{1}{|\Omega_\delta|} \int_{\Omega_\delta} f(u^\varepsilon(x,t))dx + \dot{W}(x,t), \quad x \in \Omega_\delta, \quad t > 0, \\
&\partial_n u^\varepsilon(x,t) = 0, \quad x \in \partial\Omega_\delta, \quad t > 0, \\
&u^\varepsilon(x,0) = u_0^\varepsilon(x), \quad x \in \Omega_\delta.
\end{aligned}
$$

Here $\Delta = \Delta_x$, $\dot{W}(x,t)$ is again an additive smooth in space, space-time noise defined below and ∂_n is the normal derivative to $\partial\Omega_\delta$.

1.2. **Assumptions on the noise.** The noise $\dot{W}$ is defined as the formal derivative of a Wiener process depending on ε, which is given by a Fourier series with coefficients being independent Brownian motions in time. Since $\dot{W}$ arises from a rescaling of the noise $\dot{V}$, we also could take care of the dependence on δ, but here we suppose that $\delta > 0$ is small but fixed (see Remark 1.1).

Let W be a $\mathcal{Q}$-Wiener process in the underlying Hilbert space $H := L^2(\Omega_\delta)$, where $\mathcal{Q}$ is a symmetric operator and $(e_k)_{k\in\mathbb{N}}$ is a complete $L^2(\Omega_\delta)$-orthonormal basis of eigenfunctions with corresponding eigenvalues a_k^2, so that

$$
\mathcal{Q}e_k = a_k^2 e_k.
$$

Then W is given as the Fourier series

$$
(N1) \qquad W(t) := \sum_{k=1}^\infty a_k \beta_k(t) e_k(\cdot),
$$

for a sequence of independent real-valued Brownian motions $\{\beta_k(t)\}_{t\geq 0}$, cf. DaPrato and Zabzcyck [14]. Note that, due to rescaling, a_k, $\mathcal{Q}$, and e_k will depend on δ. We suppress this dependence in our notation.

The process W is assumed to satisfy

$$
(N2) \qquad \int_{\Omega_\delta} \dot{W}(x,t)dx = 0 \quad \text{for any} \quad t \geq 0,
$$

so that the mass conservation condition (1.2) holds true.

As our approach is based on application of Itô-formula, we will always assume that the trace of the operator $\mathcal{Q}$ is finite, i.e.,

$$
\text{trace}(\mathcal{Q}) := \sum_{k=1}^\infty a_k^2 = \eta_0 < \infty.
$$

Furthermore, let $\|\mathcal{Q}\|$ be the induced L^2 operator norm, then the noise strength is defined by

$$
(N3) \qquad \|\mathcal{Q}\| = \eta_1.
$$

Here, observe that

$$
\eta_1 = \|\mathcal{Q}\| \leq \text{trace}(\mathcal{Q}) = \eta_0.
$$

The required smoothness in space of the noise is given by

$$\text{(N4)} \qquad \eta_2 = \sum_{i=1}^{\infty} a_i^2 \|\nabla e_i\|^2 = \text{trace}(Q\Delta) < \infty .$$

This assumption will be used in the sequel, when the Itô-formula will be applied for the proof of certain H^1-norm estimates.

Our results will depend on the size of (η_0, η_1, η_2) in terms of ε. The usual scenario would be that all η_i have a common prefactor in ε, which is the noise-strength, and are otherwise independent of ε.

Remark 1.1. *Note that due to the rescaling, if we assume that V did not depend on δ, then it is η that depends on δ. More specifically, since we are in two dimensions then $e_k(x) = \delta^{-1} f_k(\delta x)$, where f_k is an ONB in $L^2(\Omega)$. Thus $V = \delta W$, and all a_k are of order $\mathcal{O}(\delta)$. Hence, η_0 and η_1 are of order $\mathcal{O}(\delta^2)$, while η_2 is of order $\mathcal{O}(\delta^4)$.*

Our philosophy in this paper will be to consider δ very small but fixed, and analyze the asymptotic problem for $0 < \varepsilon \ll 1$. Thus, we suppress the explicit dependence on δ in the notation.

1.3. **The droplet.** We define, for a smooth function v, the operator

$$\mathcal{L}^\varepsilon(v) := \varepsilon^2 \Delta v - f(v) + \frac{1}{|\Omega_\delta|} \int_{\Omega_\delta} f(v) dx \quad \text{in} \quad \Omega_\delta, \quad \partial_n v = 0 \quad \text{on} \quad \partial\Omega_\delta,$$

and fix the cubic nonlinearity f as in the introduction.

Following Theorem 2.5 of [2] (p. 267), we have:

Lemma 1.2. *For any integer $K \in \mathbb{N}$ and for δ, ε sufficiently small parameters satisfying*

$$\text{(1.5)} \qquad \varepsilon \le \frac{1}{2} C_1^* \delta^2,$$

with

$$C_1^* := \frac{8\pi f'(1)}{3\sqrt{6}|\Omega| \int_{-1}^{1} \sqrt{F(s)} ds},$$

there exist a droplet like state $u = u(x, \xi, \varepsilon)$ and a scalar (velocity) field $c = c(\xi, \varepsilon)$ such that

$$\mathcal{L}^\varepsilon(u) = \varepsilon^2 c(\xi, \varepsilon) \partial_\xi u + \mathcal{O}_{L^\infty}(\varepsilon^K) \quad \text{in} \quad \Omega_\delta,$$

$$\text{(1.6)} \qquad \partial_n u = 0 \quad \text{on} \quad \partial\Omega_\delta,$$

$$\int_{\Omega_\delta} u = |\Omega_\delta| - \pi,$$

where the scalar $\xi \in (0, |\partial\Omega_\delta|)$ is the arc-length parameter of $\partial\Omega_\delta$.

Here, $\mathcal{O}_{L^\infty}(\varepsilon^K)$ denotes a term that is uniformly bounded by $C\varepsilon^K$ for some constant $C > 0$.
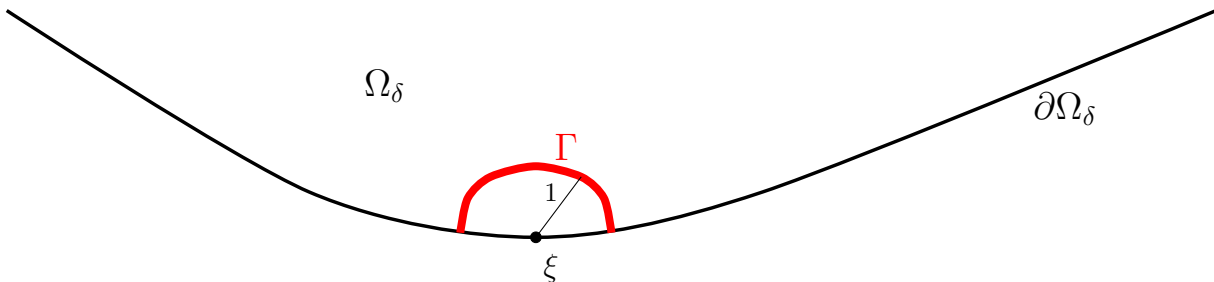

FIGURE 1.1. Droplet state in the rescaled domain Ω_δ with semicircular arc Γ

Idea of Proof. The droplet is constructed, using asymptotic expansions in ε, as a function of (r, s), with r the signed distance from the interface Γ and s the arc-length along Γ, this being an approximately semicircular curve intersecting $\partial\Omega_\delta$ orthogonally. Additional asymptotic expansions are used near the corners where the interface meets $\partial\Omega_\delta$ but these are of higher order. The first order approximation to this state is $U(r/\varepsilon)$, transverse to Γ at each point, where U is the solution to

$$(1.7) \qquad \ddot{U} - f(U) = 0, \quad U(\pm\infty) = \pm 1, \quad \text{such that} \int_{\mathbb{R}} R\dot{U}^2(R)dR = 0.$$
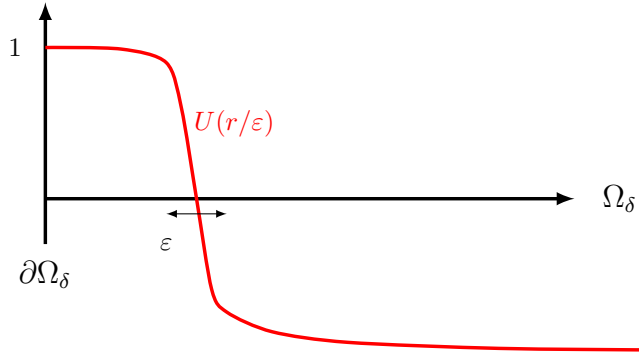
$\square$

FIGURE 1.2. Sketch of a section through the Droplet showing the local shape given by U.

In previous lemma, and for the rest of this paper $\partial_\xi u := \frac{\partial}{\partial\xi}(u(\cdot))$, while $\partial_\xi^l(u) := \frac{\partial^l}{\partial\xi^l}(u(\cdot))$ for any integer $l > 1$. Note that $\partial_\xi u$ is just the usual partial derivative when u is considered as a function of x, parameterized by ξ and ε, but when u is considered as a function of local coordinates (r, s), then one must take into account the fact that r and s both vary with ξ. This observation will be used in the final section where these derivatives are estimated.

According to [2] (p. 261 and relation (2.54) on p. 262) it follows that:

Lemma 1.3. *For c from the definition of the droplet in Lemma 1.2, we have*

$$(1.8) \qquad c = c(\xi, \varepsilon) = \mathcal{O}(\delta^2).$$

Moreover,

$$c(\xi, \varepsilon) = -\frac{4}{3\pi}g_0\mathcal{K}'_{\Omega_\delta}(\xi)\delta\left[1 + \mathcal{O}(\delta)\right] + \mathcal{O}(\delta^4)$$

where $\mathcal{K}'_{\Omega_\delta}(\xi)$ is the derivative of the curvature of $\partial\Omega_\delta$, and g_0 is a constant equal to 1 if $\varepsilon = \mathcal{O}(\delta^3)$.

1.4. **The manifold.** We define a manifold of droplet states, which approximates solutions well and captures the motion of droplets along the manifold.

Definition 1.4. *We consider the manifold*

$$\mathcal{M} := \left\{u(\cdot, \xi, \varepsilon) : \ \xi \in [0, |\partial\Omega_\delta|]\right\},$$

consisting of the smooth functions u, mentioned above, having a droplet-like structure and satisfying (1.6).

Obviously, $\mathcal{M}$ is a closed manifold without boundary which in homotopy equivalent to the boundary $\partial\Omega_\delta$. Furthermore, if Ω_δ is simply connected, then $\mathcal{M}$ is topologically a circle.

We define some tubular neighborhoods of the manifold $\mathcal{M}$ in which we shall work. For $r > 0$, let

$$
\begin{aligned}
(1.9) \qquad \mathcal{N}_{L^2}^r &:= \left\{ v \in L^2(\Omega_\delta) : \ d_{L^2}(v, \mathcal{M}) < r \right\}, \\
\mathcal{N}_{H_\varepsilon^1}^r &:= \left\{ v \in H^1(\Omega_\delta) : \ d_{H_\varepsilon^1}(v, \mathcal{M}) < r \right\},
\end{aligned}
$$

where d_{L^2} is the distance in the $L^2(\Omega_\delta)$-norm and $d_{H_\varepsilon^1}$ the distance in the norm $\| \cdot \|_{H_\varepsilon^1} : H^1(\Omega_\delta) \to \mathbb{R}^+$ defined by

$$
\|v\|_{H_\varepsilon^1} := \left(\varepsilon^2 \|\nabla v\|_{L^2(\Omega_\delta)}^2 + \|v\|_{L^2(\Omega_\delta)}^2 \right)^{\frac{1}{2}}.
$$

The usual Sobolev space $H^1(\Omega_\delta)$ equipped with the norm $\|\cdot\|_{H_\varepsilon^1}$ will be denoted by $H_\varepsilon^1 := \left\{ H^1(\Omega_\delta), \ \|\cdot\|_{H_\varepsilon^1} \right\}$. For the rest of this paper $(\cdot, \cdot)$ will denote the $L^2(\Omega_\delta)$-inner product and $\| \cdot \|$ the induced $L^2(\Omega_\delta)$-norm.

Let us finally remark, that for r sufficiently small (depending on δ) there is a well defined local coordinate system in both $\mathcal{N}_{L^2}^r$ and $\mathcal{N}_{H_\varepsilon^1}^r$ and the projection onto $\mathcal{M}$ is well defined and smooth.

1.5. **Main Results.** In Section 2, we analyze the dynamics of solutions $w := u^\varepsilon$ of (1.4) approximated by some u in $\mathcal{M}$ and written as

$$
w(t) = u(\cdot, \xi(t), \varepsilon) + v(t).
$$

Here, v is orthogonal on the manifold, i.e., $v(t) \perp_{L^2(\Omega_\delta)} \partial_\xi(u(\cdot, \xi(t), \varepsilon))$.

We suppose that $\|v\|$ is small and ξ is a diffusion process given by

$$
d\xi = b(\xi)dt + (\sigma(\xi), dW),
$$

for some scalar field $b : \mathbb{R} \to \mathbb{R}$ and some variance $\sigma : \mathbb{R} \to H$. In Remark 2.4 we shall comment on the fact that it is not restrictive to assume ξ being a diffusion.

Applying Itô calculus, we compute first b and σ exactly and then estimate their size in terms of ε, in order to determine the major contribution. See Corollary 2.8 and Remarks 2.9 and 2.10.

Under the assumption of a sufficiently smooth initial condition, by Theorem 2.5 we prove that locally in time

$$
d\xi \approx \varepsilon^2 c(\xi(t), \varepsilon)dt + d\mathcal{A}_s,
$$

where the stochastic process $\mathcal{A}_t$ is given as a diffusion process by the formula

$$
(1.10) \qquad d\mathcal{A}_s = A^{-1}\left[\frac{1}{2}(v, \partial_\xi^3(u)) - \frac{3}{2}(\partial_\xi^2 u, \partial_\xi u)\right](\mathcal{Q}\sigma, \sigma)dt + A^{-1}(\sigma, \mathcal{Q}\partial_\xi^2 u)dt + A^{-1}(\partial_\xi u, dW),
$$

for $A := \|\partial_\xi u\|^2 - (v, \partial_\xi^2 u)$.

Theorem 2.7 and its corollary estimates the effect of noise on the local in time stochastic dynamics driven by the additive term $d\mathcal{A}_s$ which supplies the deterministic dynamics of [2] with an extra deterministic drift and a noise term. We will see (cf. Remark 2.9) that this is for small v the Wiener process W projected to the manifold given by a Stratonovic differential

$$
d\mathcal{A}_s = A^{-1}(\partial_\xi u, \circ dW) .
$$

Since the deterministic dynamics are of order $\mathcal{O}(\varepsilon^2)$ then the noise is not always dominant. *Only, if the noise strength is sufficiently large, or the curvature of the boundary is constant, then noise can dominate (cf. Remark 2.10).* Also if the droplet sits in a spot with maximal curvature, then the noise dominates, at least locally around that point.

