Spectral Theory of Singular Elliptic Operators with Measurable Coefficients

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We study higher order elliptic operators with measurable coefficients acting on Euclidean domains. The coefficients may have degeneracies or singularities on the boundary or at infinity. We prove Gaussian-type bounds on the fundamental solution of the associated semigroup. These bounds are expressed in terms of a distance $d(x, y)$ that reflects the singularity or degeneracy of the coefficients. The estimates are then used to extend the semigroup to other L^p spaces and to prove that the L^p-spectrum is p-independent.   © 1998 Academic Press

1. INTRODUCTION

The aim of this paper is to develop the L^p spectral theory of a class of higher order singular and/or degenerate self-adjoint superelliptic operators with measurable coefficients. By L^p spectral theory we mean the study of heat kernels and L^p properties of the associated evolution semigroup.

Until recently, most results on L^p spectral theory of higher order elliptic operators were based on two important conditions: some kind of local regularity assumption on the coefficients and uniform ellipticity [Gu1, Gu2, R, Ko]. There are however results on other aspects of spectral theory, namely spectral asymptotics, where none of these two assumptions is made [BS1, BS2].

The study of the L^p spectral theory of higher order operators with measurable coefficients was initiated by Davies [Da3]. Under the assumption that the order $2m$ of the operator is larger than the dimension N of the underlying space he obtained Gaussian-type bounds on the fundamental solution $K(t, x, y)$ of the associated semigroup. These bounds were then used to prove L^p properties of the semigroup and, in particular, to show that the L^p spectrum is p-independent for $1 \leqslant p \leqslant \infty$. If $2m = N$ then the above mentioned heat kernel bounds are still valid, although the relevant proofs have to be modified [AMT]. If $2m < N$ then, in general, there does not exist a continuous kernel and the off-diagonal decay of the semigroup cannot be expressed in a pointwise sense but only in a suitable operator sense. Moreover, the semigroup extends

to other L^p spaces only for $p \in [2N/(N+2m), 2N/(N-2m)]$, a range that is in fact sharp [Da4]. The assumption $2m > N$ is made throughout the present paper.

Our aim is to study some spectral properties of operators with measurable coefficients that are not uniformly elliptic. We look at operators that act on a domain $\Omega \subset \mathbf{R}^N$ and are self-adjoint on $L^2(\Omega, b \, dx)$, where $b(x)$ is some weight. They are given formally by

$$Hf = (-1)^m \, b^{-1} \sum_{\substack{|\alpha| = m \\ |\beta| = m}} D^\alpha \{ b a_{\alpha\beta} D^\beta f \}, \tag{1}$$

and satisfy Dirichlet boundary conditions on $\partial\Omega$. The matrix valued function $\{a_{\alpha\beta}(x)\}$ and the weight $b(x)$ are positive and measurable and satisfy two basic hypotheses (H1) and (H2), introduced below. The first is a weighted Sobolev embedding theorem and the second a weighted interpolation inequality. In Examples 2, A, B and Proposition 3 we give sufficient conditions under which they are valid. The first, Example A, concerns functions $a(x)$ and $b(x)$ that are bounded from above and below by powers of the distance of $x \in \Omega$ from a smooth, compact manifold K of dimension M, $1 \leqslant M \leqslant N-1$; the second, Example B, deals with functions that are bounded from above and below by powers of $(1 + |x|^2)^{1/2}$ in $\mathbf{R}^N$. Although these two examples are quite general, we choose to base the whole paper on Hypotheses (H1) and (H2), not only for the sake of greater generality, but also for that of greater clarity as well as possible future applications. However, at various points we shall return to those two examples in order to illustrate the theory.

Under the above assumptions, we prove Gaussian-type bounds on the fundamental solution of the associated parabolic equation (Theorem 10). They have the form

$$|K(t, x, y)| \leqslant c_1 t^{-N/2m} \exp \left\{ -c_2 \frac{d(x, y)^{2m/(2m-1)}}{t^{1/(2m-1)}} + c_3 t \right\}, \tag{2}$$

where c_i are some positive constants. The metric $d(x, y)$ depends on the operator H and is not equivalent to the Euclidean one unless H is uniformly elliptic. In Example 12 and Proposition 13 we give precise estimates for that metric. For instance, for Example B we prove that if $a(x)$ is a function controlling the size of the matrix $\{a_{\alpha\beta}(x)\}$ (i.e., $c^{-1}a(x) \leqslant \{a_{\alpha\beta}(x)\} \leqslant ca(x)$ in the sense of matrices) and $h \in \mathbf{R}^N$ is small, then

$$c^{-1}a(x)^{-1/2m} |h| \leqslant d(x, x+h) \leqslant ca(x)^{-1/2m} |h|$$

uniformly in x. This is known to be optimal up to equivalence of metrics [T]. We do not however make any attempts to find the sharp value of the

constant c_2 in (2). For results on short time asymptotics and sharp bounds on heat kernels for uniformly elliptic operators see [EP, T] and [BaD, Ba] correspondingly.

The heat kernel bounds are then used in order to extend the semigroup to $L^p(\Omega, b\,dx)$ for $p \neq 2$ and to prove that the L^p spectrum of the generator is p-independent (Theorem 16). For this we employ the technique used in [Da3], which is based on an abstract spectral invariance theorem.

Finally, for the Examples A and B mentioned above we prove that the spectrum of H is not discrete, thus generalizing some of the results of Pang [P], who treated the second order case.

2. SETTING AND EXAMPLES

We first fix some notation. Given a multi-index $\alpha = (\alpha_1, ..., \alpha_N)$ we write $\alpha! = \alpha_1! \cdots \alpha_n!$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. We shall use the standard notation D^α for the differential expression $(\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_N)^{\alpha_N}$ and for $k \geqslant 0$ we shall denote by $\nabla^k f$ the vector $(D^\alpha f)_{|\alpha| = k}$. If g, h are two positive functions (or sequences) with common domain of definition we shall write $g \sim h$ to indicate that their ratio is bounded away from zero and infinity. We shall call such functions equivalent. Throughout the paper the letter c will denote a positive constant depending only on Ω and H, whose value may change from line to line.

We now introduce our setting. We work on $L^2(\Omega, b\,dx)$ where Ω is a domain in $\mathbf{R}^N$ and b is a positive measurable weight with

$$b^{\pm 1} \in L^\infty_{loc}(\Omega).$$

We shall work with the corresponding L^p spaces, $L^p(\Omega, b\,dx) =: L^p_b$, equipped with the norm

$$\|f\|_{L^p_b} = \left\{ \int_\Omega |f(x)|^p\, b(x)\, dx \right\}^{1/p}.$$

For the unweighted norm we shall simply write $\|\cdot\|_p$.

We assume that $2m > N$ and consider self-adjoint elliptic operators on $L^2_b(\Omega)$ of the form

$$Hf(x) = (-1)^m\, b(x)^{-1} \sum_{\substack{|\alpha| = m \\ |\beta| = m}} D^\alpha \{ a_{\alpha\beta}(x)\, b(x)\, D^\beta f(x) \} \tag{3}$$

subject to Dirichlet boundary conditions on $\partial\Omega$. In the classical case where H is uniformly elliptic with smooth coefficients and $\partial\Omega$ is smooth, this

corresponds to the requirement that functions in the domain of H as well as their derivatives $\nabla f, ..., \nabla^{m-1}f$, vanish on the boundary $\partial\Omega$. The precise definition shall be given below.

The matrix-valued function $\{a_{\alpha\beta}\}$ is assumed to be measurable and to take its values in the set of all complex, self-adjoint, positive definite $v \times v$-matrices, v being the number of multi-indices α of length $|\alpha| = m$. We assume that there exists a positive function $a(x)$ with $a^{\pm 1} \in L^\infty_{loc}(\Omega)$ that controls the magnitude of $\{a_{\alpha\beta}\}$, in the sense that

$$c^{-1}a(x)\,|p|^2 \leqslant \sum_{\substack{|\alpha|=m \\ |\beta|=m}} a_{\alpha\beta}(x)\,p_\alpha\bar{p}_\beta \leqslant ca(x)\,|p|^2 \tag{4}$$

for some constant $c < \infty$, all vectors $p = (p_\alpha)_{|\alpha|=m} \in \mathbf{C}^v$ and all $x \in \Omega$. Condition (4) is known as the *superellipticity* condition, in contrast to the weaker ellipticity condition

$$c^{-1}a(x)\,|\xi|^{2m} \leqslant \sum_{\substack{|\alpha|=m \\ |\beta|=m}} a_{\alpha\beta}(x)\,\xi^\alpha\bar{\xi}^\beta \leqslant ca(x)\,|\xi|^{2m}, \qquad \xi \in \mathbf{C}^N.$$

Under these assumptions we define a quadratic form Q with domain $C_c^\infty(\Omega)$ by

$$Q(f) = \int_\Omega \sum_{\substack{|\alpha|=m \\ |\beta|=m}} a_{\alpha\beta}(x)\,D^\alpha f(x)\,D^\beta\bar{f}(x)\,b\,dx, \qquad f \in C_c^\infty(\Omega).$$

