Stability Estimates for Resolvents, Eigenvalues, and Eigenfunctions of Elliptic Operators on Variable Domains

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Dedicated to Vladimir Maz'ya

Abstract We consider general second order uniformly elliptic operators subject to homogeneous boundary conditions on open sets $\phi(\Omega)$ parametrized by Lipschitz homeomorphisms ϕ defined on a fixed reference domain Ω. For two open sets $\phi(\Omega)$ and $\widetilde{\phi}(\Omega)$ we estimate the variation of resolvents, eigenvalues, and eigenfunctions via the Sobolev norm $\|\widetilde{\phi} - \phi\|_{W^{1,p}(\Omega)}$ for finite values of p, under natural summability conditions on eigenfunctions and their gradients. We prove that such conditions are satisfied for a wide class of operators and open sets, including open sets with Lipschitz continuous boundaries. We apply these estimates to control the variation of the eigenvalues and eigenfunctions via the measure of the symmetric difference of the open sets. We also discuss an application to the stability of solutions to the Poisson problem.

1 Introduction

This paper is devoted to the proof of stability estimates for the nonnegative selfadjoint operator

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$$Lu = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\Big(A_{ij}(x)\frac{\partial u}{\partial x_j}\Big), \quad x \in \Omega, \tag{1.1}$$

subject to homogeneous boundary conditions, upon variation of the open set Ω in $\mathbf{R}^N$. Here, A_{ij} are fixed bounded measurable real-valued functions defined in $\mathbf{R}^N$ satisfying $A_{ij} = A_{ji}$ and a uniform ellipticity condition.

The focus is on explicit quantitative estimates for the variation of the resolvents, eigenvalues, and eigenfunctions of L on a class of open sets diffeomorphic to Ω.

In the first part of the paper, we consider two diffeomorphisms ϕ and $\widetilde{\phi}$ from Ω onto $\phi(\Omega)$ and $\widetilde{\phi}(\Omega)$ respectively, and we compare the resolvents, eigenvalues, and eigenfunctions of L on the open set $\widetilde{\phi}(\Omega)$ with those of L on $\phi(\Omega)$. To compare the operators, defined on different domains $\varphi(\Omega)$ and $\widetilde{\varphi}(\Omega)$, we compare their pull-backs to the same domain Ω (cf. Section 2). The main goal is to provide stability estimates via $\|\widetilde{\phi} - \phi\|_{W^{1,p}(\Omega)}$ for finite values of p. These estimates are applied in the last part of the paper where we take $\phi = Id$ and, given a deformation $\widetilde{\Omega}$ of Ω, construct a special diffeomorphism $\widetilde{\phi}$ representing $\widetilde{\Omega}$ in the form $\widetilde{\Omega} = \widetilde{\phi}(\Omega)$, and obtain stability estimates in terms of the Lebesgue measure $|\Omega \vartriangle \widetilde{\Omega}|$ of the symmetric difference of Ω and $\widetilde{\Omega}$.

Our method allows us to treat the general case of the mixed homogeneous Dirichlet–Neumann boundary conditions

$$u = 0 \quad \text{on } \Gamma \quad \text{and} \quad \sum_{i,j=1}^{N} A_{ij}\frac{\partial u}{\partial x_j}\nu_i = 0 \quad \text{on } \partial\Omega \setminus \Gamma, \tag{1.2}$$

where $\Gamma \subset \partial\Omega$ and ν denotes the exterior unit normal to $\partial\Omega$. To our knowledge, our results are new also for the Dirichlet and for Neumann boundary conditions.

There is vast literature concerning domain perturbation problems (cf., for example, the extensive monograph [14]). The problem of finding explicit quantitative estimates for the variation of the eigenvalues of elliptic operators has been considered in [3]–[6], [8]–[10], [16, 17, 21] (cf. [7] for a survey on the results of these papers). However, less attention has been devoted to the problem of finding explicit estimates for the variation of the eigenfunctions. With regard to this, we mention the estimate in [21] concerning the first eigenfunction of the Dirichlet Laplacian and the estimates in [16, 17] concerning the variation of the eigenprojections of the Dirichlet and Neumann Laplacian. In particular, in [16, 17], the variation of the eigenvalues and eigenprojections of the Laplace operator was estimated via $\|\nabla\widetilde{\phi} - \nabla\phi\|_{L^\infty(\Omega)}$ under minimal assumptions on the regularity of Ω, ϕ and $\widetilde{\phi}$.

In all the cited papers and in this paper, perturbations of domains may be considered as in some sense regular perturbations. There is also vast literature concerning a wide range of perturbation problems of different type which may be characterized as singular perturbations (which are out of scope of this paper). Typically, formulations of such problems involve a small parameter ε and the problem degenerates in that sense or other as $\varepsilon \to 0$. Say, the domain may contain small holes, or boundaries which may include blunted angles, cones and edges, narrow slits, thin bridges etc, or the limit region may consist of subsets of different dimension, or it could be a homogenization problem. V.G. Maz'ya and his co-authors V.A. Kozlov, A.B. Movchan, S.A. Nazarov, B.A. Plamenevskii and others developed the powerful asymptotic theory which allowed to find asymptotic expansions of solutions for all aforementioned problems and can be applied in many other cases (cf., for example, [15, 18]).

In this paper, we consider the same class of transformations $\phi, \widetilde{\phi}$ as in [16, 17] ($\phi, \widetilde{\phi}$ are bi-Lipschitz homeomorphisms) and, making stronger regularity assumptions on $\phi(\Omega)$ and $\widetilde{\phi}(\Omega)$, we estimate the variation of the resolvents, eigenvalues, eigenprojections, and eigenfunctions of L via the measure of vicinity

$$\delta_p(\phi, \widetilde{\phi}) := \|\nabla\widetilde{\phi} - \nabla\phi\|_{L^p(\Omega)} + \|A \circ \widetilde{\phi} - A \circ \phi\|_{L^p(\Omega)} \tag{1.3}$$

for any $p \in]p_0, \infty]$, where $A = (A_{ij})_{i,j=1,\dots,N}$ is the matrix of coefficients.

Here, $p_0 \geqslant 2$ is a constant depending on the regularity assumptions. The best p_0 that we obtain is $p_0 = N$ which corresponds to the highest degree of regularity (cf. Remark 4.8), while the case $p_0 = \infty$ corresponds to the lowest degree of regularity in which case only the exponent $p = \infty$ can be considered. The regularity assumptions are expressed in terms of summability properties of the eigenfunctions and their gradients, see Definition 4.2. Note that if the coefficients A_{ij} of the operator L are Lipschitz continuous, then $\delta_p(\phi, \widetilde{\phi})$ does not exceed a constant independent of $\phi, \widetilde{\phi}$ multiplied by the Sobolev norm $\|\phi - \widetilde{\phi}\|_{W^{1,p}(\Omega)}$. Moreover, if the coefficients A_{ij} are constant, then the second summand on the right-hand side of (1.3) vanishes.

More precisely, we prove stability estimates for the resolvents in the Schatten classes (Theorem 4.6), stability estimates for eigenvalues (Theorem 4.11), eigenprojections (Theorem 5.2), and eigenfunctions (Theorem 5.6). In Appendix, we also consider an application to the Poisson problem (we refer to [23] for stability estimates for the solutions to the Poisson problem in the case of the Dirichlet boundary conditions obtained by a different approach). To prove the resolvent stability estimates in the Schatten classes, we follow the method developed in [1, 2].

In Section 7, we apply our general results and, for a given deformation $\widetilde{\Omega}$ of Ω, we prove stability estimates in terms of $|\Omega \vartriangle \widetilde{\Omega}|$. This is done in two cases: the case where $\widetilde{\Omega}$ is obtained by a localized deformation of the boundary of Ω and the case where $\widetilde{\Omega}$ is a deformation of Ω along its normals. We also require that the deformation $\widetilde{\Gamma}$ of Γ is induced by the deformation of Ω (cf. conditions (7.3) and (7.14)). In these cases, similarly to [5], we can construct special bi-Lipschitz transformations $\widetilde{\phi} : \Omega \to \widetilde{\Omega}$ such that $\widetilde{\phi}(\Gamma) = \widetilde{\Gamma}$ and

$$\|\nabla\widetilde{\phi} - I\|_{L^p(\Omega)} \leqslant c|\Omega \vartriangle \widetilde{\Omega}|^{1/p}, \tag{1.4}$$

where $c > 0$ is independent of Ω and $\widetilde{\Omega}$. Observe that using finite values of p is essential, since in the case $p = \infty$ the exponent on the right-hand side of (1.4) vanishes.

"Let us describe these results in the regular case in which Ω and $\widetilde{\Omega}$ are of class $C^{1,1}$, Γ, $\widetilde{\Gamma}$ are connected components of the corresponding boundaries, and the coefficients A_{ij} are Lipschitz continuous. In Theorems 7.3 and 7.6, we prove that for any $r > N$ there exists a constant $c_1 > 0$ such that

$$\left(\sum_{n=1}^{\infty}\left|\frac{1}{\lambda_n + 1} - \frac{1}{\widetilde{\lambda}_n + 1}\right|^r\right)^{1/r} \leqslant c_1|\Omega \vartriangle \widetilde{\Omega}|^{\frac{1}{r}} \tag{1.5}$$

if $|\Omega \vartriangle \widetilde{\Omega}| < c_1^{-1}$. Here, λ_n, $\widetilde{\lambda}_n$ are the eigenvalues of the operators (1.1) corresponding to the domains Ω, $\widetilde{\Omega}$ and the associated portions of the boundaries Γ, $\widetilde{\Gamma}$ respectively. Moreover, for a fixed Ω and any $r > N$ there exists $c_2 > 0$ such that if $\lambda_n = \ldots = \lambda_{n+m-1}$ is an eigenvalue of multiplicity m, then for any choice of orthonormal eigenfunctions $\widetilde{\psi}_n, \ldots, \widetilde{\psi}_{n+m-1}$ corresponding to $\widetilde{\lambda}_n, \ldots, \widetilde{\lambda}_{n+m-1}$, there exist orthonormal eigenfunctions $\psi_n, \ldots, \psi_{n+m-1}$ corresponding to $\lambda_n, \ldots, \lambda_{n+m-1}$ such that[1]

$$\|\psi_k - \widetilde{\psi}_k\|_{L^2(\Omega\cup\widetilde{\Omega})} \leqslant c_2|\Omega \vartriangle \widetilde{\Omega}|^{\frac{1}{r}} \tag{1.6}$$

for all $k = n, \ldots, n + m - 1$ provided that $|\Omega \vartriangle \widetilde{\Omega}| < c_2^{-1}$. Here, it is understood that the eigenfunctions are extended by zero outside their domains of definition.

In the general case of open sets Ω, $\widetilde{\Omega}$ with Lipschitz continuous boundaries and Γ, $\widetilde{\Gamma}$ with Lipschitz continuous boundaries in $\partial\Omega$, $\partial\widetilde{\Omega}$, our statements still hold for a possibly worse range of exponents (cf. Theorems 7.3 and 7.6).

We emphasize that, in the spirit of [16, 17], in this paper we never assume that the transformation ϕ belongs to a family of transformations ϕ_t depending analytically on one scalar parameter t, as often done in the literature (cf., for example, [14] for references). In that case, one can use proper methods of bifurcation theory in order to prove existence of branches of eigenvalues and eigenfunctions depending analytically on t. In this paper, $\widetilde{\phi}$ is an arbitrary perturbation of ϕ and this requires a totally different approach.

The paper is organized as follows. In Section 2, we describe the general setting. In Section 3, we describe our perturbation problem. In Section 4, we prove stability estimates for the resolvents and the eigenvalues. In Section 5, we prove stability estimates for the eigenprojections and eigenfunctions. In Section 6, we give sufficient conditions providing the required regularity of

[1] Note that, for a fixed Ω and variable $\widetilde{\Omega}$, one first chooses eigenfunctions in $\widetilde{\Omega}$ and then finds eigenfunctions in Ω, while the opposite is clearly not possible.

the eigenfunctions. In Section 7, we prove stability estimates via the Lebesgue measure of the symmetric difference of sets. In Appendix, we briefly discuss the Poisson problem.

2 General Setting

Let Ω be a domain, i.e., an open connected set, in $\mathbf{R}^N$ of finite measure. We consider a family of open sets $\phi(\Omega)$ in $\mathbf{R}^N$ parametrized by bi-Lipschitz homeomorphisms ϕ of Ω onto $\phi(\Omega)$. Namely, following [16], we consider the family of transformations

$$\Phi(\Omega) := \Big\{\phi \in \left(L^{1,\infty}(\Omega)\right)^N : \text{the continuous representative of } \phi \\ \text{is injective, } \operatorname*{ess\,inf}_{\Omega} |\det \nabla\phi| > 0\Big\}, \quad (2.1)$$

where $L^{1,\infty}(\Omega)$ denotes the space of the functions in $L^1_{loc}(\Omega)$ which have weak derivatives of first order in $L^\infty(\Omega)$. Note that if $\phi \in \Phi(\Omega)$, then ϕ is Lipschitz continuous with respect to the geodesic distance in Ω.

Note that if $\phi \in \Phi(\Omega)$, then $\phi(\Omega)$ is open, ϕ is a homeomorphism of Ω onto $\phi(\Omega)$, and the inverse vector-function $\phi^{(-1)}$ of ϕ belongs to $\Phi(\phi(\Omega))$. Moreover, any transformation $\phi \in \Phi(\Omega)$ allows changing variables in integrals. Accordingly, the operator C_ϕ from $L^2(\phi(\Omega))$ to $L^2(\Omega)$ defined by

$$C_\phi[v] := v \circ \phi, \quad v \in L^2(\phi(\Omega)),$$

is a linear homeomorphism which restricts to a linear homeomorphism of the space $W^{1,2}(\phi(\Omega))$ onto $W^{1,2}(\Omega)$, and of $W^{1,2}_0(\phi(\Omega))$ onto $W^{1,2}_0(\Omega)$, where $W^{1,2}(\Omega)$ denotes the standard Sobolev space and $W^{1,2}_0(\Omega)$ denotes the closure of $C^\infty_c(\Omega)$ in $W^{1,2}(\Omega)$. Furthermore, $\nabla(v \circ \phi) = \nabla v(\phi)\nabla\phi$ for all $v \in W^{1,2}(\phi(\Omega))$. Note that if $\phi \in \Phi(\Omega)$, then the measure of $\phi(\Omega)$ is finite (cf. [16] for details).

Let $A = (A_{ij})_{i,j=1,\dots,N}$ be a real symmetric matrix-valued measurable function defined on $\mathbf{R}^N$ such that for some $\theta > 0$

$$\theta^{-1}|\xi|^2 \leqslant \sum_{i,j=1}^N A_{ij}(x)\xi_i\xi_j \leqslant \theta|\xi|^2 \quad (2.2)$$

for all $x, \xi \in \mathbf{R}^N$. Note that (2.2) implies that $A_{ij} \in L^\infty(\mathbf{R}^N)$ for all $i, j = 1, \dots, N$.

Let $\phi \in \Phi(\Omega)$, and let $\mathcal{W}$ be a closed subspace of $W^{1,2}(\phi(\Omega))$ containing $W^{1,2}_0(\phi(\Omega))$. We consider the nonnegative selfadjoint operator L on $L^2(\phi(\Omega))$ canonically associated with the sesquilinear form Q_L given by

$$\mathrm{Dom}(Q_L)=\mathcal{W}\,,\quad Q_L(v_1,v_2)=\int_{\phi(\Omega)}\sum_{i,j=1}^{N}A_{ij}\frac{\partial v_1}{\partial y_i}\frac{\partial \overline{v}_2}{\partial y_j}dy\,,\quad v_1,v_2\in\mathcal{W}. \tag{2.3}$$

Recall that $v\in\mathrm{Dom}(L)$ if and only if $v\in\mathcal{W}$ and there exists $f\in L^2(\phi(\Omega))$ such that

$$Q_L(v,\psi)=\langle f,\psi\rangle_{L^2(\phi(\Omega))} \tag{2.4}$$

for all $\psi\in\mathcal{W}$, in which case $Lv=f$ (cf., for example, [11]). The choice of the space $\mathcal{W}$ determines the boundary conditions. For example, if $\mathcal{W}=W_0^{1,2}(\phi(\Omega))$ (respectively, $\mathcal{W}=W^{1,2}(\phi(\Omega))$), then the operator L satisfies the homogeneous Dirichlet (respectively, Neumann) boundary conditions.

