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SOBOLEV IMPROVEMENTS ON SHARP RELLICH INEQUALITIES

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Dedicated to E.B. Davies on the occasion of his 80th birthday

Abstract. There are two Rellich inequalities for the bilaplacian, that is for $\int(\Delta u)^2 dx$, the one involving $|\nabla u|$ and the other involving $|u|$ at the RHS. In this article we consider these inequalities with sharp constants and obtain sharp Sobolev-type improvements. More precisely, in our first result we improve the Rellich inequality with $|\nabla u|$ obtained by Beckner in dimensions $n = 3, 4$ by a sharp Sobolev term thus complementing existing results for the case $n \geq 5$. In the second theorem the sharp constant of the Sobolev improvement for the Rellich inequality with $|u|$ is obtained.

Introduction

The study of PDEs involving the bilaplacian is often related to functional inequalities for the associated energy, namely $\int(\Delta u)^2 dx$. Two important such inequalities are the Sobolev inequality and the Rellich inequality.

There are two Rellich inequalities related to the bilaplacian. The first one asserts that for $n \geq 5$ there holds

$$\int_{\mathbb{R}^n} (\Delta u)^2 dx \geq \frac{n^2(n-4)^2}{16} \int_{\mathbb{R}^n} \frac{u^2}{|x|^4} dx\,, \quad u \in C_c^\infty(\mathbb{R}^n), \tag{1}$$

and the constant is the best possible. Inequality (1) was proved by F. Rellich, see [22]. For more results on inequalities of this type and related improvements we refer to [1, 2, 4, 6, 10, 11, 12, 14, 17, 18, 19, 20, 23, 25] and references therein.

The second Rellich inequality is valid not only for $n \geq 5$ but also for $n = 3, 4$ and reads

$$\int_{\mathbb{R}^n} (\Delta u)^2 dx \geq c_n \int_{\mathbb{R}^n} \frac{|\nabla u|^2}{|x|^2} dx\,, \quad u \in C_c^\infty(\mathbb{R}^n), \tag{2}$$

where

$$c_n = \begin{cases} \frac{25}{36}, & n = 3, \\ 3, & n = 4, \\ \frac{n^2}{4}, & n \geq 5. \end{cases} \tag{3}$$

is the best possible constant. Inequality (2) was proved in [25] in case $n \geq 5$ and then by Beckner for any $n \geq 3$ [8]. An alternative proof for $n \geq 3$ was given by Cazacu [9]. We note that in cases $n = 3, 4$ there is a breaking of symmetry. For more information on Rellich inequalities in the spirit of (2) we refer to [9, 11, 13, 21, 25].

2020 *Mathematics Subject Classification.* Primary 35A23, 35G05, 35G20; Secondary 31B30, 35J30, 46E35.

Key words and phrases. Rellich inequality; Sobolev inequality; best constant

To appear in the Journal of Spectral Theory.

The Sobolev inequality for the bilaplacian in $\mathbb{R}^n$, $n \geq 5$, reads

$$\int_{\mathbb{R}^n} (\Delta u)^2 dx \geq S_{2,n} \Big(\int_{\mathbb{R}^n} |u|^{\frac{2n}{n-4}} dx \Big)^{\frac{n-4}{n}}, \quad u \in C_c^\infty(\mathbb{R}^n). \tag{4}$$

The best constant $S_{2,n}$ in (4) has been computed in [15] and is given by

$$S_{2,n} = \pi^2 (n^2 - 4n)(n^2 - 4) \Big(\frac{\Gamma(\frac{n}{2})}{\Gamma(n)} \Big)^4 .$$

The aim of this work is to improve the above Rellich inequalities by adding a Sobolev-type term. In [25] improved versions of (1) and (2) were obtained for a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 5$. More precisely, let $X(r) = (1 - \log r)^{-1}$, $0 < r < 1$, and $D = \sup_\Omega |x|$. In [25, Theorem 1.1] it was shown that for $n \geq 5$ there exist constants C_n and C_n' which depend only on n such that for any $u \in C_c^\infty(\Omega)$ there holds

$$\int_\Omega (\Delta u)^2 dx - \frac{n^2(n-4)^2}{16} \int_\Omega \frac{u^2}{|x|^4} dx \geq C_n \Big(\int_\Omega X(|x|/D)^{\frac{2(n-2)}{n-4}} |u|^{\frac{2n}{n-4}} dx \Big)^{\frac{n-4}{n}} \tag{5}$$

and

$$\int_\Omega (\Delta u)^2 dx - \frac{n^2}{4} \int_\Omega \frac{|\nabla u|^2}{|x|^2} dx \geq C_n' \Big(\int_\Omega X(|x|/D)^{\frac{2(n-1)}{n-2}} |\nabla u|^{\frac{2n}{n-2}} dx \Big)^{\frac{n-2}{n}} \tag{6}$$

The present article contains two main results. The first theorem extends inequality (6) to dimensions $n = 3, 4$.

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$, $n = 3$ or $n = 4$, be a bounded domain and let $D = \sup_{x \in \Omega} |x|$. There exists $C > 0$ such that:*
(i) *If $n = 3$ then*

$$\int_\Omega (\Delta u)^2 dx - \frac{25}{36} \int_\Omega \frac{|\nabla u|^2}{|x|^2} dx \geq C \bigg(\int_\Omega |\nabla u|^6 X^4(|x|/D) dx \bigg)^{\frac{1}{3}}, \quad u \in C_c^\infty(\Omega).$$

(ii) *If $n = 4$ then*

$$\int_\Omega (\Delta u)^2 dx - 3 \int_\Omega \frac{|\nabla u|^2}{|x|^2} dx \geq C \bigg(\int_\Omega |\nabla u|^4 dx \bigg)^{\frac{1}{2}}, \quad u \in C_c^\infty(\Omega).$$

Moreover the power X^4 in case $n = 3$ is the best possible.

It is remarkable that in case $n = 4$ no logarithmic factor is required at the RHS, as opposed to the cases $n = 3$ and $n \geq 5$.

Concerning inequality (5), let us first recall what is known for the corresponding Hardy-Sobolev problem. In [3] it was shown that for any bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$, and for any $u \in C_c^\infty(\Omega)$ there holds

$$\begin{aligned} &\int_\Omega |\nabla u|^2 dx - \Big(\frac{n-2}{2} \Big)^2 \int_\Omega \frac{u^2}{|x|^2} dx \\ &\quad \geq \quad (n-2)^{-\frac{2(n-1)}{n}} S_{1,n} \bigg(\int_\Omega X^{\frac{2(n-1)}{n-2}} \big(|x|/D\big) |u|^{\frac{2n}{n-2}} dx \bigg)^{\frac{n-2}{n}} \end{aligned}$$

where

$$S_{1,n} = \pi n(n-2) \Big(\frac{\Gamma(\frac{n}{2})}{\Gamma(n)} \Big)^{\frac{2}{n}} ,$$

is the best Sobolev constant for the standard Sobolev inequality in $\mathbb{R}^n$. Moreover the constant $(n-2)^{-\frac{2(n-1)}{n}} S_{1,n}$ is the best possible. Similarly, in the article [7] Sobolev improvements with best constants were obtained to sharp Hardy inequalities in Euclidean and hyperbolic space. We note that by slightly adapting [7, Theorem 5] we obtain that if Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 3$, then

$$\begin{aligned} &\int_\Omega |\nabla u|^2 dx - \Big(\frac{n-2}{2}\Big)^2 \int_\Omega \frac{u^2}{|x|^2} dx + \frac{(n-1)(n-3)}{4} \int_\Omega \frac{u^2}{|x|^2} X^2(|x|/D) dx \\ \geq \quad & S_{1,n} \Big(\int_\Omega X^{\frac{2(n-1)}{n-2}}(|x|/D) |u|^{\frac{2n}{n-2}} dx \Big)^{\frac{n-2}{n}} \qquad (7) \end{aligned}$$

for all $u \in C_c^\infty(\Omega)$ and the constant $S_{1,n}$ is sharp.