LEMMA 1. *The form Q is closable.*

Proof. For $f \in W^{m,2}_{loc}(\Omega)$ we define

$$\tilde{Q}(f) = \int_\Omega \sum_{\substack{|\alpha|=m \\ |\beta|=m}} a_{\alpha\beta}(x)\,D^\alpha f(x)\,D^\beta\bar{f}(x)\,b\,dx \in [0, \infty]$$

and we let

$$\mathscr{D} = \{ f \in W^{m,2}_{loc}(\Omega) \mid \tilde{Q}(f) < \infty \}.$$

Clearly $\mathscr{D} \supset C_c^\infty(\Omega)$. We shall prove that $\tilde{Q}$ is closed on $\mathscr{D}$. Indeed, let $f_n, f \in L^2_b(\Omega)$, $n = 1, 2, ...$, be such that

$$\|f_n - f\|_{L^2_b} \to 0, \qquad \tilde{Q}(f_n - f_m) \to 0$$

as $n, m \to \infty$. Let (Ω_k) be an increasing sequence of open subsets of Ω such that

$$\Omega_k \subset\subset \Omega, \qquad \bigcup_k \Omega_k = \Omega.$$

The conditions on $\{a_{\alpha\beta}(x)\}$ and $b(x)$ then imply that there exists constants c_k such that

$$\tilde{Q}(f_n - f_m) \geqslant c_k \int_{\Omega_k} |\nabla^k(f_n - f_m)|^2 \, dx.$$

Therefore (f_n) is a Cauchy sequence in $W^{m, 2}(\Omega_k)$ and its limit has to be equal to $f|_{\Omega_k}$. It follows that $f \in W^{m, 2}_{loc}(\Omega)$ and we conclude that

$$\tilde{Q}(f) = \lim_k \int_{\Omega_k} \sum a_{\alpha\beta} D^\alpha f D^\beta \bar{f} b \, dx = \lim_k \lim_n \int_{\Omega_k} \sum a_{\alpha\beta} D^\alpha f_n D^\beta \bar{f}_n b \, dx$$
$$\leqslant \limsup_n \tilde{Q}(f_n) < \infty.$$

Hence Q has a closed extension and therefore is closable. ∎

We use the same symbol, Q, for the closure of the above form and define H to be the associated self adjoint operator on L^2_b, so that (3) is valid in a weak sense. The appearance of the factor $b a_{\alpha\beta}$ in this expression may seem somehow awkward, but will lead to more nicely formulated results later.

We make two basic hypotheses on the functions a and b.

(H1) The domain $\mathrm{Dom}(Q)$ is embedded in $C_0(\Omega)$ and

$$\|f\|_\infty \leqslant c Q(f)^{s/2} \|f\|^{1-s}_{L^2_b} \tag{5}$$

for some $s \in [N/2m, 1]$ and all $f \in \mathrm{Dom}(Q)$.

(H2) There exist a constant c such that

$$\int_\Omega a^{k/m} |\nabla^k f|^2 \, b \, dx < \varepsilon \int_\Omega a |\nabla^m f|^2 \, b \, dx + c\varepsilon^{-k/(m-k)} \int_\Omega |f|^2 \, b \, dx, \tag{6}$$

for all $0 < \varepsilon < 1$, $0 \leqslant k < m$ and all $f \in C^\infty_c(\Omega)$.

Hypothesis (H1) is needed to obtain uniform (on-diagonal) bounds on the heat kernel, while (H2) will be used in order to extend those to off-diagonal Gaussian estimates. It is well known that both (H1) and (H2) are satisfied when $b(x) \sim 1$ and H is uniformly elliptic. In that case, the best value for the constant s is $s = N/2m$, and that is why we cannot expect any

value that is better (smaller) than that number. Of course, the unweighted inequality implies that (H1) is valid with $s = N/2m$ if the functions ab and b are both bounded from below by positive constants. The following example shows that more than this is possible. We set $\langle x \rangle = (1 + |x|^2)^{1/2}$, $x \in \mathbf{R}^N$.

EXAMPLE 2.   [L] Let $\Omega = \mathbf{R}^N$ with N odd and let $a(x) = \langle x \rangle^\alpha$, $b(x) = \langle x \rangle^\beta$ where α and β are real numbers satisfying

$$\beta > -N, \qquad \alpha + \beta > -N. \tag{7}$$

Assume that

$$\frac{N}{2m} \alpha + \beta \geqslant 0, \qquad \frac{N + \beta}{2m - \alpha} \leqslant 1. \tag{8}$$

Then (H1) is satisfied with $s = (N + \beta)/(2m - \alpha)$. [One easily checks that $s \geqslant N/2m$] If N is an even number, then this remains true provided inequalities (8) are replaced by strict inequalities and the exponent s is replaced by $s + \delta$ for any small $\delta > 0$. We refer to [L] for the details.

In Proposition 3 we shall prove that (H2) is valid for Examples A and B below. We shall return to these examples later on in order to illustrate some of our results.

EXAMPLE A.   Let K be a smooth compact surface in $\mathbf{R}^N$ of dimension M, where $1 \leqslant M \leqslant N - 1$, and let $\Omega = \Omega' \backslash K$, where Ω' is an open domain containing K. We let $d(x)$ be a smooth function on Ω satisfying $d(x) = \text{dist}(x, K)$ in a neighbourhood of K and such that $d(x)$ is bounded away from zero and infinity outside that neighbourhood. For fixed $\alpha, \beta \in \mathbf{R}$ we consider weights

$$a(x) \sim d(x)^\alpha, \qquad b(x) \sim d(x)^\beta, \quad x \in \Omega,$$

and assume

$$\alpha > 2m.$$

We point out that with very minor modifications this example also covers the case of a bounded domain Ω with smooth boundary, $d(x)$ being the distance from a smooth submanifold A of $\partial\Omega$. We omit the details and shall only deal with the example as stated.

EXAMPLE B.   Let $\Omega = \mathbf{R}^N$, consider weights

$$a(x) \sim \langle x \rangle^\alpha, \qquad b(x) \sim \langle x \rangle^\beta$$

and assume that

$$\alpha < 2m.$$

PROPOSITION 3. *Hypothesis* (H2) *is satisfied in both Examples A and B.*

Proof. We shall make use of the following Claim, whose proof is given in the Appendix.

Claim. Let $R = [0, h_1] \times \cdots \times [0, h_N]$ be a rectangle in $\mathbf{R}^N$ and let $h = \min_i h_i$.

There exists an absolute constant c such that the inequality

$$\int_R |\nabla^k f|^2 \, dx < \varepsilon \int_R |\nabla^m f|^2 \, dx + c\varepsilon^{-k/(m-k)} \int_R |f|^2 \, dx \qquad (9)$$

is valid for all $0 \leqslant k \leqslant m - 1$, all $\varepsilon \in (0, h^{2m-2k})$ and all $f \in C^\infty(\bar{R})$.

Note. 1. It is well known [A] that the inequality is valid for a general domain that possesses the cone property. The point here is the dependence upon the thickness h.

2. We shall often make use of the fact that in inequalities such as (9) one can replace the range $(0, h^{2m-2k})$ of ε by $(0, c_1 h^{2m-2k})$, for some constant c_1, provided the constant c is also replaced be a new constant $c' = c'(c, c_1)$.

We shall also use the following notation: given a diffeomorphism π, we set

$$s_m(\pi) = \sup_{1 \leqslant k \leqslant m} \{ \|\nabla^k \pi\|_\infty, \|\nabla^k \pi^{-1}\|_\infty \}.$$

[By $\|\nabla^k \pi\|_\infty$ we mean the maximum of the L^∞ norm of each of the components of $\nabla^k \pi$.] It is clear that inequality (9) remains valid if the rectangle R is replaced by some diffeomorphic image of itself. The constant c will then also depend on the diffeomorphism π, via the number $s_m(\pi)$ only.

We now proceed with the proofs.

Proof of Example A. There exists a neighbourhood U of K such that $\Omega \cap U$ is diffeomorphic to the product $K \times [B_{N-M} \backslash \{0\}]$, where B_{N-M} is a ball in $\mathbf{R}^{N-M}$ and for all $x \equiv (x', x'') \in U$, $x' \in K$, $x'' \in B_{N-M}$, $x'' \neq 0$, we have $|x''| = d(x)$. We use this diffeomorphism to identify U with $K \times [B_{N-M} \backslash \{0\}]$.

Let $r > 0$ be fixed. We write U as

$$U = \bigcup_{n=n_0}^{\infty} \Omega_n,$$

where Ω_n are closed "shells" enveloping K,

$$\begin{aligned}\Omega_n &= \{ x \in \Omega \mid (n+1)^{-r} \leqslant d(x) \leqslant n^{-r} \} \\ &= K \times \{ x'' \in B_{N-M} \mid (n+1)^{-r} \leqslant |x''| \leqslant n^{-r} \} =: K \times S_n.\end{aligned}$$

Now let $\{ K_j \}$ be a finite cover of K such that each K_j is diffeomorphic to some closed M-dimensional cube. The collection

$$\{ K_j \times S_n \}_{j, n}$$

is a cover of U and, moreover, it is uniformly finite; that is, denoting by $\chi_{j,n}$ the characteristic function of $K_j \times S_n$ we have

$$\sup_{x \in U} \sum_{j, n} \chi_{j, n}(x) < \infty.$$

[The various constants below may also depend on this supremum.] The thickness of each "rectangle" $K_j \times S_n$ is, for large n, approximately equal to the thickness of S_n, which is

$$n^{-r} - (n+1)^{-r} \sim n^{-(r+1)}.$$

From this follows that there exist diffeomorphisms π_{jn} that map $K_j \times S_n$ onto a rectangle R_{jn} of minimum edgelength $n^{-(r+1)}$ and such that

$$\sup_{j, n} s_m(\pi_{jn}) < \infty. \tag{10}$$

Hence, by the remark above on the invariance of (9) under diffeomorphisms, we have

$$\int_{K_j \times S_n} |\nabla^k f|^2 \, dx < \varepsilon \int_{K_j \times S_n} |\nabla^m f|^2 \, dx + c\varepsilon^{-k/(m-k)} \int_{K_j \times S_n} |f|^2 \, dx \tag{11}$$

for all j, n, all $\varepsilon \in (0, n^{-2(r+1)(m-k)})$ and some constant c which, because of (10), is independent of n.