This is in contrast to the one-dimensional stochastic Cahn-Hilliard equation (see [4]) where it has been proved that a noise of polynomial strength in ε cannot be ignored since the deterministic dynamics are exponentially small.

In Section 3, we give sufficient criteria for the noise strength such that the solution stays with high probability close to the droplet states both in L^2 and H_ε^1 norms, until times of any polynomial order $\mathcal{O}(\varepsilon^{-q})$. The previous is achieved by estimating the p-moments of v in various norms.

In our L^2-stability result of Theorem 3.4 under the assumption of $\eta_0 = \mathcal{O}(\varepsilon^{2k-2-\tilde{k}})$ for some small $\tilde{k} > 0$, we prove that if $w(0)$ lies in $\mathcal{N}_{L^2}^{\eta\varepsilon^{k-2}}$ for $k > 5$ and some $\eta > 0$ independent of ε, then with high probability the solution stays in a slightly larger neighborhood $\mathcal{N}_{L^2}^{C\varepsilon^{k-2}}$ $(C > \eta)$ for any long time of order $\mathcal{O}(\varepsilon^{-q})$, $q \in \mathbb{N}$.

Our problem is stochastic, so stability in H_ε^1-norm is investigated analytically since we can not refer to the abstract parabolic regularity argument used in [2] in the absence of noise. More specifically, in Theorem 3.11 and its corollary, imposing the additional assumption of $\eta_2 = \mathcal{O}(\varepsilon^{2k-6})$, we show that if $w(0)$ lies in $\mathcal{N}_{L^2}^{\eta\varepsilon^{k-2}}$ and $\nabla w(0)$ lies in $\mathcal{N}_{L^2}^{\eta\varepsilon^{k-4}}$ then with high probability the solution stays in $\mathcal{N}_{H_\varepsilon^1}^{C\varepsilon^{k-3-\tilde{k}}}$ for any long time of order $\mathcal{O}(\varepsilon^{-q})$, $q \in \mathbb{N}$, where $\tilde{k}$ is just some arbitrary small number. *Due to this result, the local in time stochastic dynamics derived in Section 2, are proven to be valid for very long time scales.* Nevertheless, we can not claim that the radius of stability is the optimal one.

The last independent Section 4 involves some higher order estimates needed for the stochastic dynamics. We compute these estimates by extending the analogous lower order results of [2] which were used for the deterministic problem.

Throughout this manuscript, as many of our proofs are quite technical, we present the application of Itô calculus in full details; we hope that the interested reader may gain a wider comprehension of this stochastic technique. Finally, let us remark, that we denote various constant all by C, although their value may change from line to line.

2. STOCHASTIC DYNAMICS

2.1. **The exact stochastic equation of droplet's motion.** In this section, we shall derive the stochastic motion on the manifold $\mathcal{M}$ following the main lines of Theorem 4.3 [2] (p. 297), presented for the deterministic problem. For simplicity we use the symbol w in place of u^ε, so that problem (1.4) can be written as the following stochastic PDE

$$(2.1) \qquad dw = \mathcal{L}^\varepsilon(w)dt + dW.$$

Let the position on $\partial\Omega$ be a diffusion process ξ given by

$$(2.2) \qquad d\xi = b(\xi)dt + (\sigma(\xi), dW),$$

for some scalar field $b : \mathbb{R} \to \mathbb{R}$ and some variance $\sigma : \mathbb{R} \to H$ still to be determined. We will justify this ansatz later in Remark 2.4, once we obtain equations for b and σ.

We approximate the solution w of (1.4) by some $u = u(\xi(t))$ in $\mathcal{M}$, and write

$$(2.3) \qquad w(t) = u(\cdot, \xi(t), \varepsilon) + v(t),$$

where $v \perp_{L^2(\Omega_\delta)} \partial_\xi u$. We use equation (1.4) written as (2.1) to get

$$(2.4) \qquad dw = \left[\varepsilon^2 \Delta w - f(w) + \frac{1}{|\Omega_\delta|} \int_{\Omega_\delta} f(w)dx\right] dt + dW = [\mathcal{L}^\varepsilon(u) - Lv - N(u,v)] dt + dW,$$

where L is defined, for v smooth, by

$$Lv = -\varepsilon^2 \Delta v + f'(u)v,$$

so that $-L$ is the linearization of $\mathcal{L}^\varepsilon$ at u, and $N(u,v)$ is the remaining nonlinear part. Differentiating $w = u + v$ with respect to t, we obtain by Itô calculus

$$(2.5) \qquad dw = \partial_\xi u \, d\xi + dv + \frac{1}{2}\partial_\xi^2 u \, d\xi d\xi.$$

From Itô-calculus $dt dt = dW dt = 0$ and higher order differentials all vanish.

Moreover, for the quantity $d\xi d\xi$ the following lemma holds true.

Lemma 2.1. *We have that*

$$(2.6) \qquad d\xi d\xi = (\mathcal{Q}\sigma(\xi), \sigma(\xi))dt.$$

Proof. Note that using Itô-calculus, by the definition of ξ we derive

$$d\xi d\xi = (\sigma(\xi), dW)(\sigma(\xi), dW).$$

Thus, the claim follows immediately from the definition of the covariance operator.

In more detail, we use the series expansion of W together with $d\beta_i d\beta_j = \delta_{ij} dt$, and derive from Parcevals identity for arbitrary functions a and b the relation

$$(2.7) \qquad (a, dW)(b, dW) = \sum_{i,j} a_i a_j d\beta_i d\beta_j (a, e_i)(b, e_j) = \sum_i a_i^2 (a, e_i)(b, e_i) dt = (\mathcal{Q}a, b) dt .$$

$\square$

Therefore, substituting (2.5) in (2.4) we obtain by using Lemma 2.1 above,

$$(2.8) \qquad dw = \partial_\xi u \, d\xi + \frac{1}{2} \partial_\xi^2 u \cdot (\mathcal{Q}\sigma, \sigma) dt + dv = \Big[\mathcal{L}^\varepsilon(u) - Lv - N(u, v) \Big] dt + dW.$$

We take the $L^2(\Omega_\delta)$-inner product of (2.8) with $\partial_\xi u$ and arrive at

$$(2.9) \qquad \begin{aligned} \|\partial_\xi u\|^2 d\xi + (dv, \partial_\xi u) = &\Big[(\mathcal{L}^\varepsilon(u), \partial_\xi u) - (Lv, \partial_\xi u) - (N(u, v), \partial_\xi u) \Big] dt \\ &+ (dW, \partial_\xi u) - \frac{1}{2} (\partial_\xi^2 u, \partial_\xi u) \cdot (\mathcal{Q}\sigma, \sigma) dt. \end{aligned}$$

We differentiate in t the orthogonality condition $(v, \partial_\xi u) = 0$ and obtain by applying again Itô calculus

$$(2.10) \qquad (dv, \partial_\xi u) + (v, d\partial_\xi u) + (dv, d\partial_\xi u) = 0.$$

As it is demonstrated by the following lemma, by making use of the relation above we shall eliminate $(dv, \partial_\xi u)$ in (2.9).

Lemma 2.2. *It holds that*

$$(2.11) \qquad \begin{aligned} (dv, d[\partial_\xi u]) = &- (v, \partial_\xi^2 u) d\xi - (\sigma, \mathcal{Q} \partial_\xi^2 u) dt \\ &- \frac{1}{2} (v, \partial_\xi^3 u) \cdot (\mathcal{Q}\sigma, \sigma) dt + (\partial_\xi u, \partial_\xi^2 u) \cdot (\mathcal{Q}\sigma, \sigma) dt. \end{aligned}$$

Proof. Itô calculus (recall $d\xi dt = 0$ and $dt dt = 0$) and (2.6) yields

$$d[\partial_\xi u] = \partial_\xi^2 u \, d\xi + \frac{1}{2} \partial_\xi^3 u \cdot (\mathcal{Q}\sigma, \sigma) dt .$$

Then, using this in (2.10), we obtain after some computations

$$(2.12) \qquad \begin{aligned} (\partial_\xi u, dv) &= -(v, d[\partial_\xi u]) - (dv, d\partial_\xi u) \\ &= -(v, \partial_\xi^2 u) d\xi - \frac{1}{2} (v, \partial_\xi^3 u) \cdot (\mathcal{Q}\sigma, \sigma) dt - (\partial_\xi^2 u, dv) d\xi. \end{aligned}$$

Observing that $w = u + v$ and hence, $dv = dw - du$, by (2.8), we arrive at

$$(2.13) \qquad \begin{aligned} -(\partial_\xi^2 u, dv) d\xi &= - (\partial_\xi^2 u, dw) d\xi + (\partial_\xi^2 u, du) d\xi \\ &= - (\partial_\xi^2 u, dW) d\xi + (\partial_\xi^2 u, du) d\xi \\ &= - (\partial_\xi^2 u, dW) d\xi + (\partial_\xi u, \partial_\xi^2 u) \cdot (\mathcal{Q}\sigma, \sigma) dt. \end{aligned}$$

Using our definition of ξ as a diffusion process from (2.2), i.e., $d\xi = b(\xi) dt + (\sigma(\xi), dW)$, and relation (2.7) we obtain

$$(\partial_\xi^2 u, dW) d\xi = (\partial_\xi^2 u, dW)(\sigma, dW) = (\sigma, \mathcal{Q} \partial_\xi^2 u) dt .$$

This yields

$$(2.14) \qquad - (\partial_\xi^2 u, dv) d\xi = -(\sigma, \mathcal{Q} \partial_\xi^2 u) dt + (\partial_\xi u, \partial_\xi^2 u)(\sigma, \mathcal{Q}\sigma) dt .$$

So, by replacing (2.14) in (2.12) the result follows. $\square$

Now we proceed to derive the equations of motion along the manifold. Using (2.11) in (2.9) we arrive at

$$(2.15) \quad \Big[\|\partial_\xi u\|^2 - (v, \partial_\xi^2 u)\Big] d\xi = \Big[(\mathcal{L}^\varepsilon(u), \partial_\xi u) - (Lv, \partial_\xi u) - (N(u,v), \partial_\xi u)\Big] dt$$
$$+ \Big[\frac{1}{2}(v, \partial_\xi^3 u) - \frac{3}{2}(\partial_\xi^2 u, \partial_\xi u)\Big](\mathcal{Q}\sigma, \sigma)dt + (\sigma, \mathcal{Q}\partial_\xi^2 u)dt + (\partial_\xi u, dW).$$

Obviously, if $A := \|\partial_\xi u\|^2 - (v, \partial_\xi^2 u) \neq 0$, then (2.15) finally yields:

The stochastic o.d.e. for the droplet's dynamics.

$$(2.16) \quad d\xi = A^{-1}\Big[(\mathcal{L}^\varepsilon(u), \partial_\xi u) - (Lv, \partial_\xi u) - (N(u,v), \partial_\xi u)\Big] dt$$
$$+ A^{-1}\Big[\frac{1}{2}(v, \partial_\xi^3 u) - \frac{3}{2}(\partial_\xi^2 u, \partial_\xi u)\Big](\mathcal{Q}\sigma, \sigma)dt + A^{-1}(\sigma, \mathcal{Q}\partial_\xi^2 u)dt$$
$$+ A^{-1}(\partial_\xi u, dW).$$

Note that provided v is sufficiently small, the invertibility of A is obvious. The detailed statement is proven in Lemma 2.3 presented below.

So, since $d\xi = bdt + (\sigma, dW)$, then collecting the '$dt$' terms we obtain the *formula for the drift b*

$$(2.17) \quad b = A^{-1}\Big[(\mathcal{L}^\varepsilon(u), \partial_\xi u) - (Lv, \partial_\xi u) - (N(u,v), \partial_\xi u)\Big]$$
$$+ A^{-1}\Big[\frac{1}{2}(v, \partial_\xi^3 u) - \frac{3}{2}(\partial_\xi^2 u, \partial_\xi u)\Big](\mathcal{Q}\sigma, \sigma) + A^{-1}(\sigma, \mathcal{Q}\partial_\xi^2 u),$$

while the *variance σ* is given by

$$(2.18) \quad \sigma = A^{-1}\partial_\xi u.$$

Finally, as promised, we prove the following lemma which establishes the invertibility of A and the asymptotic behavior of A^{-1} as $\varepsilon \to 0^+$.

Lemma 2.3. *For $k > 5/2$ and a fixed constant $c > 0$, if $\|v\| < c\varepsilon^{k-2}$, then there exists a constant $C_0 > 0$ such that $A \geq C_0/\varepsilon$ and therefore A^{-1} exists and*

$$(2.19) \quad |A^{-1}| = \mathcal{O}(\varepsilon) \quad as \quad \varepsilon \to 0^+.$$

Proof. From [2] (p. 297) we have the following estimates

$$(2.20) \quad \|\partial_\xi u\| \geq C^{-1}\varepsilon^{-1/2} \quad \text{and} \quad \|\partial_\xi^2 u\| \leq C\varepsilon^{-3/2},$$

so,

$$(2.21) \quad \|\partial_\xi^2 u\| \leq C\varepsilon^{-1}\varepsilon^{-1/2} \leq C^3\|\partial_\xi u\|^2\varepsilon^{-1/2}.$$

Easily, (2.20), (2.21) yield

$$(2.22) \quad |A| \geq A = \|\partial_\xi u\|^2 - (v, \partial_\xi^2 u) \geq \|\partial_\xi u\|^2 - \|v\|\|\partial_\xi^2 u\| \geq \|\partial_\xi u\|^2 - C\varepsilon^{k-2}\|\partial_\xi^2 u\|$$
$$\geq \|\partial_\xi u\|^2 - \tilde{C}\varepsilon^{k-2}\varepsilon^{-1/2}\|\partial_\xi u\|^2$$
$$\geq C\varepsilon^{-1}\Big[1 + \mathcal{O}(\varepsilon^{k-5/2})\Big].$$

Hence, provided $k > 5/2$, we obtain

$$|A^{-1}| \leq C\varepsilon\Big[1 + \mathcal{O}(\varepsilon^{k-5/2})\Big]^{-1} = \mathcal{O}(\varepsilon).$$

$\square$

Remark 2.4. *We note that the assumption of ξ being a diffusion process is not very restrictive. Following the steps of the derivation backwards, it can be established that for any pair (ξ, v) where ξ solves (2.16) and v solves*

$$dv = d[\partial_\xi u] + \mathcal{L}^\varepsilon(v + u) + f(v + u)dt + dW,$$

then $v \perp \partial_\xi u$ in L^2 and the function $w = u + v$ solves the mass conserving Allen-Cahn equation with noise. See also the analytical analogous results for the stochastic Cahn-Hilliard equation in [4], or the detailed discussion in [29] for Allen-Cahn equation without mass-conservation.