The second theorem of this article provides an estimate with best Sobolev constant for a slightly modified version of (5) which is in the spirit of (7).

Theorem 2. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 5$, be a bounded domain and let $D = \sup_\Omega |x|$. For any $u \in C_c^\infty(\Omega)$ there holds*

$$\begin{aligned} &\int_\Omega (\Delta u)^2 dx - \frac{n^2(n-4)^2}{16} \int_\Omega \frac{u^2}{|x|^4} dx + \frac{n^2(n-4)^2}{16} \int_\Omega \frac{u^2}{|x|^4} X^{\frac{2(n-2)}{n-1}} dx \\ &\quad \geq S_{2,n} \Big(\int_\Omega X^{\frac{2(n-2)}{n-4}} |u|^{\frac{2n}{n-4}} dx \Big)^{\frac{n-4}{n}}; \end{aligned}$$

here $X = X(|x|/D)$. Moreover the constant $S_{2,n}$ is the best possible.

The proof of Theorem 1 is in Section 1 and the proof of Theorem 2 is in Section 2.

1. Rellich-Sobolev inequality I

In this section we shall prove Theorem 1. An important tool will be the decomposition of functions in spherical harmonics [24, Section IV.2].

We recall that the eigenvalues of the Laplace-Beltrami operator on the unit sphere S^{n-1} are given by

$$\mu_k = k(k+n-2), \qquad k = 0, 1, 2 \ldots,$$

Each μ_k has multiplicity

$$d_k = \binom{n+k-1}{k} - \binom{n+k-3}{k-2}, \quad k \geq 2,$$

while $d_0 = 1$ and $d_1 = n$.

Let $\{\phi_{kj}\}_{j=1}^{d_k}$ be an orthonormal basis of eigenfunctions for the eigenvalue μ_k. Then any function $u \in L^2(\mathbb{R}^n)$ can be decomposed as

$$u(x) = \sum_{k=0}^\infty \sum_{j=1}^{d_k} u_{kj}(x) = \sum_{k=0}^\infty \sum_{j=1}^{d_k} f_{kj}(r) \phi_{kj}(\omega) \qquad (8)$$

where $x = r\omega$, $r > 0$, $\omega \in \mathrm{S}^{n-1}$, and

$$f_{kj}(r) = \int_{\mathrm{S}^{n-1}} u(r\omega) \phi_{kj}(\omega) dS(\omega).$$

We note that each ϕ_{kj} is the restriction on the unit sphere of a harmonic homogeneous polynomial of degree k [24].

Assume now that $u\in C_c^\infty(\mathbb{R}^n)$. Since any homogeneous polynomial can be written as a linear combination of harmonic homogeneous polynomials, taking the Taylor expansion of u near the origin we easily infer that

$$f_{kj}(r)=O(r^k),\quad f'_{kj}(r)=O(r^{k-1})\ ,\qquad \text{as } r\to 0. \tag{9}$$

for any $k\geq 1$ and any $j=1,\ldots,d_k$.

We note that

$$\mu_k\geq n-1\ ,\qquad \forall\, k\geq 1, \tag{10}$$

an estimate that will be used several times in what follows.

In what follows we shall use $\sum_{k,j}$ as a shorthand for $\sum_{k=0}^\infty\sum_{j=1}^{d_k}$.

For simplicity we shall denote by u_0 (instead of u_{01}) the first (radial) term in the decomposition (8) of u into spherical harmonics. We note the relation

$$\int_{\mathbb{R}^n}(\Delta u-\Delta u_0)^2dx=\sum_{k=1}^\infty\sum_{j=1}^{d_k}\int_{\mathbb{R}^n}(\Delta u_{kj})^2dx\,. \tag{11}$$

Lemma 1. *Let $n\geq 3$. For any $u\in C_c^\infty(\mathbb{R}^n)$ there holds*

(i) $$\int_{\mathbb{R}^n}(\Delta u)^2dx=\sum_{k,j}\Big\{\int_0^\infty r^{n-1}f''^2_{kj}dr$$
$$+(n-1+2\mu_k)\int_0^\infty r^{n-3}f'^2_{kj}dr+\big(2(n-4)\mu_k+\mu_k^2\big)\int_0^\infty r^{n-5}f^2_{kj}dr\Big\}$$

(ii) $$\int_{\mathbb{R}^n}\frac{|\nabla u|^2}{|x|^2}dx=\sum_{k,j}\Big\{\int_0^\infty r^{n-3}f'^2_{kj}dr+\mu_k\int_0^\infty r^{n-5}f^2_{kj}dr\Big\}$$

Proof. Using the orthonormality of the set $\{\phi_{kj}\}$ we have

$$\begin{aligned}\int_{\mathbb{R}^n}(\Delta u)^2dx &= \sum_{k,j}\int_{\mathbb{R}^n}(\Delta u_{kj})^2dx\\ &= \sum_{k,j}\int_0^\infty\Big(f''_{kj}+\frac{n-1}{r}f'_{kj}-\frac{\mu_k}{r^2}f_{kj}\Big)^2r^{n-1}dr.\end{aligned}$$

Part (i) then follows by expanding the square and integrating by parts. Estimates (9) ensure that no terms appear from $r=0$. The proof of (ii) is similar and is omitted. □

For $n\geq 3$ we set

$$\mathbb{I}[u]=\int_{\mathbb{R}^n}(\Delta u)^2dx-c_n\int_{\mathbb{R}^n}\frac{|\nabla u|^2}{|x|^2}dx$$

where the constant c_n is given by (3).