Now, let a_n, b_n be positive constants. Summing over all j, replacing ε by $\varepsilon a_n^{(m-k)/m}$ and multiplying both sides by $a_n^{k/m} b_n$, we conclude that

$$a_n^{k/m} b_n \int_{\Omega_n} |\nabla^k f|^2 \, dx < \varepsilon a_n b_n \int_{\Omega_n} |\nabla^m f|^2 \, dx + c\varepsilon^{-k/(m-k)} b_n \int_{\Omega_n} |f|^2 \, dx, \tag{12}$$

for all $f \in C^\infty(\overline{\Omega_n})$ and all ε such that $0 < \varepsilon < n^{-2(m-k)(r+1)} a_n^{-(m-k)/m}$. Taking $a_n = n^{-r\alpha}$ and $b_n = n^{-r\beta}$ we have

$$c^{-1} d(x)^\alpha \leqslant a_n \leqslant c d(x)^\alpha, \qquad \text{all} \quad x \in \Omega_n,$$

and

$$c^{-1} d(x)^\beta \leqslant b_n \leqslant c d(x)^\beta, \qquad \text{all} \quad x \in \Omega_n.$$

Hence

$$\int_{\Omega_n} d(x)^{(\alpha k/m) + \beta} |\nabla^k f|^2 \, dx < \varepsilon \int_{\Omega_n} d(x)^{\alpha + \beta} |\nabla^m f|^2 \, dx$$

$$+ c\varepsilon^{-k/(m-k)} \int_{\Omega_n} d(x)^\beta |f|^2 \, dx$$

for all $f \in C^\infty(\overline{\Omega_n})$. A similar inequality is valid away from K, where the coefficients of H do not have singularities or degeneracies. Adding all the above inequalities we conclude that

$$\int_\Omega a(x)^{k/m} |\nabla^k f|^2 \, b(x) \, dx$$

$$< \varepsilon \int_\Omega a(x) |\nabla^m f|^2 \, b(x) \, dx + c\varepsilon^{-k/(m-k)} \int_\Omega |f|^2 \, b(x) \, dx$$

for all $f \in C_c^\infty(\Omega)$ and all ε such that

$$0 < \varepsilon < c \inf_{n \in \mathbf{N}} n^{-2(m-k)(r+1)} a_n^{-(m-k)/m} = \inf_n n^{((m-k) \, r\alpha/m) - 2(m-k)(r+1)}.$$

Hence, in order to have a non-trivial range of $\varepsilon > 0$ we need the above infimum to be positive, that is we need

$$\alpha \geqslant \frac{2m(r+1)}{r}.$$

Since $\alpha > 2m$, this is true provided $r > 0$ is chosen large enough.

Proof of Example B. The proof is very similar to that of Example A, so we only give an outline. We let $r > 0$ be arbitrary but fixed and define

$$\Omega_n = \{ x \in \Omega \mid n^r < \langle x \rangle < (n+1)^r \}.$$

Each Ω_n has thickness approximately n^{r-1} and therefore

$$\int_{\Omega_n} |\nabla^k f|^2 \, dx < \varepsilon \int_{\Omega_n} |\nabla^m f|^2 \, dx + c\varepsilon^{-k/(m-k)} \int_{\Omega_n} |f|^2 \, dx$$

for all $\varepsilon < n^{2(m-k)(r-1)}$ and all $f \in C^\infty(\overline{\Omega_n})$. Taking $a_n = n^{r\alpha}$ and $b_n = n^{r\beta}$ and following the same reasoning as in Example A we conclude that

$$\int_{\mathbf{R}^N} \langle x \rangle^{(\alpha k/m) + \beta} |\nabla^k f|^2 \, dx$$

$$< \varepsilon \int_{\mathbf{R}^N} \langle x \rangle^{\alpha + \beta} |\nabla^m f|^2 \, dx + c\varepsilon^{-k/(m-k)} \int_{\mathbf{R}^N} \langle x \rangle^{\beta} |f|^2 \, dx$$

for all $f \in C_c^\infty(\mathbf{R}^N)$ and all ε such that

$$0 < \varepsilon < \inf_{n \in \mathbf{N}} n^{2(m-k)(r-1)} a_n^{-(m-k)/m}.$$

The assumption $\alpha < 2m$ implies that this range is non-empty provided we take r to be large enough. ∎

3. HEAT KERNEL ESTIMATES

Diagonal Bounds

We start by proving uniform bounds on the heat kernel. The way these follow from the Sobolev embedding (H1) is standard, but we include the proof for the sake of completeness. Given an operator T that acts on different L_b^p spaces we shall denote by $\|T\|_{p \to q}$ its norm when regarded as an operator from L_b^p to L_b^q. When $p = q = 2$ we shall simply write $\|T\|$. We always assume that $2m > N$.

PROPOSITION 4. *The semigroup e^{-Hz}, $\mathrm{Re}\, z > 0$, has a jointly continuous kernel $K(z, x, y)$ that satisfies*

$$|K(t + iu, x, y)| \leqslant ct^{-s}, \tag{13}$$

for all $t > 0$, $u \in \mathbf{R}$ and $x, y \in \Omega$. Moreover for fixed x, y the kernel is analytic as a function of z on $\{z \mid \mathrm{Re}\, z > 0\}$.

Proof. Let $f \in L_b^2$ be fixed. It follows from (H1) that for $t > 0$ we have

$$\|e^{-Ht}f\|_\infty \leqslant cQ(e^{-Ht}f)^{s/2} \|e^{-Ht}f\|_{L_b^2}^{1-s}$$
$$= c \|H^{1/2}e^{-Ht}f\|_{L_b^2}^{s} \|e^{-Ht}f\|_{L_b^2}^{1-s} \leqslant ct^{-s/2} \|f\|_{L_b^2}.$$

Hence for $t > 0$ and $u \in \mathbf{R}$ we can write

$$e^{-H(t + iu)} = A \cdot B \cdot C$$

where $A = e^{-Ht/2} : L_b^2 \to L^\infty$ satisfies $\|A\|_{2 \to \infty} \leqslant ct^{-s/2}$, $B = e^{-iHu}$ is unitary on L_b^2 and $C = A^* = e^{-Ht/2} : L_b^1 \to L_b^2$. Using a standard theorem on integral operators we conclude that $e^{-H(t + iu)}$ has an (x, y)-measurable integral kernel $K(t + iu, x, y)$ which satisfies

$$\sup_{x, \, y \in \Omega} |K(t + iu, x, y)| = \|e^{-H(t + iu)}\|_{1 \to \infty} \leqslant \|A\| \, \|B\| \, \|C\| \leqslant ct^{-s}.$$

The analyticity in z of the kernel follows, for example, from [Da5] where the analyticity of kernels of semigroups is proved under very general conditions. To prove the joint continuity we define the map

$$\phi : \Omega \to L_b^2(\Omega)$$

by

$$[(H + 1)^{-1/2} f](x) = \langle f, \phi(x) \rangle, \qquad \text{all} \quad f \in L_b^2.$$

Let $x \in \Omega$ be fixed, let D be a bounded open neighbourhood of x with $\bar{D} \subset \Omega$ and let $g \in C_c^\infty(D)$ be such that $g = 1$ near x. The compactness of the unweighted embedding $W_0^{m, \, 2}(D) \subset C_0(D)$ implies that the set

$$\{ g(H + 1)^{-1/2} f \mid f \in L_b^2(\Omega), \|f\|_{L_b^2} \leqslant 1 \}$$

is precompact in $C_0(\Omega)$, and therefore equicontinuous by the Arzela–Ascoli theorem.

Hence ϕ is continuous at x. The joint continuity of the heat kernel then follows by means of the formula

$$K(z, x, y) = \langle (1 + H) \, e^{-Hz} \phi(y), \phi(x) \rangle. \quad \blacksquare$$

COROLLARY 5. *If the weight $b(x)$ is integrable then the semigroup e^{-Ht} is trace-class and*

$$|\mathrm{Tr}(e^{-Ht})| \leqslant ct^{-s}$$

for all $t > 0$.

Proof. We have $\mathrm{Tr}(e^{-Ht}) = \int_\Omega K(t, x, x) \, b(x) \, dx$. $\quad \blacksquare$

Off-Diagonal Bounds

We now proceed to prove off-diagonal bounds for the kernel $K(z, x, y)$. These bounds will then be used to extend the semigroup to L_b^p for $p \neq 2$ and

also to prove that the L^p-spectrum of H is independent of p. Hypotheses (H1) and (H2) are assumed for the rest of the paper.