We also consider the operator H on $L^2(\Omega)$ obtained by pulling-back L to $L^2(\Omega)$ as follows. Let $v\in W^{1,2}(\phi(\Omega))$ be given, and let $u=v\circ\phi$. Note that

$$\int_{\phi(\Omega)}|v|^2dy=\int_{\Omega}|u|^2|\det\nabla\phi|\,dx\,.$$

Moreover, a simple computation shows that

$$\int_{\phi(\Omega)}\sum_{i,j=1}^{N}A_{ij}\frac{\partial v}{\partial y_i}\frac{\partial \overline{v}}{\partial y_j}dy=\int_{\Omega}\sum_{i,j=1}^{N}a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial \overline{u}}{\partial x_j}|\det\nabla\phi|\,dx\,,$$

where $a=(a_{ij})_{i,j=1,\dots,N}$ is the symmetric matrix-valued function defined in Ω by

$$a_{ij}=\sum_{r,s=1}^{N}\left(A_{rs}\frac{\partial\phi_i^{(-1)}}{\partial y_r}\frac{\partial\phi_j^{(-1)}}{\partial y_s}\right)\circ\phi=((\nabla\phi)^{-1}A(\phi)(\nabla\phi)^{-t})_{ij}\,. \tag{2.5}$$

The operator H is defined as the nonnegative selfadjoint operator on the Hilbert space $L^2(\Omega,|\det\nabla\phi|\,dx)$ associated with the sesquilinear form Q_H given by

$$\mathrm{Dom}(Q_H)=C_\phi[\mathcal{W}],\ \ Q_H(u_1,u_2)$$
$$=\int_{\Omega}\sum_{i,j=1}^{N}a_{ij}\frac{\partial u_1}{\partial x_i}\frac{\partial \overline{u}_2}{\partial x_j}|\det\nabla\phi|\,dx,\ \ u_1,u_2\in C_\phi[\mathcal{W}].$$

Formally,

$$Hu=-\frac{1}{|\det\nabla\phi|}\sum_{i,j=1}^{N}\frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}|\det\nabla\phi|\right).$$

Alternatively, the operator H can be defined as

$$H = C_\phi L C_{\phi^{(-1)}}.$$

In particular, H and L are unitarily equivalent and the operator H has compact resolvent if and only if L has compact resolvent. (Note that the embedding $\mathcal{W} \subset L^2(\phi(\Omega))$ is compact if and only if the embedding $C_\phi[\mathcal{W}] \subset L^2(\Omega)$ is compact.)

We set $g(x) := |\det \nabla\phi(x)|$, $x \in \Omega$, and denote by $\langle \cdot, \cdot \rangle_g$ the inner product in $L^2(\Omega, g\,dx)$ and also in $(L^2(\Omega, g\,dx))^N$.

Let $T : L^2(\Omega, g\,dx) \to (L^2(\Omega, g\,dx))^N$ be the operator defined by

$$\mathrm{Dom}(T) = C_\phi[\mathcal{W}], \qquad Tu = a^{1/2}\nabla u, \quad u \in C_\phi[\mathcal{W}].$$

Then it is easy to see that

$$H = T^* T,$$

where the adjoint T^* of T is understood with respect to the inner products of $L^2(\Omega, g\,dx)$ and $(L^2(\Omega, g\,dx))^N$.

3 Perturbation of ϕ

In this paper, we study the variation of the operator L defined by (2.3) upon variation of ϕ. Our estimates depend on $\operatorname{ess\,inf}_\Omega |\det \nabla\phi|$ and $\|\nabla\phi\|_{L^\infty(\Omega)}$. Thus, in order to obtain uniform estimates it is convenient to consider the families of transformations

$$\Phi_\tau(\Omega) = \left\{\phi \in \Phi(\Omega) : \ \tau^{-1} \leqslant \operatorname*{ess\,inf}_{\Omega} |\det \nabla\phi| \ \text{ and } \ \|\nabla\phi\|_{L^\infty(\Omega)} \leqslant \tau\right\}$$

for all $\tau > 0$, as in [16]. Hereinafter, for a matrix-valued function $B(x)$, $x \in \Omega$, we set $\|B\|_{L^p(\Omega)} = \| \, |B| \, \|_{L^p(\Omega)}$, where $|B(x)|$ denotes the operator norm of the matrix $B(x)$.

Let $\phi, \widetilde{\phi} \in \Phi_\tau(\Omega)$. Let $\mathcal{W}$ and $\widetilde{\mathcal{W}}$ be closed subspaces of $W^{1,2}(\phi(\Omega))$, $W^{1,2}(\widetilde{\phi}(\Omega))$ respectively, containing $W^{1,2}_0(\phi(\Omega))$, $W^{1,2}_0(\widetilde{\phi}(\Omega))$ respectively. We use tildes to distinguish objects induced by $\widetilde{\phi}$, $\widetilde{\mathcal{W}}$ from those induced by ϕ, $\mathcal{W}$. We consider the operators L and $\widetilde{L}$ defined on $L^2(\phi(\Omega))$, $L^2(\widetilde{\phi}(\Omega))$ respectively, as in Section 2.

In order to compare L and $\widetilde{L}$, we make a "compatibility" assumption on the respective boundary conditions; namely, we assume that

$$C_\phi[\mathcal{W}] = C_{\widetilde{\phi}}[\widetilde{\mathcal{W}}]. \tag{3.1}$$

This means that $\mathrm{Dom}(Q_H) = \mathrm{Dom}(Q_{\widetilde{H}})$, a property which is important in what follows. It is clear that (3.1) holds if either L and $\widetilde{L}$ both satisfy the homogeneous Dirichlet boundary conditions or they both satisfy homogeneous Neumann boundary conditions.

We always assume that the spaces $\mathcal{W}$, $\widetilde{\mathcal{W}}$ are compactly embedded in $L^2(\phi(\Omega))$, $L^2(\widetilde{\phi}(\Omega))$ respectively, or equivalently that the space $\mathcal{V} := C_\phi[\mathcal{W}] = C_{\widetilde{\phi}}[\widetilde{\mathcal{W}}]$ is compactly embedded in $L^2(\Omega)$.

Moreover, we require that the nonzero eigenvalues λ_n of the Laplace operator associated in $L^2(\Omega)$ with the quadratic form $\int_\Omega |\nabla u|^2 dx$, $u \in \mathcal{V}$, defined on $\mathcal{V}$, satisfy the condition

$$c^* := \sum_{\lambda_n \neq 0} \lambda_n^{-\alpha} < \infty \tag{3.2}$$

for some fixed $\alpha > 0$. (This is in fact a very weak condition on the regularity of the set Ω and the associated boundary conditions.)

For brevity, we refer to assumption (A) as the following set of conditions which summarize the setting described above:

$$(\mathrm{A}): \quad \begin{cases} \phi, \widetilde{\phi} \in \Phi_\tau(\Omega), \\ \mathcal{V} := C_\phi[\mathcal{W}] = C_{\widetilde{\phi}}[\widetilde{\mathcal{W}}] \ \text{ is compactly embedded in } L^2(\Omega), \\ \text{condition (3.2) holds} . \end{cases}$$

Remark 3.1. We note that if Ω is a domain of class $C^{0,1}$, i.e., Ω is locally a subgraph of Lipschitz continuous functions, then the inequality (3.2) holds for any $\alpha > N/2$ (cf., for example, [3, 20] and also Remark 6.5 below). We also note that by the Min-Max Principle [11, p. 5] and by comparing with the Dirichlet Laplacian on a ball contained in Ω, the condition (3.2) does not hold for $\alpha \leqslant N/2$ (no matter whether Ω is regular or not).

To compare L and $\widetilde{L}$, we compare the respective pull-backs H and $\widetilde{H}$. Since they act on different Hilbert spaces – $L^2(\Omega, g\,dx)$ and $L^2(\Omega, \widetilde{g}\,dx)$ – we use the canonical unitary operator

$$w : L^2(\Omega, g\,dx) \longrightarrow L^2(\Omega, \widetilde{g}\,dx) \ , \qquad u \mapsto wu \, ,$$

defined as the multiplication by the function $w := g^{1/2}\widetilde{g}^{-1/2}$. We also introduce the multiplication operator S on $(L^2(\Omega))^N$ by the symmetric matrix

$$w^{-2} a^{-1/2} \widetilde{a} a^{-1/2} \, , \tag{3.3}$$

where the matrix a is defined by (2.5) and the matrix $\widetilde{a}$ is defined in the same way with $\widetilde{\phi}$ replacing ϕ. If there is no ambiguity, we also denote by S the matrix (3.3).

As we see in the sequel, in order to compare H and $\widetilde{H}$, we need an auxiliary operator. Namely, we consider the operator T^*ST, which is the nonnegative selfadjoint operator in $L^2(\Omega, g\,dx)$ canonically associated with the sesquilinear form

$$\int_{\Omega} (\widetilde{a}\nabla u_1 \cdot \nabla \overline{u}_2)\widetilde{g}dx\ , \quad u_1, u_2 \in \mathcal{V}\,.$$

It is easily seen that the operator T^*ST is the pull-back to Ω via $\widetilde{\phi}$ of the operator

$$\widehat{L} := \frac{\widetilde{g} \circ \widetilde{\phi}^{(-1)}}{g \circ \widetilde{\phi}^{(-1)}} \widetilde{L} \tag{3.4}$$

defined on $L^2(\widetilde{\phi}(\Omega))$. Thus, in the sequel, we deal with the operators $L, \widetilde{L}$ and $\widehat{L}$ and the respective pull-backs H, $\widetilde{H}$, and T^*ST. We repeatedly use the fact that these are pairwise unitarily equivalent.

We denote by $\lambda_n[E]$, $n \in \mathbf{N}$, the eigenvalues of a nonnegative selfadjoint operator E with compact resolvent, arranged in nondecreasing order and repeated according to multiplicity, and by $\psi_n[E]$, $n \in \mathbf{N}$, a corresponding orthonormal sequence of eigenfunctions.

Lemma 3.2. *Let* (A) *be satisfied. Then the operators* L, $\widetilde{L}$, $\widehat{L}$, H, $\widetilde{H}$, *and* T^*ST *have compact resolvents and the corresponding nonzero eigenvalues satisfy the inequality*

$$\sum_{\lambda_n[E]\neq 0} \lambda_n[E]^{-\alpha} \leqslant cc^* \tag{3.5}$$

for $E = L$, $\widetilde{L}$, $\widehat{L}$, H, $\widetilde{H}$, T^*ST, *where* c *depends only on* N, τ, *and* θ.

Proof. We prove the inequality (3.5) only for $E = T^*ST$, the other cases being similar. Note that the Rayleigh quotient corresponding to T^*ST is given by

$$\frac{\langle T^*STu, u\rangle_g}{\langle u, u\rangle_g} = \frac{\langle STu, Tu\rangle_g}{\langle u, u\rangle_g} = \frac{\displaystyle\int_{\Omega} (\widetilde{a}\nabla u \cdot \nabla \overline{u})\widetilde{g}dx}{\displaystyle\int_{\Omega} |u|^2 g dx}, \quad u \in \mathcal{V}.$$

Then the inequality (3.5) easily follows by observing that

$$\widetilde{a}\nabla u \cdot \nabla \overline{u} \geqslant \theta^{-1}|(\nabla\widetilde{\phi})^{-1}\nabla u|^2 \geqslant \theta^{-1}\tau^{-2}|\nabla u|^2,$$

$$|\mathrm{det}\nabla\phi| \leqslant N!|\nabla\phi|^N \tag{3.6}$$

and using the Min-Max Principle [11, p. 5]. □

4 Stability Estimates for the Resolvents and Eigenvalues

The following lemma is based on the well-known commutation formula (4.3) (cf. [12]). We denote by $\sigma(E)$ the spectrum of an operator E.

Lemma 4.1. *Let* (A) *be satisfied. Then for all* $\xi \in \mathbf{C} \setminus (\sigma(H) \cup \sigma(\widetilde{H}) \cup \sigma(T^*ST))$

$$(w^{-1}\widetilde{H}w - \xi)^{-1} - (H - \xi)^{-1} = A_1 + A_2 + A_3 + B\,, \tag{4.1}$$

where

$$A_1 = (1-w)(wT^*STw - \xi)^{-1},$$

$$\begin{aligned} A_2 &= w(wT^*STw - \xi)^{-1}(1-w), \\ A_3 &= -\xi(T^*ST - \xi)^{-1}(w - w^{-1})(wT^*STw - \xi)^{-1}w, \\ B &= T^*S^{1/2}(S^{1/2}TT^*S^{1/2} - \xi)^{-1}S^{1/2}(S^{-1} - I)(TT^* - \xi)^{-1}T\,. \end{aligned}$$

Proof. It suffices to prove (4.1) for $\xi \neq 0$ since the case $\xi = 0 \notin \sigma(H) \cup \sigma(\widetilde{H}) \cup \sigma(T^*ST)$ is then obtained by letting $\xi \to 0$.

Recall that $T^*T = H$. Similarly $\widetilde{T}^{\widetilde{*}}\widetilde{T} = \widetilde{H}$, where we have emphasized the dependence of the adjoint operation on the specific inner-product used. In this respect we note that the two adjoints of an operator E are related by the conjugation relation $E^{\widetilde{*}} = w^2E^*w^{-2}$. This allows us to use only $*$ and not $\widetilde{*}$.

Note that

$$\widetilde{H} = (\widetilde{a}^{1/2}\nabla)^{\widetilde{*}}\widetilde{a}^{1/2}\nabla = w^2(\widetilde{a}^{1/2}\nabla)^*w^{-2}\widetilde{a}^{1/2}\nabla = w^2T^*ST\,. \tag{4.2}$$

Therefore, by simple computations, we obtain

$$\begin{aligned} &(w^{-1}\widetilde{H}w - \xi)^{-1} - (H-\xi)^{-1} \\ &= w^{-1}(\widetilde{H} - \xi)^{-1}w - (H-\xi)^{-1} \\ &= w^{-1}(w^2T^*ST - \xi)^{-1}w - (T^*T - \xi)^{-1} \\ &= w^{-1}(w^2T^*ST - \xi)^{-1}w - (T^*ST - \xi w^{-2})^{-1}w^{-1} \\ &\quad + (T^*ST - \xi w^{-2})^{-1}w^{-1} - (T^*ST - \xi w^{-2})^{-1} \\ &\quad + (T^*ST - \xi w^{-2})^{-1} - (T^*ST - \xi)^{-1} \\ &\quad + (T^*ST - \xi)^{-1} - (T^*T - \xi)^{-1} \\ &= A_1 + A_2 + A_3 + ((T^*ST - \xi)^{-1} - (T^*T - \xi)^{-1})\,. \end{aligned}$$

To compute the last term we use the commutation formula

$$-\xi(E^*E - \xi)^{-1} + E^*(EE^* - \xi)^{-1}E = I \tag{4.3}$$

which holds for any closed and densely defined operator E (cf. [12]). We write (4.3) first for $E = T$, then for $E = S^{1/2}T$, and then we subtract the two relations. After some simple calculations we obtain $(T^*ST - \xi)^{-1} - (T^*T - \xi)^{-1} = B$, as required. □

We now introduce a regularity property which is important for our estimates. Sufficient conditions for its validity are given in Section 6.

Definition 4.2. Let U be an open set in $\mathbf{R}^N$, and let E be a nonnegative selfadjoint operator on $L^2(U)$ with compact resolvent and $\mathrm{Dom}(E) \subset W^{1,2}(U)$. We say that E satisfies property (P) if there exist $q_0 > 2$, $\gamma \geqslant 0$, $C > 0$ such that the eigenfunctions $\psi_n[E]$ of E satisfy the following conditions:

$$\|\psi_n[E]\|_{L^{q_0}(U)} \leqslant C\lambda_n[E]^{\gamma} \tag{P1}$$

and

$$\|\nabla\psi_n[E]\|_{L^{q_0}(U)} \leqslant C\lambda_n[E]^{\gamma+\frac{1}{2}} \tag{P2}$$

for all $n \in \mathbf{N}$ such that $\lambda_n[E] \neq 0$.