Lemma 2. *Assume that $n=3$ or $n=4$. There exists $c>0$ such that for any $u\in C_c^\infty(\mathbb{R}^n)$ there holds*

$$\mathbb{I}[u]\geq\mathbb{I}[u_0]+\sum_{j=1}^n\mathbb{I}[u_{1j}]+c\int_{\mathbb{R}^n}\Big(\Delta u-\Delta u_0-\sum_{j=1}^n\Delta u_{1j}\Big)^2dx. \tag{12}$$

Proof. Let $u \in C_c^\infty(\mathbb{R}^n)$. Because of the relation

$$\mathbb{I}[u] = \mathbb{I}[u_0] + \sum_{j=1}^{n} \mathbb{I}[u_{1j}] + \sum_{k=2}^{\infty}\sum_{j=1}^{d_k} \mathbb{I}[u_{kj}],$$

inequality (12) will follow if we establish the existence of $c > 0$ such that

$$\mathbb{I}[u_{kj}] \geq c\int_{\mathbb{R}^n} (\Delta u_{kj})^2 dx\,, \qquad k \geq 2\,,\ \ 1 \leq j \leq d_k. \tag{13}$$

Assume first that $n = 3$. Let $\lambda > 0$ be fixed. For $k \geq 2$ we have $\mu_k \geq 6$ and therefore

$$\begin{aligned}
&\int_{\mathbb{R}^3} (\Delta u_{kj})^2 dx \\
= \;& \int_0^\infty r^2 f_{kj}''^2 dr + (2+2\mu_k)\int_0^\infty f_{kj}'^2 dr + (-2\mu_k + \mu_k^2)\int_0^\infty r^{-2} f_{kj}^2 dr \\
\geq \;& \Big(\frac{9}{4} + 2\lambda\mu_k\Big)\int_0^\infty f_{kj}'^2 dr + \Big(2(1-\lambda)\frac{1}{4}\mu_k - 2\mu_k + \mu_k^2\Big)\int_0^\infty r^{-2} f_{kj}^2 dr \\
\geq \;& \Big(\frac{9}{4} + 12\lambda\Big)\int_0^\infty f_{kj}'^2 dr + \Big(\frac{9}{2} - \frac{\lambda}{2}\Big)\mu_k \int_0^\infty r^{-2} f_{kj}^2 dr.
\end{aligned}$$

Choosing $\lambda = 9/50$ we arrive at

$$\int_{\mathbb{R}^3} (\Delta u_{kj})^2 dx \geq \frac{441}{100}\int_{\mathbb{R}^3} \frac{|\nabla u_{kj}|^2}{|x|^2} dx,$$

and (13) follows. In case $n = 4$ we argue similarly. We now have $\mu_k \geq 8$, hence

$$\begin{aligned}
\int_{\mathbb{R}^4} (\Delta u_{kj})^2 dx &= \int_0^\infty r^3 f_{kj}''^2 dr + (3+2\mu_k)\int_0^\infty r f_{kj}'^2 dr + \mu_k^2 \int_0^\infty r^{-1} f_{kj}^2 dr \\
&\geq (4+2\mu_k)\int_0^\infty r f_{kj}'^2 dr + \mu_k^2\int_0^\infty r^{-1} f_{kj}^2 dr \\
&\geq 8\int_{\mathbb{R}^4} \frac{|\nabla u_{kj}|^2}{|x|^2} dx,
\end{aligned}$$

as required. □

Lemma 3. *Let $n = 3$ or $n = 4$. Then there exists $c > 0$ such that*

$$\mathbb{I}[u_0] \geq c\bigg(\int_{B_1} |\nabla u_0|^{\frac{2n}{n-2}} dx\bigg)^{\frac{n-2}{n}}. \tag{14}$$

Additionally for $n = 3$ we have

$$\mathbb{I}[u_{1j}] \geq c\bigg(\int_{B_1} |\nabla u_{1j}|^6 X^4 dx\bigg)^{\frac{1}{3}}, \qquad j = 1,2,3, \tag{15}$$

while for $n = 4$

$$\mathbb{I}[u_{1j}] \geq c\bigg(\int_{B_1} |\nabla u_{1j}|^4 dx\bigg)^{\frac{1}{2}}, \qquad j = 1,2,3,4. \tag{16}$$

Here $X = X(|x|)$.

Proof. From Lemma 1 (i) and the standard Sobolev inequality we obtain

$$\begin{aligned}\mathbb{I}[u_0] &\geq \int_0^1 f_0''^2\, r^{n-1}dr \\ &\geq c\Big(\int_0^1 |f_0'|^{\frac{2n}{n-2}} r^{n-1}dr\Big)^{\frac{n-2}{n}} \\ &= c\Big(\int_{B_1} |\nabla u_0|^{\frac{2n}{n-2}} dx\Big)^{\frac{n-2}{n}}\end{aligned}$$

as required.

Assume now that $n=3$. By Lemma 1 and the improved Hardy-Sobolev inequality of [3] we have

$$\begin{aligned}\mathbb{I}[u_{1j}] &= \int_0^1 f_{1j}''^2 r^2 dr - \frac{1}{4}\int_0^1 f_{1j}'^2 dr \\ &\qquad + \frac{50}{9}\Big(\int_0^1 f_{1j}'^2 dr - \frac{1}{4}\int_0^1 r^{-2} f_{1j}^2 dr\Big) \\ &\geq c\Big(\int_0^1 |f_{1j}'|^6 X^4\, r^2 dr\Big)^{\frac{1}{3}} + c\Big(\int_0^1 |f_{1j}|^6 X^4\, dr\Big)^{\frac{1}{3}} \\ &\geq c\Big(\int_{B_1} |\nabla u_{1j}|^6 X^4 dx\Big)^{\frac{1}{3}}.\end{aligned}$$

In case $n=4$ we argue similarly applying again Lemma 1 and, now, the standard Sobolev inequality; we obtain

$$\begin{aligned}\mathbb{I}[u_{1j}] &= \int_0^1 f_{1j}''^2\, r^3 dr + 6\int_0^1 f_{1j}'^2\, r\, dr \\ &\geq c\Big(\int_0^1 |f_{1j}'|^4\, r^3 dr\Big)^{\frac{1}{2}} + c\Big(\int_0^1 |f_{1j}|^4\, r\, dr\Big)^{\frac{1}{2}} \\ &\geq c\Big(\int_{B_1} |\nabla u_{1j}|^4 dx\Big)^{\frac{1}{2}},\end{aligned}$$

as required. □

Proof of Theorem 1. We first note that by the standard Sobolev inequality we have

$$\int_\Omega (\Delta u - \Delta u_0 - \sum_{j=1}^n \Delta u_{1j})^2 dx \geq c\Big(\int_\Omega |\nabla u - \nabla u_0 - \sum_{j=1}^n \nabla u_{1j}|^{\frac{2n}{n-2}}\, dx\Big)^{\frac{1}{3}};$$

In case $n=3$ we apply (12), (14), (15) and the triangle inequality to obtain

$$\begin{aligned}
\mathbb{I}[u] &\geq \mathbb{I}[u_0]+\sum_{j=1}^{n}\mathbb{I}[u_{1j}]+c\int_{\mathbb{R}^n}\big(\Delta u-\Delta u_0-\sum_{j=1}^{n}\Delta u_{1j}\big)^2dx\\
&\geq c\Big(\int_{\Omega}|\nabla u_0|^6X^4dx\Big)^{\frac13}+c\sum_{j=1}^{n}\Big(\int_{B_1}|\nabla u_{1j}|^6X^4dx\Big)^{\frac13}\\
&\quad +c\Big(\int_{\Omega}|\nabla u-\nabla u_0-\sum_{j=1}^{n}\nabla u_{1j}|^6dx\Big)^{\frac13}\\
&\geq c\Big(\int_{\Omega}|\nabla u|^6X^4dx\Big)^{\frac13}.
\end{aligned}$$

In case $n=4$ we argue similarly, the only difference being that we use (16) instead of (15).