Our estimates will be expressed in terms of a metric $d(x, y)$ induced canonically by the function $a(x)$ as follows: Let

$$\mathscr{E} = \{\phi \in C^m(\Omega) \cap L^\infty(\Omega) \mid |\nabla^k \phi(x)| \leqslant a(x)^{-k/2m}, 1 \leqslant k \leqslant m\}.$$

We define the distance $d(x, y)$ on Ω by

$$d(x, y) = \sup\{\phi(x) - \phi(y) \mid \phi \in \mathscr{E}\}. \tag{14}$$

We shall discuss this metric in more detail later in this section (Example 12 and Proposition 13) where, in specific cases, we shall give explicit lower bounds on $d(x, y)$.

LEMMA 6. *Given k, l such that $0 \leqslant k, l \leqslant m, k + l < 2m$, there exists a constant c so that*

$$(1 + \lambda^{2m-k-l}) \int_\Omega a^{(k+l)/2m} |\nabla^k f| \, |\nabla^l f| \, b \, dx$$

$$< \varepsilon Q(f) + c\varepsilon^{-(k+l)/(2m-k-l)}(1 + \lambda^{2m}) \|f\|_{L_b^2}^2, \tag{15}$$

for all $0 < \varepsilon < 1$, $\lambda > 0$ and all $f \in C_c^\infty(\Omega)$.

Proof. We shall first prove (15) for $\lambda = 1$. Let $0 < \varepsilon < 1$ be given. If both k and l are smaller than m then using (H2) we have

$$\int_\Omega a^{(k+l)/2m} |\nabla^k f| \, |\nabla^l f| \, b \, dx$$

$$\leqslant \left(\int_\Omega a^{k/m} |\nabla^k f|^2 \, b \, dx\right)^{1/2} \left(\int_\Omega a^{l/m} |\nabla^l f|^2 \, b \, dx\right)^{1/2}$$

$$\leqslant c(\delta_1 Q(f) + \delta_1^{-k/(m-k)} \|f\|_{L_b^2}^2)^{1/2} (\delta_2 Q(f) + \delta_2^{-l/(m-l)} \|f\|_{L_b^2}^2)^{1/2}$$

and choosing

$$\delta_1 = \varepsilon^{2(m-k)/(2m-k-l)}, \qquad \delta_2 = \varepsilon^{2(m-l)/(2m-k-l)},$$

we conclude that

$$\int_\Omega a^{(k+l)/2m} |\nabla^k f| \, |\nabla^l f| \, b \, dx < c(\varepsilon Q(f) + \varepsilon^{-(k+l)/(2m-k-l)} \|f\|_{L_b^2}^2), \tag{16}$$

as required. If $l = m$, say, then we have

$$\int_\Omega a^{(m+k)/2m} \, |\nabla^k f| \, |\nabla^m f| \, b \, dx$$

$$\leqslant \left(\int_\Omega a^{k/m} \, |\nabla^k f|^2 \, b \, dx \right)^{1/2} \left(\int_\Omega a \, |\nabla^m f|^2 \, b \, dx \right)^{1/2}$$

$$\leqslant (\varepsilon^2 Q(f) + \varepsilon^{-2k/(m-k)})^{1/2} \, Q(f)^{1/2}$$

and (16) again follows, completing the proof of the case $\lambda = 1$. Suppose now that $\lambda \neq 1$. If $\lambda < 1$, then (15) follows immediately from (16); if $\lambda > 1$, we replace ε in (16) by $\varepsilon \lambda^{k+l-2m}$ (which is smaller than one) and (15) follows after multiplying both sides by λ^{2m-k-l}. ∎

Given $\phi \in \mathscr{E}$ and a multi-index γ, $|\gamma| \leqslant m$, we define the function

$$P_{\gamma, \phi}(x) = e^{-\phi(x)} D^\gamma [e^{\phi(x)}].$$

This is a polynomial in various derivatives of ϕ and a simple induction argument shows that

$$|P_{\gamma, \lambda\phi}(x)| \leqslant c(1 + \lambda^{|\gamma|}) \, a(x)^{-|\gamma|/2m} \tag{17}$$

for all $\phi \in \mathscr{E}$, $\lambda > 0$ and $x \in \Omega$.

LEMMA 7. *For $\phi \in \mathscr{E}$ and $\lambda > 0$ the map $f \mapsto e^{\lambda\phi} f$ maps $\mathrm{Dom}(Q)$ into $\mathrm{Dom}(Q)$.*

Proof. Let $f \in \mathrm{Dom}(Q)$ and let $(f_n) \subset C_c^\infty(\Omega)$ be such that

$$\|f_n - f\|_{L_b^2} \to 0, \qquad \text{as} \quad n \to \infty$$

$$Q(f_n - f_m) \to 0, \qquad \text{as} \quad n, m \to \infty.$$

Clearly $\|e^{\lambda\phi} f_n - e^{\lambda\phi} f\|_{L_b^2} \to 0$ and therefore it is enough to prove that $Q(e^{\lambda\phi}(f_n - f_m)) \to 0$ as $n, m \to \infty$. Letting $f_{nm} = f_n - f_m$ and $c_\gamma^\alpha = \alpha!/\gamma!(\alpha - \gamma)!$ we have

$$Q(e^{\lambda\phi} f_{nm}) = \int_\Omega \sum_{\substack{|\alpha| = m \\ |\beta| = m}} a_{\alpha\beta} D^\alpha(e^{\lambda\phi} f_{nm}) \, D^\beta(e^{\lambda\phi} \bar{f}_{nm}) \, b \, dx$$

$$= \int_\Omega e^{2\lambda\phi} \sum_{\substack{|\alpha| = m \\ |\beta| = m}} a_{\alpha\beta} \sum_{\substack{\gamma \leqslant \alpha \\ \delta \leqslant \beta}} c_\gamma^\alpha c_\delta^\beta P_{\gamma, \lambda\phi} P_{\delta, \lambda\phi} \, D^{\alpha-\gamma} f_{nm} \, D^{\beta-\delta} \bar{f}_{nm} \, b \, dx$$

$$=: \int_\Omega e^{2\lambda\phi} \sum_{\substack{|\alpha| = m \\ |\beta| = m}} a_{\alpha\beta} D^\alpha f_{nm} \, D^\beta \bar{f}_{nm} \, b \, dx + I_{nm}.$$

Here the first of the two terms has $(\gamma, \delta) = (\alpha, \beta)$ while the second has $(\gamma, \delta) \neq (\alpha, \beta)$. Now, the first term is smaller than $e^{2\lambda\,\|\phi\|_\infty} Q(f_{nm})$ and therefore converges to zero as $n, m \to \infty$. For the second, using Lemma 6 and (17) we have

$$
\begin{aligned}
|I_{nm}| &= \left| \int e^{2\lambda\phi} \sum_{\substack{|\alpha|=m \\ |\beta|=m}} a_{\alpha\beta} \sum_{\substack{\gamma \leqslant \alpha;\, \delta \leqslant \beta \\ |\gamma+\delta| < 2m}} c_\gamma^\alpha c_\delta^\beta P_{\alpha-\gamma,\,\lambda\phi} P_{\beta-\delta,\,\lambda\phi} \, |D^\gamma f_{nm}| \, |D^\delta f_{nm}| \, b \, dx \right| \\
&\leqslant c e^{2\lambda\,\|\phi\|_\infty} \int a \sum_{\substack{|\gamma|,\,|\delta| \leqslant m \\ |\gamma+\delta| < 2m}} (1 + \lambda^{2m-|\gamma+\delta|}) \, a^{-(2m-|\gamma+\delta|)/2m} \\
&\quad \times |D^\gamma f_{nm}| \, |D^\delta f_{nm}| \, b \, dx \\
&\leqslant c e^{2\lambda\,\|\phi\|_\infty} (1 + \lambda^{2m}) \{ Q(f_{nm}) + \|f_{nm}\|_{L_b^2}^2 \} \to 0, \\
&\qquad \text{as} \quad n, m \to \infty.
\end{aligned}
$$

This concludes the proof. ∎

Given $\phi \in \mathscr{E}$ and $\lambda > 0$ Lemma 7 allows us to define a non-symmetric sesquilinear form $Q_{\lambda\phi}$ with domain $\mathrm{Dom}(Q)$ by

$$
Q_{\lambda\phi}(f, g) = Q(e^{\lambda\phi}f, e^{-\lambda\phi}g),
$$

where $Q(f, g)$ is the sesquilinear form induced by the quadratic form $Q(f)$. The associated operator is given by

$$
H_{\lambda\phi} f = e^{-\lambda\phi} H e^{\lambda\phi} f
$$

for all $f \in \mathrm{Dom}(H_{\lambda\phi}) = \{ f \in L_b^2 \mid e^{\lambda\phi}f \in \mathrm{Dom}(H) \}$. Since the form $Q_{\lambda\phi}$ has the same highest order terms as Q, we can use Lemma 6 to obtain

Lemma 8. *We have*

$$
|Q_{\lambda\phi}(f) - Q(f)| < \varepsilon Q(f) + c\varepsilon^{-2m+1}(1 + \lambda^{2m}) \, \|f\|_{L_b^2}^2 \tag{18}
$$

for all $\phi \in \mathscr{E}$, $\lambda > 0$, $0 < \varepsilon < 1$ and all $f \in C_c^\infty(\Omega)$.