Remark 4.3. It is known that if Ω, A_{ij} and Γ are sufficiently smooth, then for the operator L in (1.1), subject to the boundary conditions (1.2), property (P) is satisfied with $q_0 = \infty$ and $\gamma = N/4$ (cf. Theorem 6.3 and the proof of Theorem 7.3).

By interpolation, it follows that if conditions (P1) and (P2) are satisfied, then

$$\begin{aligned} \|\psi_n[E]\|_{L^q(U)} &\leqslant C^{\frac{q_0(q-2)}{q(q_0-2)}}\lambda_n[E]^{\frac{q_0(q-2)\gamma}{q(q_0-2)}}, \\ \|\nabla\psi_n[E]\|_{L^q(U)} &\leqslant C^{\frac{q_0(q-2)}{q(q_0-2)}}\lambda_n[E]^{\frac{1}{2}+\frac{q_0(q-2)\gamma}{q(q_0-2)}} \end{aligned} \tag{4.4}$$

for all $q \in [2, q_0]$.

In the sequel, we require that property (P) is satisfied by the operators H, $\widetilde{H}$ and T^*ST which, according to the following lemma, is equivalent to requiring that property (P) is satisfied by the operators L, $\widetilde{L}$ and $\widehat{L}$ respectively.

Lemma 4.4. *Let* (A) *be satisfied. Then the operators* H, $\widetilde{H}$, *and* T^*ST *respectively, satisfy property* (P) *for some* $q_0 > 2$ *and* $\gamma \geqslant 0$ *if and only if the operators* L, $\widetilde{L}$, *and* $\widehat{L}$ *respectively, satisfy property* (P) *for the same* q_0 *and* γ.

Let E be a nonnegative selfadjoint operator on a Hilbert space the spectrum of which consists of isolated positive eigenvalues of finite multiplicity and may also contain zero as an eigenvalue of possibly infinite multiplicity. Let $s > 0$. For a function $g : \sigma(E) \to \mathbf{C}$ we define

$$\begin{aligned} |g(E)|_{p,s} &= \Big(\sum_{\lambda_n[E]\neq 0} |g(\lambda_n[E])|^p \lambda_n[E]^s\Big)^{1/p}, \ 1 \leqslant p < \infty, \\ |g(E)|_{\infty,s} &= \sup_{\lambda_n[E]\neq 0} |g(\lambda_n[E])|, \end{aligned}$$

where, as usual, each positive eigenvalue is repeated according to its multiplicity.

The next lemma involves the Schatten norms $\|\cdot\|_{\mathcal{C}^r}$, $1 \leqslant r \leqslant \infty$. For a compact operator E on a Hilbert space they are defined by $\|E\|_{\mathcal{C}^r} = (\sum_n \mu_n(E)^r)^{1/r}$, if $r < \infty$, and $\|E\|_{\mathcal{C}^\infty} = \|E\|$, where $\mu_n(E)$ are the singular values of E, i.e., the nonzero eigenvalues of $(E^*E)^{1/2}$; recall that the Schatten space $\mathcal{C}^r$, defined as the space of those compact operators for which the Schatten norm $\|\cdot\|_{\mathcal{C}^r}$ is finite, is a Banach space (cf. [22] or [24] for details).

Let $F := TT^*$. Recall that $\sigma(F) \setminus \{0\} = \sigma(H) \setminus \{0\}$ (cf. [12]). In the next lemma, $g(H)$ and $g(F)$ are operators defined in the standard way by functional calculus. The following lemma is a variant of Lemma 8 of [2].

Lemma 4.5. *Let $q_0 > 2$, $\gamma \geqslant 0$, $p \geqslant q_0/(q_0-2)$, $2 \leqslant r < \infty$ and $s = 2q_0\gamma/[p(q_0-2)]$. Then the following statements hold.*

(i) *If the eigenfunctions of H satisfy* (P1)*, then for any measurable function $R : \Omega \to \mathbf{C}$ and function $g : \sigma(H) \to \mathbf{C}$ we have*

$$\|Rg(H)\|_{\mathcal{C}^r} \leqslant \|R\|_{L^{pr}(\Omega)}\Big(|\Omega|^{-\frac{1}{pr}}|g(0)| + C^{\frac{2q_0}{pr(q_0-2)}}|g(H)|_{r,s}\Big). \tag{4.5}$$

(ii) *If the eigenfunctions of H satisfy* (P2)*, then for any measurable matrix-valued function R in Ω and function $g : \sigma(F) \to \mathbf{C}$ such that if $0 \in \sigma(F)$, then $g(0) = 0$, we have*

$$\|Rg(F)\|_{\mathcal{C}^r} \leqslant C^{\frac{2q_0}{pr(q_0-2)}}\|a\|^{\frac{1}{r}}_{L^\infty(\Omega)}\|R\|_{L^{pr}(\Omega)}|g(F)|_{r,s}\,. \tag{4.6}$$

Proof. We only prove statement (ii) since the proof of (i) is simpler. It is enough to consider the case where R is bounded and g has finite support: the general case then follows by approximating R in $\|\cdot\|_{L^{pr}(\Omega)}$ by a sequence R_n, $n \in \mathbf{N}$, of bounded matrix-valued functions and g in $|\cdot|_{r,s}$ by a sequence g_n, $n \in \mathbf{N}$, of functions with finite support, and observing that, by (4.6), the sequence $R_n g_n(F)$, $n \in \mathbf{N}$, is then a Cauchy sequence in $\mathcal{C}^r$.

Since R is bounded and g has finite support, $Rg(F)$ is compact. Hence the inequality (4.6) is trivial for $r = \infty$. Thus, it is enough to prove (4.6) for $r = 2$ since the general case follows by interpolation (cf. [24]). It is easily seen that $z_n := T\psi_n[H]/\|T\psi_n[H]\| = \lambda_n[H]^{-1/2}T\psi_n[H]$, for all $n \in \mathbf{N}$ such that $\lambda_n[H] \neq 0$, are orthonormal eigenfunctions of F, $Fz_n = \lambda_n[H]z_n$, $n \in \mathbf{N}$, and $\mathrm{span}\{z_n\} = \mathrm{Ker}(F)^\perp$. Since $g(0) = 0$,

$$\|Rg(F)\|^2_{\mathcal{C}^2} = \sum_{n=1}^\infty \|Rg(F)z_n\|^2_{L^2(\Omega)} = \sum_{n=1}^\infty |g(\lambda_n[H])|^2\|Rz_n\|^2_{L^2(\Omega)}$$
$$= \sum_{n=1}^\infty \lambda_n[H]^{-1}|g(\lambda_n[H])|^2\|Ra^{1/2}\nabla\psi_n[H]\|^2_{L^2(\Omega)}$$

$$\leqslant \|a^{1/2}\|^2_{L^\infty(\Omega)}\|R\|^2_{L^{2p}(\Omega)}\sum_{n=1}^{\infty}\lambda_n[H]^{-1}|g(\lambda_n[H])|^2\|\nabla\psi_n[H]\|^2_{L^{2p/(p-1)}(\Omega)}$$

$$\leqslant C^{\frac{2q_0}{p(q_0-2)}}\|a^{1/2}\|^2_{L^\infty(\Omega)}\|R\|^2_{L^{2p}(\Omega)}\sum_{n=1}^{\infty}|g(\lambda_n[H])|^2\lambda_n[H]^{\frac{2q_0\gamma}{p(q_0-2)}}, \qquad (4.7)$$

where for the last inequality we used (4.4). This proves (4.6) for $r=2$, thus completing the proof of the lemma. □

Recall that $\delta_p(\phi,\widetilde{\phi})$, $1\leqslant p\leqslant\infty$, is defined in (1.3).

Theorem 4.6 (stability of resolvents)**.** *Let* (A) *be satisfied. Let* $\xi\in\mathbf{C}\setminus(\sigma(H)\cup\{0\})$. *Then the following statements hold.*

(i) *There exists* $c_1>0$ *depending only on* N, τ, θ, α, c^*, *and* ξ *such that if* $\delta_\infty(\phi,\widetilde{\phi})\leqslant c_1^{-1}$, *then* $\xi\notin\sigma(\widetilde{H})$ *and*

$$\|(w^{-1}\widetilde{H}w-\xi)^{-1}-(H-\xi)^{-1}\|_{\mathcal{C}^\alpha}\leqslant c_1\delta_\infty(\phi,\widetilde{\phi}). \qquad (4.8)$$

(ii) *Let, in addition,* (P) *be satisfied by the operators* H, $\widetilde{H}$, *and* T^*ST *for the same* q_0, γ, *and* C. *Let* $p\geqslant q_0/(q_0-2)$ *and* $r\geqslant\max\{2,\alpha+\frac{2q_0\gamma}{p(q_0-2)}\}$. *Then there exists* $c_2>0$ *depending only on* N, τ, θ, α, c^*, r, p, q_0, C, γ, $|\Omega|$, *and* ξ *such that if* $\delta_{pr}(\phi,\widetilde{\phi})\leqslant c_2^{-1}$, *then* $\xi\notin\sigma(\widetilde{H})$ *and*

$$\|(w^{-1}\widetilde{H}w-\xi)^{-1}-(H-\xi)^{-1}\|_{\mathcal{C}^r}\leqslant c_2\delta_{pr}(\phi,\widetilde{\phi}). \qquad (4.9)$$

Remark 4.7. Let $s=[q_0/(q_0-2)]\max\{2,\alpha+2\gamma\}$. It follows by Theorem 4.6 (ii) (choosing $p=q_0/(q_0-2)$) that if $\delta_s(\phi,\widetilde{\phi})\leqslant c_2^{-1}$, then $\xi\notin\sigma(\widetilde{H})$ and

$$\|(w^{-1}\widetilde{H}w-\xi)^{-1}-(H-\xi)^{-1}\|\leqslant c_2\delta_s(\phi,\widetilde{\phi}). \qquad (4.10)$$

Remark 4.8. As we will see in Section 7 below, the best range for s in (4.10) used in our applications is $s>N$; this corresponds to the case where $q_0=\infty$, $\gamma=N/4$, and $\alpha>N/2$ (cf. Remarks 3.1 and 4.3).

Proof of Theorem 4.6. In this proof, we denote by c a positive constant depending only on N, τ, θ, α, and c^* the value of which may change along the proof. When dealing with statement (ii) constant c may depend also on r, p, q_0, C, γ, and $|\Omega|$. We divide the proof into two steps.

Step 1. We assume first that $\xi\notin\sigma(\widetilde{H})\cup\sigma(T^*ST)$ and set

$$d_\sigma(\xi)=\operatorname{dist}(\xi,\sigma(H)\cup\sigma(\widetilde{H})\cup\sigma(T^*ST)).$$

At this first step, we prove (4.8) and (4.9) without any smallness assumptions on $\delta_\infty(\phi,\widetilde{\phi})$ and $\delta_{pr}(\phi,\widetilde{\phi})$ respectively.

We first prove (4.8). We use Lemma 4.1, and to do so we first estimate the terms A_1, A_2, A_3 in the identity (4.1). It is clear that

$$\frac{\lambda_n[\widetilde{H}]}{|\lambda_n[\widetilde{H}]-\xi|} \leqslant \left(1+\frac{|\xi|}{d(\xi,\sigma(\widetilde{H}))}\right). \tag{4.11}$$

Since the eigenvalues of the operator wT^*STw coincide with the eigenvalues of $\widetilde{H}$ (cf. (4.2)), it follows that

$$\begin{aligned} \|(wT^*STw-\xi)^{-1}\|_{\mathcal{C}^\alpha}^\alpha &= \sum_{n=1}^\infty \frac{1}{|\lambda_n[\widetilde{H}]-\xi|^\alpha} \\ &\leqslant \frac{1}{|\xi|^\alpha} + \left(1+\frac{|\xi|}{d(\xi,\sigma(\widetilde{H}))}\right)^\alpha \sum_{\lambda_n[\widetilde{H}]\neq 0} \lambda_n[\widetilde{H}]^{-\alpha} \\ &= \frac{1}{|\xi|^\alpha} + c\left(1+\frac{|\xi|}{d(\xi,\sigma(\widetilde{H}))}\right)^\alpha. \end{aligned} \tag{4.12}$$

Taking into account (3.6) and observing that

$$\left|\det\nabla\phi - \det\nabla\widetilde{\phi}\right| \leqslant N!N\left|\nabla\phi-\nabla\widetilde{\phi}\right| \max\left\{\,|\nabla\phi|\,,\,\left|\nabla\widetilde{\phi}\right|\,\right\}^{N-1}, \tag{4.13}$$

we find

$$|1-w|, |w-w^{-1}| \leqslant c|\nabla\phi-\nabla\widetilde{\phi}|. \tag{4.14}$$

Combining the inequalities (4.12) and (4.14), we obtain

$$\begin{aligned} &\|A_1\|_{\mathcal{C}^\alpha}, \|A_2\|_{\mathcal{C}^\alpha} \leqslant c\left(1+\frac{1}{|\xi|}+\frac{|\xi|}{d_\sigma(\xi)}\right)\|\nabla\phi-\nabla\widetilde{\phi}\|_{L^\infty(\Omega)}, \\ &\|A_3\|_{\mathcal{C}^\alpha} \leqslant c\left(\frac{1+|\xi|}{d_\sigma(\xi)}+\frac{|\xi|^2}{d_\sigma(\xi)^2}\right)\|\nabla\phi-\nabla\widetilde{\phi}\|_{L^\infty(\Omega)}. \end{aligned} \tag{4.15}$$

We now estimate the term B in (4.1). We recall that $F=TT^*$ and set $F_S = S^{1/2}TT^*S^{1/2}$. Then, by polar decomposition, there exist partial isometries $Y, Y_S : L^2(\Omega, g\,dx) \to (L^2(\Omega, g\,dx))^N$ such that $T = F^{1/2}Y$ and $S^{1/2}T = F_S^{1/2}Y_S$. We have

$$B = Y_S^* F_S^{1/2}(F_S-\xi)^{-1}S^{1/2}(S^{-1}-I)(F-\xi)^{-1}F^{1/2}Y.$$

Hence, by the Hölder inequality for the Schatten norms (cf. [22, p. 41]), it follows that

$$\|B\|_{\mathcal{C}^\alpha} \leqslant \|F_S^{1/2}(F_S-\xi)^{-1}\|_{\mathcal{C}^{2\alpha}}\|S^{1/2}(S^{-1}-I)\|_{L^\infty(\Omega)}\|(F-\xi)^{-1}F^{1/2}\|_{\mathcal{C}^{2\alpha}}. \tag{4.16}$$

Since $\sigma(F)\setminus\{0\}=\sigma(H)\setminus\{0\}$, we may argue as before and obtain

$$\begin{aligned}\|(F-\xi)^{-1}F^{1/2}\|_{\mathcal{C}^{2\alpha}}^{2\alpha} &\leqslant c\left(1+\frac{|\xi|}{d(\xi,\sigma(H))}\right)^{2\alpha},\\ \|(F_S-\xi)^{-1}F_S^{1/2}\|_{\mathcal{C}^{2\alpha}}^{2\alpha} &\leqslant c\left(1+\frac{|\xi|}{d(\xi,\sigma(T^*ST))}\right)^{2\alpha}.\end{aligned} \tag{4.17}$$

Now, it is easy to see that

$$\begin{aligned}|S^{-1}-I| &\leqslant |(w^2-1)a^{1/2}\widetilde{a}^{-1}a^{1/2}|+|a^{1/2}(\widetilde{a}^{-1}-a^{-1})a^{1/2}|\\ &\leqslant c(|\nabla\phi-\nabla\widetilde{\phi}|+|A\circ\phi-A\circ\widetilde{\phi}|).\end{aligned} \tag{4.18}$$