We next prove the optimality of the power X^4 in (i), that is in case $n=3$. So let us assume instead that there exist $\mu<4$ and $c>0$ so that

$$\int_{\Omega}(\Delta u)^2dx-\frac{25}{36}\int_{\Omega}\frac{|\nabla u|^2}{|x|^2}dx\geq c\Big(\int_{\Omega}|\nabla u|^6X^{\mu}(|x|/D)dx\Big)^{\frac13}, \tag{17}$$

for all $u\in C_c^{\infty}(\Omega)$. Without loss of generality we assume that $B_1\subset\Omega$. We consider small positive numbers ϵ and δ and define the functions

$$u_{\epsilon,\delta}(x)=f_{\epsilon,\delta}(r)\phi_1(\omega):=r^{\frac12+\epsilon}X(r)^{-\frac12+\delta}\psi(r)\phi_1(\omega)$$

where $\phi_1(\omega)$ is a normalized eigenfunction for the first non-zero eigenvalue of the Laplace-Beltrami operator on S^2 and $\psi(r)$ is a smooth radially symmetric function supported in B_1 and equal to one near $r=0$.

Applying Lemma 1 we see that $\int(\Delta u_{\epsilon,\delta})^2dx-\frac{25}{36}\int\frac{|\nabla u_{\epsilon,\delta}|^2}{|x|^2}dx$ is a linear combination of the integrals

$$I_{\epsilon,\delta}^{(j)}=\int_0^1 r^{-1+2\epsilon}X^{-1+j+2\delta}\psi^2dr,\quad 0\leq j\leq 4,$$

and of integrals that contain at least one derivative of ψ and are, therefore, uniformly bounded. Moreover simple computations yield that for $j=3,4$ the integrals $I_{\epsilon,\delta}^{(j)}$ are also uniformly bounded for small $\epsilon,\delta>0$.

Restricting attention to a small neighbourhood of the origin where $\psi=1$ we find

$$f'_{\epsilon,\delta}(r)=r^{-\frac12+\epsilon}\Big(\big(\frac12+\epsilon\big)X^{-\frac12+\delta}+\big(-\frac12+\delta\big)X^{\frac12+\delta}\Big)$$

and

$$f''_{\epsilon,\delta}(r)=r^{-\frac32+\epsilon}\Big(\big(\epsilon^2-\frac14\big)X^{-\frac12+\delta}+2\epsilon\big(-\frac12+\delta\big)X^{\frac12+\delta}+\big(\delta^2-\frac14\big)X^{\frac32+\delta}\Big)$$

Hence we arrive at

$$\begin{aligned}\int_{B_1}(\Delta u_{\epsilon,\delta})^2dx-\frac{25}{36}\int_{B_1}\frac{|\nabla u_{\epsilon,\delta}|^2}{|x|^2}dx\\ =\ &\Big(\frac{191}{36}\epsilon+\frac{173}{36}\epsilon^2+\epsilon^4\Big)I^{(0)}_{\epsilon,\delta}\\ &-\Big(\frac{191}{72}-\frac{191}{36}\delta+\big(\frac{173}{36}-\frac{173}{18}\delta\big)\epsilon+(2-4\delta)\epsilon^3\Big)I^{(1)}_{\epsilon,\delta}\\ &+\Big(\frac{209}{144}-\frac{191}{36}\delta+\frac{173}{36}\delta^2+\big(\frac{1}{2}-4\delta+6\delta^2\big)\epsilon^2\Big)I^{(2)}_{\epsilon,\delta}+O(1).\end{aligned}$$

It is easily seen that

$$I^{(j)}_{\epsilon,0}=\frac{1}{2\epsilon}+O(1)\ ,\quad j=0,1,2.$$

Hence, rearranging also terms we obtain

$$\begin{aligned}\int_{B_1}(\Delta u_{\epsilon,\delta})^2dx-\frac{25}{36}\int_{B_1}\frac{|\nabla u_{\epsilon,\delta}|^2}{|x|^2}dx &= \frac{191}{72}\big(2\epsilon I^{(0)}_{\epsilon,\delta}-(1-2\delta)I^{(1)}_{\epsilon,\delta}\big)\\ &\quad+\Big(\frac{209}{144}-\frac{191}{36}\delta+\frac{173}{36}\delta^2\Big)I^{(2)}_{\epsilon,\delta}+O(1).\end{aligned}$$

Now, by [5, p181] we have

$$2\epsilon I^{(0)}_{\epsilon,\delta}-(1-2\delta)I^{(1)}_{\epsilon,\delta}=O(1).$$

Hence, letting $\epsilon\to 0$ we obtain

$$\begin{aligned}\int_{B_1}(\Delta u_{\epsilon,\delta})^2dx-\frac{25}{36}\int_{B_1}\frac{|\nabla u_{\epsilon,\delta}|^2}{|x|^2}dx &\to \Big(\frac{209}{144}-\frac{191}{36}\delta+\frac{173}{36}\delta^2\Big)I^{(2)}_{0,\delta}+O(1)\\ &= \frac{209}{144}\int_0^1 r^{-1}X^{1+2\delta}\psi^2dr+O(1),\end{aligned}$$

which is finite for any $\delta>0$ and diverges to infinity as $\delta\to 0+$.

Now, for $\delta>(4-\mu)/6$ we have

$$\int_{B_1}|\nabla u_{\epsilon,\delta}|^6X^{\mu}dx\geq c\int_0^{1/2}r^{-1+6\epsilon}X^{\mu-3+6\delta}dr.$$

Letting first $\epsilon\to 0$ and then $\delta\to\frac{4-\mu}{6}+$ the last integral tends to infinity. Hence the Rayleigh quotient tends to zero, which implies that the constant c in (17) should be zero. This concludes the proof. □

2. Rellich-Sobolev inequality II

In this section we shall prove Theorem 2. Throughout the proof we shall make use of spherical coordinates (r,ω), $r>0$, $\omega\in \mathrm{S}^{n-1}$. We denote by ∇_ω and Δ_ω the gradient and Laplacian on S^{n-1}.