Proof. We have

$$
\begin{aligned}
Q_{\lambda\phi}(f) &= \int \sum_{\substack{|\alpha|=m \\ |\beta|=m}} a_{\alpha\beta} D^\alpha(e^{\lambda\phi}f) \, D^\beta(e^{-\lambda\phi}\bar{f}) \, b \, dx \\
&= \int \sum_{\substack{|\alpha|=m \\ |\beta|=m}} a_{\alpha\beta} \sum_{\substack{\gamma \leqslant \alpha \\ \delta \leqslant \beta}} c_\gamma^\alpha c_\delta^\beta P_{\gamma,\,\lambda\phi} P_{\delta,\,-\lambda\phi} \, D^{\alpha-\gamma}f \, D^{\beta-\delta}\bar{f} \, b \, dx
\end{aligned}
$$

and therefore, applying Lemma 6 and recalling (17),

$$
\begin{aligned}
&|Q_{\lambda\phi}(f) - Q(f)| \\
&= \left| \int \sum_{\substack{|\alpha| = m \\ |\beta| = m}} a_{\alpha\beta} \sum_{\substack{\gamma \leqslant \alpha;\ \delta \leqslant \beta \\ \gamma + \delta \neq \alpha + \beta}} c_\gamma^\alpha c_\delta^\beta P_{\alpha - \gamma,\ \lambda\phi} P_{\beta - \delta,\ -\lambda\phi}\, D^\gamma f\, D^\delta \bar{f} b\, dx \right| \\
&\leqslant c \sum_{\substack{|\gamma|,\ |\delta| \leqslant m \\ |\gamma + \delta| < 2m}} (1 + \lambda^{2m - |\gamma + \delta|}) \int a^{|\gamma + \delta|/2m}\, |D^\gamma f|\, |D^\delta f|\, b\, dx \\
&\leqslant \varepsilon Q(f) + c\varepsilon^{-2m+1}(1 + \lambda^{2m}) \|f\|_{L_b^2}^2
\end{aligned}
$$

as required. ∎

LEMMA 9. *There exists a constant* $k' < +\infty$ *such that*

$$
\text{(i)} \quad \|e^{-H_{\lambda\phi}t}\| \leqslant c \exp\{k'(1 + \lambda^{2m})t\}
$$

$$
\text{(ii)} \quad \|H_{\lambda\phi} e^{-H_{\lambda\phi}t}\| \leqslant \frac{c}{t} \exp\{k'(1 + \lambda^{2m})t\}
$$

for all $\phi \in \mathscr{E}$ *and* $\lambda > 0$.

Proof. It follows from (18) and the non-negativity of $Q(f)$ that there exists a constant k_1 such that

$$
\operatorname{Re} Q_{\lambda\phi}(f) \geqslant -k_1(1 + \lambda^{2m}) \|f\|_{L_b^2}^2,
$$

for all $f \in C_c^\infty(\Omega)$. Hence, letting $f \in L_b^2$ and $f_t = e^{-H_{\lambda\phi}t}f$ we have

$$
\frac{d}{dt} \|f_t\|_{L_b^2}^2 = -\langle H_{\lambda\phi} f_t, f_t \rangle - \langle f_t, H_{\lambda\phi} f_t \rangle \leqslant -2k_1(1 + \lambda^{2m}) \|f_t\|_{L_b^2}^2
$$

and (i) follows by integration.

Now, let $f \in L_b^2$ and $|\theta| < \pi/3$ be given and for $r > 0$ set

$$
f_r = e^{-H_{\lambda\phi} re^{i\theta}}f.
$$

Then, using also (18) for small but fixed $\varepsilon > 0$ we get

$$
\begin{aligned}
\frac{d}{dr} \|f_r\|_{L_b^2}^2 &= -2\cos\theta Q(f_r) + 2\operatorname{Re}[e^{i\theta}(Q - Q_{\lambda\phi})(f_r)] \\
&\leqslant -Q(f_r) + 2\{\varepsilon Q(f_r) + c_\varepsilon(1 + \lambda^{2m}) \|f_r\|_{L_b^2}^2\} \\
&\leqslant 2k_2(1 + \lambda^{2m}) \|f_r\|_{L_b^2}^2
\end{aligned}
$$

for some positive constant k_2 so that

$$\|e^{-[H_{\lambda\phi} + 2k_2(1 + \lambda^{2m})] \, re^{i\theta}}\| \leqslant 1$$

by integration. Lemma 2.38 of [Da1] then implies that

$$\|[H_{\lambda\phi} + 2k_2(1 + \lambda^{2m})] \, e^{-[H_{\lambda\phi} + 2k_2(1 + \lambda^{2m})] t}\| \leqslant c/t$$

for all $t > 0$, from which (ii) follows with $k' = \max\{k_2 + 1, k_1\}$ by means of the triangle inequality and (i). ∎

We can now prove our main heat kernel estimate.

THEOREM 10. *Suppose hypotheses* (H1) *and* (H2) *are valid. Then there exist positive constants c_1, c_2 and k such that the kernel $K(t, x, y)$ of e^{-Ht} satisfies*

$$|K(t, x, y)| \leqslant c_1 t^{-s} \exp\{-c_2 d(x, y)^{2m/(2m-1)} \, t^{-1/(2m-1)} + kt\} \qquad (19)$$

for all $t > 0$ and all $x, y \in \Omega$.

Proof. Let $f \in L_b^2$ and for $t > 0$ set $f_t = e^{-H_{\lambda\phi} t} f$. Then, using (H1), (18) and Lemma 9 we have

$$\begin{aligned}
\|f_t\|_\infty &\leqslant cQ(f_t)^{s/2} \|f_t\|_{L_b^2}^{1-s} \\
&\leqslant c\{\operatorname{Re} Q_{\lambda\phi}(f_t) + (1 + \lambda^{2m}) \|f_t\|_{L_b^2}^2\}^{s/2} \|f_t\|_{L_b^2}^{1-s} \\
&\leqslant c\{\|H_{\lambda\phi} f_t\|_{L_b^2}^2 \|f_t\|_{L_b^2}^2 + (1 + \lambda^{2m}) \|f_t\|_{L_b^2}^2\}^{s/2} \|f_t\|_{L_b^2}^{1-s} \\
&\leqslant c\{t^{-1} + (1 + \lambda^{2m})\}^{s/2} \exp\{k'(1 + \lambda^{2m}) t\} \|f\|_{L_b^2}^2 \\
&\leqslant ct^{-s/2} \exp\{(k' + 1)(1 + \lambda^{2m}) t\} \|f\|_{L_b^2}^2,
\end{aligned}$$

that is $\|e^{-H_{\lambda\phi} t}\|_{2 \to \infty} \leqslant ct^{-s/2} \exp\{k(1 + \lambda^{2m}) t\}$, where $k = k' + 1$. By duality we obtain a similar bound on $\|e^{-H_{\lambda\phi} t}\|_{1 \to 2}$ (note that $H_{\lambda\phi}^* = H_{-\lambda\phi}$) and the semigroup property then implies that

$$\|e^{-H_{\lambda\phi} t}\|_{1 \to \infty} \leqslant ct^{-s} \exp\{k(1 + \lambda^{2m}) t\}.$$

Hence $e^{-H_{\lambda\phi} t}$ has a kernel $K_{\lambda\phi}(t, x, y)$ that satisfies

$$|K_{\lambda\phi}(t, x, y)| \leqslant ct^{-s} \exp\{k(1 + \lambda^{2m}) t\}$$

for all $t > 0$ and $x, y \in \Omega$. Since the two kernels are related by

$$K(t, x, y) = e^{\lambda\phi(x)} K_{\lambda\phi}(t, x, y) \, e^{-\lambda\phi(y)}$$

we conclude that

$$|K(t, x, y)| \leqslant ct^{-s} \exp\{\lambda\phi(x) - \lambda\phi(y) + k(1 + \lambda^{2m})\, t\}.$$

The stated bound then follows by optimizing first with respect to $\phi \in \mathscr{E}$ and then with respect to $\lambda > 0$. $\blacksquare$

We shall give below some examples to illustrate the above proposition, but first we prove a lemma which gives an alternative description of $d(x, y)$:

LEMMA 11.   *Let*

$$\mathscr{E}^* = \{\phi \in C^m(\Omega) \mid |\nabla^k\phi(x)| \leqslant a(x)^{-k/2m}, 1 \leqslant k \leqslant m\}.$$

Then

$$d(x, y) = \sup\{\phi(x) - \phi(y) \mid \phi \in \mathscr{E}^*\}. \tag{20}$$

Note.   The only difference between $\mathscr{E}$ and $\mathscr{E}^*$ is the boundedness requirement.