Combining (4.16), (4.17), and (4.18), we conclude that

$$\|B\|_{\mathcal{C}^\alpha}\leqslant c\left(1+\frac{|\xi|}{d_\sigma(\xi)}\right)^2\delta_\infty(\phi,\widetilde{\phi}). \tag{4.19}$$

By Lemma 4.1 and the estimates (4.15) and (4.19), we deduce that

$$\|(w^{-1}\widetilde{H}w-\xi)^{-1}-(H-\xi)^{-1}\|_{\mathcal{C}^\alpha}\leqslant c_1\left(1+\frac{1}{|\xi|}+\frac{1}{d_\sigma(\xi)}+\frac{|\xi|^2}{d_\sigma(\xi)^2}\right)\delta_\infty(\phi,\widetilde{\phi}). \tag{4.20}$$

We now prove (4.9). In order to estimate A_1, A_2, and A_3, we use the estimate (4.5) and get

$$\|A_1\|_{\mathcal{C}^r},\|A_2\|_{\mathcal{C}^r}\leqslant c\left(1+\frac{1}{|\xi|}+\frac{|\xi|}{d_\sigma(\xi)}\right)\|\nabla\phi-\nabla\widetilde{\phi}\|_{L^{pr}(\Omega)}, \tag{4.21}$$

$$\|A_3\|_{\mathcal{C}^r}\leqslant c\left(\frac{1+|\xi|}{d_\sigma(\xi)}+\frac{|\xi|^2}{d_\sigma(\xi)^2}\right)\|\nabla\phi-\nabla\widetilde{\phi}\|_{L^{pr}(\Omega)}. \tag{4.22}$$

We now estimate B. We assume without loss of generality that $S^{-1}-I\geqslant 0$. Thus, in order to estimate the $\mathcal{C}^r$ norm of B, we estimate the $\mathcal{C}^{2r}$ norms of $F_S^{1/2}(F_S-\xi)^{-1}S^{1/2}(S^{-1}-I)^{1/2}$ and $(S^{-1}-I)^{1/2}(F-\xi)^{-1}F^{1/2}$. By Lemma 4.5, it follows that

$$\begin{aligned}&\|(S^{-1}-I)^{1/2}(F-\xi)^{-1}F^{1/2}\|_{\mathcal{C}^{2r}}^{2r}\\ &\quad\leqslant c\|(S^{-1}-I)^{1/2}\|_{L^{2pr}(\Omega)}^{2r}\sum_{n=1}^{\infty}\left|\frac{\lambda_n[H]}{\lambda_n[H]-\xi}\right|^{2r}\lambda_n[H]^{\frac{2q_0\gamma}{p(q_0-2)}-r}\\ &\quad\leqslant c\|S^{-1}-I\|_{L^{pr}(\Omega)}^r\left(1+\frac{|\xi|}{d_\sigma(\xi)}\right)^{2r}.\end{aligned} \tag{4.23}$$

The same estimate holds also for the operator $F_S^{1/2}(F_S-\xi)^{-1}S^{1/2}(S^{-1}-I)^{1/2}$. Thus, by the Hölder inequality for the Schatten norms, it follows that

$$\|B\|_{\mathcal{C}^r} \leqslant c\left(1+\frac{|\xi|}{d_\sigma(\xi)}\right)^2 \|S^{-1}-I\|_{L^{pr}(\Omega)}. \tag{4.24}$$

Using Lemma 4.1 and combining the estimates (4.18), (4.22), and (4.24), we deduce that

$$\|(w^{-1}\widetilde{H}w-\xi)^{-1}-(H-\xi)^{-1}\|_{\mathcal{C}^r} \leqslant c_1\left(1+\frac{1}{|\xi|}+\frac{1}{d_\sigma(\xi)}+\frac{|\xi|^2}{d_\sigma(\xi)^2}\right)\delta_{pr}(\phi,\widetilde{\phi}). \tag{4.25}$$

Step 2. We prove statement (i). First of all, we prove that there exists $c>0$ such that if

$$\delta_\infty(\phi,\widetilde{\phi}) < \frac{d(\xi,\sigma(H))}{c(1+|\xi|^2+d(\xi,\sigma(H))^2)}, \tag{4.26}$$

then $\xi \notin \sigma(\widetilde{H}) \cup \sigma(T^*ST)$ and

$$d(\xi,\sigma(\widetilde{H})),\ d(\xi,\sigma(T^*ST)) > \frac{d(\xi,\sigma(H))}{2}. \tag{4.27}$$

We begin with T^*ST. By recalling that $B=(T^*ST-\xi)^{-1}-(T^*T-\xi)^{-1}$ (cf. the proof of Lemma 4.1) and using the estimate (4.19) with $\xi=-1$ and the inequality (4.36), we find that there exists $C_1>0$ such that for all $n\in\mathbf{N}$

$$\left|\frac{1}{\lambda_n[T^*ST]+1}-\frac{1}{\lambda_n[H]+1}\right| \leqslant C_1\delta_\infty(\phi,\widetilde{\phi}). \tag{4.28}$$

Assume that $n\in\mathbf{N}$ is such that

$$\lambda_n[T^*ST] \leqslant |\xi| + d(\xi,\sigma(H)).$$

By (4.28), it follows that if

$$C_1(1+|\xi|+d(\xi,\sigma(H)))\delta_\infty(\phi,\widetilde{\phi}) < \frac{|\xi|+d(\xi,\sigma(H))}{2(|\xi|+d(\xi,\sigma(H)))+1},$$

then

$$\begin{aligned}\lambda_n[H] &\leqslant \frac{|\xi|+d(\xi,\sigma(H))+C_1[1+|\xi|+d(\xi,\sigma(H))]\delta_\infty(\phi,\widetilde{\phi})}{1-C_1[1+|\xi|+d(\xi,\sigma(H))]\delta_\infty(\phi,\widetilde{\phi})} \\ &\leqslant 2(|\xi|+d(\xi,\sigma(H))),\end{aligned} \tag{4.29}$$

(the elementary inequality $(A+t)(1-t)^{-1}<2A$ if $0<t<A(2A+1)^{-1}$ was used). Thus, by (4.28) and (4.29), it follows that if

$$\delta_\infty(\phi,\widetilde{\phi}) \leqslant \frac{d(\xi,\sigma(H))}{2C_1[1+|\xi|+d(\xi,\sigma(H))][1+2(|\xi|+d(\xi,\sigma(H)))]}$$

then

$$\begin{aligned}|\xi-\lambda_n[T^*ST]| &\geqslant |\xi-\lambda_n[H]| - |\lambda_n[H]-\lambda_n[T^*ST]|\\ &\geqslant d(\xi,\sigma(H)) - C_1[1+|\xi|+d(\xi,\sigma(H))][1+2(|\xi|+d(\xi,\sigma(H)))]\delta_\infty(\phi,\widetilde{\phi})\\ &\geqslant \frac{d(\xi,\sigma(H))}{2}\end{aligned} \tag{4.30}$$

for all $n\in\mathbf{N}$ such that $\lambda_n[T^*ST]\leqslant|\xi|+d(\xi,\sigma(H))$. Thus, the inequality (4.27) for $d(\xi,\sigma(T^*ST))$ follows by (4.30) and by observing that if $n\in\mathbf{N}$ is such that $\lambda_n[T^*ST]>|\xi|+d(\xi,\sigma(H))$, then

$$|\xi-\lambda_n[T^*ST]|>d(\xi,\sigma(H)).$$

The inequality (4.27) for $d(\xi,\sigma(\widetilde{H}))$ can be proved in the same way. Indeed, it suffices to observe that, by Step 1, there exists $C_2>0$ such that

$$\left|\frac{1}{\lambda_n[\widetilde{H}]+1}-\frac{1}{\lambda_n[H]+1}\right|\leqslant C_2\delta_\infty(\phi,\widetilde{\phi}); \tag{4.31}$$

we then proceed exactly as above.

By (4.20) and (4.27), there exists $c>0$ such that if

$$\delta_\infty(\phi,\widetilde{\phi})\leqslant\frac{d(\xi,\sigma(H))}{c(1+|\xi|^2+d(\xi,\sigma(H))^2)}, \tag{4.32}$$

then $\xi\notin\sigma(\widetilde{H})\cup\sigma(T^*ST)$ and

$$\begin{aligned}&\|(w^{-1}\widetilde{H}w-\xi)^{-1}-(H-\xi)^{-1}\|_{\mathcal{C}^\alpha}\\ &\leqslant c\Big(1+\frac{1}{|\xi|}+\frac{1}{d(\xi,\sigma(H))}+\frac{|\xi|^2}{d(\xi,\sigma(H))^2}\Big)\delta_\infty(\phi,\widetilde{\phi}).\end{aligned} \tag{4.33}$$

This completes the proof of statement (i).

The argument above works word by word also for the proof of statement (ii) provided that $\delta_\infty(\phi,\widetilde{\phi})$ is replaced with $\delta_{pr}(\phi,\widetilde{\phi})$. □

Remark 4.9. The proof of Theorem 4.6 gives some information about the dependence of the constants c_1 and c_2 on ξ, which is useful in the sequel. For instance, in the case of statement (i), in fact it was proved that there exists c depending only on N, τ, θ, α, and c^* such that if (4.32) holds, then (4.33) holds. Exactly the same holds for statement (ii) where c depends also on r, p, q_0, C, γ, and $|\Omega|$. Moreover, for such ϕ and $\widetilde{\phi}$, if $0\notin\sigma(H)$, then $0\notin\sigma(\widetilde{H})$ and the summand $1/|\xi|+1/d(\xi,\sigma(H)$ can be removed from the right-hand side of (4.33). Furthermore, in this case, statements (i) and (ii) also hold for $\xi = 0$. This can be easily seen by looking closely at the proofs of (4.15) and (4.21).

Remark 4.10. By the proof of Theorem 4.6, for a fixed $\xi \in \mathbf{C} \setminus [0, \infty[$ no smallness conditions on $\delta_\infty(\phi, \widetilde{\phi})$ and $\delta_{pr}(\phi, \widetilde{\phi})$ are required for the validity of (4.8) and (4.9) respectively.

Theorem 4.11 (stability of eigenvalues)**.** *Let* (A) *be satisfied. Then the following statements hold.*

(i) *There exists* $c_1 > 0$ *depending only on* N, τ, θ, α, *and* c^* *such that if* $\delta_\infty(\phi, \widetilde{\phi}) \leqslant c_1^{-1}$, *then*

$$\left(\sum_{n=1}^{\infty} \left| \frac{1}{\lambda_n[\widetilde{L}] + 1} - \frac{1}{\lambda_n[L] + 1} \right|^\alpha \right)^{1/\alpha} \leqslant c_1 \delta_\infty(\phi, \widetilde{\phi}). \tag{4.34}$$

(ii) *Let, in addition,* (P) *be satisfied by the operators* L, $\widetilde{L}$, *and* $\widehat{L}$ *for the same* $q_0 > 2$, $\gamma \geqslant 0$, *and* $C > 0$. *Let* $p \geqslant q_0/(q_0 - 2)$ *and* $r \geqslant \max\{2, \alpha + \frac{2q_0\gamma}{p(q_0-2)}\}$. *Then there exists* $c_2 > 0$ *depending only on* N, τ, θ, α, c^*, r, p, q_0, C, γ, *and* $|\Omega|$ *such that if* $\delta_{pr}(\phi, \widetilde{\phi}) \leqslant c_2^{-1}$, *then*

$$\left(\sum_{n=1}^{\infty} \left| \frac{1}{\lambda_n[\widetilde{L}] + 1} - \frac{1}{\lambda_n[L] + 1} \right|^r \right)^{1/r} \leqslant c_2 \delta_{pr}(\phi, \widetilde{\phi}). \tag{4.35}$$

Proof. The theorem follows by Theorem 4.6 and by applying the inequality

$$\left(\sum_{n=1}^{\infty} \left| \frac{1}{\lambda_n[E_1] + 1} - \frac{1}{\lambda_n[E_2] + 1} \right|^r \right)^{1/r} \leqslant \|(E_1+1)^{-1} - (E_2+1)^{-1}\|_{\mathcal{C}^r}, \tag{4.36}$$

with $E_1 = w^{-1}\widetilde{H}w$, $E_2 = H$ (cf. [24, p. 20]). □

Remark 4.12. We note that, in the case of the Dirichlet boundary conditions, i.e., $\mathcal{V} = W_0^{1,2}(\Omega)$, the inequality (4.34) directly follows from Lemma 6.1 in [5] the proof of which is based on the Min-Max Principle.

5 Stability Estimates for Eigenfunctions

Definition 5.1. Let E be a nonnegative selfadoint operator with compact resolvent on a Hilbert space $\mathcal{H}$. For a finite subset G of $\mathbf{N}$ we denote by

$P_G(E)$ the orthogonal projection from $\mathcal{H}$ onto the linear space generated by all the eigenfunctions corresponding to the eigenvalues $\lambda_k[E]$ with $k \in G$.

Note that the dimension of the range of $P_G(E)$ coincides with the number of elements of G if and only if no eigenvalue with index in G coincides with an eigenvalue with index in $\mathbf{N} \setminus G$; this will always be the case in what follows.

In the following statements, it is understood that whenever $n = 1$ the term λ_{n-1} has to be dropped.

Theorem 5.2. *Let* (A) *be satisfied. Let λ be a nonzero eigenvalue of H of multiplicity m, let $n \in \mathbf{N}$ be such that $\lambda = \lambda_n[H] = \ldots = \lambda_{n+m-1}[H]$, and let $G = \{n, n+1, \ldots, n+m-1\}$. Then the following statements hold.*

(i) *There exists $c_1 > 0$ depending only on N, τ, θ, α, c^*, $\lambda_{n-1}[H]$, λ, and $\lambda_{n+m}[H]$ such that if $\delta_\infty(\phi, \widetilde{\phi}) \leqslant c_1^{-1}$, then* $\dim \operatorname{ran} P_G(w^{-1}\widetilde{H}w) = m$ *and*

$$\|P_G(H) - P_G(w^{-1}\widetilde{H}w)\| \leqslant c_1 \delta_\infty(\phi, \widetilde{\phi}). \tag{5.1}$$

(ii) *Let, in addition,* (P) *be satisfied by the operators H, $\widetilde{H}$ and T^*ST for the same q_0, γ and C. Let $s = [q_0/(q_0-2)] \max\{2, \alpha + 2\gamma\}$. Then there exists $c_2 > 0$ depending only on N, τ, θ, α, c^*, q_0, C, γ, $|\Omega|$, $\lambda_{n-1}[H]$, λ, and $\lambda_{n+m}[H]$ such that if $\delta_s(\phi, \widetilde{\phi}) \leqslant c_2^{-1}$, then* $\dim \operatorname{ran} P_G(w^{-1}\widetilde{H}w) = m$ *and*

$$\|P_G(H) - P_G(w^{-1}\widetilde{H}w)\| \leqslant c_2 \delta_s(\phi, \widetilde{\phi}). \tag{5.2}$$

Proof. We set $\rho = \frac{1}{2} \operatorname{dist}(\lambda, (\sigma(H) \cup \{0\}) \setminus \{\lambda\})$ and $\lambda^* = \lambda$ if λ is the first nonzero eigenvalue of H, and $\lambda^* = \lambda_{n-1}[H]$ otherwise.