Lemma 4. *Let $\theta \in \mathbb{R}$. For any $v \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$ there holds*

$$\begin{aligned}
&\int_{\mathbb{R}^n} (\Delta v)^2 |x|^\theta dx \\
= \quad & \int_0^\infty \int_{\mathrm{S}^{n-1}} v_{rr}^2 r^{n+\theta-1} dS\, dr + (n-1)(1-\theta) \int_0^\infty \int_{\mathrm{S}^{n-1}} v_r^2 r^{n+\theta-3} dS\, dr \\
& + \int_0^\infty \int_{\mathrm{S}^{n-1}} (\Delta_\omega v)^2 r^{n+\theta-5} dS\, dr + 2 \int_0^\infty \int_{\mathrm{S}^{n-1}} |\nabla_\omega v_r|^2 r^{n+\theta-3}\, dS\, dr \\
& -(\theta-2)(n+\theta-4) \int_0^\infty \int_{\mathrm{S}^{n-1}} |\nabla_\omega v|^2 r^{n+\theta-5} dS\, dr\,.
\end{aligned}$$

Proof. This follows by writing

$$\Delta v = v_{rr} + \frac{n-1}{r} v_r + \frac{1}{r^2} \Delta_\omega v$$

and integrating by parts; we omit the details. □

In the next lemma and also later, we shall use subscripts R and NR to denote the radial and non-radial component of a given functional.

Lemma 5. *Let $n \geq 5$, $\beta > 0$ and define*

$$A = \frac{1}{\beta^2} \big(2n - 4 - \beta(n-4+\beta) \big).$$

Let $u \in C_c^\infty(\mathbb{R}^n)$. Changing variables by $u(r,\omega) = y(t,\omega)$, $t = r^\beta$, we have

$$\frac{\displaystyle\int_{\mathbb{R}^n} (\Delta u)^2 dx}{\Big(\displaystyle\int_{\mathbb{R}^n} |u|^{\frac{2n}{n-4}} dx \Big)^{\frac{n-4}{n}}} = \beta^{\frac{4(n-1)}{n}} \frac{\mathcal{A}_{\mathrm{R}}[y] + \mathcal{A}_{\mathrm{NR}}[y]}{\Big(\displaystyle\int_0^\infty \int_{\mathrm{S}^{n-1}} t^{\frac{n-\beta}{\beta}} |y|^{\frac{2n}{n-4}} dS\, dt \Big)^{\frac{n-4}{n}}}$$

where

$$\begin{aligned}
\mathcal{A}_{\mathrm{R}}[y] &= \int_0^\infty \int_{\mathrm{S}^{n-1}} \big(t^{\frac{3\beta+n-4}{\beta}} y_{tt}^2 + A t^{\frac{\beta+n-4}{\beta}} y_t^2 \big) dS\, dt \\
\mathcal{A}_{\mathrm{NR}}[y] &= \int_0^\infty \int_{\mathrm{S}^{n-1}} \Big(\frac{1}{\beta^4} t^{\frac{n-\beta-4}{\beta}} (\Delta_\omega y)^2 + \frac{2}{\beta^2} t^{\frac{\beta+n-4}{\beta}} |\nabla_\omega y_t|^2 \\
&\qquad + \frac{2(n-4)}{\beta^4} t^{\frac{n-\beta-4}{\beta}} |\nabla_\omega y|^2 \Big) dS\, dt
\end{aligned}$$

Proof. After some lengthy but otherwise elementary computations we find

$$\int_0^\infty \big(u_{rr} + \frac{n-1}{r} u_r \big)^2 r^{n-1} dr = \beta^3 \int_0^\infty \big(t^{\frac{3\beta+n-4}{\beta}} y_{tt}^2 + A t^{\frac{\beta+n-4}{\beta}} y_t^2 \big) dt$$

and

$$\int_0^\infty |u|^{\frac{2n}{n-4}} r^{n-1} dr = \frac{1}{\beta} \int_0^\infty |y|^{\frac{2n}{n-4}} t^{\frac{n-\beta}{\beta}} dt\,.$$

Similar computations involving the non-radial (tangential) derivatives yield the term $\mathcal{A}_{\mathrm{NR}}[y]$. We omit the details. □

We now consider the Rayleigh quotient for the Rellich-Sobolev inequality (5). Changing variables by $u(x)=|x|^{-\frac{n-4}{2}}v(x)$ we obtain (cf. [25, Lemma 2.3 (ii)])

$$\begin{aligned}
&\int_\Omega (\Delta u)^2 dx - \frac{n^2(n-4)^2}{16}\int_\Omega \frac{u^2}{|x|^4}dx \qquad (18)\\
= \;& \int_\Omega \Big(|x|^{4-n}(\Delta v)^2 + \frac{n(n-4)}{2}|x|^{2-n}|\nabla v|^2 - n(n-4)|x|^{-n}(x\cdot\nabla v)^2\Big)dx\,.\\
=: \;& J[v] \qquad (19)
\end{aligned}$$

Applying Lemma 4 we find that

$$\begin{aligned}
J[v] \;=\;& \int_0^1\int_{S^{n-1}} r^3 v_{rr}^2 dS\,dr + \frac{n^2-4n+6}{2}\int_0^1\int_{S^{n-1}} r v_r^2 dS\,dr\\
&+\int_0^1\int_{S^{n-1}} r^{-1}(\Delta_\omega v)^2 dS\,dr + 2\int_0^1\int_{S^{n-1}} |\nabla_\omega v_r|^2 r\,dS\,dr\\
&+\frac{n(n-4)}{2}\int_0^1\int_{S^{n-1}} r^{-1}|\nabla_\omega v|^2 dS\,dr\,. \qquad (20)
\end{aligned}$$

In view of (20) we set

$$\begin{aligned}
J_{\rm R}[v] \;=\;& \int_0^1\int_{S^{n-1}} r^3 v_{rr}^2 dS\,dr + \frac{n^2-4n+6}{2}\int_0^1\int_{S^{n-1}} r v_r^2 dS\,dr\\
J_{\rm NR}[v] \;=\;& \int_0^1\int_{S^{n-1}} r^{-1}(\Delta_\omega v)^2 dS\,dr + 2\int_0^1\int_{S^{n-1}} r\,|\nabla_\omega v_r|^2 dS\,dr\\
&+\frac{n(n-4)}{2}\int_0^1\int_{S^{n-1}} r^{-1}|\nabla_\omega v|^2 dS\,dr,
\end{aligned}$$

the radial and non-radial parts of $J[v]$, so that,

$$J[v] = J_{\rm R}[v] + J_{\rm NR}[v].$$

We shall change variables once more and for this we define the functions

$$g(r) = \exp\left(1 - X(r)^{-\frac{n}{2(n-1)}}\right), \qquad \alpha(r) = X(r)^{-\frac{3(n-2)}{4(n-1)}} g(r)^{\frac{n-4}{2\beta}}. \qquad (21)$$