Proof.   Let $d^*(x, y)$ denote the RHS of (20). Clearly $d^*(x, y) \geqslant d(x, y)$. For the converse, let $\phi \in \mathscr{E}^*$ and for $n = 1, 2, \ldots$ define the bounded functions

$$\phi_n(x) = n \tanh(n^{-1}\phi(x)). \tag{21}$$

Then simple calculations, similar to those that prove (17), show that there exists a sequence δ_n that converges to zero and such that

$$|\nabla^k\phi_n(x)| \leqslant (1 + \delta_n)\, a(x)^{-k/2m}, \qquad k = 1, \ldots, m,$$

so $(1 + \delta_n)^{-1} \phi_n \in \mathscr{E}$. Since for fixed $x, y \in \Omega$ we have

$$\lim_n (1 + \delta_n)^{-1} [\phi_n(x) - \phi_n(y)] = \phi(x) - \phi(y),$$

we conclude that $d^*(x, y) \leqslant d(x, y)$. $\blacksquare$

We now proceed with our examples. We recall that given two positive functions f, g on Ω we write $f \sim g$ to indicate that their ratio is bounded away from zero and infinity.

EXAMPLE 12.   If $a(x) \geqslant c_1$ then $d(x, y) \geqslant c_1^{1/2m} |x - y|$. This follows from the fact that given $x, y \in \mathbf{R}^N$ there exists an affine function ϕ such that $|\nabla\phi| = c_1^{1/2m}$ and $\phi(x) - \phi(y) = c_1^{1/2m} |x - y|$. That function lies in $\mathscr{E}^*$.

If in addition $2m = 4$ and $\partial\Omega$ is C^2 then $d(x, y) \geqslant c d_\Omega(x, y)$, where $d_\Omega(x, y)$ is the geodesic distance on Ω. This follows from the above and Theorem 4.19 of $[O]$.

In the proposition below we denote by $d_K(\cdot, \cdot)$ the geodesic distance on the compact manifold K of Example A.

PROPOSITION 13.   *For the two examples* (*Examples A and B*) *of Section* 2, *the distance function* $d(x, y)$ *satisfies*:

Example A:

$$d(x, y) \geqslant \min\{d(x)^{-\alpha/2m}, d(y)^{-\alpha/2m}\}\, |x'' - y''|$$
$$+ \min\{d(x)^{1-\alpha/2m}, d(y)^{1-\alpha/2m}\}\, d_K(x', y'),$$

for all $x = (x', x'')$ *and* $y = (y', y'')$ *near K*.

Example B:

$$d(x, y) \geqslant c \min\{\langle x \rangle^{-\alpha/2m}, \langle y \rangle^{-\alpha/2m}\}\, |x - y|$$

for all $x, y \in \mathbf{R}^N$.

Proof of Example A.   As mentioned in the proof of Proposition 3, the smoothness of K implies that there exists a δ-neighbourhood U of K that is diffeomorphic to the product $K \times [B_{N-M} \backslash \{0\}]$, where B_{N-M} is an $(N-M)$-dimensional ball. $B_{N-M} \backslash \{0\}$ itself is diffeomorphic to $S^{N-M-1} \times (0, \delta)$, $\delta > 0$. Hence, we shall write points x in U as

$$x = (x', \omega_x, r_x), \qquad x' \in K, \qquad \omega_x = \in S^{N-M-1}, \qquad r_x \in (0, \delta)$$

and, taking $\delta > 0$ to be small enough, we may further assume that the diffeomorphisms are such that $r_x = d(x)$.

We consider functions on Ω that on U have the form

$$\phi(x) = \tau \psi(x')\, \chi(\omega_x)\, g(r_x),$$

where $\tau > 0$, $g(r) = r^{1-\alpha/2m}$, $\psi \in C^\infty(K)$ is such that

$$\|\psi\|_{W^{m,\infty}(K)} := \sup_{0 \leqslant i \leqslant k} \sup_{x' \in K} |\nabla^i \psi(x')| \leqslant 1,$$

and, similarly, $\chi \in C^\infty(S^{N-M-1})$ satisfies $\|\chi\|_{W^{m,\infty}(S^{N-M-1})} \leqslant 1$. The behaviour of ϕ outside U (where $a(x) \sim 1$) can be prescribed by means of a cut-off function and poses non problem. Using subindices to indicate the variable

with respect to which differentiation is performed, one can see that for $k \geqslant 1$ we have

$$|\nabla_x^k \psi(x')| = |\nabla_{x'}^k \psi(x')|, \tag{22}$$

$$|\nabla_x^k \chi(\omega_x)| \leqslant cr^{-k}\{|\nabla_\omega \chi| + \cdots + |\nabla_\omega^k \chi|\} \tag{23}$$

and

$$|\nabla_x^k g(r)| \leqslant c\{|g^{(k)}| + r^{-1}|g^{(k-1)}| + \cdots + r^{-k+1}|g'|\}. \tag{24}$$

It follows that for $x \in U$

$$\begin{aligned}
|\nabla_x^k \phi(x)| &\leqslant c\tau \sum_{i+j+l=k} |\nabla_x^l \psi(x')|\,|\nabla_x^i \chi(\omega_x)|\,|\nabla_x^j g(r_x)| \\
&\leqslant c\tau \sum_{i+j \leqslant k} r_x^{-i}\{|g^{(j)}| + r_x^{-1}|g^{(j-1)}| + \cdots + r_x^{-j+1}|g'|\} \\
&\leqslant c\tau \sum_{s=0}^k r_x^{-k+s}|g^{(s)}(r_x)| \qquad \text{(since } r_x \text{ is bounded)} \\
&\leqslant c\tau \sum_{s=0}^k r_x^{-k+s} r_x^{1-\alpha/2m-s} = c\tau\, d(x)^{-k+1-\alpha/2m}.
\end{aligned}$$

The condition $\alpha > 2m$ implies that for $1 \leqslant k \leqslant m$ we have

$$d(x)^{-k+1-\alpha/2m} \leqslant cd(x)^{-\alpha k/2m}$$

and we conclude that $\phi \in \mathscr{E}^*$ provided τ is small enough. Hence $d(x, y) \geqslant \phi(x) - \phi(y)$. We shall now estimate this difference. Without any loss of generality we assume that $d(x) \leqslant d(y)$ (i.e., $r_x \leqslant r_y$).

Now, by a compactness argument we have

$$d_K(x', y') \sim \sup\{\psi(x') - \psi(y') \mid \|\psi\|_{W^{m,\infty}(K)} \leqslant 1\}.$$

Hence we can take the function ψ to be such that $\psi(x') - \psi(y') \geqslant \frac{1}{2}d_K(x', y')$, and, moreover, $\psi(x') = 1$. Similarly we can find a sufficiently small constant c_1 and choose the function χ so that it satisfies

$$\chi(\omega_x) - \chi(\omega_y) \geqslant c_1 d_{S^{N-M-1}}(\omega_x, \omega_y)$$

(where $d_{S^{N-M-1}}$ is the standard metric on S^{N-M-1}) as well as

$$\chi(\omega_x) \geqslant c_1, \qquad \chi(\omega_y) \geqslant c_1.$$

Moreover, there exists $\xi \in [r_x, r_y]$ such that $g(r_x) - g(r_y) = g'(\xi)(r_x - r_y)$, and therefore, since $g'(\xi) < 0$,

$$g(r_x) - g(r_y) = g'(\xi)(r_x - r_y) \geq c r_y^{-\alpha/2m}(r_y - r_x).$$

It follows that

$$
\begin{aligned}
\phi(x) - \phi(y) &= \psi(x')\,\chi(\omega_x)\,g(r_x) - \psi(y')\,\chi(\omega_y)\,g(r_y) \\
&= \psi(x')\,\chi(\omega_x)[\,g(r_x) - g(r_y)\,] + \psi(x')[\chi(\omega_x) - \chi(\omega_y)]\,g(r_y) \\
&\quad + [\,\psi(x') - \psi(y')\,]\,\chi(\omega_y)\,g(r_y) \\
&\geq c r_y^{-\alpha/2m}(r_y - r_x) + c_1 r_y^{1-\alpha/2m} d_{S^{N-M-1}}(\omega_x, \omega_y) \\
&\quad + c_1 r_y^{1-\alpha/2m}\, d_K(x', y').
\end{aligned}
$$

But on $\mathbf{R}^{N-M}$, using spherical coordinates, $x'' = (\omega_x, r_x)$, we have the equivalence

$$|x'' - y''| \sim |r_x - r_y| + \sqrt{r_x r_y}\, d_{S^{N-M-1}}(\omega_x, \omega_y)$$

from which follows that

$$
\begin{aligned}
r_y^{-\alpha/2m}&(r_y - r_x) + r_y^{1-\alpha/2m} d_{S^{N-M-1}}(\omega_x, \omega_y) \\
&\geq c \min\left\{ r_y^{-\alpha/2m}, \frac{r_y^{1-\alpha/2m}}{\sqrt{r_x r_y}} \right\} |x'' - y''| = c r_y^{-\alpha/2m} |x'' - y''|. \quad (25)
\end{aligned}
$$

A combination of the above implies the required inequality. ∎

Proof of Example B.   The proof is essentially contained in that of Example A and we therefore only give a sketch of it. We use spherical coordinates $x = (r, \omega)$ where

$$r = |x| \in [0, \infty), \qquad \text{and} \qquad \omega = x/|x| \in S^{N-1}.$$