By Theorem 4.11 (i), it follows that

$$|\lambda_k[H] - \lambda_k[\widetilde{H}]| \leqslant c(\lambda_k[H] + 1)(\lambda_k[\widetilde{H}] + 1)\delta_\infty(\phi, \widetilde{\phi})\,. \tag{5.3}$$

This implies that there exists $c > 0$ such that if

$$\delta_\infty(\phi, \widetilde{\phi}) < c^{-1}\lambda_k[H]/(\lambda_k[H] + 1)^2,$$

then

$$\lambda_k[\widetilde{H}] \leqslant 2\lambda_k[H].$$

This together with (5.3) implies the existence of $c > 0$ such that if

$$\delta_\infty(\phi, \widetilde{\phi}) < c^{-1} \min\{\rho, \lambda_k[H]\}/(\lambda_k[H] + 1)^2,$$

then

$$|\lambda_k[H] - \lambda_k[\widetilde{H}]| < \rho/2.$$

Applying this inequality for $k = n-1, \ldots, n+m$, we deduce that if

$$\delta_\infty(\phi, \widetilde{\phi}) < \frac{\min\{\rho, \lambda^*\}}{c(\lambda_{n+m}[H]+1)^2},$$

then

$$\begin{aligned} |\lambda_k[\widetilde{H}] - \lambda| &\leqslant \rho/2 \quad \forall\, k \in G\,, \\ |\lambda_k[\widetilde{H}] - \lambda| &\geqslant 3\rho/2 \quad \forall\, k \in \mathbf{N} \setminus G\,. \end{aligned} \tag{5.4}$$

Hence $\dim \operatorname{ran} P_G(w^{-1}\widetilde{H}w) = m$ and, by the well-known Riesz formula, we have

$$P_G[H] = -\frac{1}{2\pi i}\int_\Gamma (H-\xi)^{-1} d\xi\,, \tag{5.5}$$

$$P_G[w^{-1}\widetilde{H}w] = -\frac{1}{2\pi i}\int_\Gamma (w^{-1}\widetilde{H}w-\xi)^{-1} d\xi\,, \tag{5.6}$$

where $\Gamma(\theta) = \lambda + \rho e^{i\theta}$, $0 \leqslant \theta < 2\pi$. Hence

$$\|P_G[H] - P_G[w^{-1}\widetilde{H}w]\| \leqslant \rho \sup_{\xi\in\Gamma} \|(w^{-1}\widetilde{H}w-\xi)^{-1} - (H-\xi)^{-1}\|. \tag{5.7}$$

Let c_1 be the same as in Theorem 4.6 (i). Using Theorem 4.6 (i) and Remark 4.9 and observing that $\lambda - \rho \leqslant |\xi| \leqslant \lambda + \rho$ and $1/|\xi| \leqslant 1/\rho$ for all $\xi \in \Gamma$, we find that

$$\delta_\infty(\phi, \widetilde{\phi}) < \frac{\rho}{c_1(1+\lambda_{n+m}^2[H]+\rho^2)}$$

implies

$$\|(w^{-1}\widetilde{H}w-\xi)^{-1} - (H-\xi)^{-1}\| \leqslant c_1\left(1 + \frac{1}{\rho} + \frac{\lambda^2}{\rho^2}\right)\delta_\infty(\phi, \widetilde{\phi}). \tag{5.8}$$

The proof of statement (i) then follows by combining (5.7) and (5.8). The proof of (ii) is similar. □

Remark 5.3. The proof of Theorem 5.2 gives some information about the dependence of the constants c_1, c_2 on $\lambda_{n-1}[H], \lambda, \lambda_{n+m}[H]$ which is useful in the sequel. For instance, in the case of statement (i), in fact we proved that there exists $c > 0$ depending only on N, τ, θ, α, and c^* such that

$$\delta_\infty(\phi, \widetilde{\phi}) \leqslant \frac{\min\{\rho, \lambda^*\}}{c(1+\rho^2+\lambda_{n+m}[H]^2)}$$

implies

$$\|P_G(H) - P_G(w^{-1}\widetilde{H}w)\| \leqslant c\left(1 + \rho + \frac{\lambda^2}{\rho}\right)\delta_\infty(\phi, \widetilde{\phi}). \tag{5.9}$$

Exactly the same is true for statement (ii), where c depends also on q_0, C, γ, and $|\Omega|$.

We are going to apply the stability estimates of Theorem 5.2 to obtain stability estimates for eigenfunctions. For this purpose, we need the following lemma.

Lemma 5.4 (selection lemma). *Let U and V be finite dimensional subspaces of a Hilbert space $\mathcal{H}$, $\dim U = \dim V = m$, and let $u_1, \dots, u_m$ be an orthonormal basis for U. Then there exists an orthonormal basis $v_1, \dots, v_m$ for V such that*

$$\|u_k - v_k\| \leqslant 5^k \|P_U - P_V\|\,, \quad k = 1, \dots, m\,, \tag{5.10}$$

where P_U and P_V are the orthogonal projections onto U and V respectively.

Proof. Step 1. It is clear that $\|P_U - P_V\| \leqslant 2$. If $1 \leqslant \|P_U - P_V\| \leqslant 2$, then the estimate (5.10) obviously holds for any choice of an orthonormal basis $v_1, \dots, v_m$ for V. So, we assume that $\|P_U - P_V\| < 1$. Let $u \in U$, $\|u\| = 1$. Then

$$\|P_V u\| = \|u + (P_V - P_U)u\| \geqslant 1 - \|P_U - P_V\| > 0\,. \tag{5.11}$$

Letting $z = P_V u / \|P_V u\|$, we have $\|z\| = 1$ and

$$\langle u, z\rangle = \frac{\langle u, P_V^2 u\rangle}{\|P_V u\|} = \|P_V u\|\,.$$

Hence

$$\|P_U - P_V\|^2 \geqslant \|(P_U - P_V)u\|^2 = \|u\|^2 - \|P_V u\|^2 \geqslant \frac{\|u - z\|^2}{2},$$

and therefore

$$\|u - z\| \leqslant \sqrt{2}\|P_U - P_V\|. \tag{5.12}$$

Step 2. Assume that $\|P_U - P_V\| \leqslant 1/6$. Let

$$z_k = \frac{P_V u_k}{\|P_V u_k\|}\ ,\ k = 1, \dots, m.$$

We prove that

$$|\langle z_k, z_l\rangle| \leqslant 3\|P_U - P_V\|\,, \quad k, l = 1, \dots, m,\ k \neq l. \tag{5.13}$$

Indeed, for $k \neq l$ we have

$$\begin{aligned}
|\langle P_V u_k, P_V u_l\rangle| &= |\langle P_V u_k - u_k, P_V u_l\rangle + \langle u_k, u_l\rangle + \langle u_k, P_V u_l - u_l\rangle| \\
&= |\langle (P_V - P_U)u_k, P_V u_l\rangle + \langle u_k, (P_V - P_U)u_l\rangle| \\
&\leqslant 2\|P_U - P_V\|,
\end{aligned}$$

and the claim is proved by recalling (5.11).

Step 3. It is easy to see that since $\|P_U - P_V\| < 1$, the vectors $z_1, \ldots, z_m$ are linearly independent. Thus, we can apply the Gram–Schmidt orthogonalization procedure, i.e., define

$$v_1 = z_1 \ , \quad v_k = \frac{z_k - \sum_{l=1}^{k-1} \langle z_k, v_l \rangle v_l}{\|z_k - \sum_{l=1}^{k-1} \langle z_k, v_l \rangle v_l\|}, \quad k = 2, \ldots, m.$$

Note that for $k = 2, \ldots, m$,

$$v_k - z_k = \left(\frac{1}{\|z_k - \sum_{l=1}^{k-1} \langle z_k, v_l \rangle v_l\|} - 1 \right) z_k - \frac{\sum_{l=1}^{k-1} \langle z_k, v_l \rangle v_l}{\|z_k - \sum_{l=1}^{k-1} \langle z_k, v_l \rangle v_l\|}$$

and

$$1 \geqslant \left\| z_k - \sum_{l=1}^{k-1} \langle z_k, v_l \rangle v_l \right\| \geqslant 1 - \sum_{l=1}^{k-1} |\langle z_k, v_l \rangle| \ .$$

Hence if

$$\sum_{l=1}^{k-1} |\langle z_k, v_l \rangle| < 1, \tag{5.14}$$

then

$$\|v_k - z_k\| \leqslant \frac{2 \sum_{l=1}^{k-1} |\langle z_k, v_l \rangle|}{1 - \sum_{l=1}^{k-1} |\langle z_k, v_l \rangle|}. \tag{5.15}$$

Also, for $s = k, \ldots, m$

$$|\langle z_s, v_k \rangle| \leqslant \frac{|\langle z_s, z_k \rangle| + \sum_{l=1}^{k-1} |\langle z_k, v_l \rangle|}{1 - \sum_{l=1}^{k-1} |\langle z_k, v_l \rangle|}. \tag{5.16}$$

Step 4. We prove that for all $k = 2, \ldots, m$

$$\|v_k - z_k\| \leqslant 3 \cdot 5^{k-1} \|P_U - P_V\|, \tag{5.17}$$

$$|\langle z_s, v_k \rangle| \leqslant 3 \cdot 5^{k-1} \|P_U - P_V\| \, , \quad s = k+1, \ldots, m, \tag{5.18}$$

provided that

$$\|P_U - P_V\| \leqslant \frac{2}{3} 5^{-k+1} \, . \tag{5.19}$$

We prove this by induction. If $k = 2$, then, by (5.13) and (5.19),

$$|\langle z_2, v_1 \rangle| = |\langle z_2, z_1 \rangle| \leqslant 3 \|P_U - P_V\| \leqslant \tfrac{2}{5}.$$

Hence, by (5.15),

$$\|v_2 - z_2\| \leqslant \frac{6 \|P_U - P_V\|}{1 - 3\|P_U - P_V\|} \leqslant 15 \|P_U - P_V\|$$

and, by (5.16) and (5.13), for $s = 3, \dots, m$

$$|\langle z_s, v_2\rangle| \leqslant \frac{6\|P_U - P_V\|}{1 - 3\|P_U - P_V\|} \leqslant 15\|P_U - P_V\|.$$

Let $2 \leqslant k \leqslant m-1$. Suppose that the inequalities (5.17) and (5.18) under the assumption (5.19) are satisfied for all $2 \leqslant j \leqslant k$. By assuming the validity of (5.19) for $k+1$ and using (5.15), we obtain

$$\|v_{k+1} - z_{k+1}\| \leqslant \frac{6(\sum_{j=1}^k 5^{j-1})\|P_U - P_V\|}{1 - 3(\sum_{j=1}^k 5^{j-1})\|P_U - P_V\|}.$$

Since $\sum_{l=1}^k 5^{l-1} \leqslant 5^k/4$, by (5.19) with k replaced by $k+1$,

$$3\sum_{l=1}^{k} 5^{l-1}\|P_U - P_V\| \leqslant 1/2.$$

Hence

$$\|v_{k+1} - z_{k+1}\| \leqslant 3 \cdot 5^k \|P_U - P_V\|.$$

Similarly, by (5.16) and (5.13), for all $s = k+2, \dots m$

$$|(z_s, v_{k+1})| \leqslant \frac{3\|P_U - P_V\| + 3(\sum_{j=1}^k 5^{j-1})\|P_U - P_V\|}{1 - 3(\sum_{j=1}^k 5^{j-1})\|P_U - P_V\|} \leqslant 3 \cdot 5^k \|P_U - P_V\|.$$

Step 5. To complete the proof, we note that, by (5.12),

$$\|u_1 - v_1\| \leqslant \sqrt{2}\|P_U - P_V\|.$$

For $k \geqslant 2$ we have that (5.19) implies

$$\|u_k - v_k\| \leqslant \|u_k - z_k\| + \|z_k - v_k\| \leqslant (\sqrt{2} + 3 \cdot 5^{k-1})\|P_U - P_V\|,$$

while if (5.19) does not hold, then $\|P_U - P_V\| > 10/(3 \cdot 5^k)$, and therefore

$$\|u_k - v_k\| \leqslant 2 \leqslant 3 \cdot 5^{k-1}\|P_U - P_V\|.$$

This completes the proof of the lemma. □

Lemma 5.5. *Let* (A) *be satisfied. Let λ be a nonzero eigenvalue of H of multiplicity m, and let $n \in \mathbf{N}$ be such that $\lambda = \lambda_n[H] = \ldots = \lambda_{n+m-1}[H]$. Then the following statements hold.*

(i) *There exists $c_1 > 0$ depending only on N, τ, θ, α, c^*, $\lambda_{n-1}[H]$, λ, and $\lambda_{n+m}[H]$ such that the following is true*: *if*

$$\delta_\infty(\phi, \widetilde{\phi}) \leqslant c_1^{-1}$$

and $\psi_n[\widetilde{H}], \dots, \psi_{n+m-1}[\widetilde{H}]$ *are orthonormal eigenfunctions of* $\widetilde{H}$ *in the space* $L^2(\Omega, \widetilde{g}dx)$*, then there are orthonormal eigenfunctions* $\psi_n[H], \dots, \psi_{n+m-1}[H]$ *of* H *in* $L^2(\Omega, gdx)$ *such that*

$$\|\psi_k[H] - \psi_k[\widetilde{H}]\|_{L^2(\Omega)} \leqslant c_1 \delta_\infty(\phi, \widetilde{\phi}), \tag{5.20}$$

for all $k = n, \dots, n+m-1$.

(ii) *Let, in addition,* (P) *be satisfied by the operators* H*,* $\widetilde{H}$*, and* T^*ST *for the same* q_0*,* γ *and* C*. Let* $s = [q_0/(q_0-2)] \max\{2, \alpha + 2\gamma\}$*. Then there exists* $c_2 > 0$ *depending only on* $N, \tau, \theta, \alpha, c^*, q_0, C, \gamma, |\Omega|, \lambda_{n-1}[H], \lambda, \lambda_{n+m}[H]$ *such that the following is true*: *if*

$$\delta_s(\phi, \widetilde{\phi}) \leqslant c_2^{-1}$$

and $\psi_n[\widetilde{H}], \dots, \psi_{n+m-1}[\widetilde{H}]$ *are orthonormal eigenfunctions of* $\widetilde{H}$ *in the space* $L^2(\Omega, \widetilde{g}dx)$*, then there are orthonormal eigenfunctions* $\psi_n[H], \dots, \psi_{n+m-1}[H]$ *of* H *in* $L^2(\Omega, gdx)$ *such that*

$$\|\psi_k[H] - \psi_k[\widetilde{H}]\|_{L^2(\Omega)} \leqslant c_2 \delta_s(\phi, \widetilde{\phi}) \tag{5.21}$$

for all $k = n, \dots, n+m-1$.

Proof. We prove only statement (ii) since the proof of (i) is similar. We first note that $f_k := w^{-1}\psi_k[\widetilde{H}]$, $k = n, \dots, n+m-1$, are orthonormal eigenfunctions in $L^2(\Omega, g\,dx)$ of $w^{-1}\widetilde{H}w$ corresponding to the eigenvalues $\lambda_n[\widetilde{H}], \dots, \lambda_{n+m-1}[\widetilde{H}]$. By Theorem 5.2 and Lemma 5.4, there exists $c > 0$ such that if $\delta_s(\phi, \widetilde{\phi}) < c^{-1}$, then there exist eigenfunctions $\psi_n[H], \dots, \psi_{n+m-1}[H]$ of H corresponding to the eigenvalue λ such that

$$\|\psi_k[H] - f_k\|_{L^2(\Omega)} \leqslant c\delta_s(\phi, \widetilde{\phi})\,. \tag{5.22}$$

To complete the proof, it suffices to note that

$$\begin{aligned}\|f_k - \psi_k[\widetilde{H}]\|_{L^2(\Omega)} &\leqslant \|1 - w^{-1}\|_{L^s(\Omega)} \|\psi_k[\widetilde{H}]\|_{L^{2s/(s-2)}(\Omega)} \\ &\leqslant c\|\nabla\phi - \nabla\widetilde{\phi}\|_{L^s(\Omega)}\,.\end{aligned}$$

The lemma is proved. □

In the following theorem, we estimate the deviation of the eigenfunctions $\psi_k[\widetilde{L}]$ of $\widetilde{L}$ from the eigenfunctions $\psi_k[L]$ of L. We adopt the convention that $\psi_k[L]$ and $\psi_k[\widetilde{L}]$ are extended by zero outside $\phi(\Omega)$ and $\widetilde{\phi}(\Omega)$ respectively.