Lemma 6. *Let $n\geq 5$, $\beta>0$ and set*

$$s = \frac{n-4}{2\beta}.$$

Let $v\in C_c^\infty(B_1\setminus\{0\})$. Changing variables by

$$v(r,\omega) = \alpha(r) w(t,\omega)\,, \qquad t = g(r), \qquad (22)$$

we have

$$\text{(i)}\quad J_{\rm R}[v]=\int_0^1\int_{{\rm S}^{n-1}}\Big\{\Big(\frac{n}{2(n-1)}\Big)^3 t^{\frac{3\beta+n-4}{\beta}}w_{tt}^2+t^{\frac{\beta+n-4}{\beta}}G(t)w_t^2 + t^{\frac{-\beta+n-4}{\beta}}H(t)w^2\Big\}dS\,dt$$

$$\text{(ii)}\quad J_{\rm NR}[v]=\frac{2(n-1)}{n}\int_0^1\int_{{\rm S}^{n-1}}t^{\frac{n-\beta-4}{\beta}}X(t)^{\frac{8-4n}{n}}(\Delta_\omega w)^2dS\,dt + \frac{n}{n-1}\int_0^1\int_{{\rm S}^{n-1}}t^{\frac{n+\beta-4}{\beta}}X(t)^{\frac{4-2n}{n}}|\nabla_\omega w_t|^2dS\,dt + \int_0^1\int_{{\rm S}^{n-1}}t^{\frac{n-\beta-4}{\beta}}|\nabla_\omega w|^2K(t)dS\,dt$$

$$\text{(iii)}\quad \int_0^1\int_{{\rm S}^{n-1}}r^{-1}X(r)^{\frac{2n-4}{n-4}}|v|^{\frac{2n}{n-4}}dS\,dr=\frac{2(n-1)}{n}\int_0^1\int_{{\rm S}^{n-1}}|w|^{\frac{2n}{n-4}}t^{\frac{n-\beta}{\beta}}dS\,dt\,,$$

where the functions $G(t)$, $H(t)$ and $K(t)$ are given by

$$G(t)=\frac{n(n^2-4n+8)}{4(n-1)}X(t)^{\frac{4-2n}{n}}-\frac{n^3(2s^2+2s+1)}{8(n-1)^3}+\frac{5n(n-2)(3n-2)}{16(n-1)^3}X(t)^2$$

$$\begin{aligned}H(t)=&-\frac{s^2n(n^2-4n+8)}{4(n-1)}X(t)^{\frac{4-2n}{n}}+\frac{s(n-2)(n^2-4n+8)}{2(n-1)}X(t)^{\frac{4-n}{n}}\\&+\frac{s^4n^3}{8(n-1)^3}+\frac{3(n^2-4)(n^2-4n+8)}{16n(n-1)}X(t)^{\frac{4}{n}}\\&-\frac{5s^2n(n-2)(3n-2)}{16(n-1)^3}X(t)^2-\frac{5sn(n-2)(3n-2)}{8(n-1)^3}X(t)^3\\&-\frac{9(3n-2)(5n-2)(n^2-4)}{128n(n-1)^3}X(t)^4\end{aligned}$$

$$\begin{aligned}K(t)=&(n-1)(n-4)X(t)^{\frac{8-4n}{n}}-\frac{n(n-4)^2}{4(n-1)\beta^2}X(t)^{\frac{4-2n}{n}}\\&+\frac{(n-2)(n-4)}{(n-1)\beta}X(t)^{\frac{4-n}{n}}+\frac{3(n^2-4)}{4n(n-1)}X(t)^{\frac{4}{n}}.\end{aligned}$$

Proof. To prove (i) we set for simplicity

$$J_{\rm R}^*[v]=\int_0^1 r^3v_{rr}^2dr+\frac{n^2-4n+6}{2}\int_0^1 rv_r^2dr\,.$$

We first note that r and $t=g(r)$ are also related by the relation

$$X(t)=X(r)^{\frac{n}{2(n-1)}} \tag{23}$$

and that

$$dt=\frac{n}{2(n-1)}\frac{g(r)}{r}X(r)^{\frac{n-2}{2(n-1)}}dr\,.$$

Expressing $J_{\rm R}^*[v]$ in terms of the function $w(t)$ involves some lengthy computations, of which we include only the main steps.

From (22) we have

$$v_r = \alpha g' w_t + \alpha' w$$
$$v_{rr} = \alpha g'^2 w_{tt} + (2\alpha' g' + \alpha g'')w_t + \alpha'' w\,.$$

Substuting in $J^*_{\rm R}[v]$ and expanding we find that

$$\begin{aligned} J^*_{\rm R}[v] &= \Big(\frac{n}{2(n-1)}\Big)^3 \int_0^1 t^{\frac{3\beta+n-4}{\beta}} w_{tt}^2 dt + \int_0^1 B(t) w_t^2 dt + \int_0^1 C(t) w^2 dt \\ &\quad + \int_0^1 D(t) w_{tt}\, w_t dt + \int_0^1 E(t) w_{tt} w\, dt + \int_0^1 F(t) w_t\, w\, dt \end{aligned} \tag{24}$$

where the functions $B(t),\dots,F(t)$ will be described below in terms of the variable r. Integrating by parts we obtain from (24) that

$$J^*_{\rm R}[v] = \Big(\frac{n}{2(n-1)}\Big)^3 \int_0^1 t^{\frac{3\beta+n-4}{\beta}} w_{tt}^2 dt + \int_0^1 P(t) w_t^2 dt + \int_0^1 Q(t) w^2 dt$$

where

$$P(t) = B(t) - \frac{1}{2} D_t(t) - E(t)\,, \qquad Q(t) = C(t) + \frac{1}{2} E_{tt}(t) - \frac{1}{2} F_t(t)\,. \tag{25}$$

To compute the functions $P(t)$ and $Q(t)$ it is convenient to regard them as functions of the variable r. To do this we consider the functions B, C, D, E and F also as functions of r and indicate this with tildes; we shall thus write $B(t) = \tilde{B}(r)$, etc. Relations (25) then take the form

$$\tilde{P}(r) = \tilde{B} - \frac{1}{2g'}\tilde{D}_r - \tilde{E}\,, \qquad \tilde{Q}(r) = \tilde{C} + \frac{1}{2}\Big(\frac{\tilde{E}_{rr}}{g'^2} - \frac{g''\tilde{E}_r}{g'^3}\Big) - \frac{1}{2g'}\tilde{F}_r\,. \tag{26}$$

After some computations we eventually find

$$\begin{aligned} \tilde{B}(r) &= \frac{r^3}{g'}\big(2\alpha' g' + \frac{n-1}{r}\alpha g' + \alpha g''\big)^2 - \frac{n(n-4)}{2} r\alpha^2 g' \\ \tilde{C}(r) &= \frac{r^3}{g'}\big(\alpha'' + \frac{n-1}{r}\alpha'\big)^2 - \frac{n(n-4)}{2}\frac{r}{g'}\alpha'^2 \\ \tilde{D}(r) &= 2r^3 \alpha g'\big(2\alpha' g' + \frac{n-1}{r}\alpha g' + \alpha g''\big) \\ \tilde{E}(r) &= 2r^3 \alpha g'\big(\alpha'' + \frac{n-1}{r}\alpha'\big) \\ \tilde{F}(r) &= 2r^3\big(2\alpha' + \frac{n-1}{r}\alpha + \alpha\frac{g''}{g'}\big)\big(\alpha'' + \frac{n-1}{r}\alpha'\big) - n(n-4) r\alpha\alpha'\,. \end{aligned}$$