We define the function $g(r)$ by

$$g(r) = (1 + r^2)^{1/2 - \alpha/4m}$$

and let $\phi(x)$ be such that

$$\phi(x) = \tau\psi(\omega)\,g(r), \qquad \text{all} \quad |x| > 1,$$

where $\psi \in C^\infty(S^{N-1})$ satisfies $\|\psi\|_{W^{m,\infty}(S^{N-1})} \leqslant 1$. Using relations (23) and (24) (with S^{N-M-1} replaced now by S^{N-1}) we conclude that for $k \in \mathbf{N}$ we have

$$|\nabla_x^k \phi(x)| \leqslant c\tau \, \|\psi\|_{W^{m,\infty}} \sum_{j=0}^k \frac{1}{r^{k-j}} \, |g^{(j)}(r)| \leqslant c\tau \sum_{j=0}^k \frac{1}{r^{k-j}} r^{m-j} \langle r \rangle^{1-\alpha/2m-m}$$

$$\leqslant c\tau \langle r \rangle^{-k+1-\alpha/2m}$$

for all $x \in \mathbf{R}^N$, $x \neq 0$. The assumption $\alpha < 2m$ then implies

$$|\nabla^k \phi(x)| \leqslant c\tau \langle x \rangle^{-\alpha k/2m}$$

for all $1 \leqslant k \leqslant m-1$. Therefore $\phi \in \mathscr{E}^*$ if τ is small enough. Hence $d(x, y) \geqslant \phi(y) - \phi(x)$. The rest of the proof is exactly as in Example A, the main point being estimate (25). ∎

4. L^p THEORY

In this section we shall use the heat kernel estimates obtained in Section 3 to extend the semigroup e^{-Ht} from $L_b^2(\Omega)$ to $L_b^p(\Omega)$ for $p \neq 2$. Moreover we give sufficient conditions for the spectrum of the corresponding generator $-H_p$ to be independent of p. We shall make use of the following two lemmas from [Da3].

LEMMA 14. *Let $g(z)$ by analytic on $\{z \mid \operatorname{Re} z > 0\}$ and suppose that*

$$|g(re^{i\theta})| \leqslant c(r \cos \theta)^{-\beta}$$

$$|g(r)| \leqslant cr^{-\beta} \exp\{-kr^{-\alpha}\}$$

for some constants k, α and β and all $r > 0$, $|\theta| < \pi/2$. Then there exists a constant k' such that

$$|g(re^{i\theta})| \leqslant c(r \cos \theta)^{-\beta} \exp\{-k'r^{-\alpha} \cos \theta\}$$

for all $r > 0$ and $|\theta| < \pi/2$.

LEMMA 15. *Let $T_p(z)$ and $T_q(z)$ be two consistent strongly continuous holomorphic semigroups of angle $\pi/2$ on the spaces L_b^p and L_b^q respectively. If they both satisfy estimates of the form*

$$\|T(re^{i\theta})\| \leqslant c(\cos \theta)^{-M} \exp\{kr \cos \theta\}$$

for some positive M and k and all $r > 0$ and $|\theta| < \pi/2$, then

$$\mathrm{Sp}(H_p) = \mathrm{Sp}(H_q),$$

where $-H_i$ is the generator of $T_i(z)$.

We are now in a position to prove the main theorem of this section. We shall only deal with the extension of the semigroup on L_p^1. Intermediate values of p are then treated by means of the interpolation inequality

$$\|T\|_{p \to p} \leqslant \|T\|_{1 \to 1}^{(2-p)/p} \|T\|_{2 \to 2}^{(2p-2)/p}.$$

We set

$$\rho_t(x) = \exp\{-t\,|x|^{2m/(2m-1)}\}, \qquad x \in \mathbf{R}^N, \quad t > 0.$$

Moreover we denote by $\widetilde{b}$ the function on $\mathbf{R}^N$ which is equal to $b(x)$ on Ω and zero on $\mathbf{R}^N \backslash \Omega$.

THEOREM 16.   *Suppose hypotheses* (H1) *and* (H2) *are valid. Suppose further that* (i) *there exists a constant $c_1 > 0$ such that*

$$a(x) \leqslant c_1, \qquad all \quad x \in \Omega, \tag{26}$$

and (ii) *that the convolution $\rho_t * \widetilde{b}$ is a bounded function for all $t > 0$. Then the operator e^{-Hz} on $L_b^2 \cap L_b^1$ can be extended to a bounded operator $T_1(z)$ on L_b^1 and we have*

$$\|T_1(re^{i\theta})\|_{1 \to 1} \leqslant c(r \cos \theta)^{-s} e^{kr \cos \theta} \|\rho_{t(z)} * \widetilde{b}\|_\infty \tag{27}$$

for all $z = re^{i\theta}$, $|\theta| < \pi/2$, where

$$t(z) = c_2 r^{-1/(2m-1)} \cos \theta. \tag{28}$$

Proof.   Let

$$g(z) = K(z, x, y)\, e^{-kz},$$

where k is as in Theorem 10. Inequalities (13) and (19) tell us that the conditions of Lemma 14 are satisfied, and we thus conclude that

$|K(re^{i\theta}, x, y)|$

$$\leqslant c(r \cos \theta)^{-s} \exp\{-c_2\, d(x, y)^{2m/(2m-1)}\, r^{-1/(2m-1)} \cos \theta + kr \cos \theta\}.$$

Since we also have $d(x, y) \geqslant c |x - y|$ (by Example 12) it follows that

$$\begin{aligned}
\|T_1(z)\|_{1 \to 1} &= \sup_{x \in \Omega} \int_\Omega |K(z, x, y)| \, b(y) \, dy \\
&\leqslant c(r \cos \theta)^{-s} e^{kr \cos \theta} \\
&\quad \times \sup_{x \in \Omega} \int_\Omega \exp\{-c_2 |x - y|^{2m/(2m-1)} r^{-1/(2m-1)} \cos \theta\} \, b(y) \, dy \\
&\leqslant c(r \cos \theta)^{-s} e^{kr \cos \theta} \|\rho_{t(z)} * \widetilde{b}\|_\infty
\end{aligned}$$

as required. ∎

COROLLARY 17.   *If $a(x) \geqslant c_1$ and $b \in L^q(\Omega)$ for some $1 \leqslant q \leqslant \infty$ then*

$$\|T_1(z)\|_{1 \to 1} \leqslant cr^{(N/2mq')-s}(\cos \theta)^{-s-((2m-1)N/2mq')} e^{kr \cos \theta}.$$

If, further, b is bounded and $s = N/2m$, then the spectrum of the generator $-H_p$ of $T_p(z)$ is independent of $p \in [1, \infty]$.

Proof.   The first statement follows from Theorem 16 and the inequality

$$\|\rho_t * \widetilde{b}\|_\infty \leqslant \|\rho_t\|_{q'} \|\widetilde{b}\|_q \leqslant ct^{-(2m-1)N/2mq'} \|b\|_q$$

applied for $t = t(z)$ and $q = \infty$. The second statement is then an immediate consequence of Lemma 15. ∎

We finally prove a proposition about the spectrum of the operator H in either of the two Examples A and B. The case $2m = 2$ of Example B has been first proved in [P]. We also point out that the main theorem of [BS1] contains what is almost the converse for Example A, in the special case where K is the boundary of a domain Ω: It is proved there that if $\alpha < 2m$ then H has discrete spectrum.

PROPOSITION 18.   *For the two examples (Examples A and B) of Section 2 we have $0 \in \mathrm{EssSp}(H)$.*

Proof of Example A.   Let $v: \mathbf{R} \to [0, 1]$ be a smooth function such that $v(t) = 1$ for $|t| \leqslant 1$ and $v(t) = 0$ if $|t| \geqslant 2$. We fix a parameter $\gamma \in (0, 1)$ and for $k \in \mathbf{N}$ we let ϕ_k be a function in $C_c^\infty(\Omega)$ such that $\phi_k(x) = 0$ on $\Omega \backslash U$ and

$$\phi_k(x) = v\left(\frac{d(x)^{-2m} - k}{k^\gamma}\right), \qquad x \in U, \quad k = 1, 2, \dots.$$

Then

$$\mathrm{supp}(\phi_k) \subset A_k := \{x \in \Omega \mid (k + k^\gamma)^{-1/2m} \leqslant d(x) \leqslant (k + 2k^\gamma)^{-1/2m}\}.$$

For $g \in L_b^2$ an application of the Cauchy–Schwarz inequality yields

$$\|\phi_k\|_{L_b^2}^{-2} |\langle \phi_k, g \rangle|^2 \leqslant c \int_{A_k} |g|^2 \, d(x)^\beta \, dx \to 0,$$

that is, the sequence $\|\phi_k\|_{L_b^2}^{-1} \phi_k$ converges weakly to zero. Moreover, we observe that for any ρ we have

$$\int_{A_k} d(x)^\rho \, dx = \int_{(k+k\gamma)^{-1/2m}}^{(k+2k\gamma)^{-1/2m}} r^\rho r^{N-M-1} \, dr$$

$$= (k+2k\gamma)^{-(\rho+N-M)/2m} - (k+k\gamma)^{-(\rho+N-M)/2m}$$

and therefore

$$\int_{A_k} d(x)^\rho \, dx \sim k^{-(\rho+N-M)/2m+\gamma-1}. \tag{29}$$

Further, a simple argument shows that for any multiindex α we have

$$D^\alpha \phi_k(x) = \sum_{j=1}^{|\alpha|} k^{-j\gamma} w_{\alpha,j}(x) \, d(x)^{-2jm-|\alpha|}$$

where $w_{\alpha,j}(x)$ are uniformly bounded functions. Hence

$$|\nabla^m \phi_k(x)|^2 \leqslant c \sum_{j=1}^m k^{-2j\gamma} d(x)^{-2m(2j+1)}.$$