Theorem 5.6 (stability of eigenfunctions)**.** *Let* (A) *be satisfied. Let* λ *be a nonzero eigenvalue of* L *of multiplicity* m*, and let* $n \in \mathbf{N}$ *be such that* $\lambda = \lambda_n[L] = \ldots = \lambda_{n+m-1}[L]$*. Then the following statements hold.*

(i) *There exists* $c_1 > 0$ *depending only on* N, τ, θ, α, c^*, $\lambda_{n-1}[L]$, λ, $\lambda_{n+m}[L]$ *such that if* $\delta_\infty(\phi, \widetilde{\phi}) \leqslant c_1^{-1}$ *and* $\psi_n[\widetilde{L}], \ldots, \psi_{n+m-1}[\widetilde{L}]$ *are orthonormal eigenfunctions of* $\widetilde{L}$ *in* $L^2(\widetilde{\phi}(\Omega))$*, then there exist orthonormal eigenfunctions* $\psi_n[L], \ldots, \psi_{n+m-1}[L]$ *of* L *in* $L^2(\phi(\Omega))$ *such that*

$$\|\psi_k[L] - \psi_k[\widetilde{L}]\|_{L^2(\phi(\Omega)\cup\widetilde{\phi}(\Omega))} \leqslant c\big(\delta_\infty(\phi, \widetilde{\phi}) + \\ + \|\psi_k[L] \circ \phi - \psi_k[L] \circ \widetilde{\phi}\|_{L^2(\Omega)} + \|\psi_k[\widetilde{L}] \circ \phi - \psi_k[\widetilde{L}] \circ \widetilde{\phi}\|_{L^2(\Omega)}\big), \quad (5.23)$$

for all $k = n, \ldots, n+m-1$.

(ii) *Let, in addition,* (P) *be satisfied by the operators* L, $\widetilde{L}$, *and* $\widehat{L}$ *for the same* q_0, γ, *and* C*. Let* $s = [q_0/(q_0-2)] \max\{2, \alpha + 2\gamma\}$*. Then there exists* $c_2 > 0$ *depending only on* N, τ, θ, α, c^*, q_0, C, γ, $|\Omega|$, $\lambda_{n-1}[L]$, λ, *and* $\lambda_{n+m}[L]$ *such that the following is true*: *if* $\delta_s(\phi, \widetilde{\phi}) \leqslant c_1^{-1}$ *and* $\psi_n[\widetilde{L}], \ldots, \psi_{n+m-1}[\widetilde{L}]$ *are orthonormal eigenfunctions of* $\widetilde{L}$ *in* $L^2(\widetilde{\phi}(\Omega))$*, then there exist orthonormal eigenfunctions* $\psi_n[L], \ldots, \psi_{n+m-1}[L]$ *of* L *in* $L^2(\phi(\Omega))$ *such that*

$$\|\psi_k[L] - \psi_k[\widetilde{L}]\|_{L^2(\phi(\Omega)\cup\widetilde{\phi}(\Omega))} \leqslant c\big(\delta_s(\phi, \widetilde{\phi}) + \|\psi_k[L] \circ \phi \\ - \psi_k[L] \circ \widetilde{\phi}\|_{L^2(\Omega)} + \|\psi_k[\widetilde{L}] \circ \phi - \psi_k[\widetilde{L}] \circ \widetilde{\phi}\|_{L^2(\Omega)}\big), \quad (5.24)$$

for all $k = n, \ldots, n+m-1$.

Remark 5.7. We note that if, in addition, the semigroup e^{-Lt} is ultracontractive, then the eigenfunctions are bounded hence

$$\|\psi_k[L] \circ \phi - \psi_k[L] \circ \widetilde{\phi}\|_{L^2(\Omega)} + \|\psi_k[\widetilde{L}] \circ \phi - \psi_k[\widetilde{L}] \circ \widetilde{\phi}\|_{L^2(\Omega)} \leqslant c(\lambda)|\mathcal{D}|^{1/2},$$

where $\mathcal{D} = \{x \in \Omega : \phi(x) \neq \widetilde{\phi}(x)\}$.

Proof of Theorem 5.6. We set

$$\psi_k[\widetilde{H}] = \psi_k[\widetilde{L}] \circ \widetilde{\phi}$$

for all $k = n, \ldots, n+m-1$, so that $\psi_n[\widetilde{H}], \ldots, \widetilde{\psi}_{n+m-1}[\widetilde{H}]$ are orthonormal eigenfunctions in $L^2(\Omega, \widetilde{g}dx)$ of the operator $\widetilde{H}$ corresponding to the eigenvalues $\lambda_n[\widetilde{H}], \ldots, \lambda_{n+m-1}[\widetilde{H}]$. By Lemma 5.5 (i), there exists $c_1 > 0$ such that if $\delta_\infty(\phi, \widetilde{\phi}) < c_1^{-1}$, then there exist orthonormal eigenfunctions $\psi_n[H], \ldots, \psi_{n+m-1}[H]$ in $L^2(\Omega, gdx)$ of H corresponding to the eigenvalue λ such that the inequality (5.20) is satisfied. We now set

$$\psi_k[L] = \psi_k[H] \circ \ \phi^{(-1)}$$

for all $k = n, \dots, n+m-1$, so that $\psi_n[L], \dots, \psi_{n+m-1}[L]$ are orthonormal eigenfunctions in $L^2(\phi(\Omega))$ of L corresponding to the eigenvalue λ. Changing variables in integrals, we obtain

$$\begin{aligned}\|\psi_k[L] - \psi_k[\widetilde{L}]\|_{L^2(\widetilde{\phi}(\Omega))} &\leqslant \|\psi_k[L] \circ \widetilde{\phi} - \psi_k[\widetilde{L}] \circ \widetilde{\phi}\|_{L^2(\Omega)} \\ &\leqslant c\big(\|\psi_k[L] \circ \widetilde{\phi} - \psi_k[L] \circ \phi\|_{L^2(\Omega)} + \|\psi_k[L] \circ \phi - \psi_k[\widetilde{L}] \circ \widetilde{\phi}\|_{L^2(\Omega)}\big) \\ &= c\big(\|\psi_k[L] \circ \widetilde{\phi} - \psi_k[L] \circ \phi\|_{L^2(\Omega)} + \|\psi_k[H] - \psi_k[\widetilde{H}]\|_{L^2(\Omega)}\big).\end{aligned}$$

In the same way,

$$\begin{aligned}\|\psi_k[L] - \psi_k[\widetilde{L}]\|_{L^2(\phi(\Omega))} \leqslant c(&\|\psi_k[\widetilde{L}] \circ \widetilde{\phi} - \psi_k[\widetilde{L}] \circ \phi\|_{L^2(\Omega)} \\ &+ \|\psi_k[H] - \psi_k[\widetilde{H}]\|_{L^2(\Omega)}).\end{aligned}$$

Hence (5.23) and (5.24) follow from (5.20) and (5.21) respectively. □

6 On the Regularity of Eigenfunctions

In this section, we obtain sufficient conditions for the validity of conditions (P1) and (P2). We begin by recalling the following known result based on the notion of ultracontractivity which guarantees the validity of property (P1) under rather general assumptions, namely under the assumption that a Sobolev-type embedding theorem holds for the space $\mathcal{V}$.

Lemma 6.1. *Let Ω be a domain in $\mathbf{R}^N$ of finite measure and $\mathcal{V}$ a closed subspace of $W^{1,2}(\Omega)$ containing $W_0^{1,2}(\Omega)$. Assume that there exist $p > 2$ and $D > 0$ such that*

$$\|u\|_{L^p(\Omega)} \leqslant D\|u\|_{W^{1,2}(\Omega)} \tag{6.1}$$

for all $u \in \mathcal{V}$. Then the following statements hold.

(i) *The condition* (3.2) *is satisfied for any $\alpha > \frac{p}{p-2}$.*

(ii) *The eigenfunctions of the operators H, $\widetilde{H}$ and T^*ST satisfy* (P1) *with $q_0 = \infty$, $\gamma = \frac{p}{2(p-2)}$, where C depends only on p, D, τ, θ, and c^*.*

Proof. For the proof of statement (i) we refer to [3, Theorem 7], where the case $\mathcal{V} = W^{1,2}(\Omega)$ is considered. The proof works word by word also in the slightly more general case considered here. The proof of statement (ii) is the same as in [3, Theorem 7], where it is proved that for the Neumann Laplacian property (P1) is satisfied if (6.1) holds: this proof can be easily adapted to the operators H, $\widetilde{H}$, and T^*ST. □

We now give conditions for the validity of property (P2). We consider first the case where an a priori estimate holds for the operators L and $\widetilde{L}$, which is typically the case of sufficiently smooth open sets and coefficients. Then we consider a more general situation based on an approach which goes back to Meyers [19].

The regular case. Recall that an open set in $\mathbf{R}^N$ satisfies the interior cone condition with the parameters $R > 0$ and $h > 0$ if for all $x \in \Omega$ there exists a cone $K_x \subset \Omega$ with the point x as vertex congruent to the cone

$$K(R,h) = \left\{ x \in \mathbf{R}^N : \ 0 < \left(\sum_{i=1}^{N-1} x_i^2 \right)^{1/2} < \frac{Rx_N}{h} < R \right\}.$$

In this paper, the cone condition is used in order to guarantee the validity of the standard Sobolev embedding.

The next theorem is a simplified version of Theorem 5.1 in [6].

Theorem 6.2. *Let $R > 0$, $h > 0$. Let U be an open set in $\mathbf{R}^N$ satisfying the interior cone condition with the parameters R and h, and let E be an operator in $L^2(U)$ satisfying the following a priori estimate: there exists $B > 0$ such that if $2 \leqslant p < N+2$ and $u \in \mathrm{Dom}(E)$, $Eu \in L^p(U)$, then $u \in W^{2,p}(U)$ and*

$$\|u\|_{W^{2,p}(U)} \leqslant B \left(\|Eu\|_{L^p(U)} + \|u\|_{L^2(U)} \right). \tag{6.2}$$

Assume that $E\psi = \lambda\psi$ for some $\psi \in \mathrm{Dom}(E)$ and $\lambda \in \mathbf{C}$. Then there exists $c > 0$ depending only on R, h, N, and B such that for $\mu = 0, 1$,

$$\|\psi\|_{W^{\mu,\infty}(U)} \leqslant c(1 + |\lambda|)^{\frac{N}{4} + \frac{\mu}{2}} \|\psi\|_{L^2(U)}. \tag{6.3}$$

Theorem 6.3. *Let* (A) *be satisfied, and let $\phi(\Omega)$ and $\widetilde{\phi}(\Omega)$ be open sets satisfying the interior cone condition with the same parameters R, h. If the operators L, $\widetilde{L}$ satisfy the a priori estimate* (6.2) *with the same B, then the operators H, $\widetilde{H}$, and T^*ST satisfy property* (P) *with $q_0 = \infty$, $\gamma = N/4$ and C depending only on τ, R, h, c^*, θ, and B.*

Proof. Recall that H, $\widetilde{H}$, and T^*ST are the operators obtained by pulling-back to Ω the operators L, $\widetilde{L}$, and $\widehat{L}$ respectively. It is clear that $\widehat{L}$ also satisfies the a priori estimate (6.2). Thus, by Theorem 6.2, the eigenfunctions of the operators L, $\widetilde{L}$, and $\widehat{L}$ satisfy the condition (6.3). Hence, by pulling such eigenfunctions back to Ω, it follows that the eigenfunctions of H, $\widetilde{H}$, and T^*ST satisfy (P1) and (P2) with $q_0 = \infty$, $\gamma = N/4$ and C as in the statement. □

The general case. Here, we assume that $\mathcal{V} = \mathrm{cl}_{W^{1,2}(\Omega)}\mathcal{V}_0$, where $\mathcal{V}_0$ is a space of functions defined in Ω such that $C_c^\infty(\Omega) \subset \mathcal{V}_0 \subset W^{1,\infty}(\Omega)$. Moreover, for all $1 < q < \infty$ we set

$$V_q = \mathrm{cl}_{W^{1,q}(\Omega)} \mathcal{V}_0.$$

Let $-\Delta_q : V_q \to (V_{q'})'$ be the operator defined by

$$(-\Delta_q u, \psi) = \int_\Omega \nabla u \cdot \nabla \psi dx$$

for all $u \in V_q$, $\psi \in V_{q'}$.

The following theorem is a variant of a result of Gröger [13] (cf. also [2]).

Theorem 6.4. *Let* (A) *be satisfied. Assume that there exists $q_1 > 2$ such that the operator $I - \Delta_q : V_q \to (V_{q'})'$ has a bounded inverse for all $2 \leqslant q \leqslant q_1$. Then there exist $q_0 > 2$ and $c > 0$ depending only on $\mathcal{V}_0$, τ, and θ such that if u is an eigenfunction of one of the operators H, $\widetilde{H}$, T^*ST and λ is the corresponding eigenvalue, then*

$$\|\nabla u\|_q \leqslant c(1+\lambda)\|u\|_q \tag{6.4}$$

for all $2 \leqslant q \leqslant q_0$.

Moreover, if Ω is such that the interior cone condition holds, then there exists $c > 0$ depending only on $\mathcal{V}_0$, τ, and θ such that

$$\|\nabla u\|_q \leqslant c(1+\lambda)\|u\|_{\frac{Nq}{N+q}} \tag{6.5}$$

for all $2 < q \leqslant q_0$.

Proof. We prove the statement for the operator T^*ST, the other cases being similar. We divide the proof into three steps.

Step 1. We define

$$Q(u,\psi) = \int_\Omega u\psi g dx + \int_\Omega \widetilde{a} \nabla u \cdot \nabla \psi \widetilde{g} dx,$$

$$Q_0(u,\psi) = \int_\Omega u\psi dx + \int_\Omega \nabla u \cdot \nabla \psi dx$$

for all $u \in V_q$, $\psi \in V_{q'}$. Since[2]

$$\begin{aligned} |Q_0(u,\psi) - \beta Q(u,\psi)| \leqslant \max\{&\|1-\beta g\|_{L^\infty(\Omega)}, \|I - \beta \widetilde{a}\widetilde{g}\|_{L^\infty(\Omega)}\} \\ &\times \|u\|_{W^{1,q}(\Omega)} \|\psi\|_{W^{1,q'}(\Omega)}, \end{aligned}$$

there exist $\beta > 0$ and $0 < c < 1$ depending only on N, τ, and θ such that

$$|Q_0(u,\psi) - \beta Q(u,\psi)| \leqslant c\|u\|_{W^{1,q}(\Omega)} \|\psi\|_{W^{1,q'}(\Omega)} \tag{6.6}$$

for all $u \in W^{1,q}(\Omega)$ and $\psi \in W^{1,q'}(\Omega)$.

[2] Here, we use $\|f\|^p_{W^{1,p}(\Omega)} = \|f\|^p_{L^p(\Omega)} + \| |\nabla f| \|^p_{L^p(\Omega)}$ as the norm in $W^{1,p}(\Omega)$.

Step 2. Using the fact that $\|(I-\Delta_2)^{-1}\| = 1$ and that $q \mapsto \|(I-\Delta_q)^{-1}\|$ is continuous and taking into account that $2/(c+1) > 1$, we find that there exists $q_0 > 2$ such that

$$\|(I-\Delta_q)^{-1}\| < \frac{2}{c+1} \tag{6.7}$$

for all $2 \leqslant q \leqslant q_0$. By (6.6), for all $2 \leqslant q \leqslant q_0$

$$\begin{aligned} &\inf_{\|u\|_{W^{1,q}(\Omega)}=1} \sup_{\|\psi\|_{W^{1,q'}(\Omega)}=1} Q(u,\psi) \\ &\quad \geqslant \frac{1}{\beta} \inf_{\|\psi\|_{W^{1,q'}(\Omega)}=1} \sup_{\|u\|_{W^{1,q}(\Omega)}=1} Q_0(u,\psi) - \frac{c}{\beta} \\ &\quad = \frac{1}{\beta}\|(I-\Delta_q)^{-1}\|^{-1} - \frac{c}{\beta} > \frac{1-c}{2\beta} > 0\,. \end{aligned} \tag{6.8}$$

Step 3. By (6.8), the operator $I + (T^*ST)_q$ from V_q to $V'_{q'}$ defined by

$$(I + (T^*ST)_q u, \psi) = Q(u,\psi) \tag{6.9}$$

has a bounded inverse such that

$$\|(I + (T^*ST)_q)^{-1}\| = \left(\inf_{\|u\|_{W^{1,q}(\Omega)}=1} \sup_{\|\psi\|_{W^{1,q'}(\Omega)}=1} Q(u,\psi) \right)^{-1} < \frac{2\beta}{1-c}. \tag{6.10}$$

Then (6.4) follows from (6.9), (6.10), and the relation

$$Q(u,\psi) = (1+\lambda)\int_\Omega u\psi g\,dx \tag{6.11}$$

for all $\psi \in V_{q'}$.