2.2. **The approximate stochastic o.d.e. for the droplet's motion.** Now we proceed by proving the main theorems of Section 2 that analyze the droplet's exact dynamics and approximations thereof, in terms of ε.

Here, we need to assume bounds on the H^1-norm of v, as we are not able to bound the nonlinearity otherwise, since the L^2-bound of v cannot control the cubic nonlinearity.

Theorem 2.5. *For some $k > 5/2$ and fixed small $\tilde{\kappa} > 0$, suppose that $w(0) \in \mathcal{N}_{L^2}^{\eta \varepsilon^{k-2}}$ and that $w(0) \in \mathcal{N}_{H^1_\varepsilon}^{\eta \varepsilon^{k-3-\tilde{\kappa}}}$ for any fixed constant η. Then, locally in time (as long as $w(t) \in \mathcal{N}_{H^1_\varepsilon}^{C\varepsilon^{k-3-\tilde{\kappa}}} \cap \mathcal{N}_{L^2}^{C\varepsilon^{k-2}}$ for any fixed large constant $C > \eta$), it holds that*

$$(2.23) \qquad d\xi = \varepsilon^2 c(\xi(t), \varepsilon)dt + \mathcal{O}(\varepsilon^{k-1/2}\delta^2)dt + \mathcal{O}(\varepsilon^{\min\{2k-11/2-2\tilde{\kappa}, 3k-17/2-2\tilde{\kappa}, k-3/2\}})dt + d\mathcal{A}_t,$$

where the stochastic process $\mathcal{A}_t$ defined in (1.10) is the part in the equation for ξ, which arises due to the presence of noise.

Recall Lemma 1.3, where $c = \mathcal{O}(\delta^2)$. Note that we need $k \geq 15/4$ to have the error term in (2.23) above of order $\mathcal{O}(\varepsilon^2)$.

Using the results of p. 297 of [2], we obtain:

Lemma 2.6. *Under the assumptions of Theorem 2.5, it holds that*

$$\left|(\mathcal{L}^\varepsilon(u), \partial_\xi u) - (Lv, \partial_\xi(u)) - (N(u, v), \partial_\xi u)\right| = \left[\varepsilon^2 c + \mathcal{O}(\varepsilon^{2k-11/2-2\tilde{\kappa}}) + \mathcal{O}(\varepsilon^{3k-17/2-2\tilde{\kappa}}) + \mathcal{O}(\varepsilon^{k-3/2})\right]\|\partial_\xi u\|^2.$$

Proof. Observe first that by definition (1.6) and for some large $K > 0$, we get

$$(\mathcal{L}^\varepsilon(u), \partial_\xi u) = (c\varepsilon^2\|\partial_\xi u\|^2 + \mathcal{O}(\varepsilon^K)\|\partial_\xi u\|_{L^1}).$$

Secondly, we will use the following interpolations

$$\|v\|_{L^4}^2 \leq \|v\|_{H^1}\|v\| \qquad \text{and} \qquad \|v\|_{L^6}^3 \leq \|v\|_{H^1}^2\|v\|.$$

Since f is cubic and u is uniformly bounded, we arrive at

$$|(N(u, v), \partial_\xi u)| \leq C(\|v\|_{H^1} + \|v\|_{H^1}^2)\|v\|\|\partial_\xi u\| \leq C(\varepsilon^{2k-6-2\tilde{\kappa}} + \varepsilon^{3k-9-2\tilde{\kappa}})\varepsilon^{1/2}\|\partial_\xi u\|^2,$$

where we used that

$$\|v\|_{H^1} \leq \varepsilon^{-1}\|v\|_{H^1_\varepsilon}.$$

Note that this is the only argument in this proof where the H^1-norm appears.

For the third term, using that $\|\partial_\xi u\| \geq C\varepsilon^{-1/2}$ and $\|\Delta\partial_\xi u\| \sim \|u_\xi^{(3)}\| \leq C\varepsilon^{-5/2}$ (cf. Appendix) we obtain

$$
\begin{aligned}
(2.24) \qquad |(Lv, \partial_\xi u)| &= |(v, L[\partial_\xi u])| = |-\varepsilon^2(v, \Delta\partial_\xi u) + (v, f'(u)\partial_\xi u)| \\
&\leq \varepsilon^2\|v\|\varepsilon^{-5/2} + C\|v\|\|\partial_\xi u\| \\
&\leq C\varepsilon^{k-2-1/2} + C\varepsilon^{k-2}\|\partial_\xi u\| \\
&\leq C\varepsilon^{k-3/2}\|\partial_\xi u\|^2.
\end{aligned}
$$

$\square$

Proof of Theorem 2.5. Setting

$$B := \left[\frac{1}{2}(v, \partial_\xi^3 u) - \frac{3}{2}(\partial_\xi^2 u, \partial_\xi u)\right],$$

we get by (2.15)

$$(2.25) \qquad A d\xi = \left[(\mathcal{L}^\varepsilon(u), \partial_\xi u) - (Lv, \partial_\xi u) - (N(u, v), \partial_\xi u)\right]dt$$
$$+ B(\mathcal{Q}\sigma, \sigma)dt + (\sigma, \mathcal{Q}\partial_\xi^2 u)dt + (\partial_\xi u, dW).$$

Lemma 2.6 yields

$$A d\xi = \left[\varepsilon^2 c(\xi, \varepsilon) + \mathcal{O}(\varepsilon^{2k-11/2-2\tilde{\kappa}}) + \mathcal{O}(\varepsilon^{3k-17/2-2\tilde{\kappa}}) + \mathcal{O}(\varepsilon^{k-3/2})\right]\|\partial_\xi u\|^2 dt$$
$$+ B(\mathcal{Q}\sigma, \sigma)dt + (\sigma, \mathcal{Q}\partial_\xi^2 u)dt + (\partial_\xi u, dW).$$

Therefore, we get by (1.10)

$$(2.26) \qquad d\xi = A^{-1}\left[\varepsilon^2 c(\xi, \varepsilon) + \mathcal{O}(\varepsilon^{2k-11/2-2\tilde{\kappa}}) + \mathcal{O}(\varepsilon^{3k-17/2-2\tilde{\kappa}}) + \mathcal{O}(\varepsilon^{k-3/2})\right]\|\partial_\xi(u)\|^2 dt + d\mathcal{A}_s.$$

By Lemma 2.3 we have locally in time, as long as $w(t) \in \mathcal{N}_{L^2}^{\eta \varepsilon^{k-2}}$,

$$(2.27) \qquad |A^{-1}|\|\partial_\xi u\|^2 \le \left[1 + C\varepsilon^{k-5/2}\right]^{-1} = 1 + \mathcal{O}(\varepsilon^{k-5/2}).$$

So, from (2.26) analogously to the arguments in p. 297 of [2], we have

$$(2.28) \qquad d\xi = \left[\varepsilon^2 c(\xi, \varepsilon) + \mathcal{O}(\varepsilon^{2k-11/2-2\tilde{\kappa}}) + \mathcal{O}(\varepsilon^{3k-17/2-2\tilde{\kappa}}) + \mathcal{O}(\varepsilon^{k-3/2})\right]\left[1 + \mathcal{O}(\varepsilon^{k-5/2})\right]dt + d\mathcal{A}_s$$
$$= \left[\varepsilon^2 c(\xi, \varepsilon) + \mathcal{O}(\varepsilon^{k-1/2}\delta^2) + \mathcal{O}(\varepsilon^{\min\{2k-11/2-2\tilde{\kappa}, 3k-17/2-2\tilde{\kappa}, k-3/2\}})\right]dt + d\mathcal{A}_s.$$

Here, recall (1.8), i.e., $c(\xi, \varepsilon) = \mathcal{O}(\delta^2)$. $\qquad\qquad\square$

In the following theorem we shall evaluate the noise effect in the local in time stochastic dynamics (2.23) driven by the additive term $d\mathcal{A}_s$ defined by (1.10), which supplies the deterministic dynamics of [2] with an extra deterministic drift and a noise term.

Theorem 2.7. *Assume that in* (1.6) $k > 5/2$ *and* $w(0) \in \mathcal{N}_{L^2}^{\eta \varepsilon^{k-2}}$. *Then, locally in time (as long as $w(t) \in \mathcal{N}_{L^2}^{C\varepsilon^{k-2}}$) the noise induced terms appearing in* (2.23) *given by*

$$d\mathcal{A}_s = A^{-1}\left[\frac{1}{2}(v, \partial_\xi^3(u)) - \frac{3}{2}(\partial_\xi^2 u, \partial_\xi u)\right](\mathcal{Q}\sigma, \sigma)dt + A^{-1}(\sigma, \mathcal{Q}\partial_\xi^2 u)dt + A^{-1}(\partial_\xi u, dW),$$

are estimated by

$$(2.29) \qquad d\mathcal{A}_s = \mathcal{O}(\eta_1)dt + (\mathcal{O}_{L^2}(\varepsilon^{\frac{1}{2}}), dW).$$

Here, $\eta_1 = \|Q\|$ is the noise strength.

The leading order term in $\mathcal{O}(\eta_1)$ is

$$A^{-2}(\partial_\xi u, \mathcal{Q}\partial_\xi^2 u) + \mathcal{O}(\eta_1 \varepsilon^{\min\{1, k-5/2\}}).$$

We can summarize the last two theorems in the next result.

Corollary 2.8. *Under the assumptions of Theorem 2.5 it holds that*

$$(2.30) \qquad d\xi = \varepsilon^2 c(\xi(t), \varepsilon)dt + A^{-2}(\partial_\xi u, \mathcal{Q}\partial_\xi^2 u)dt + A^{-1}(\partial_\xi u, dW)$$
$$+ \mathcal{O}(\varepsilon^{\min\{2k-11/2-2\tilde{\kappa}, 3k-17/2-2\tilde{\kappa}, k-1/2\}} + \eta_1 \varepsilon^{\min\{1, k-5/2\}})dt.$$

Proof. We shall use some estimates proven in Section 4. For ε small, by (4.1), (4.3), and (4.4) given in Theorems 4.1 and 4.2, it holds that

$$|(\partial_\xi^2 u, \partial_\xi u)| = \mathcal{O}(\varepsilon^{-1}), \quad \|\partial_\xi u\| = \mathcal{O}(\varepsilon^{-\frac{1}{2}}),$$

$$\|\partial_\xi^2 u\| = \mathcal{O}(\varepsilon^{-\frac{3}{2}}), \quad \|\partial_\xi^3 u\| = \mathcal{O}(\varepsilon^{-\frac{5}{2}}), \quad \|u\|_{L^\infty} = \mathcal{O}(1).$$

We need to estimate, by means of upper bounds in terms of ε, all the new terms appearing in the stochastic dynamics. By Lemma 2.3 we have

$$|A^{-1}| = \mathcal{O}(\varepsilon),$$

as locally in time we assumed $\|v\|_{L^2} \leq C\varepsilon^{k-2}$.

Recall from (2.27)

$$|A^{-1}|\|\partial_\xi u\|^2 = 1 + \mathcal{O}(\varepsilon^{k-5/2}).$$

We estimate the variance σ by

$$\|\sigma\|^2 = |A^{-1}|^2\|\partial_\xi u\|^2 = |A^{-1}||A^{-1}|\|\partial_\xi u\|^2 \leq C|A^{-1}| \leq C\varepsilon,$$

and therefore, deduce that

$$\|\sigma\| \leq C\varepsilon^{\frac{1}{2}}.$$

The term $A^{-1}(\partial_\xi^2 u, \partial_\xi(u))(\mathcal{Q}\sigma, \sigma)$ is estimated as follows:
For the induced $L^2(\Omega_\delta)$-operator norm $\| \cdot \|$, we get

$$|(\mathcal{Q}\sigma, \sigma)| \leq \|\mathcal{Q}\sigma\|\|\sigma\| \leq \|\mathcal{Q}\|\|\sigma\|^2 \leq C\eta_1\varepsilon.$$

In addition, we have

$$A^{-1}(\partial_\xi^2 u, \partial_\xi u)(\mathcal{Q}\sigma, \sigma) \leq C\varepsilon\varepsilon^{-1}\|\mathcal{Q}\|\varepsilon = C\varepsilon\eta_1.$$

Further, the following estimate holds true

$$|A^{-1}(\sigma, \mathcal{Q}\partial_\xi^2 u)| \leq |A^{-1}||A^{-1}|\|\partial_\xi u\|\|\mathcal{Q}\|\|\partial_\xi^2 u\| \leq C\varepsilon^2\varepsilon^{-1/2}\|\mathcal{Q}\|\varepsilon^{-3/2} = C\eta_1.$$

The remaining term depending on v is bounded (as long as $\|v\|_{L^2} \leq C\varepsilon^{k-2}$) by

$$\frac{1}{2}|A^{-1}(v, \partial_\xi^3(u))(\mathcal{Q}\sigma, \sigma)| \leq C\varepsilon\|v\|\varepsilon^{-5/2}\|Q\|\|\sigma\|^2 = C\varepsilon^{k-5/2}\eta_1.$$

This finishes the proof of the theorem. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

2.3. Interpretation of the result.

Here, we present some comments on the results derived in the section above.

Remark 2.9 (Itô-Stratonovich correction)**.** *Let us look more closely on Corollary 2.8. Using the Itô-Stratonovich correction term, we obtain for a function $g(\xi)$ that the Stratonovich differential is*

$$(g, \circ dW) = \frac{1}{2}(g, \mathcal{Q}\partial_\xi g)dt + (g, dW),$$

where $\mathcal{Q}$ is the covariance operator of W.

Thus, in our case we use that $\partial_\xi v = \partial_\xi u$, as the solution w is independent of ξ, in order to obtain after some calculation

$$d\mathcal{A}_t = A^{-1}(\partial_\xi u, \circ dW),$$

which is the Wiener process W projected onto the manifold of the small droplets. Here $\circ dW$ denotes the Stratonovic differential.

Remark 2.10 (Boundaries of constant curvature). *If the noise is small, which we will need for the attractivity result of the manifold, then the motion of the droplet is to first order given by the $d\xi = \varepsilon^2 c(\xi, \varepsilon)dt$, which is the deterministic result of [2]. On timescales of order ε^{-2} the droplet moves with velocity determined by the changes in the curvature of the boundary.*

If our domain has parts of constant curvature, like a circle or the sides on a square, then $c(\xi, \varepsilon)$, which depends essentially on the derivative of the curvature, is 0 and we expect the droplet to move with a Wiener process projected to the manifold, i.e., locally like a Brownian motion. Nevertheless, here we need to look more closely into the higher order (in δ) corrections to $c(\xi, \varepsilon)$, in order to prove such a claim.