Substituting in (26) we arrive at

$$\begin{aligned} \tilde{P}(r) &= 2r^3\alpha'^2 g' - 6r^2\alpha\alpha' g' + \frac{n^2-4n+6}{2} r\alpha^2 g' - 3r^2\alpha^2 g'' \\ &\quad - 4r^3\alpha\alpha'' g' - 2r^3\alpha\alpha' g'' - r^3\alpha^2 g''' \\ \tilde{Q}(r) &= \frac{1}{g'}\Big(6r^2\alpha\alpha''' - \frac{n^2-4n-6}{2} r\alpha\alpha'' - \frac{n^2-4n+6}{2}\alpha\alpha' + r^3\alpha\alpha^{(4)}\Big). \end{aligned}$$

Now, some more computations give

$$
\begin{aligned}
g'(r) &= \frac{n}{2(n-1)}\frac{g(r)}{r}X(r)^{\frac{n-2}{2(n-1)}}\ ,\\
g''(r) &= \Big(-\frac{n}{2(n-1)}X^{\frac{n-2}{2(n-1)}}+\frac{n^2}{4(n-1)^2}X(r)^{\frac{n-2}{n-1}}+\frac{n(n-2)}{4(n-1)^2}X(r)^{\frac{3n-4}{2(n-1)}}\Big)\frac{g(r)}{r^2}\\
g'''(r) &= \Big(-\frac{3n(n-2)}{4(n-1)^2}X(r)^{\frac{3n-4}{2(n-1)}}+\frac{3n^2(n-2)}{8(n-1)^3}X^{\frac{2n-3}{n-1}}+\frac{n(n-2)(3n-4)}{8(n-1)^3}X^{\frac{5n-6}{2(n-1)}}\\
&\quad +\frac{n}{n-1}X(r)^{\frac{n-2}{2(n-1)}}-\frac{3n^2}{4(n-1)^2}X(r)^{\frac{n-2}{n-1}}+\frac{n^3}{8(n-1)^3}X(r)^{\frac{3n-6}{2(n-1)}}\Big)\frac{g(r)}{r^3}.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\alpha'(r) &= \frac{g(r)^s}{r}\Big(\frac{s}{2(n-1)}X^{\frac{2-n}{4(n-1)}}-\frac{3(n-2)}{4(n-1)}X(r)^{\frac{n+2}{4(n-1)}}\Big)\\
\alpha''(r) &= \frac{g(r)^s}{r^2}\Big(-\frac{sn}{2(n-1)}X(r)^{\frac{2-n}{4(n-1)}}+\frac{s^2n^2}{4(n-1)^2}X(r)^{\frac{n-2}{4(n-1)}}\\
&\quad +\frac{3(n-2)}{4(n-1)}X(r)^{\frac{n+2}{4(n-1)}}-\frac{sn(n-2)}{2(n-1)^2}X(r)^{\frac{3n-2}{4(n-1)}}-\frac{3(n^2-4)}{16(n-1)^2}X(r)^{\frac{5n-2}{4(n-1)}}\Big)\\
\alpha'''(r) &= \frac{g(r)^s}{r^3}\Big(\frac{sn}{n-1}X^{\frac{2-n}{4(n-1)}}-\frac{3s^2n^2}{4(n-1)^2}X(r)^{\frac{n-2}{4(n-1)}}\\
&\quad -\frac{3(n-2)}{2(n-1)}X(r)^{\frac{n+2}{4(n-1)}}+\frac{s^3n^3}{8(n-1)^3}X^{\frac{3n-6}{4(n-1)}}+\frac{3sn(n-2)}{2(n-1)^2}X^{\frac{3n-2}{4(n-1)}}\\
&\quad -\frac{3s^2n^2(n-2)}{16(n-1)^3}X(r)^{\frac{5n-6}{4(n-1)}}+\frac{9(n^2-4)}{16(n-1)^2}X(r)^{\frac{5n-2}{4(n-1)}}\\
&\quad -\frac{sn(n-2)(15n-2)}{32(n-1)^3}X(r)^{\frac{7n-6}{4(n-1)}}-\frac{3(n^2-4)(5n-2)}{64(n-1)^3}X(r)^{\frac{9n-6}{4(n-1)}}\Big) \qquad (27)
\end{aligned}
$$

and

$$
\begin{aligned}
\alpha^{(4)}(r) = \frac{g(r)^s}{r^4}\Big(&\frac{3sn}{n-1}X(r)^{\frac{2-n}{4(n-1)}}-\frac{11s^2n^2}{4(n-1)^2}X(r)^{\frac{n-2}{4(n-1)}}\\
&-\frac{9(n-2)}{2(n-1)}X(r)^{\frac{n+2}{4(n-1)}}+\frac{3s^3n^3}{4(n-1)^3}X(r)^{\frac{3n-6}{4(n-1)}}\\
&+\frac{11sn(n-2)}{2(n-1)^2}X(r)^{\frac{3n-2}{4(n-1)}}-\frac{s^4n^4}{16(n-1)^4}X(r)^{\frac{5n-10}{4(n-1)}}\\
&-\frac{9s^2n^2(n-2)}{8(n-1)^3}X(r)^{\frac{5n-6}{4(n-1)}}+\frac{33(n^2-4)}{16(n-1)^2}X(r)^{\frac{5n-2}{4(n-1)}}\\
&-\frac{3sn(n-2)(15n-2)}{16(n-1)^3}X(r)^{\frac{7n-6}{4(n-1)}}+\frac{5s^2n^2(n-2)(3n-2)}{32(n-1)^4}X(r)^{\frac{9n-10}{4(n-1)}}\\
&-\frac{9(5n-2)(n^2-4)}{32(n-1)^3}X(r)^{\frac{9n-6}{4(n-1)}}+\frac{5sn^2(n-2)(3n-2)}{16(n-1)^4}X(r)^{\frac{11n-10}{4(n-1)}}\\
&+\frac{9(3n-2)(5n-2)(n^2-4)}{256(n-1)^4}X(r)^{\frac{13n-10}{4(n-1)}}\Big).
\end{aligned}
$$

Combining the above we eventually arrive at

$$\begin{aligned}\tilde{P}(r) &= g(r)^{\frac{\beta+n-4}{\beta}}\Big(\frac{n(n^2-4n+8)}{4(n-1)}X(r)^{\frac{2-n}{n-1}} - \frac{n^3(2s^2+2s+1)}{8(n-1)^3} \\ &\qquad + \frac{5n(n-2)(3n-2)}{16(n-1)^3}X(r)^{\frac{n}{n-1}}\Big)\end{aligned}$$

and

$$\begin{aligned}\tilde{Q}(r) = g(r)^{\frac{-\beta+n-4}{\beta}}\Big(&-\frac{s^2n(n^2-4n+8)}{4(n-1)}X(r)^{\frac{2-n}{n-1}} + \frac{s(n-2)(n^2-4n+8)}{2(n-1)}X(r)^{\frac{4-n}{2(n-1)}} \\ &+ \frac{s^4n^3}{8(n-1)^3} + \frac{3(n^2-4)(n^2-4n+8)}{16n(n-1)}X(r)^{\frac{2}{n-1}} \\ &- \frac{5s^2n(n-2)(3n-2)}{16(n-1)^3}X(r)^{\frac{n}{n-1}} - \frac{5sn(n-2)(3n-2)}{8(n-1)^3}X(r)^{\frac{3n}{2(n-1)}} \\ &- \frac{9(3n-2)(5n-2)(n^2-4)}{128n(n-1)^3}X(r)^{\frac{2n}{n-1}}\Big).\end{aligned}$$