Now, if Ω satisfies the interior cone condition, then the standard Sobolev embedding holds. Thus, if $q > 2$, then $q' < 2 \leqslant N$. Hence $V_{q'}$ is continuously embedded into $L^{\frac{Nq'}{N-q'}}(\Omega)$. By (6.11), we have

$$\begin{aligned} \|u\|_{W^{1,q}(\Omega)} &\leqslant (1+\lambda)\|(I + (T^*ST)_q)^{(-1)}\| \sup_{\|\psi\|_{W^{1,q'}(\Omega)}=1} \Big| \int_\Omega u\psi g\,dx \Big| \\ &\leqslant \frac{2\beta}{1-c}(1+\lambda)\|g\|_{L^\infty(\Omega)}\|u\|_{L^{\frac{Nq}{N+q}}(\Omega)} \sup_{\|\psi\|_{W^{1,q'}(\Omega)}=1} \|\psi\|_{L^{\frac{Nq'}{N-q'}}(\Omega)}\,, \end{aligned} \tag{6.12}$$

and the last supremum is finite due to the Sobolev embedding. □

Remark 6.5. If Ω satisfies the interior cone condition, then the inequality (6.1) is satisfied with $p = 2N/(N-2)$ if $N \geqslant 3$ and with any $p > 2$ if $N = 2$. Then, by Lemma 6.1, the condition (3.2) holds for any $\alpha > N/2$ and the operators H, $\widetilde{H}$, T^*ST, L, $\widetilde{L}$, and $\widehat{L}$ satisfy property (P1) with $q_0 = \infty$, $\gamma = N/4$ if $N \geqslant 3$ and any $\gamma > 1/2$ if $N = 2$. In fact, if $N = 2$, property (P1) is also satisfied for $\gamma = 1/2$. This follows from [11, Theorem 2.4.4] and [3, Lemma 10]. Thus, by the second part of Theorem 6.4, both properties (P1) and (P2) are satisfied for some $q_0 > 2$ and $\gamma = N(q_0 - 2)/(4q_0)$ for any $N \geqslant 2$.

If Ω is of class $C^{0,\nu}$ (i.e., Ω is locally a subgraph of $C^{0,\nu}$ functions) with $0 < \nu < 1$, then the inequality (6.1) is satisfied with $p = 2(N+\nu-1)/(N-\nu-1)$ for any $N \geqslant 2$ (cf. also [3]). Thus, Lemma 6.1 implies that the condition (3.2) holds for any $\alpha > (N+\nu-1)/(2\nu)$ and the operators H, $\widetilde{H}$, T^*ST, L, $\widetilde{L}$, $\widehat{L}$ satisfy property (P1) with $q_0 = \infty$ and $\gamma = (N+\nu-1)/(4\nu)$.

7 Estimates via Lebesgue Measure

In this section, we consider two examples to which we apply the results of the previous sections in order to obtain stability estimates via the Lebesgue measure.

Let $A_{ij} \in L^\infty(\mathbf{R}^N)$ be real-valued functions satisfying $A_{ij} = A_{ji}$ for all $i, j = 1, \ldots, N$ and the condition (2.2). Let Ω be a bounded domain in $\mathbf{R}^N$ of class $C^{0,1}$, and let Γ be an open subset of $\partial\Omega$ with Lipschitz boundary in $\partial\Omega$ (cf. Definition 7.1 below). We consider the eigenvalue problem with the mixed Dirichlet–Neumann boundary conditions

$$\begin{cases} -\sum_{i,j=1}^N \frac{\partial}{\partial x_i}\big(A_{ij}(x)\frac{\partial u}{\partial x_j}\big) = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma, \\ \sum_{i,j=1}^N A_{ij}\frac{\partial u}{\partial x_j}\nu_i = 0 & \text{on } \partial\Omega \setminus \Gamma, \end{cases} \tag{7.1}$$

where ν denotes the exterior unit normal to $\partial\Omega$. Note that our analysis comprehends the "simpler" cases $\Gamma = \partial\Omega$ (Dirichlet boundary conditions) or $\Gamma = \varnothing$ (Neumann boundary conditions), as well as all other cases where Γ is a connected component of $\partial\Omega$ (the boundary of Γ in $\partial\Omega$ is empty) (cf. [13] for details).

We denote by $\lambda_n[\Omega, \Gamma]$ the sequence of eigenvalues of the problem (7.1) and by $\psi_n[\Omega, \Gamma]$ the corresponding orthonormal system of eigenfunctions in $L^2(\Omega)$. In this section, we compare the eigenvalues and the eigenfunctions corresponding to open sets Ω and $\widetilde{\Omega}$ and the associate portions of the boundaries $\Gamma \subset \partial\Omega$ and $\widetilde{\Gamma} \subset \partial\widetilde{\Omega}$. To do so, we think of Ω as a fixed reference domain and we apply the results of the previous sections to transformations ϕ and

$\widetilde{\phi}$ defined in Ω, where $\phi = Id$ and $\widetilde{\phi}$ is a suitably constructed bi-Lipschitz homeomorphism such that $\widetilde{\Omega} = \widetilde{\phi}(\Omega)$ and $\widetilde{\Gamma} = \widetilde{\phi}(\Gamma)$.

Before doing so, we recall the weak formulation of the problem (7.1) in Ω. For $\Gamma \subset \partial\Omega$ we consider the space $W^{1,2}_{\Gamma}(\Omega)$ obtained by taking the closure of $C^{\infty}_{\Gamma}(\overline{\Omega})$ in $W^{1,2}(\Omega)$, where $C^{\infty}_{\Gamma}(\overline{\Omega})$ denotes the space of functions in $C^{\infty}(\overline{\Omega})$ vanishing in a neighborhood of Γ. Then the eigenvalues and eigenfunctions of the problem (7.1) in Ω are the eigenvalues and eigenfunctions of the operator L associated with the sesquilinear form Q_L defined on $\mathcal{W} := W^{1,2}_{\Gamma}(\Omega)$ as in (2.3).

Definition 7.1. Let Ω be a bounded open set in $\mathbf{R}^N$ of class $C^{0,1}$, and let Γ be an open subset of $\partial\Omega$. We say that Γ has *Lipschitz continuous boundary* $\partial\Gamma$ in $\partial\Omega$ if for all $x \in \partial\Gamma$ there exists an open neighborhood U of x in $\mathbf{R}^N$ and $\phi \in \Phi(U)$ such that

$$\begin{aligned}\phi(U \cap (\Omega \cup \Gamma)) = \{x \in \mathbf{R}^N : |x| < 1,\, x_N < 0\} \\ \cup \{x \in \mathbf{R}^N : |x| < 1,\, x_N \leqslant 0,\, x_1 > 0\}\,.\end{aligned}$$

7.1 Local perturbations

In this subsection, we consider open sets belonging to the following class.

Definition 7.2. Let V be a bounded open cylinder, i.e., there exists a rotation R such that $R(V) = W \times]a,b[$, where W is a bounded convex open set in $\mathbf{R}^{N-1}$. Let $M, \rho > 0$. We say that a bounded open set $\Omega \subset \mathbf{R}^N$ belongs to $\mathcal{C}^{m,1}_M(V,R,\rho)$ if Ω is of class $C^{m,1}$ (i.e., Ω is locally a subgraph of $C^{m,1}$ functions) and there exists a function $g \in C^{m,1}(\overline{W})$ such that $a+\rho \leqslant g \leqslant b$, $|g|_{m,1} := \sum_{0<|\alpha|\leqslant m+1} \|D^\alpha g\|_{L^\infty(W)} \leqslant M$, and

$$R(\Omega \cap V) = \{(\overline{x}, x_N) \ : \ \overline{x} \in W\,,\, a < x_N < g(\overline{x})\}. \tag{7.2}$$

Let $\Omega, \widetilde{\Omega} \in \mathcal{C}^{0,1}_M(V,R,\rho)$ be such that $\Omega \cap (V_\rho)^c = \widetilde{\Omega} \cap (V_\rho)^c$. We assume that the corresponding sets $\Gamma \subset \partial\Omega$, $\widetilde{\Gamma} \subset \partial\widetilde{\Omega}$, where the Dirichlet boundary conditions are imposed, are such that

$$\Gamma \cap V^c = \widetilde{\Gamma} \cap V^c \quad \text{and} \quad P_{R^{(-1)}W}(\Gamma \cap V) = P_{R^{(-1)}W}(\widetilde{\Gamma} \cap V), \tag{7.3}$$

where $P_{R^{(-1)}W}$ denotes the orthogonal projection onto $R^{(-1)}W$. Given Γ, the condition (7.3) uniquely determines $\widetilde{\Gamma}$.

Theorem 7.3. *Let $\Omega \in \mathcal{C}^{0,1}_M(V,R,\rho)$, and let Γ be an open subset of $\partial\Omega$ with Lipschitz continuous boundary in $\partial\Omega$. Then there exists $2 < q_0 \leqslant \infty$ such that for any $r > \max\{2, N(q_0-1)/q_0\}$ the following statements hold.*

(i) *There exists* $c_1 > 0$ *such that*

$$\left(\sum_{n=1}^{\infty} \left| \frac{1}{\lambda_n[\widetilde{\Omega}, \widetilde{\Gamma}] + 1} - \frac{1}{\lambda_n[\Omega, \Gamma] + 1} \right|^r \right)^{1/r} \leqslant c_1 |\Omega \vartriangle \widetilde{\Omega}|^{\frac{q_0 - 2}{r q_0}} \tag{7.4}$$

for all $\widetilde{\Omega} \in \mathcal{C}^{0,1}_M(V, R, \rho)$ *such that* $\widetilde{\Omega} \cap (V_\rho)^c = \Omega \cap (V_\rho)^c$, $|\Omega \vartriangle \widetilde{\Omega}| \leqslant c_1^{-1}$, *where* $\widetilde{\Gamma} \subset \partial \widetilde{\Omega}$ *is determined by the condition* (7.3).

(ii) *Let* $\lambda[\Omega, \Gamma]$ *be an eigenvalue of multiplicity* m, *and let* $n \in \mathbf{N}$ *be such that* $\lambda[\Omega, \Gamma] = \lambda_n[\Omega, \Gamma] = \cdots = \lambda_{n+m-1}[\Omega, \Gamma]$. *There exists* $c_2 > 0$ *such that the following is true*: *if* $\widetilde{\Omega} \in \mathcal{C}^{0,1}_M(V, R, \rho)$, $\Omega \cap (V_\rho)^c = \widetilde{\Omega} \cap (V_\rho)^c$, $|\Omega \vartriangle \widetilde{\Omega}| \leqslant c_2^{-1}$, *and* $\widetilde{\Gamma} \subset \partial \widetilde{\Omega}$ *is determined by* (7.3), *then, given orthonormal eigenfunctions* $\psi_n[\widetilde{\Omega}, \widetilde{\Gamma}], \ldots, \psi_{n+m-1}[\widetilde{\Omega}, \widetilde{\Gamma}]$, *there exist corresponding orthonormal eigenfunctions* $\psi_n[\Omega, \Gamma], \ldots, \psi_{n+m-1}[\Omega, \Gamma]$ *such that*

$$\|\psi_n[\Omega, \Gamma] - \psi_n[\widetilde{\Omega}, \widetilde{\Gamma}]\|_{L^2(\Omega \cup \widetilde{\Omega})} \leqslant c_2 |\Omega \vartriangle \widetilde{\Omega}|^{\frac{q_0 - 2}{r q_0}}.$$

If, in addition, $A_{ij} \in C^{0,1}(\mathbf{R}^N)$, $\Omega, \widetilde{\Omega} \in \mathcal{C}^{1,1}_M(V, R, \rho)$ *and* Γ *is a connected component of* $\partial \Omega$, *then statements* (i) *and* (ii) *hold with* $q_0 = \infty$.

For the proof we need the following variant of Lemma 4.1 in [5].

Lemma 7.4. *Let* W *be a bounded convex open set in* $\mathbf{R}^{N-1}$, *and let* $M > 0$. *Let* $0 < \rho < b - a$ *and* g_1, g_2 *be Lipschitz continuous functions from* $\overline{W}$ *to* $\mathbf{R}$ *such that*

$$a + \rho < g_1(\overline{x}), \quad g_2(\overline{x}) < b \tag{7.5}$$

for all $\overline{x} \in \overline{W}$ *and such that* $\mathrm{Lip} g_1$, $\mathrm{Lip} g_2 \leqslant M$. *Suppose that* $\delta = \frac{\rho}{2(b-a)}$, $g_3 = \min\{g_1, g_2\} - \delta |g_1 - g_2|$, *and*

$$\mathcal{O}_k := \{(\overline{x}, x_N) : \ \overline{x} \in W, \ a < x_N < g_k(\overline{x})\} \tag{7.6}$$

for $k = 1, 2, 3$. *Let* Φ *be the map from* $\overline{\mathcal{O}}_1$ *into* $\overline{\mathcal{O}}_2$ *defined as follows*:

if $g_2(\overline{x}) \leqslant g_1(\overline{x})$, *then*

$$\Phi(\overline{x}, x_N) \equiv \begin{cases} (\overline{x}, x_N) & \text{if } \ (\overline{x}, x_N) \in \overline{\mathcal{O}}_3, \\ \left(\overline{x}, g_2(\overline{x}) + \frac{\delta}{\delta+1}(x_N - g_1(\overline{x}))\right) & \text{if } \ (\overline{x}, x_N) \in \overline{\mathcal{O}}_1 \setminus \overline{\mathcal{O}}_3; \end{cases} \tag{7.7}$$

if $g_2(\overline{x}) > g_1(\overline{x})$, *then*

$$\Phi(\overline{x}, x_N) \equiv \begin{cases} (\overline{x}, x_N) & \text{if } \ (\overline{x}, x_N) \in \overline{\mathcal{O}}_3, \\ \left(\overline{x}, g_2(\overline{x}) + \frac{\delta+1}{\delta}(x_N - g_1(\overline{x}))\right) & \text{if } \ (\overline{x}, x_N) \in \overline{\mathcal{O}}_1 \setminus \overline{\mathcal{O}}_3. \end{cases} \tag{7.8}$$

Then $\varnothing \neq \mathcal{O}_3 \subset \mathcal{O}_1 \cap \mathcal{O}_2$,

$$|\{x \in \mathcal{O}_1 : \ \Phi(x) \neq x\}| = |\mathcal{O}_1 \setminus \mathcal{O}_3| \leqslant 2 |\mathcal{O}_1 \vartriangle \mathcal{O}_2|, \tag{7.9}$$

and Φ is a bi-Lipschitz homeomorphism of $\overline{\mathcal{O}}_1$ onto $\overline{\mathcal{O}}_2$. Moreover, $\Phi \in \Phi_\tau(\Omega)$, where τ depends only on N, M, δ.