Remark 2.11 (Extremal points of curvature). *Similarly to Remark 2.10, we could study the random fluctuations at the stationary points of maximal or minimal curvature. In this case, at least locally, we expect also the Brownian motion to dominate. At points of minimal curvature the noise drives the droplet away from this unstable stationary situation. At points of maximal curvature the droplet is deterministically attracted, and noise only induces fluctuations around that point.*

Moreover, an exit result of large deviation type from this point of maximal curvature should hold for the droplet. But it is an interesting question, in which direction the droplet will exit. For small amplitude noise, we conjecture that it moves along the manifold and not exits in normal direction away from boundary. Our present stability result does not answer this question as the exponential time-scales present in large deviation problems are too long for our result.

Remark 2.12 (Large Noise). *Our approximation of ξ is valid for a general noise strength $\eta_1 := \|\mathcal{Q}\|$ as long as $\|v\|_{H^1_\varepsilon} = \mathcal{O}(\varepsilon^{k-(3^+)})$. If the noise strength is large, we expect the droplet to move randomly, independent of the curvature of the boundary.*

But as we shall see in the sequel, we are not yet able to verify the stability of the slow manifold for relatively large noise strength. Only for a sufficiently smooth in space noise of sufficiently small strength, the attractivity result is established on arbitrarily long time intervals. As we prove, the restrictions that we must impose on the noise strength for maintaining $\|v\|_{H^1_\varepsilon} = \mathcal{O}(\varepsilon^{k-3-})$, lead to a noise that does not dominate the deterministic dynamics.

More specifically, as we prove, for any $k > 5$ one of the restrictions is given by

$$\eta_1 \leq \eta_0 = \mathcal{O}(\varepsilon^{2k-2-}),$$

resulting in a noise which is small when compared with the original dynamics of the deterministic problem.

This is mainly due to the fact that we are not able yet to find an efficient way to control the nonlinearity further away from the manifold of droplet states. Moreover, the linear attractivity of the manifold is pretty weak. If one could overcome either one of these difficulties, then the case of a much larger noise strength could be analyzed; this is a difficult open problem to be investigated in a future work.

3. Stochastic stability

In this section, we establish the stability of our droplet-manifold in the L^2-norm for any polynomial times of order $\mathcal{O}(\varepsilon^{-q})$, $q > 0$. Note that we could not apply standard large deviation type estimates, since we are not exiting from a single fixed point, and as we only have a manifold of approximate solutions. Moreover, results in the spirit of Berglund and Gentz [8] are not yet developed in the infinite dimensional setting.

3.1. L^2-bounds.

Here, we follow the proof of [2] (p. 296) with some significant changes related to the additive noise.

We wish to show that some small tubular neighborhood of our manifold is positively invariant. *Obviously, in the stochastic setting any solution will leave the neighborhood at some point. The question is, how long does this take.* Here, we are going to present a relatively simple and direct proof for the bound on the exit time.

Recall that

$$w(t) = u(\cdot, \xi(t), \varepsilon) + v(t) \, ,$$

where $v(t) \perp_{L^2(\Omega_\delta)} \partial_\xi u$. Relation (2.8) gives

$$(3.1) \qquad dw = \Big[\mathcal{L}^\varepsilon(u) - Lv - N(u,v) \Big] dt + dW,$$

and on the other hand

$$(3.2) \qquad dw = du + dv = \partial_\xi u d\xi + \frac{1}{2} \partial_\xi^2 u(Q\sigma, \sigma) dt + dv .$$

Recall the definitions

$$\mathcal{L}^\varepsilon(u) := \varepsilon^2 \Delta u - f(u) + \frac{1}{|\Omega_\delta|} \int_{\Omega_\delta} f(u) dx, \qquad Lv := -\varepsilon^2 \Delta v + f'(u)v,$$

$$N(u,v) := f(u+v) - f(u) - f'(u)v - \frac{1}{|\Omega_\delta|} \int_{\Omega_\delta} [f(u+v) - f(u)] dx.$$

Recall also from Lemma 1.2 that (for any large $K > k$)

$$(3.3) \qquad \mathcal{L}^\varepsilon(u) = c(\xi, \varepsilon) \varepsilon^2 \partial_\xi u + \mathcal{B}, \quad \text{with } \mathcal{B} = \mathcal{O}_{L^\infty}(\varepsilon^K).$$

Solving (3.1) and (3.2) for dv and substituting (3.3), we obtain the equation for motion orthogonal to the manifold.

Lemma 3.1. *Consider a solution $w(t) = u(\cdot, \xi(t), \varepsilon) + v(t)$ with $v(t) \perp \partial_\xi u$ and ξ being our diffusion process, then*

$$(3.4) \qquad dv = \Big[c\varepsilon^2 \partial_\xi u + \mathcal{B} - Lv - N(u,v) \Big] dt - \partial_\xi u d\xi - \frac{1}{2} \partial_\xi^2 u(Q\sigma, \sigma) dt + dW.$$

Let us now turn to the estimate of $\|v\|^2$. First we obtain since $(\partial_\xi u, v) = 0$

$$(3.5) \qquad (dv, v) = \Big[(\mathcal{B}, v) - (Lv, v) - (N(u,v), v) \Big] dt - \frac{1}{2} (\partial_\xi^2(u), v)(Q\sigma, \sigma) dt + (v, dW).$$

In view of (3.5) we first observe that Itô calculus gives

$$d\|v\|^2 = d(v,v) = 2(v, dv) + (dv, dv).$$

Since $d\xi = bdt + (\sigma, dW)$ and

$$(dW, dW) = \text{trace}(Q) dt = \eta_0 dt,$$

again by Itô-calculus we derive

$$\begin{aligned} (dv, dv) &= (dW, dW) - 2(dW, \partial_\xi u) d\xi + (\partial_\xi u, \partial_\xi u) d\xi d\xi \\ &= \eta_0 dt - 2(\partial_\xi(u), Q\sigma) dt + \|\partial_\xi u\|^2 (\sigma, Q\sigma) dt \\ &= \mathcal{O}(\eta_0) dt, \end{aligned}$$

where we used in the last step that $\|\partial_\xi u\| \leq C\varepsilon^{-1/2}$, $\|Q\| = \eta_1 \leq \eta_0$ and $\|\sigma\| \leq C\varepsilon^{1/2}$.

Finally, we have

$$(3.6) \qquad \frac{1}{2} d\|v\|^2 = (v, dv) + \mathcal{O}(\eta_0) dt.$$

In order to proceed, we now bound the terms in (3.5). Obviously, we have

$$(3.7) \qquad (\mathcal{B}, v) dt \leq C\varepsilon^K \|v\| dt.$$

Furthermore, for the quadratic form of the linearized operator the following lemma based on [2] holds true.

Lemma 3.2. *There exists some $\nu_0 > 0$ such that for all $v \perp \partial_\xi u$*

$$(3.8) \qquad -(Lv, v) \leq -\nu_0 \varepsilon^2 \|v\|_{H^1_\varepsilon}^2.$$

Let us remark that any improvement in the spectral gap immediately yields an improvement in the noise-strength we can study.

Proof. From the main spectral theorem of [2] we have

$$-(Lv, v) \leq -\tilde{\nu}\varepsilon^2 \|v\|^2.$$

As u is bounded, this implies that for any $\gamma \in (0, 1)$

$$-(Lv, v) \leq -\gamma\varepsilon^2 \|\nabla v\|^2 + \gamma\|f'(u)\|_\infty \|v\|^2 - (1-\gamma)\tilde{\nu}\varepsilon^2 \|v\|^2.$$

Choosing $\gamma = \varepsilon^2$ yields the claim. $\qquad \square$

We consider finally the term $-(N(u, v), v)dt$. Since u is uniformly bounded, we obtain

$$(N(u, v), v) = \int_{\Omega_\delta} (3uv^3 + v^4)dx - \frac{1}{|\Omega_\delta|} \int_{\Omega_\delta} (3u^2v + 3uv^2 + v^3)dx \int_{\Omega_\delta} vdx \geq -C \int_{\Omega_\delta} |v|^3 dx + \int_{\Omega_\delta} v^4 dx,$$

where we used that the mass conservation together with the definition of u gives

$$\frac{1}{|\Omega_\delta|} \int_{\Omega_\delta} wdx = \frac{1}{|\Omega_\delta|} \int_{\Omega_\delta} udx, \qquad \text{and therefore} \qquad \int_{\Omega_\delta} vdx = 0.$$

Hence, we arrive at

$$(3.9) \qquad -(N(u, v), v)dt \leq \left(C \int_{\Omega_\delta} |v|^3 dx - \int_{\Omega_\delta} v^4 dx \right) dt.$$

Also we have

$$(3.10) \qquad (\partial_\xi^2 u, v)(\mathcal{Q}\sigma, \sigma)dt \leq C\varepsilon^{-3/2} \eta_1 \varepsilon^{1/2} \varepsilon^{1/2} \|v\|dt = C\varepsilon^{-1/2} \eta_1 \|v\|dt.$$

Using now the relations (3.6), (3.7), (3.8), (3.9) and (3.10) in (3.5) we get

$$(3.11) \quad d\|v\|^2 \leq \left[C\eta_0 + C\varepsilon^K \|v\| - 2\nu_0\varepsilon^2 \|v\|_{H_\varepsilon^1}^2 + C \int_{\Omega_\delta} |v|^3 dx - 2 \int_{\Omega_\delta} v^4 dx + C\varepsilon^{-1/2} \eta_1 \|v\| \right] dt + 2(v, dW).$$

By Nirenberg's inequality and since $\varepsilon \ll 1$, it follows that

$$(3.12) \qquad \begin{aligned} C \int_{\Omega_\delta} |v|^3 dx &= C\|v\|_{L^3}^3 \leq \tilde{C}\|v\|_{H_1} \|v\|^2 \\ &\leq C_2 \frac{1}{\varepsilon} \|v\|_{H_\varepsilon^1} \|v\|^2 \leq \|v\|_{H_\varepsilon^1} \|v\| \nu_0\varepsilon^2 \leq \nu_0\varepsilon^2 \|v\|_{H_\varepsilon^1}^2, \end{aligned}$$

provided that for $\hat{C} \geq C_2$

$$(3.13) \qquad \|v\| \leq \frac{\nu_0\varepsilon^3}{\hat{C}}.$$

Note that this is the point where the condition $k > 5$ will finally appear, since if $\|v\| \leq C\varepsilon^{k-2}$ for $k > 5$ then indeed $\|v\| \leq \frac{\nu_0\varepsilon^3}{\hat{C}}$ (observe that we do not have any sharp estimate of the constant ν_0 even if $\hat{C}$ could be determined explicitly by the Nirenberg's inequality constant).

Using (3.12) in (3.11) and the fact that $-\|v\|_{H_\varepsilon^1}^2 \leq -\|v\|^2$ we arrive at

$$\begin{aligned} d\|v\|^2 &\leq \left[C\eta_0 + C\varepsilon^K \|v\| - \nu_0\varepsilon^2 \|v\|_{H_\varepsilon^1}^2 + C\varepsilon^{-1/2} \eta_1 \|v\| \right] dt + 2(v, dW) \\ &\leq \left[C\eta_0 + \frac{C_0^2}{2\nu_0\varepsilon^2} (\varepsilon^K + \varepsilon^{-1/2}\eta_1)^2 - \frac{\nu_0\varepsilon^2}{2} \|v\|^2 \right] dt + 2(v, dW). \end{aligned}$$

Let us summarize the result proven so far.

Lemma 3.3. *As long as* $\|v\| \leq \frac{\nu_0\varepsilon^3}{\hat{C}}$, *then*

$$(3.14) \qquad d\|v\|^2 \leq \left[C_\varepsilon - \frac{\nu_0\varepsilon^2}{2} \|v\|^2 \right] dt + 2(dW, v),$$

for

$$C_\varepsilon := C\eta_0 + \frac{C_0^2}{2\nu_0\varepsilon^2}(\varepsilon^K + \varepsilon^{-1/2}\eta_1)^2,$$

where C_0, $\hat{C}$ and C are the specific constants appearing in the proof.

3.2. **Long-time L^2-stability.** Let us define the stopping time $\tau^\star$ as the exit-time of a neighborhood of the manifold before time T

$$\tau^\star := \inf\{t \in [0,T] \ : \ \|v(t)\| > B\},$$

with the convention that

$$\tau^\star = T, \quad \text{if} \quad \|v(t)\| \le B \text{ for all } t \in [0,T],$$

and thus, the solution did not exit before T.

Recall that v satisfies an inequality of the form

$$d\|v\|^2 \le [C_\varepsilon - a\|v\|^2]dt + 2(v, dW),$$

for all $t \le \tau^\star$, provided that $B \le \frac{\nu_0\varepsilon^3}{\hat{C}}$. Thus from now on, we fix

$$B = C\varepsilon^{k-2} \quad \text{for } k > 5 .$$

More specifically, from (3.14)

$$a := \frac{\nu_0\varepsilon^2}{2} \quad \text{and} \quad C_\varepsilon := C\eta_0 + \frac{C_0^2}{2\nu_0\varepsilon^2}(\varepsilon^K + \varepsilon^{-1/2}\eta_1)^2$$

So, for all $t \le \tau^\star$, it holds that

$$\|v(t)\|^2 + a\int_0^t \|v(s)\|^2 ds \le \|v(0)\|^2 + C_\varepsilon t + 2\int_0^t (v, dW).$$

Using the fact that stopped stochastic integrals still have mean value 0 (referring to optimal stopping of martingales), we obtain

$$(3.15) \qquad \mathbb{E}\|v(\tau^\star)\|^2 + a\mathbb{E}\int_0^{\tau^\star} \|v(s)\|^2 ds \le \|v(0)\|^2 + C_\varepsilon T,$$

where we used that $\tau^\star \le T$ by definition.

We can extend (3.15) above to higher powers using Itô calculus as follows:

$$\begin{aligned} d\|v\|^{2p} &= p\|v\|^{2p-2}d\|v\|^2 + p(p-1)\|v\|^{2p-4}d\|v\|^2 d\|v\|^2 \\ &= p\|v\|^{2p-2}d\|v\|^2 + 4p(p-1)\|v\|^{2p-4}(v,Qv)dt \\ &\le p\|v\|^{2p-2}[C_\varepsilon - a\|v\|^2]dt + 4p(p-1)\|v\|^{2p-2}\|Q\|dt + 2p\|v\|^{2p-2}(v,dW). \end{aligned}$$

Hence, for all integers $p > 1$, we arrive at

$$\|v(t)\|^{2p} + pa\int_0^t \|v(s)\|^{2p}ds \le \|v(0)\|^{2p} + C(C_\varepsilon + \|Q\|)\int_0^t \|v\|^{2p-2}dt + 2p\int_0^t \|v\|^{2p-2}(v,dW),$$

provided $t \le \tau^\star$. Here and in the sequel, we denote all constants depending explicitly on p only by C.