Therefore, making use of (29) with $\rho = -2m(2j+1)+\alpha+\beta$,

$$Q(\phi_k) \leqslant c \sum_{j=1}^m k^{-2j\gamma} \int_{A_k} d(x)^{-2m(2j+1)} d(x)^{\alpha+\beta} \, dx$$

$$\leqslant c \sum_{j=1}^m k^{-2j\gamma} k^{2j+\gamma-(\alpha+\beta+N-M)/2m}$$

while, again from (29), with $\rho = \beta$ this time,

$$\|\phi_k\|_{L_b^2}^2 \geqslant c k^{-(\beta+N-M)/2m+\gamma-1}.$$

It follows that

$$\|\phi_k\|_{L_b^2}^{-2} Q(\phi_k) \leqslant c \sum_{j=1}^m k^{2j(1-\gamma)+1-(\alpha/2m)}.$$

Since $\alpha > 2m$, this tends to zero provided we choose γ to be close enough to one. ∎

Proof of Example B. As before, let $v: \mathbf{R} \to [0, 1]$ be a smooth function such that $v(t) = 1$ if $|t| \leqslant 1$ and $v(t) = 0$ of $|t| > 2$. We fix $\gamma \in (0, 1)$ and define the functions

$$\phi_k(x) = v\left(\frac{|x|^{2m} - k}{k^\gamma}\right), \qquad k = 1, 2, ...;$$

Then

$$\operatorname{supp}(\phi_k) \subset A_k := \{x \mid (k + k^\gamma)^{1/2m} \leqslant |x| \leqslant (k + 2k^\gamma)^{1/2m}\}.$$

Clearly for any $g \in L_b^2$ we have

$$\|\phi_k\|_{L_b^2}^{-2} |\langle \phi_k, g \rangle|^2 \leqslant c \int_{A_k} |g|^2 \langle x \rangle^\beta,$$

and so $\|\phi_k\|_{L_b^2}^{-1} \phi_k$ converges weakly to zero.

First we observe that for k large enough we have

$$\begin{aligned}
\int_{A_k} \langle x \rangle^\rho \, dx &\sim c \int_{(k + k^\gamma)^{1/2m}}^{(k + 2k^\gamma)^{1/2m}} r^{\rho + N - 1} \, dr \\
&\sim c k^{(\rho + N)/2m} \{(1 + 2k^{\gamma - 1})^{(\rho + N)/2m} - (1 + k^{\gamma - 1})^{(\rho + N)/2m}\} \\
&\sim c k^{(\rho + N)/2m + \gamma - 1}.
\end{aligned} \tag{30}$$

Similar calculations, but now restricting the integral on the set where $\phi_k = 1$, show that

$$\|\phi_k\|_{L_b^2}^2 \geqslant c k^{-(\beta + N)/2m - \gamma + 1}, \qquad \text{all } k.$$

Now it is easily seen that if k is large enough then

$$|\nabla^m \phi_k(x)| \leqslant c \sum_{j=1}^m k^{-j\gamma} |x|^{m(2j-1)}, \qquad \text{all } x \in A_k.$$

Hence, putting $\rho = 2m(2j - 1) + \alpha + \beta$ in (30),

$$\begin{aligned}
\|\phi_k\|_{L_b^2}^{-2} Q(\phi_k) &\leqslant c k^{-(\beta + N)/2m - \gamma + 1} \int_{\mathbf{R}^N} |\nabla^m \phi_k(x)|^2 \langle x \rangle^{\alpha + \beta} \, dx \\
&\leqslant c k^{-(\beta + N)/2m - \gamma + 1} \sum_{j=1}^m k^{-2j\gamma} \int_{A_k} |x|^{2m(2j-1)} |x|^{\alpha + \beta} \, dx \\
&\leqslant \sum_{j=1}^m k^{-(\beta + N)/2m - \gamma + 1 - 2j\gamma} k^{2j-1 + (\alpha + \beta + N)/2m + \gamma - 1} \\
&= \sum_{j=1}^m k^{2j(1-\gamma) - 1 + \alpha/2m}.
\end{aligned}$$

Since $\alpha < 2m$ the exponent of k can be made negative (for each j) by taking γ to be close enough enough to one. Hence for such γ we have $\|\phi_k\|_{L_b^2}^{-2} Q(\phi_k) \to 0$, proving that $0 \in \text{EssSp}(H)$. ∎

APPENDIX

In this Appendix we prove the Claim that is used in the proof of Proposition 3. For the proof we follow closely a similar proof in [F].

Claim. Let $R = [0, h_1] \times \cdots \times [0, h_N]$ be a rectangle in $\mathbf{R}^N$ and let $h = \min_i h_i$. There exists an absolute constant c such that the inequality

$$\int_R |\nabla^k f|^2 \, dx < \varepsilon \int_R |\nabla^m f|^2 \, dx + c\varepsilon^{-k/(m-k)} \int_R |f|^2 \, dx \tag{31}$$

is valid for all $0 \leqslant k \leqslant m-1$, all $\varepsilon \in (0, h^{2m-2k})$ and all $f \in C^\infty(\bar{R})$.

Proof. We start from the standard inequality [A]

$$\int_0^1 |u'(x)|^2 \, dx < \varepsilon \int_0^1 |u''(x)|^2 \, dx + c\varepsilon^{-1} \int_0^1 |u(x)|^2 \, dx,$$

which is valid for some absolute constant c and all $\varepsilon \in (0, 1)$ and $u \in C([0, 1])$. A simple scaling argument implies that for any $h > 0$ the inequality

$$\int_0^h |u'(x)|^2 \, dx < \varepsilon \int_0^h |u''(x)|^2 \, dx + c\varepsilon^{-1} \int_0^h |u(x)|^2 \, dx, \tag{32}$$

is valid for all $\varepsilon \in (0, h^2)$ and all $u \in C([0, h])$. Suppose now that $R = [0, h_1] \times \cdots \times [0, h_N]$ is a rectangle in $\mathbf{R}^N$ and let $h = \min_i h_i$. Let $u \in C^\infty(\bar{R})$ be given. For any fixed values of $x_2, ..., x_N$ we have from (32)

$$\int_0^{h_1} \left|\frac{\partial u}{\partial x_1}\right|^2 \, dx_1 < \varepsilon \int_0^{h_1} \left|\frac{\partial^2 u}{\partial x_1^2}\right|^2 \, dx_1 + c\varepsilon^{-1} \int_0^{h_1} |u|^2 \, dx_1,$$

for all $\varepsilon \in (0, h_1^2)$. Integrating with respect to the other variables it follows that

$$\int_R \left|\frac{\partial u}{\partial x_1}\right|^2 \, dx < \varepsilon \int_R \left|\frac{\partial^2 u}{\partial x_1^2}\right|^2 \, dx + c\varepsilon^{-1} \int_R |u|^2 \, dx, \qquad 0 < \varepsilon < h_1^2.$$

Doing the same with the other variables and adding the resulting inequalities we conclude that

$$\|\nabla u\|_{L^2(R)}^2 < \varepsilon \|\nabla^2 u\|_{L^2(R)}^2 + c\varepsilon^{-1} \|u\|_{L^2(R)}^2 \tag{33}$$

for all $\varepsilon \in (0, h^2)$, and all $u \in C^\infty(\bar{R})$.

To generalize this to the higher order case we use induction on m. So, suppose (31) is true for all $0 \leqslant k \leqslant m-1$ and for all $\varepsilon \in (0, h^{2m-2k})$. We shall prove that it is true when m is replaced by $m+1$ and for all $0 \leqslant k \leqslant m$ and $\varepsilon \in (0, h^{2m+2-2k})$. If $k = m$, then

$$\|\nabla^m u\|_{L^2(R)}^2 < \varepsilon \|\nabla^{m+1} u\|_{L^2(R)}^2 + c\varepsilon^{-1} \|u\|_{L^2(R)}^2, \qquad \varepsilon \in (0, h^2), \tag{34}$$

simply by replacing u by $\nabla^{m-1} u$ in (33). If $0 \leqslant k \leqslant m-1$ and $\varepsilon < h^{2m+2-2k}$ are given, we define

$$\varepsilon_1 = \varepsilon^{(m-k)/(m+1-k)}, \qquad \varepsilon_2 = \varepsilon^{1/(m+1-k)}.$$

Then $\varepsilon_1 < h^{2m-2k}, \varepsilon_2 < h^2$ and therefore, using the induction hypothesis and (34),

$$\begin{aligned}
\|\nabla^k u\|_{L^2(R)}^2 &< \varepsilon_1 \|\nabla^m u\|_{L^2(R)}^2 + c\varepsilon_1^{k/(m-k)} \|u\|_{L^2(R)}^2 \\
&< \varepsilon_1 \{\varepsilon \|\nabla^{m+1} u\|_{L^2(R)}^2 + c\varepsilon^{-1} \|u\|_{L^2(R)}^2\} + c\varepsilon_1^{k/(m-k)} \|u\|_{L^2(R)}^2 \\
&= \varepsilon \|\nabla^{m+1} u\|_{L^2(R)}^2 + c'\varepsilon^{k/(m+1-k)} \|u\|_{L^2(R)}^2,
\end{aligned}$$

as claimed. ∎

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