Proof. The proof is the same as that of Lemma 4.1. in [5], where the case $g_2 \leqslant g_1$ was considered: here we simply replace $g_1 - g_2$ with $|g_1 - g_2|$. □

Proof of Theorem 7.3. We apply Theorems 4.11 and 5.6 with $\phi = Id$ and $\widetilde{\phi}$ given by

$$\widetilde{\phi}(x) = \begin{cases} x, & x \in \overline{\Omega} \setminus V, \\ R^{(-1)} \circ \Phi \circ R(x), & x \in \overline{\Omega} \cap V. \end{cases} \tag{7.10}$$

Here, Φ is defined as in Lemma 7.4 for $g_1 = g$ and $g_2 = \widetilde{g}$, where $g, \widetilde{g}$ are the functions describing the boundaries in V of Ω, $\widetilde{\Omega}$ respectively, as in Definition 7.2. Then clearly $\phi, \widetilde{\phi} \in \Phi_\tau(\Omega)$, where τ depends only on N, V, M, ρ. It is clear that $\phi(\Omega) = \Omega$ and $\widetilde{\phi}(\Omega) = \widetilde{\Omega}$. Moreover, $\widetilde{\phi}(\Gamma) = \widetilde{\Gamma}$. Hence

$$C_{\widetilde{\phi}}[W^{1,2}_{\widetilde{\Gamma}}(\widetilde{\Omega})] = C_\phi[W^{1,2}_\Gamma(\Omega)].$$

Moreover, the condition (3.2) is satisfied for any $\alpha > N/2$ (cf. Remark 3.1). Hence assumption (A) is satisfied. Note that, by (7.9) and the boundedness of the coefficients A_{ij},

$$\begin{aligned} \delta_p(\phi, \widetilde{\phi})^p &\leqslant c \int_{\{x \in \Omega : \phi(x) \neq \widetilde{\phi}(x)\}} \big(|\nabla\phi - \nabla\widetilde{\phi}|^p + |A \circ \phi - A \circ \widetilde{\phi}|^p\big) dx \\ &\leqslant c|\Omega \vartriangle \widetilde{\Omega}|. \end{aligned} \tag{7.11}$$

By [13, Theorem 3], the assumption of Theorem 6.4 is satisfied for the space $\mathcal{V}_0 = C^\infty_\Gamma(\overline{\Omega})$ for some $2 < q_1 < \infty$. Thus, by Remark 6.5, the operators L, $\widetilde{L}$, and $\widehat{L}$ satisfy properties (P1) and (P2) for some $2 < q_0 < \infty$ and $\gamma = N(q_0 - 2)/(4q_0)$. Thus, statement (i) follows from Theorem 4.11 (ii) with $p = q_0/(q_0 - 2)$. Moreover, Theorem 5.6 (ii) provides the existence of orthonormal eigenfunctions $\psi_k[\Omega, \Gamma]$ satisfying the estimate (5.24) with $s = [q_0/(q_0-2)] \max\{2, N(q_0-1)/q_0\}$. By Lemma 6.1, the functions $\psi_k[\Omega, \Gamma]$, $\psi_k[\widetilde{\Omega}, \widetilde{\Gamma}]$ are bounded. Hence, by (7.9),

$$\|\psi_k[\Omega, \Gamma]\circ\phi - \psi_k[\Omega, \Gamma]\circ\widetilde{\phi}\|^2_{L^2(\Omega)}, \|\psi_k[\widetilde{\Omega}, \widetilde{\Gamma}]\circ\phi - \psi_k[\widetilde{\Omega}, \widetilde{\Gamma}]\circ\widetilde{\phi}\|^2_{L^2(\Omega)} \leqslant c|\Omega \vartriangle \widetilde{\Omega}|. \tag{7.12}$$

Thus, statement (ii) follows from the estimates (5.24) and (7.12).

Finally, if $A_{ij} \in C^{0,1}(\mathbf{R}^N)$, $\Omega, \widetilde{\Omega} \in \mathcal{C}^{1,1}_M(V, R, \rho)$, and Γ is a connected component of $\partial\Omega$, by Troianiello [25, Thm. 3.17 (ii)], the operators L and $\widetilde{L}$ satisfy the a priori estimate (6.2) in Ω and $\widetilde{\Omega}$ respectively. Thus, by Theorem 6.3, the operators L, $\widetilde{L}$, and $\widehat{L}$ satisfy properties (P1) and (P2) with $q_0 = \infty$ and $\gamma = N/4$, and the result follows as above. □

7.2 Global normal perturbations

Let Ω be a bounded domain with C^2 boundary. By the Tubular Neighborhood Theorem, there exists $t > 0$ such that for each $x \in (\partial\Omega)^t := \{x \in \mathbf{R}^N : \mathrm{dist}(x, \partial\Omega) < t\}$ there exists a unique couple $(\overline{x}, s) \in \partial\Omega\times]-t, t[$ such that $x = \overline{x} + s\nu(\overline{x})$; moreover, $\overline{x}$ is the (unique) nearest to x point of the boundary and $s = \mathrm{dist}(x, \partial\Omega)$. One can see that, by possibly reducing the value of t, the map $x \mapsto (\overline{x}, s)$ is a bi-Lipschitz homeomorphism of $(\partial\Omega)^t$ onto $\partial\Omega\times]-t, t[$. Accordingly, we often use the coordinates $(\overline{x}, s)$ to represent the point $x \in (\partial\Omega)^t$.

In this subsection, we consider deformations $\widetilde{\Omega}$ of Ω of the form

$$\widetilde{\Omega} = (\Omega \setminus (\partial\Omega)^t) \cup \{(\overline{x}, s) \in (\partial\Omega)^t \ : \ s < g(\overline{x})\} \tag{7.13}$$

for appropriate functions g on $\partial\Omega$.

Definition 7.5. Let Ω and t be as above. Let $0 < \rho < t$ and $M > 0$. We say that the domain $\widetilde{\Omega}$ belongs to the class $\mathcal{C}_M^{m,1}(\Omega, t, \rho)$, $m = 0$ or 1, if $\widetilde{\Omega}$ is given by (7.13) for some $C^{m,1}$ function g on $\partial\Omega$ which takes values in $]-t+\rho, t[$ and satisfies $|g|_{m,1} \leqslant M$.

For $\Gamma \subset \partial\Omega$ and $\widetilde{\Omega} \in \mathcal{C}_M^{m,1}(\Omega, t, \rho)$ the set $\widetilde{\Gamma} \subset \partial\widetilde{\Omega}$, where the homogeneous Dirichlet boundary conditions are imposed, is given by

$$\widetilde{\Gamma} = \{(\overline{x}, g(\overline{x})) : \ \overline{x} \in \Gamma\}. \tag{7.14}$$

Theorem 7.6. *Let Ω be an open set of class C^2, and let $t > 0$ be as above. Let Γ be an open subset of $\partial\Omega$ with Lipschitz continuous boundary in $\partial\Omega$. Then there exists $2 < q_0 \leqslant \infty$ such that for any $r > \max\{2, N(q_0-1)/q_0\}$ the following statements hold.*

(i) *There exists $c_1 > 0$ such that*

$$\left(\sum_{n=1}^{\infty} \left| \frac{1}{\lambda_n[\widetilde{\Omega}, \widetilde{\Gamma}] + 1} - \frac{1}{\lambda_n[\Omega, \Gamma] + 1} \right|^r \right)^{1/r} \leqslant c_1 |\Omega \vartriangle \widetilde{\Omega}|^{\frac{q_0 - 2}{r q_0}} \tag{7.15}$$

for all $\widetilde{\Omega} \in \mathcal{C}_M^{0,1}(\Omega, t, \rho)$ such that, $|\Omega \vartriangle \widetilde{\Omega}| \leqslant c_1^{-1}$, where $\widetilde{\Gamma} \subset \widetilde{\Omega}$ is given by (7.14).

(ii) *Let $\lambda[\Omega, \Gamma]$ be an eigenvalue of multiplicity m, and let $n \in \mathbf{N}$ be such that $\lambda[\Omega, \Gamma] = \lambda_n[\Omega, \Gamma] = \cdots = \lambda_{n+m-1}[\Omega, \Gamma]$. There exists $c_2 > 0$ such that the following is true: if $\widetilde{\Omega} \in \mathcal{C}_M^{0,1}(\Omega, t, \rho)$, $|\Omega \vartriangle \widetilde{\Omega}| \leqslant c_2^{-1}$, and $\widetilde{\Gamma} \subset \partial\widetilde{\Omega}$ is given by (7.14), then, given orthonormal eigenfunctions $\psi_n[\widetilde{\Omega}, \widetilde{\Gamma}], \ldots, \psi_{n+m-1}[\widetilde{\Omega}, \widetilde{\Gamma}]$, there exist orthonormal eigenfunctions $\psi_n[\Omega, \Gamma], \ldots, \psi_{n+m-1}[\Omega, \Gamma]$ such that*

$$\|\psi_n[\Omega, \Gamma] - \psi_n[\widetilde{\Omega}, \widetilde{\Gamma}]\|_{L^2(\Omega \cup \widetilde{\Omega})} \leqslant c_2 |\Omega \vartriangle \widetilde{\Omega}|^{\frac{q_0-2}{rq_0}}.$$

If, in addition, $A_{ij} \in C^{0,1}(\mathbf{R}^N)$*,* $\widetilde{\Omega} \in \mathcal{C}^{1,1}_M(\Omega, t, \rho)$ *and* Γ *is a connected component of* $\partial\Omega$*, then statements* (i) *and* (ii) *hold with* $q_0 = \infty$*.*

Proof. The proof is essentially a repetition of the proof of Theorem 7.3: the transformation Φ is defined as in Lemma 7.4, with $\partial\Omega$ replacing W and curvilinear coordinates $(\overline{x}, s)$ replacing the local euclidean coordinates $(\overline{x}, x_N)$. □

8 Appendix

In this section, we briefly discuss how Theorem 4.6 can be used to obtain stability estimates for the solutions of the Poisson problem.

Theorem 8.1. *Let* (A) *be satisfied. Let the operators* L*,* $\widetilde{L}$*, and* $\widehat{L}$ *satisfy* (P)*, and let* Ω *satisfy the interior cone condition. Let* $f \in L^2(\mathbf{R}^N)$*, and let* $v \in \mathcal{W}, \widetilde{v} \in \widetilde{\mathcal{W}}$ *be such that*

$$(L+1)v = f \quad \text{in} \ \ \phi(\Omega),$$
$$(\widetilde{L}+1)\widetilde{v} = f \quad \text{in} \ \ \widetilde{\phi}(\Omega).$$

Let $s = [q_0/(q_0-2)]\max\{2, \alpha + 2\gamma\}$*. If* $N \geqslant 3$*, then there exists* $c > 0$ *depending only on* N*,* τ*,* α*,* c^**,* q_0*,* C*,* γ*,* Ω *such that*

$$\|v - \widetilde{v}\|_{L^2(\phi(\Omega)\cup\widetilde{\phi}(\Omega))} \leqslant c\big((|\mathcal{D}|^{1/N} + \delta_s(\phi, \widetilde{\phi}))\|f\|_{L^2(\mathbf{R}^N)} + \|f\circ\phi - f\circ\widetilde{\phi}\|_{L^2(\Omega)}\big),$$

where $\mathcal{D} = \{x \in \Omega : \phi(x) \neq \widetilde{\phi}(x)\}$*. The same is true if* $N = 2$ *provided that* $|\mathcal{D}|^{1/N}$ *is replaced with* $|\mathcal{D}|^{\frac{1}{2}-\epsilon}$*,* $\epsilon > 0$*.*

Proof. Note that

$$(H+1)(v\circ\phi) = f\circ\phi \quad \text{in} \ \ \Omega, \qquad (\widetilde{H}+1)(\widetilde{v}\circ\widetilde{\phi}) = f\circ\widetilde{\phi} \quad \text{in} \ \ \Omega.$$

Hence

$$\|v\circ\phi - \widetilde{v}\circ\widetilde{\phi}\|_{L^2(\Omega)} \leqslant \|f\circ\phi - f\circ\widetilde{\phi}\|_{L^2(\Omega)} + \|(\widetilde{H}+1)^{-1} - (H+1)^{-1}\|\|f\circ\widetilde{\phi}\|_{L^2(\Omega)}.$$

Proceeding as in the proof of Theorem 5.6, it is easy to see that

$$\|v - \widetilde{v}\|_{L^2(\phi(\Omega)\cup\widetilde{\phi}(\Omega))} \leqslant c\big(\|v\circ\phi - v\circ\widetilde{\phi}\|_{L^2(\Omega)} + \|\widetilde{v}\circ\phi - \widetilde{v}\circ\widetilde{\phi}\|_{L^2(\Omega)}$$

$$+ \|f \circ \phi - f \circ \widetilde{\phi}\|_{L^2(\Omega)} + \|(\widetilde{H}+1)^{-1} - (H+1)^{-1}\| \|f \circ \widetilde{\phi}\|_{L^2(\Omega)}\big).$$

By the Sobolev embedding, it follows that if $N \geqslant 3$

$$\|v \circ \phi - v \circ \widetilde{\phi}\|_{L^2(\Omega)}, \|\widetilde{v} \circ \phi - \widetilde{v} \circ \widetilde{\phi}\|_{L^2(\Omega)} \\ \leqslant c|\mathcal{D}|^{1/N}(\|u\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)}) \leqslant c|\mathcal{D}|^{1/N}\|f\|_{L^2(\mathbf{R}^N)}.$$

The same is true for $N = 2$ provided $|\mathcal{D}|^{1/N}$ is replaced with $|\mathcal{D}|^{\frac{1}{2}-\epsilon}$, $\epsilon > 0$. Moreover, by Theorem 4.6,

$$\|(\widetilde{H}+1)^{-1} - (H+1)^{-1}\| \|f \circ \widetilde{\phi}\|_{L^2(\Omega)} \leqslant c\delta_s(\phi, \widetilde{\phi})\|f\|_{L^2(\mathbf{R}^N)}.$$

Thus, the statement follows by combining the estimates above. □

We now apply the previous theorem in order to estimate $\|u - \widetilde{u}\|_{L^2(\Omega\cup\widetilde{\Omega})}$, where u and $\widetilde{u}$ are the solutions to the following mixed boundary valued problems and $\widetilde{\Omega}$ is either a local perturbation of Ω as in Section 7.1 or a global normal perturbation as in Section 7.2:

$$\begin{cases} -\sum_{i,j=1}^N \frac{\partial}{\partial x_i}\big(A_{ij}(x)\frac{\partial u}{\partial x_j}\big) = f & \text{in } \ \Omega, \\ u = 0 & \text{on } \ \Gamma, \\ \sum_{i,j=1}^N A_{ij}\frac{\partial u}{\partial x_j}\nu_i = 0 & \text{on } \ \partial\Omega \setminus \Gamma, \end{cases}$$

$$\begin{cases} -\sum_{i,j=1}^N \frac{\partial}{\partial x_i}\big(A_{ij}(x)\frac{\partial \widetilde{u}}{\partial x_j}\big) = f & \text{in } \ \widetilde{\Omega}, \\ \widetilde{u} = 0 & \text{on } \ \widetilde{\Gamma}, \\ \sum_{i,j=1}^N A_{ij}\frac{\partial \widetilde{u}}{\partial x_j}\nu_i = 0 & \text{on } \ \partial\widetilde{\Omega} \setminus \widetilde{\Gamma}. \end{cases}$$

For any $s > 0$ we set

$$\mathcal{M}_f(s) = \sup_{\substack{A\subset\mathbf{R}^N \\ |A|\leqslant s}} \left(\int_A |f|^2 dx\right)^{1/2}.$$

The next theorem is a simple consequence of Theorem 8.1 and the inequality (7.11).

Theorem 8.2. *Let Ω, $\widetilde{\Omega}$, Γ, $\widetilde{\Gamma}$ be either as in Theorem* 7.3 *or as in* Theorem 7.6. *Then the following is true: there exists $2 < q_0 \leqslant \infty$ such that for any $r > \max\{2, N(q_0-1)/q_0\}$ there exists $c > 0$ such that $|\Omega \triangle \widetilde{\Omega}| < c^{-1}$ implies*

$$\|u - \widetilde{u}\|_{L^2(\Omega\cup\widetilde{\Omega})} \leqslant c\big(|\Omega \vartriangle \widetilde{\Omega}|^{\frac{q_0-2}{rq_0}}\|f\|_{L^2(\mathbf{R}^N)} + \mathcal{M}_f(c|\Omega \vartriangle \widetilde{\Omega}|)\big). \tag{8.1}$$

If, in addition, $A_{ij} \in C^{0,1}(\mathbf{R}^N)$, $\Omega, \widetilde{\Omega} \in C^{1,1}$ and Γ is a connected component of $\partial\Omega$, then the estimate (8.1) *holds with $q_0 = \infty$.*

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