Therefore, we obtain

$$(3.16) \qquad \mathbb{E}\|v(\tau^\star)\|^{2p} \le \|v(0)\|^{2p} + C(C_\varepsilon + \|Q\|)\mathbb{E}\int_0^{\tau^\star} \|v\|^{2p-2}dt,$$

and

$$(3.17) \qquad a\mathbb{E}\int_0^{\tau^\star} \|v(s)\|^{2p}ds \le \frac{1}{p}\|v(0)\|^{2p} + C(C_\varepsilon + \|Q\|)\mathbb{E}\int_0^{\tau^\star} \|v\|^{2p-2}dt.$$

Let us now assume that

$$q = \frac{C_\varepsilon + \|Q\|}{a} \ll 1 \qquad \text{and} \qquad \|v(0)\|^2 \le q \ll B^2.$$

An induction argument yields

$$\frac{1}{p}\mathbb{E}\|v(\tau^\star)\|^{2p} \overset{(3.16)}{\leq} \frac{1}{p}\|v(0)\|^{2p} + C(C_\varepsilon + \|Q\|)\mathbb{E}\int_0^{\tau^\star}\|v\|^{2p-2}dt$$

$$= \frac{1}{p}\|v(0)\|^{2p} + C \cdot qa \cdot \mathbb{E}\int_0^{\tau^\star}\|v\|^{2p-2}dt$$

$$\overset{(3.17)}{\leq} \frac{1}{p}\|v(0)\|^{2p} + Cq\|v(0)\|^{2p-2} + Cq^2a\mathbb{E}\int_0^{\tau^\star}\|v\|^{2p-4}dt$$

$$\leq Cq\|v(0)\|^{2p-2} + Cq^2a\mathbb{E}\int_0^{\tau^\star}\|v\|^{2p-4}dt$$

$$\leq \dots$$

$$\leq Cq^{p-2}\|v(0)\|^4 + Cq^{p-1}a\mathbb{E}\int_0^{\tau^\star}\|v\|^2dt$$

$$\overset{(3.15)}{\leq} Cq^{p-2}\|v(0)\|^4 + Cq^{p-1}[\|v(0)\|^2 + C_\varepsilon T]$$

$$\leq Cq^p + Caq^pT,$$

as $C_\varepsilon \leq aq$. Using Chebychev's inequality, finally, we arrive at

$$\mathbb{P}(\tau^\star < T) = \mathbb{P}(\|v(\tau^\star)\| \geq B) \leq B^{-2p}\mathbb{E}\|v(\tau^\star)\|^{2p}$$

(3.18)
$$\leq CB^{-2p}[q^p + aq^pT] = C\left(\frac{q}{B^2}\right)^p + Ca\left(\frac{q}{B^2}\right)^pT.$$

Therefore, we obtain the following L^2-stability theorem.

Theorem 3.4. *Consider the exit time*

$$\tau^\star := \inf\{t \in [0, T_\varepsilon] \; : \; \|v(t)\| > C\varepsilon^{k-2}\},$$

with $T_\varepsilon := \varepsilon^{-N}$ for any fixed large $N > 0$. Fix

$$k > 5, \quad \text{and} \quad \|v(0)\| < \eta\varepsilon^{k-2}.$$

Also, assume that the noise satisfies

$$\eta_0 \leq C\varepsilon^{2k-2+\tilde{k}},$$

for some $\tilde{k} > 0$ very small. Then the probability $\mathbb{P}(\tau^\star < T_\varepsilon)$ is smaller than any power of ε, as $\varepsilon \to 0$. And thus for very large times with high probability the solution stays close to the manifold.

Proof. The claim follows from inequality (3.18) if $q/B^2 = \mathcal{O}(\varepsilon^{\tilde{\kappa}})$.
Indeed, using the definitions of C_ε, $a = \mathcal{O}(\varepsilon^2)$ and $B = C\varepsilon^{k-2}$, we have (note $K > k$)

$$q = \frac{C_\varepsilon + \|Q\|}{a} \leq C[\eta_0\varepsilon^{-2} + \varepsilon^{-4}(\varepsilon^{2K} + \varepsilon^{-1}\eta_1^2)]$$

Furthermore, $B^2 = \mathcal{O}(\varepsilon^{2k-4})$. So, indeed we get

$$q/B^2 = \eta_0\varepsilon^{2-2k} + \varepsilon^{2(K-k)} + \varepsilon^{-1-2k}\eta_1^2 = \mathcal{O}(\varepsilon^{\tilde{\kappa}}),$$

since $\eta_1 \leq \eta_0$ and $k > 5$. $\qquad\square$

Remark 3.5. *The stability result presented so far does not state that the local in time stochastic dynamics for ξ given by Theorem 2.5 and Theorem 2.7 hold with high probability for a long time. For this we need to prove stochastic stability in the H_ε^1-norm. As we rely for simplicity of presentation on the direct application of Itô's formula, this will only be achieved for a noise $\dot{W}$ sufficiently regular in space.*

Remark 3.6. *The presence of the very small $\tilde{\kappa}$ in the conditions of η_0 is only for simplicity. Instead of having the $\tilde{\kappa}$ here, we could use $B := \varepsilon^{2k-4-\tilde{\kappa}}$. This would yield the same result but for a slightly smaller neighborhood for v.*

We need the gap created by $\tilde{\kappa}$ in order to control the probability and to obtain very large time-scales in the stability result. This is the reason, why $k = 5$ (included in the deterministic case result of [2]) is out of reach in our approach, and we can only consider $k > 5$.

3.3. Estimates in H_ε^1-norm.

As we can not rely on bounds of the linearized operator in H_ε^1-norm, we shall use instead the previously established L^2-stability result given by Theorem 3.4. Nevertheless, in order to bound the H_ε^1-norm of stochastic solution over a very long time-scale, we can allow the use of a larger tube bounding ∇v. But, as we shall see in the sequel, this will further limit the noise we can consider since we obtain additional restrictions for the size of η_2.

Let w be the solution of the mass conserving stochastic Allen-Cahn equation (1.4), then

$$w = u + v, \quad u \in \mathcal{M}, \quad u \perp_{L^2(\Omega_\delta)} \partial_\xi u.$$

Moreover, recall (3.4)

$$dv = \left[c\varepsilon^2 \partial_\xi u + \mathcal{B} - Lv - N(u,v)\right]dt - \partial_\xi u d\xi - \frac{1}{2}\partial_\xi^2 u(\mathcal{Q}\sigma, \sigma)dt + dW,$$

with

$$Lv + N(u,v) = -\varepsilon^2 \Delta v + f(u+v) - f(u) - \frac{1}{|\Omega_\delta|}\int_{\Omega_\delta}[f(u+v) - f(u)]dx.$$

We consider first the following relation

$$(3.19) \qquad \begin{aligned} d\|\nabla v\|^2 &= 2(\nabla v, d\nabla v) + (\nabla dv, \nabla dv) \\ &= -2(\Delta v, dv) + (\nabla dv, \nabla dv), \end{aligned}$$

where we used integration by parts, as v satisfies a Neumann boundary condition.

Observe that by series expansion of W and since $\|e_k\| = 1$, $\|\sigma\| = \mathcal{O}(\varepsilon^{1/2})$, $\|\nabla \partial_\xi u\| = \mathcal{O}(\varepsilon^{-3/2})$, we obtain

$$\begin{aligned} (\nabla \partial_\xi u, \nabla dW)(\sigma, dW) &= \sum_k \alpha_k^2 (\nabla \partial_\xi u, \nabla e_k)(\sigma, e_k)dt \\ &\leq \sum_k \alpha_k \|\nabla e_k\| \alpha_k \|\nabla \partial_\xi u\| \|\sigma\| dt \\ &\leq \eta_2^{1/2} \eta_0^{1/2} \|\nabla \partial_\xi u\| \|\sigma\| dt \\ &\leq C(\varepsilon^{-2}\eta_0 + \eta_2)dt. \end{aligned}$$

Thus, since $\eta_1 \leq \eta_0$ then for the Itô-correction term we have

$$\begin{aligned} (\nabla dv, \nabla dv) &= \|\nabla \partial_\xi u\|^2 (d\xi)^2 - 2(\nabla \partial_\xi u, \nabla dW)d\xi + (\nabla dW, \nabla dW) \\ &= \|\nabla \partial_\xi u\|^2 (Q\sigma, \sigma)dt - 2(\nabla \partial_\xi u, \nabla dW)(\sigma, dW) + \text{trace}(\Delta Q) \\ &= \mathcal{O}(\varepsilon^{-2}\eta_1 + \eta_2 + \varepsilon^{-2}\eta_0 + \eta_2)dt \\ &= \mathcal{O}(\varepsilon^{-2}\eta_0 + \eta_2)dt. \end{aligned}$$

Considering the other mixed term in (3.19) and using that $(1, \Delta v) = 0$, we obtain the following relation

$$(3.20) \qquad \begin{aligned} -(\Delta v, dv) = &- \left(\Delta v, \mathcal{B} + \varepsilon^2 \Delta v\right)dt - \left(\Delta v, -f(u+v) + f(u)\right)dt \\ &+ \left(\Delta v, \partial_\xi u[b - c\varepsilon^2] + \frac{1}{2}\partial_\xi^2 u(\mathcal{Q}\sigma, \sigma)\right)dt - \left(\Delta v, -\partial_\xi u(\sigma, dW) + dW\right) \\ =&\, T_1 + T_2 + T_3 + T_4\,. \end{aligned}$$

First, we estimate the martingale term T_4. Note that we only integrate by parts once, as we only know that v satisfies Neumann boundary conditions.

$$T_4 = (\Delta v, \partial_\xi u)(\sigma, dW) - (\Delta v, dW) = -(\nabla v, \nabla \partial_\xi u)(\sigma, dW) + (\nabla v, \nabla dW)$$
$$= (\mathcal{O}(\varepsilon^{-1}\|\nabla v\|), dW) + (\mathcal{O}(\|\nabla v\|), d\nabla W).$$

For T_1 and for $K > 0$ sufficiently large, we obtain by Lemma 1.2 that

$$T_1 = -(\Delta v, [\mathcal{B} + \varepsilon^2 \Delta v])dt = -(\Delta v, \mathcal{B})dt - \varepsilon^2\|\Delta v\|^2 dt$$
$$= \mathcal{O}(\varepsilon^K \|\Delta v\|)dt - \varepsilon^2 \|\Delta v\|^2 dt.$$

For T_3 we have

$$T_3 = \left(\Delta v, \partial_\xi u[b - c\varepsilon^2] + \frac{1}{2}\partial_\xi^2 u(\mathcal{Q}\sigma, \sigma)\right)dt = (\Delta v, \partial_\xi u)[b - c\varepsilon^2]dt + \frac{1}{2}(\partial_\xi^2 u, \Delta v)(\mathcal{Q}\sigma, \sigma)dt$$
$$= \mathcal{O}(\|\Delta v\|\varepsilon^{-1/2}[b - c\varepsilon^2])dt + \mathcal{O}(\varepsilon^{-1/2}\eta_0\|\Delta v\|)dt.$$

Here, note that we can bound b by Corollary 2.8 only up to a stopping time.

For T_2 we derive

$$T_2 = -\left(\Delta v, -f(u+v) + f(u)\right)dt = \left(\Delta v, -v + 3u^2 v + 3uv^2 + v^3\right)dt$$
$$= \left(\|\nabla v\|^2 - \int \nabla(3u^2 v + 3uv^2 + v^3) \cdot \nabla v dx\right)dt$$
$$= \left(\|\nabla v\|^2 - \int (3u^2 + 6uv + 3v^2)|\nabla v|^2 dx - \int (6uv + 3v^2)(\nabla u \cdot \nabla v)dx\right)dt$$
$$\leq \left(\|\nabla v\|^2 - \int (6uv + 3v^2)(\nabla u \cdot \nabla v)dx\right)dt.$$

We observe that $\|\nabla u\|_\infty \sim \|\partial_\xi u\|_\infty$ (cf. Appendix). So, we have $\|\nabla u\|_\infty \leq C\varepsilon^{-1}$ (since $\|\partial_\xi u\|_\infty \leq C\varepsilon^{-1}$, cf. [2]). In addition, u is uniformly bounded in ε. Furthermore, the Nirenberg's inequality gives

$$\|v^2\| = \|v\|_4^2 \leq (C\|v\|_{H_1}^{1/2}\|v\|^{1/2})^2 = C\|v\|_{H_1}\|v\| \leq C\|v\|^2 + C\|\nabla v\|\|v\|,$$

while the following interpolation inequality holds true

$$\|\nabla v\|^2 = -(\Delta v, v) \leq \|\Delta v\|\|v\|.$$

Using the previous estimates we arrive at

$$T_2 \leq \left(\|\nabla v\|^2 + 3\|v^2\|\|\nabla u\|_\infty\|\nabla v\| + 6\|v\|\|u\|_\infty\|\nabla u\|_\infty\|\nabla v\|\right)dt$$
(3.21)
$$\leq \left(\|\nabla v\|^2 + C\varepsilon^{-1}\|v^2\|\|\nabla v\| + C\varepsilon^{-1}\|v\|\|\nabla v\|\right)dt$$
$$\leq \left(\|\nabla v\|^2 + C\varepsilon^{-1}\|v\|\|\nabla v\|^2 + C\varepsilon^{-1}\|v\|^2\|\nabla v\| + C\varepsilon^{-1}\|v\|\|\nabla v\|\right)dt.$$

So, by relation (3.20), we derive

$$d\|\nabla v\|^2 = \mathcal{O}(\varepsilon^K\|\Delta v\|)dt - 2\varepsilon^2\|\Delta v\|^2 dt + T_2$$
$$+ \mathcal{O}(\|\Delta v\|\varepsilon^{-1/2}|b - c\varepsilon^2|)dt + \mathcal{O}(\varepsilon^{-1/2}\eta_0\|\Delta v\|)dt + \mathcal{O}(\varepsilon^{-2}\eta_0 + \eta_2)dt$$
$$+ (\mathcal{O}(\varepsilon^{-1}\|\nabla v\|), dW) + \mathcal{O}(\|\nabla v\|, \nabla dW).$$

for T_2 estimated by (3.21). Therefore, Young's inequality yields

$$d\|\nabla v\|^2 = -\varepsilon^2\|\Delta v\|^2 dt + T_2$$
(3.22)
$$+ \mathcal{O}(\varepsilon^{2K-2} + \varepsilon^{-3}|b - c\varepsilon^2|^2 + \varepsilon^{-3}\eta_0^2 + \varepsilon^{-2}\eta_0 + \eta_2)dt$$
$$+ (\mathcal{O}(\varepsilon^{-1}\|\nabla v\|), dW) + \mathcal{O}(\|\nabla v\|, \nabla dW).$$

In order to proceed with the estimate of T_2 given by (3.21) where the L^2-norm of v is also involved, we shall rely on the L^2-stability result proven so far, observing evolution in time as long as $\|v\|$ is not too large.

Definition 3.7. *Let $k > 5$ and $\tilde{\kappa} > 0$ small. For some given large T_ε, we define the stopping time*

$$(3.23) \qquad \tau_\varepsilon = \inf\{t \in [0, T_\varepsilon] \ : \ \|\nabla v(t)\| > C_0 \varepsilon^{k-4-\tilde{\kappa}} \ or \ \|v(t)\| > C_0 \varepsilon^{k-2}\} \,.$$

Here, C_0 is a large fixed positive constant. Obviously, we set $\tau_\varepsilon = T_\varepsilon$ if none of the conditions is satisfied for all $t < T_\varepsilon$.

From the previous definition, it follows that

$$\sup_{t \in [0,T_\varepsilon]} \|v(t)\|_{H^1_\varepsilon} \le C_0 \varepsilon^{k-3-\tilde{\kappa}}.$$

Hence, for this stopping time considered, the bound in L^2-norm provided by Theorem 3.4 (i.e., $\|v(t)\| < C\varepsilon^{k-2}$ for all $t \le \tau_\varepsilon$ with high probability, as long as T_ε is a polynomial in ε^{-1}) is much stronger than the L^2-bound given in the H^1_ε-norm.

For the rest of the arguments we assume that $t \le \tau_\varepsilon$ and let $K \ge k$. We know by Corollary 2.8 that

$$\sup_{[0,\tau_\varepsilon]} |b - c\varepsilon^2| = \mathcal{O}(\varepsilon^{k-1/2}).$$

Thus, relation (3.22) yields

$$(3.24) \qquad \begin{aligned} d\|\nabla v\|^2 = & -\varepsilon^2 \|\Delta v\|^2 dt + T_2 \\ & + \mathcal{O}(\varepsilon^{2k-4} + \varepsilon^{-3}\eta_0^2 + \varepsilon^{-2}\eta_0 + \eta_2)dt \\ & + (\mathcal{O}(\varepsilon^{-1}\|\nabla v\|), dW) + (\mathcal{O}(\|\nabla v\|), \nabla dW) \,. \end{aligned}$$

In order to estimate the term T_2, we use the bound given by (3.21) and the following relations

$$\|v(t)\| \le C\varepsilon^{k-2}, \qquad \|\nabla v\|^2 \le \|\Delta v\| \|v\|.$$

Therefore, for $k \ge 3$, we obtain by Young's inequality

$$(3.25) \qquad \begin{aligned} T_2 & \le \Big(\|\nabla v\|^2 + C\varepsilon^{-1}\|v\|\|\nabla v\|^2 + C\varepsilon^{-1}\|v\|^2\|\nabla v\| + C\varepsilon^{-1}\|v\|\|\nabla v\| \Big)dt \\ & \le \Big(4\|\nabla v\|^2 + C\varepsilon^{2k-6} \Big)dt \\ & \le \Big(\frac{1}{2}\varepsilon^2\|\Delta v\|^2 + 8\varepsilon^{-2}\|v\|^2 + C\varepsilon^{2k-6} \Big)dt \\ & \le \Big(\frac{1}{2}\varepsilon^2\|\Delta v\|^2 + C\varepsilon^{2k-6} \Big)dt \,. \end{aligned}$$

To close the argument, we need a Poincaré type estimate. More specifically, for any function satisfying Neumann boundary conditions there exists some positive constant $c > 0$ such that

$$(3.26) \qquad\qquad 2c\|\nabla v\|^2 \le \|\Delta v\|^2 \,.$$

To prove the statement above, first let us denote by $\bar{v}$ the spatial average of v, and then use interpolation and the standard Poincaré inequality. So, we have

$$\|\nabla v\|^2 = \|\nabla(v - \bar{v})\|^2 \le \|v - \bar{v}\| \|\Delta(v - \bar{v})\| \le C\|\nabla(v - \bar{v})\| \|\Delta(v - \bar{v})\| = C\|\nabla v\| \|\Delta v\| \,.$$

Therefore, by (3.24), we obtain the following lemma.

Lemma 3.8. *If $k \ge 3$ and $t \le \tau_\varepsilon$, with τ_ε given by (3.23), then for $c > 0$ the constant appearing in Poincaré inequality (3.26), the following relation holds true*

$$(3.27) \qquad d\|\nabla v\|^2 + c\varepsilon^2\|\nabla v\|^2 dt = \Gamma_\varepsilon dt + (Z, dW) + (\Psi, d\nabla W),$$

for

$$(3.28) \qquad\qquad \Gamma_\varepsilon := \mathcal{O}(\varepsilon^{2k-6} + \varepsilon^{-3}\eta_0^2 + \varepsilon^{-2}\eta_0 + \eta_2),$$

and

$$(3.29) \qquad \|Z\|^2 := \mathcal{O}(\varepsilon^{-2}\|\nabla v\|^2), \quad \|\Psi\|^2 := \mathcal{O}(\|\nabla v\|^2).$$

Furthermore, it holds that

$$(3.30) \qquad \|\nabla v(t)\|^2 + c\varepsilon^2 \int_0^t \|\nabla v(s)\|^2 ds \leq \|\nabla v(0)\|^2 + \Gamma_\varepsilon T_\varepsilon + \int_0^t [(Z, dW) + (\Psi, d\nabla W)]ds,$$

and thus

$$(3.31) \qquad \mathbb{E}\|\nabla v(t)\|^2 + c\varepsilon^2 \mathbb{E}\int_0^t \|\nabla v(s)\|^2 ds \leq \|\nabla v(0)\|^2 + \Gamma_\varepsilon T_\varepsilon.$$

Note that by a slight abuse of notation, we identify Γ_ε defined in (3.28) with the corresponding $\mathcal{O}$-term.

We remark, that as in the L^2-case this lemma is only the first step for proving stability; higher moments will be derived by an induction argument in the following section.

3.4. **Long-Time H^1-stability.** Keeping the same assumptions as in Lemma 3.8, we proceed similar to the L^2-stability result by estimating for any integer $p > 1$ the pth-moment of $\|\nabla v\|^2$. Itô calculus yields

$$(3.32) \qquad d\|\nabla v\|^{2p} = p\|\nabla v\|^{2p-2}d\|\nabla v\|^2 + p(p-1)\|\nabla v\|^{2p-4}[d\|\nabla v\|^2]^2 .$$

Using now (3.27), we obtain

$$(3.33) \qquad [d\|\nabla v\|^2]^2 \leq (Z, \mathcal{Q}Z)dt + (\Psi, \Delta\mathcal{Q}\Psi)dt + 2(Z, dW)(\Psi, d\nabla W).$$

Observing now that $d\beta_i d\beta_j = \delta_{ij}$, we get by series expansion and Cauchy-Schwarz

$$(3.34) \qquad \begin{aligned} (Z, dW)(\psi, d\nabla W) &= \sum a_i^2(Z, e_i)(\psi, \nabla e_i) \leq \sum a_i^2\|e_i\|\|\nabla e_i\|\|Z\|\|\psi\| \\ &\leq \|Z\|\|\psi\|\sqrt{\eta_0\eta_2} \leq \|Z\|^2\eta_0 + \|\psi\|^2\eta_2 . \end{aligned}$$

By (3.33) and (3.34), we arrive at

$$(3.35) \qquad \begin{aligned} [d\|\nabla v\|^2]^2 &\leq (\|\mathcal{Q}\|\|Z\|^2 + \|\Psi\|^2\|\Delta\mathcal{Q}\|)dt + 2\|Z\|\|\Psi\|\sqrt{\eta_0\eta_2}dt \\ &\leq C(\|Z\|^2\eta_0 + \|\Psi\|^2\eta_2)dt. \end{aligned}$$

Replacing (3.35) in (3.32) and using (3.27), we get

$$\begin{aligned} d\|\nabla v\|^{2p} \leq &p\|\nabla v\|^{2p-2}d\|\nabla v\|^2 + Cp(p-1)\|\nabla v\|^{2p-4}(\|Z\|^2\eta_0 + \|\Psi\|^2\eta_2)dt \\ \leq &p\|\nabla v\|^{2p-2}(\Gamma_\varepsilon - c\varepsilon^2\|\nabla v\|^2)dt \\ &+ p\|\nabla v\|^{2p-2}[(Z, dW) + (\Psi, d\nabla W)] \\ &+ Cp(p-1)\|\nabla v\|^{2p-4}(\|Z\|^2\eta_0 + \|\Psi\|^2\eta_2)dt. \end{aligned}$$

In the relation above, we now use (3.29) and the fact that $\|v\| \leq C\varepsilon^{k-2}$. Hence, we obtain for any $t \leq \tau_\varepsilon$

$$(3.36) \qquad \begin{aligned} d\|\nabla v\|^{2p} \leq &p\|\nabla v\|^{2p-2}(\Gamma_\varepsilon - c\varepsilon^2\|\nabla v\|^2)dt + p\|\nabla v\|^{2p-2}[(Z, dW) + (\Psi, d\nabla W)] \\ &+ C\|\nabla v\|^{2p-2}\varepsilon^{-2}\eta_0 dt + C\|\nabla v\|^{2p-2}\eta_2 dt. \end{aligned}$$

Here, all appearing constants may depend on p. Integrating (3.36) we derive the following lemma:

Lemma 3.9. *Under the assumptions of Lemma 3.8, then for any integer $p \geq 2$, the following estimate holds true*

$$(3.37) \qquad \mathbb{E}\|\nabla v(\tau_\varepsilon)\|^{2p} + c\varepsilon^2 p\mathbb{E}\int_0^{\tau_\varepsilon} \|\nabla v(t)\|^{2p}dt \leq \|\nabla v(0)\|^{2p} + C[\Gamma_\varepsilon + \varepsilon^{-2}\eta_0 + \eta_2] \cdot \mathbb{E}\int_0^{\tau_\varepsilon} \|\nabla v(t)\|^{2p-2}dt.$$

Here, $c, C > 0$ are constants that may depend on $p \in \mathbb{N}$ and Γ_ε was defined in (3.28).

Keeping the same assumptions as these of Lemma 3.8, we consider a sufficiently small noise such that

(3.38) $$\varepsilon^{-2}\eta_0 + \eta_2 = \mathcal{O}(\varepsilon^{2k-6}),$$

i.e., the main order term of $\Gamma_\varepsilon + \varepsilon^{-2}\eta_0 + \eta_2$ is ε^{2k-6}.

Furthermore, in (3.37) we define

$$K_p := \mathbb{E}\int_0^{\tau_\varepsilon} \|\nabla v(t)\|^{2p} dt,$$

and thus, we obtain for $p \geq 2$

$$\begin{aligned}
K_p &\leq \varepsilon^{-2}\|\nabla v(0)\|^{2p} + C\varepsilon^{-2}[\Gamma_\varepsilon + \varepsilon^{-2}\eta_0 + \eta_2]K_{p-1} \\
&\leq \varepsilon^{-2}\|\nabla v(0)\|^{2p} + C\varepsilon^{2k-8}K_{p-1} \\
&=: \varepsilon^{-2}A^p + aK_{p-1},
\end{aligned}$$

for

$$A := \|\nabla v(0)\|^2, \quad a := C\varepsilon^{2k-8} < 1.$$

So, we have inductively

$$K_p \leq \varepsilon^{-2}A^p + aK_{p-1} \leq \varepsilon^{-2}A^p + \varepsilon^{-2}aK_{p-2}A^{p-1} + 2a^2 K_{p-2}$$

$$\leq \ldots \leq C\varepsilon^{-2}\sum_{i=2}^{p} a^{p-i}A^i + Ca^{p-1}K_1.$$

By definition and (3.31) we get

$$K_1 \leq \varepsilon^{-2}A + \varepsilon^{-2}\Gamma_\varepsilon T_\varepsilon \leq \varepsilon^{-2}A + CaT_\varepsilon.$$

Hence, we obtain

(3.39) $$K_p \leq C\varepsilon^{-2}\sum_{i=1}^{p} a^{p-i}A^i + Ca^p T_\varepsilon \leq C[\varepsilon^{-2} + T_\varepsilon]a^p + C\varepsilon^{-2}A^p,$$

for $C > 0$ a constant depending on p.

This yields the following lemma. Note that here we need only $k > 3$. It is the L^2-bound that needs $k > 5$.

Lemma 3.10. *Let $k \geq 3$ and τ_ε given by (3.23). If*

$$\Gamma_\varepsilon + \varepsilon^{-2}\eta_0 + \eta_2 \leq C\varepsilon^{2k-6},$$

and if

$$\|\nabla v(0)\|^2 \leq a := C\varepsilon^{2k-8},$$

then for any integer $p > 1$ it holds that

(3.40) $$\mathbb{E}\|\nabla v(\tau_\varepsilon)\|^{2p} \leq C\varepsilon^{-2}[\varepsilon^{-2} + T_\varepsilon]a^p.$$

Proof. Using the definitions of a, A, and K_p and relation (3.37), we have

$$\begin{aligned}
\mathbb{E}\|\nabla v(\tau_\varepsilon)\|^{2p} &\leq A^p + C\varepsilon^{-2}aK_{p-1} \\
&\leq A^p + C\varepsilon^{-2}[\varepsilon^{-2} + T_\varepsilon]a^p + C\varepsilon^{-4}aA^{p-1} \\
&\leq C\varepsilon^{-2}[\varepsilon^{-2} + T_\varepsilon]a^p.
\end{aligned}$$

$\square$

We proceed now to the proof of the following main stability result.

Theorem 3.11. *Consider the exit time*

$$\tau_\varepsilon := \inf\{t \in [0, T_\varepsilon] \; : \; \|\nabla v(t)\| > C_0 \varepsilon^{k-4-\tilde\kappa} \; or \; \|v(t)\| > C_0 \varepsilon^{k-2}\},$$

where $T_\varepsilon := \varepsilon^{-N}$ for arbitrary and fixed large $N > 0$ and for arbitrary and small $\tilde\kappa > 0$. Let also,

$$k > 5\,, \quad \|v(0)\| < \eta \varepsilon^{k-2} \quad and \quad \|\nabla v(0)\| < \eta \varepsilon^{k-4}.$$

In addition, let

$$\eta_0 \le C\varepsilon^{2k-2+\tilde k} \quad and \quad \eta_2 \le C\varepsilon^{2k-6},$$

be the assumptions for the noise, for $\tilde k > 0$ small. Then the probability $\mathbb{P}(\tau_\varepsilon < T_\varepsilon)$ is smaller than any power of ε, as $\varepsilon \to 0$. So, for very large times and with high probability, the solution stays close to the manifold in the H^1_ε-norm.

Proof. Obviously, we have

$$\mathbb{P}(\tau_\varepsilon < T_\varepsilon) \le \mathbb{P}(\|\nabla v(\tau_\varepsilon)\| \ge C_0 \varepsilon^{k-4-\tilde\kappa}) + \mathbb{P}(\|v(\tau_\varepsilon)\| \ge C_0 \varepsilon^{k-2}).$$

Using now Theorem 3.4 for any large $\ell > 1$ and Chebychev's inequality, we obtain

$$\mathbb{P}(\tau_\varepsilon < T_\varepsilon) \le C\varepsilon^{-2p(k-4-\tilde\kappa)}\mathbb{E}\|\nabla v(\tau_\varepsilon)\|^{2p} + C\varepsilon^\ell$$

$$\le C\varepsilon^{2p\tilde\kappa}\varepsilon^{-2}[\varepsilon^{-2} + T_\varepsilon] + C\varepsilon^\ell,$$

where Lemma 3.10 was applied. Choosing $p \gg 1/2\tilde\kappa$ yields the result. $\qquad\square$

Let us rephrase Theorem 3.11 slightly:

Corollary 3.12. *Under the assumptions of Theorem 3.11 and if $w(0) \in \mathcal{N}_{L^2}^{\eta\varepsilon^{k-2}}$ and $\nabla w(0) \in \mathcal{N}_{L^2}^{\eta\varepsilon^{k-4}}$ for any $\eta > 0$, then for any sufficiently large $C > \eta$ and any $q \in \mathbb{N}$ there exists a constant $C_q > 0$ such that*

$$\mathbb{P}\Big(w(t) \in \mathcal{N}_{H^1_\varepsilon}^{C\varepsilon^{k-3-\tilde k}} \; for \; all \; t \in [0, \varepsilon^{-q}]\Big) \ge 1 - C_q \varepsilon^q.$$

4. Estimates

In this section, we will present the estimates of $(\partial_\xi^2 u, \partial_\xi u)$ and of $\partial_\xi u, \partial_\xi^2 u, \partial_\xi^3 u$ in various norms, used throughout the previous sections. Recall that Γ was the small semicircle where $u^\xi = 0$, and apart from a small neighborhood, $u^\xi \approx 1$ inside and $u^\xi \approx -1$ outside. Our proof extends certain lower order results of [2] derived for the deterministic problem. First, we estimate the scalar product between $\partial_\xi^2 u$ and $\partial_\xi u$, which can be bounded much better than via Cauchy-Schwarz.

Theorem 4.1. *The following estimate holds true*

(4.1) $$(\partial_\xi^2 u, \partial_\xi u) = \mathcal{O}(\varepsilon^{-1}).$$

Proof. We may consider the case that Ω_δ is a normal graph over the unit sphere, so that any $x \in \Omega_\delta$ is represented by $x = (r\cos(\theta), r\sin(\theta))$ for any $0 \le \theta < 2\pi$ and any $0 \le r < R(\theta)$, where $R(\theta)$ is the distance of the point of the boundary $\partial\Omega_\delta$ from the origin, at the angle θ. This is not restrictive since, as we shall see, our integral vanishes outside a neighborhood of a point on $\partial\Omega_\delta$. The coordinate r here should not be confused with the local coordinate near Γ. Therefore, we have

$$(\partial_\xi^2 u, \partial_\xi u) = \int_{\Omega_\delta} \partial_\xi^2 u \partial_\xi u dx = \int_0^{2\pi} \int_0^{R(\theta)} \frac{1}{2}\frac{d}{d\xi}(\partial_\xi u)^2 r dr\, d\theta.$$

Observe that if $\partial\Omega_\delta = (a(\theta), b(\theta)) = (R(\theta)\cos(\theta), R(\theta)\sin(\theta))$ for $t \in [0, 2\pi]$ then, the arc-length parameter ξ of $\partial\Omega_\delta$ is given by

$$\xi(\theta) = \int_0^\theta (a'(t)^2 + b'(t)^2)^{1/2}dt.$$

Therefore, $\xi_\theta = (R'(\theta)^2 + R(\theta)^2)^{1/2}$ and thus

$$d\xi = \xi_\theta d\theta = (R'(\theta)^2 + R(\theta)^2)^{1/2}d\theta.$$

Setting $L := |\partial\Omega_\delta|$ and using that the boundary is a closed curve, it follows for $\hat{R}(\xi) = R(\theta)$ that

$$
\begin{aligned}
(\partial_\xi^2 u, \partial_\xi(u)) &= \int_0^L \int_0^{\hat{R}(\xi)} \frac{1}{2}\frac{d}{d\xi}(\partial_\xi(u))^2 r dr (R'(\theta)^2 + R(\theta)^2)^{-1/2} d\xi \\
&= \int_0^L \frac{1}{2}\frac{d}{d\xi}\Big(\int_0^{\hat{R}(\xi)}(\partial_\xi u)^2 r dr\Big)(R'(\theta)^2 + R(\theta)^2)^{-1/2}d\xi \\
&\quad - \int_0^L \frac{1}{2}\hat{R}_\xi(\xi)(\partial_\xi(u)(\hat{R},\xi))^2\hat{R}(\xi)(R'(\theta)^2 + R(\theta)^2)^{-1/2}d\xi \\
&= \Big[\frac{1}{2}\Big(\int_0^{\hat{R}(\xi)}(\partial_\xi u)^2 r dr\Big)(R'(\theta)^2 + R(\theta)^2)^{-1/2}\Big]_0^L \\
&\quad - \int_0^L \frac{1}{2}\Big(\int_0^{\hat{R}(\xi)}(\partial_\xi u)^2 r dr\Big)\partial_\xi\Big((R'(\theta)^2 + R(\theta)^2)^{-1/2}\Big)d\xi \\
&\quad - \int_0^L \frac{1}{2}\hat{R}_\xi(\xi)(\partial_\xi(u)(\hat{R},\xi))^2\hat{R}(\xi)(R'(\theta)^2 + R(\theta)^2)^{-1/2}d\xi \\
&= 0 - \int_0^L \frac{1}{2}\Big(\int_0^{\hat{R}(\xi)}(\partial_\xi u)^2 r dr\Big)\partial_\xi\Big((R'(\theta)^2 + R(\theta)^2)^{-1/2}\Big)d\xi \\
&\quad - \int_0^L \frac{1}{2}\hat{R}_\xi(\xi)(\partial_\xi(u)(\hat{R},\xi))^2\hat{R}(\xi)(R'(\theta)^2 + R(\theta)^2)^{-1/2}d\xi.
\end{aligned}
$$

Note that the construction of u in [2] shows that $\partial_\xi u$ vanishes outside a neighborhood of Γ of width $2\varepsilon\log^2\varepsilon$, which allows us to use the representation of $\partial\Omega_\delta$.

We will use the notation $M \sim \mathcal{O}(\delta^s)$ to mean that there are constants $C_1, C_2 > 0$ such that $C_1\delta^s \le M \le C_2\delta^s$ for δ sufficiently small.

Returning to the original set $\Omega = \delta\Omega_\delta$ with arc-length parameter for its boundary $\tilde{\xi} = \delta\xi$. Moreover, we frequently use indices to denote derivatives to stay close to the notation of [2]. Then for the distance, $\tilde{R}(\tilde{\xi})$, of the boundary, $\partial\Omega$, from the origin, we have

$$\tilde{R}(\tilde{\xi}) = \delta\hat{R}(\xi) \sim \mathcal{O}(1), \quad \tilde{R}_{\tilde{\xi}}(\tilde{\xi}) \sim \mathcal{O}(1), \quad \tilde{R}_{\tilde{\xi}\tilde{\xi}}(\tilde{\xi}) \sim \mathcal{O}(1).$$

So

$$
\begin{aligned}
\hat{R}(\xi) &\sim \mathcal{O}(\delta^{-1}), \\
\hat{R}_\xi(\xi) &\sim \mathcal{O}(\delta^{-1})\tilde{R}_{\tilde{\xi}}(\tilde{\xi})\tilde{\xi}_\xi \sim \mathcal{O}(\delta^{-1})\mathcal{O}(1)\delta \sim \mathcal{O}(1), \\
\hat{R}_{\xi\xi}(\xi) &\sim \mathcal{O}(\delta^{-1})\tilde{R}_{\tilde{\xi}\tilde{\xi}}(\tilde{\xi})\tilde{\xi}_\xi^2 + \mathcal{O}(\delta^{-1})\tilde{R}_{\tilde{\xi}}(\tilde{\xi})\tilde{\xi}_{\xi\xi} \sim \mathcal{O}(\delta^{-1})\mathcal{O}(1)\delta^2 \sim \mathcal{O}(\delta).
\end{aligned}
$$

If $\bar{R}(\theta)$ is the distance from the origin to the boundary $\partial\Omega$ at angle θ, we have

$$
\begin{aligned}
\hat{R}(\xi) &= R(\theta), \\
\delta\hat{R}(\xi) &= \tilde{R}(\tilde{\xi}) = \bar{R}(\theta) \sim \mathcal{O}(1), \\
\bar{R}_\theta(\theta) &\sim \mathcal{O}(1), \\
\bar{R}_{\theta\theta}(\theta) &\sim \mathcal{O}(1),
\end{aligned}
$$

and thus

$$\bar{R}(\theta) = \delta R(\theta) \sim \mathcal{O}(1),$$
$$R(\theta) \sim \mathcal{O}(\delta^{-1}),$$
$$R_\theta(\theta) = \delta^{-1}\bar{R}_\theta(\theta) \sim \mathcal{O}(\delta^{-1}),$$
$$R_{\theta\theta}(\theta) = \delta^{-1}\bar{R}_{\theta\theta}(\theta) \sim \mathcal{O}(\delta^{-1}).$$

But $R(\theta) = \hat{R}(\xi)$ so $R_\theta(\theta) = \hat{R}_\xi(\xi)\xi_\theta = \hat{R}_\xi(\xi)(R_\theta^2 + R^2)^{1/2}$. So

$$\partial_\xi\left((R'(\theta)^2 + R(\theta)^2)^{-1/2}\right) = \partial_\xi\left(\frac{\hat{R}_\xi(\xi)}{R_\theta(\theta)}\right) = \partial_\xi\frac{\hat{R}_{\xi\xi}R_\theta - \hat{R}R_{\theta\theta}\theta_\xi}{R_\theta^2}.$$

In Ω we have

$$\frac{\partial\theta}{\partial\tilde{\xi}} \sim \mathcal{O}(1) \quad \text{and} \quad \tilde{\xi} = \delta\xi .$$

Thus

$$\theta_\xi \sim \mathcal{O}(\delta).$$

So we get

$$\partial_\xi\left((R'(\theta)^2 + R(\theta)^2)^{-1/2}\right) \sim \frac{\mathcal{O}(\delta)\mathcal{O}(\delta^{-1}) - \mathcal{O}(1)\mathcal{O}(\delta^{-1})\mathcal{O}(\delta)}{\delta^{-2}} \sim \mathcal{O}(\delta^2).$$

But (cf. [2], p. 294) in a neighborhood of Γ of width $\varepsilon\log^2\varepsilon$, using local coordinates (r, s) with r being signed distance from Γ and s being arclength along Γ,

$$(4.2) \qquad \partial_\xi u = -\frac{1}{\varepsilon}\left\{\cos(\pi s/|\Gamma|)\dot{U}(\frac{r}{\varepsilon}) + \mathcal{O}(\varepsilon)\right\} = \mathcal{O}(\varepsilon^{-1}),$$

where U is the heteroclinic solution to the one-dimensional problem connecting ± 1, given by (1.7).

The smooth cut-off function maintains this estimate on the support of $\partial_\xi u$, being a neighborhood of Γ of width $2\varepsilon\log^2\varepsilon$. Furthermore, $\dot{U}$ decays exponentially, as shown in the classical work of Fife and McLeod [18].

So we obtain

$$|(\partial_\xi^2 u, \partial_\xi u)| = \int_0^L \frac{1}{2}\left(\int_0^{\hat{R}(\xi)} |\partial_\xi(u)|^2 r dr\right)\left|\left((R'(\theta)^2 + R(\theta)^2)^{-1/2}\right)_\xi\right| d\xi$$
$$+ \int_0^L \frac{1}{2}|\hat{R}_\xi(\xi)||\partial_\xi(u)(\hat{R},\xi)|^2\hat{R}(\xi)(R'(\theta)^2 + R(\theta)^2)^{-1/2}d\xi$$
$$= \mathcal{O}(\varepsilon^{-1})\mathcal{O}(\delta^2) + \mathcal{O}(\varepsilon^{-1}) + \text{ higher order terms}$$
$$= \mathcal{O}(\varepsilon^{-1}).$$

$\square$

We now give estimates for various derivatives of u with respect to ξ. Some of these are already given in [2] but are repeated here for completeness of presentation.

Theorem 4.2. *It holds that*

$$(4.3) \qquad \|\partial_\xi u\|_{L^1} = \mathcal{O}(1), \qquad \|\partial_\xi u\| = \mathcal{O}(\varepsilon^{-\frac{1}{2}}),$$

$$(4.4) \qquad \|\partial_\xi^2 u\| = \mathcal{O}(\varepsilon^{-\frac{3}{2}}), \quad \text{and} \quad \|\partial_\xi^3 u\| = \mathcal{O}(\varepsilon^{-\frac{5}{2}}).$$

Proof. Following [2], p. 294, in a neighborhood of Γ of width $\varepsilon \log^2 \varepsilon$,

$$(4.5) \qquad \partial_\xi u = -\frac{1}{\varepsilon}\Big\{ \cos(\pi s/|\Gamma|)\dot{U}(R) + \mathcal{O}(\varepsilon)\Big\}.$$

where $R = \frac{r}{\varepsilon}$, and (r, s) are local coordinates in this neighborhood, with r being signed distance and s being arc length along Γ. Recall that U is defined as the heteroclinic solution to (1.7). See also (4.2).

The function u is the sum of two terms written in local coordinates, an interior expansion u^I and a corner layer expansion $u^{B\pm}$:

$$u = u^I + u^B,$$

where (from [2] pp. 251, 254)

$$(4.6) \qquad u^I = U(R) + \varepsilon \sum_{j \geq 0} \varepsilon^j u_j^I(R, \cdots),$$

and

$$(4.7) \qquad u^B = \sum_{j \geq 1} \varepsilon^j u_j^{B\pm}(R, \cdots).$$

Let D_a^k denote the k partial derivative with respect to variable a. For the interior layer expansion it holds that (see [2] p. 252)

$$(4.8) \qquad |D_R^m D_s^n D_\xi^l u_j^I| = \mathcal{O}(1),$$

for any integer $m, n, l \geq 0$. Since the interior expansion in local coordinates has a smooth extension to the whole domain (by (4.6), this means that u_j^I have smooth extensions to the whole domain), then (4.8), which is true for u_j^I in local coordinates, is true for their smooth extensions too. On the other hand for a given ξ, the construction of $u^{B\pm}$ in local coordinates (see [2] p. 241) permits as to derive that $D_R^m D_s^n D_\xi^l u_j^{B\pm}$ are bounded uniformly in ε, i.e.,

$$(4.9) \qquad |D_R^m D_s^n D_\xi^l u_j^{B\pm}| = \mathcal{O}(1),$$

for any integer $m, n, l \geq 0$.

Furthermore, u is smoothly extended to the whole domain, cf. p. 257 of [2], and thus, is given as

$$u = \tilde{u^I} + \tilde{u^B},$$

for $\tilde{u^I}$ and $\tilde{u^B}$ suitable modifications of u^I and u^B respectively. More specifically, for the cut-off function $\zeta \in C^\infty$ with $\zeta(s) = 1$ if $s > 1$, $\zeta(s) = 0$ if $s < 0$ and $s\zeta'(s) > 0$ on $\mathbb{R}$, $\tilde{u^I}$ is given as

$$(4.10) \qquad \begin{aligned} \tilde{u^I} :=&(1 - \zeta^+ - \zeta^-)\Big(U(R) + \varepsilon \sum_{j \geq 0} \varepsilon^j u_j^I(R, \cdots)\Big) + \zeta^+ u^+(\xi) + \zeta^- u^-(\xi) \\ =&(1 - \zeta^+ - \zeta^-)u^I + \zeta^+ u^+(\xi) + \zeta^- u^-(\xi), \end{aligned}$$

where

$$\zeta^\pm = \zeta(\pm\tfrac{r(\xi,\cdots)}{\varepsilon \ln \varepsilon^2} - 1), \quad \text{and} \quad u^\pm = \pm 1 + \varepsilon \sum_{j \geq 0} \varepsilon^j u_j^\pm.$$

Note that $u_j^\pm$ are smooth and uniformly bounded in ε, and the same holds for their derivatives of any order. Easily, by taking the derivative in ξ we obtain

$$\begin{aligned} \partial_\xi(\tilde{u^I}) =&\partial_\xi(1 - \zeta^+ - \zeta^-)u^I + (1 - \zeta^+ - \zeta^-)\partial_\xi(u^I) + \partial_\xi(\zeta^+)u^+ + \zeta^+ \partial_\xi(u^+) + \partial_\xi(\zeta^-)u^- + \zeta^- \partial_\xi(u^-) \\ =&\frac{1}{\varepsilon \ln \varepsilon^2}\mathcal{O}(u^I) + \frac{1}{\varepsilon}\mathcal{O}(u^I) + \frac{1}{\varepsilon \ln \varepsilon^2}\mathcal{O}(1) + \mathcal{O}(u^I) \sim \mathcal{O}(\partial_\xi(u^I)), \end{aligned}$$

where we used that $\frac{1}{\varepsilon}u^I = \mathcal{O}(\partial_\xi u^I)$ (to be proved in the sequel). So, the order of $\partial_\xi(\tilde{u^I})$ is this of $\partial_\xi(u^I)$, while by induction the same is happening for the derivatives of higher order i.e., $\partial_\xi^m(\tilde{u^I}) \sim \mathcal{O}(\partial_\xi^m(u^I))$ for $m = 1, 2, 3$. In the sequel, we compute $\mathcal{O}(\partial_\xi^m(u^I))$ in detail.

An analogous construction for $\tilde{u^B}$ will give the same result, which means that $\partial_\xi^m(\tilde{u^B}) \sim \mathcal{O}(\partial_\xi^m(u^B))$, so it is sufficient if we estimate $\partial_\xi^m(u^B)$ for $m = 1, 2, 3$.

In addition, $D_\xi^l s$, $D_\xi^l r$ exist for $l \geq 0$, and is uniformly bounded for any ε (for convenience cf. in [2] p. 294, the formulas $r_\xi = -\cos(\pi s/|\Gamma|) + \mathcal{O}(\delta)$ and $s_\xi = \mathcal{O}(\varepsilon + h + |r|)$) so,

$$(4.11) \qquad |D_\xi^l s|, \;\; |D_\xi^l r| < C,$$

for any $l \geq 0$ uniformly in ε.

But, cf. [2] p. 258

$$\frac{\partial x}{\partial(R, s)} = \varepsilon(1 + \varepsilon R\mathcal{K}),$$

thus, for any $g = g(R, \cdots)$ of compact support, it holds that (as in p. 258 of [2])

$$\int_{\Omega_\delta} g(R, \cdots)dx = \varepsilon \int_{-\infty}^{+\infty} dR \int_{S^-(R,\xi,\varepsilon)}^{S^+(R,\xi,\varepsilon)} (1 + \varepsilon R\mathcal{K})g(R, \cdots)ds,$$

for the resulting S^- and S^+, where the change of variables from local to global coordinates is valid. Since Ω_δ is bounded and $R = \frac{r}{\varepsilon}$ then if $g(R, \cdots)$ is bounded uniformly for any ε for any $R \in \mathbb{R}$ then

$$(4.12) \qquad \int_{\Omega_\delta} g(R, \cdots)dx = \mathcal{O}(1).$$

Using the above considerations then we can derive the estimates of $\|\partial_\xi u\|_{L^1}$, $\|\partial_\xi u\|$, $\|\partial_\xi^2 u\|$ and $\|\partial_\xi^3 u\|$, where $\partial_\xi u = \frac{\partial}{\partial \xi}(u(r, s, \xi, \varepsilon))$ and so on.

By using (4.5) we obtain

$$\begin{aligned} \int_{\Omega_\delta} |\partial_\xi(u)|dx =& \mathcal{O}\Big(\varepsilon \int_{-\infty}^{+\infty} dR \int_{S^-(R,\xi,\varepsilon)}^{S^+(R,\xi,\varepsilon)} (1 + \varepsilon R\mathcal{K})|\partial_\xi u|ds\Big) \\ =& \mathcal{O}\Big(\varepsilon \int_{-\infty}^{+\infty} dR \int_{S^-(R,\xi,\varepsilon)}^{S^+(R,\xi,\varepsilon)} (1 + \varepsilon R\mathcal{K})\frac{1}{\varepsilon}|\cos(\pi s/|\Gamma|)||\dot{U}(R)|ds\Big) \\ &+ \mathcal{O}\Big(\varepsilon \int_a^b dR \int_{S^-(R,\xi,\varepsilon)}^{S^+(R,\xi,\varepsilon)} (1 + \varepsilon R\mathcal{K})\frac{1}{\varepsilon}\mathcal{O}(\varepsilon)ds\Big) \\ =& \mathcal{O}(1), \end{aligned}$$

i.e., we proved the left equation in (4.3). Here, we used that $\int_{-\infty}^{+\infty} |S^+(R, \xi, \varepsilon) - S^-(R, \xi, \varepsilon)||\dot{U}(R)|dR$ is bounded uniformly in ε and that the order for $\partial_\xi u$ is given by the order of the local representation $\partial_\xi(u^I + u^B)$, estimated by (4.5).

Further it follows that

$$\begin{aligned} \int_{\Omega_\delta} |\partial_\xi(u)|^2 dx &= \mathcal{O}\Big(\varepsilon \int_{-\infty}^{+\infty} dR \int_{S^-(R,\xi,\varepsilon)}^{S^+(R,\xi,\varepsilon)} (1 + \varepsilon R\mathcal{K})|\partial_\xi u|^2 ds\Big) \\ &= \mathcal{O}\Big(\varepsilon \int_{-\infty}^{+\infty} dR \int_{S^-(R,\xi,\varepsilon)}^{S^+(R,\xi,\varepsilon)} (1 + \varepsilon R\mathcal{K})\frac{1}{\varepsilon^2}|\cos(\pi s/|\Gamma|)|^2|\dot{U}(R)|^2 ds + C\Big) = \mathcal{O}(\varepsilon^{-1}), \end{aligned}$$

so, we get the right equality in (4.3). Here, we used that $\int_{-\infty}^{+\infty} |S^+(R, \xi, \varepsilon) - S^-(R, \xi, \varepsilon)||\dot{U}(R)|^2 dR$ is bounded uniformly in ε.

Obviously, since

$$\partial_\xi u = u_r r_\xi + u_s s_\xi + u_\xi,$$

it holds that

$$\partial_\xi^2 u = (u_{rr}r_\xi + u_{rs}s_\xi + u_{r\xi}\xi_\xi)r_\xi + u_r r_{\xi\xi}$$
$$+ (u_{sr}r_\xi + u_{ss}s_\xi + u_{s\xi}\xi_\xi)s_\xi + u_s s_{\xi\xi}$$
$$+ u_{\xi r}r_\xi + u_{\xi s}s_\xi + u_{\xi\xi}\xi_\xi.$$

Hence, the worst term is u_{rr} multiplied by a uniformly (in ε) bounded quantity. More specifically, using (4.6), (4.7), (4.8), (4.9) and (4.11), we observe that any derivative in r gives ε^{-1} since $R = \frac{r}{\varepsilon}$, while all the other derivatives of s, r are uniformly bounded in ε and the same is true for the derivatives of u in s, ξ. Further, considering the term $\partial_\xi^3(u)$, the chain rule analogously gives that the worst order is given by u_{rrr}.

In details, (4.6) and (4.8) give for $m, n, l \geq 0$

(4.13)
$$D_r^m D_s^n D_\xi^l u^I = D_r U(R) + \varepsilon \varepsilon^{-m} \sum_{j \geq 0} \varepsilon^j D_R^m D_s^n D_\xi^l u_j^I$$
$$\leq \varepsilon^{-m} D_R^m U(R) + C \varepsilon \varepsilon^{-m},$$

while by (4.7) and (4.9) we obtain

(4.14)
$$D_r^m D_s^n D_\xi^l u^B = \varepsilon^{-m} \sum_{j \geq 1} \varepsilon^j D_R^m D_s^n D_\xi^l u_j^{B\pm}$$
$$\leq C\varepsilon^{-m} \sum_{j \geq 1} \varepsilon^j \leq C\varepsilon^{-m}\varepsilon.$$

Since $u_{rr} = u_{rr}^I + u_{rr}^B$, then taking $m = 2$ we get using (4.13) and (4.14), and the regularity of the second derivative of the heteroclinic,

$$\int_{\Omega_\delta} |\partial_\xi^2 u|^2 dx \leq C\varepsilon \int_{-\infty}^{+\infty} dR \int_{S^-}^{S^+} (1 + \varepsilon R\mathcal{K})|u_{rr}^I|^2 ds + C\varepsilon \int_{-\infty}^{+\infty} dR \int_{S^-}^{S^+} (1 + \varepsilon R\mathcal{K})|u_{rr}^B|^2 ds$$

$$\leq C\varepsilon \int_{-\infty}^{+\infty} \varepsilon^{-2\cdot 2} dR \int_{S^-}^{S^+} (1 + \varepsilon R\mathcal{K})|D_R^2 U(R)|^2 ds + C\varepsilon^2 \varepsilon^{-2\cdot 2} + C\varepsilon^{-2\cdot 2}\varepsilon^2$$

$$\leq C\varepsilon^{-3},$$

and thus, in accordance to [2] p. 297, we proved the left statement in (4.4), i.e.,

$$\|\partial_\xi^2 u\| = \mathcal{O}(\varepsilon^{-\frac{3}{2}}).$$

For the previous relation we used that $\int_{-\infty}^{+\infty} |S^+(R, \xi, \varepsilon) - S^-(R, \xi, \varepsilon)||D_R^2 U(R)|^2 dR$ is bounded uniformly in ε.

The analogous computation by taking $m = 3$ in (4.13) and (4.14), since the third derivative of the heteroclinic is regular, yields

$$\int_{\Omega_\delta} |\partial_\xi^3 u|^2 dx \leq C\varepsilon \int_{-\infty}^{+\infty} dR \int_{S^-}^{S^+} (1 + \varepsilon R\mathcal{K})|u_{rrr}^I|^2 ds + C\varepsilon \int_{-\infty}^{+\infty} dR \int_{S^-}^{S^+} (1 + \varepsilon R\mathcal{K})|u_{rrr}^B|^2 ds$$

$$\leq C\varepsilon \int_{-\infty}^{+\infty} \varepsilon^{-2\cdot 3} dR \int_{S^-}^{S^+} (1 + \varepsilon R\mathcal{K})|D_R^3 U(R)|^2 ds + C\varepsilon^2 \varepsilon^{-2\cdot 3} + C\varepsilon^{-2\cdot 3}\varepsilon^2 \leq C\varepsilon^{-5}.$$

Thus, we proved the right inequality in (4.4), i.e., that

$$\|\partial_\xi^3 u\| = \mathcal{O}(\varepsilon^{-\frac{5}{2}}).$$

Here, we used that $\int_{-\infty}^{+\infty} |S^+(R, \xi, \varepsilon) - S^-(R, \xi, \varepsilon)||D_R^3 U(R)|^2 dR$ is bounded uniformly in ε.   □

Remark 4.3. *We note that by [2, p. 297] also $\|\partial_\xi u\| \geq C\varepsilon^{-\frac{1}{2}}$. Thus there exist $0 < C_1 \leq C_2$ such that*

$$C_1\varepsilon^{-\frac{1}{2}} \leq \|\partial_\xi u\| \leq C_2\varepsilon^{-\frac{1}{2}}.$$

Remark 4.4. *We used regularity for the heteroclinic $U(R)$ up to derivatives of third order, see* [18].

Remark 4.5. *From the proof of Theorem 4.2 it is obvious that for any $k \geq 1$*

$$\|\partial_x^k u\| \sim \|\partial_\xi^k(u)\|, \quad and \quad \|\partial_x^k u\|_\infty \sim \|\partial_\xi^k(u)\|_\infty.$$

Furthermore, by [2]*, it holds that*

$$\|\partial_\xi u\|_\infty \leq C\varepsilon^{-1}, \quad and \ thus \|\nabla u\|_\infty \leq C\varepsilon^{-1}.$$

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