Part (i) now follows by recalling (23) and noting that

$$P(t) = t^{\frac{\beta+n-4}{\beta}}G(t)\,, \qquad Q(t) = t^{\frac{-\beta+n-4}{\beta}}H(t).$$

To prove part (ii) we first note that

$$\begin{aligned}\int_0^1\int_{\mathbb{S}^{n-1}} r^{-1}(\Delta_\omega v)^2 dS\,dr &= \int_0^1\int_{\mathbb{S}^{n-1}} r^{-1}\alpha(r)^2(\Delta_\omega w)^2\frac{1}{g'(r)}dS\,dt \\ &= \frac{2(n-1)}{n}\int_0^1\int_{\mathbb{S}^{n-1}} t^{\frac{n-\beta-4}{\beta}}X(t)^{\frac{8-4n}{n}}(\Delta_\omega w)^2 dS\,dt\end{aligned}$$

and similarly

$$\int_0^1\int_{\mathbb{S}^{n-1}} r^{-1}|\nabla_\omega v|^2 dS\,dr = \frac{2(n-1)}{n}\int_0^1\int_{\mathbb{S}^{n-1}} t^{\frac{n-\beta-4}{\beta}}X(t)^{\frac{8-4n}{n}}|\nabla_\omega w|^2 dS\,dt\,.$$

For the remaining term in $J_{\rm NR}[v]$ we compute

$$\begin{aligned}&\int_0^1\int_{\mathbb{S}^{n-1}} r|\nabla_\omega v_r|^2 dS\,dr \\ =\ &\int_0^1\int_{\mathbb{S}^{n-1}} r\alpha^2 g'|\nabla_\omega w_t|^2\,dS\,dt - \int_0^1\int_{\mathbb{S}^{n-1}}|\nabla_\omega w|^2\frac{1}{g'}(\alpha\alpha'' r+\alpha\alpha')\,dS\,dt\end{aligned}$$

On the one hand we have

$$\int_0^1\int_{\mathbb{S}^{n-1}} \alpha^2 g' r|\nabla_\omega w_t|^2\,dS\,dt = \frac{n}{2(n-1)}\int_0^1\int_{\mathbb{S}^{n-1}} t^{\frac{n+\beta-4}{\beta}}X(t)^{\frac{4-2n}{n}}|\nabla_\omega w_t|^2 dS\,dt$$

and on the other hand, recalling (27),

$$\begin{aligned}
&\int_0^1\int_{\mathbb{S}^{n-1}} |\nabla_\omega w|^2 \frac{1}{g'}(\alpha\alpha'' r+\alpha\alpha')\,dS\,dt\\
&= \frac{n(n-4)^2}{8(n-1)\beta^2}\int_0^1\int_{\mathbb{S}^{n-1}} t^{\frac{n-\beta-4}{\beta}} X(t)^{\frac{4-2n}{n}}|\nabla_\omega w|^2\,dS\,dt\\
&\quad -\frac{(n-2)(n-4)}{2(n-1)\beta}\int_0^1\int_{\mathbb{S}^{n-1}} t^{\frac{n-\beta-4}{\beta}} X(t)^{\frac{4-n}{n}}|\nabla_\omega w|^2\,dS\,dt\\
&\quad -\frac{3(n^2-4)}{8n(n-1)}\int_0^1\int_{\mathbb{S}^{n-1}} t^{\frac{n-\beta-4}{\beta}} X(t)^{\frac{4}{n}}|\nabla_\omega w|^2\,dS\,dt.
\end{aligned}$$

Combining the above we obtain (ii). The proof of (iii) is much simpler and is omitted. □

To proceed we define

$$G^{\#}(t)=G(t)-\Big(\frac{n}{2(n-1)}\Big)^3 A\,,\qquad t\in(0,1),$$

where we recall that A has been defined in Lemma 5.

Lemma 7. *Let $v\in C_c^\infty(B_1\setminus\{0\})$ and let w be defined by (22). There holds*

$$\begin{aligned}
J_{\rm R}[v]=&\Big(\frac{n}{2(n-1)}\Big)^3\mathcal{A}_{\rm R}[w]\\
&+\int_0^1\int_{\mathbb{S}^{n-1}} t^{\frac{\beta+n-4}{\beta}} w_t^2 G^{\#}(t)dS\,dt+\int_0^1\int_{\mathbb{S}^{n-1}} t^{\frac{-\beta+n-4}{\beta}} w^2 H(t)dS\,dt.
\end{aligned}$$

Proof. This is a direct consequence of Lemma 6 (i). □

Lemma 8. *Let $n\geq 5$. If*

$$\beta\geq\beta_n:=n\Big(\frac{n^2-4n+8}{4n^4-24n^3+83n^2-120n+52}\Big)^{1/2} \tag{28}$$

then the function $G^{\#}(t)$ is non-negative in $(0,1)$.

Proof. We first note that

$$\begin{aligned}
G^{\#}(t) &= \frac{n(n^2-4n+8)}{4(n-1)}X(t)^{\frac{4-2n}{n}}-\frac{n^3(n^2-4n+8)}{16(n-1)^3\beta^2}+\frac{5n(n-2)(3n-2)}{16(n-1)^3}X(t)^2\\
&=: p_1X(t)^{\frac{4-2n}{n}}+p_2+p_3X(t)^2 \qquad (29)
\end{aligned}$$

Now, it easily follows from (29) that $G^{\#}(t)$ is monotone decreasing in $(0,1]$. Hence its minimum equal to

$$p_1+p_2+p_3=\frac{n(4n^4-24n^3+83n^2-120n+52)}{16(n-1)^3}-\frac{n^3(n^2-4n+8)}{16(n-1)^3\beta^2},$$

which is non-negative if $\beta\geq\beta_n$. □

Lemma 9. *Let $n\geq 5$ and $\beta\geq\beta_n$. For any $w\in C_c^\infty(0,1)$ there holds*

$$\int_0^1 t^{\frac{\beta+n-4}{\beta}}G^{\#}(t)w_t^2dt+\int_0^1 t^{\frac{-\beta+n-4}{\beta}}H^{\#}(t)w^2dt\geq 0$$

where

$$
\begin{aligned}
H^{\#}(t) = & -\frac{n(n-4)^2(n^2-4n+8)}{16(n-1)\beta^2}X^{\frac{4-2n}{n}} + \frac{(n-2)(n-4)(n^2-4n+8)}{4(n-1)\beta}X^{\frac{4-n}{n}} \\
& + \frac{n^3(n-4)^2(n^2-4n+8)}{64(n-1)^3\beta^4} + \frac{3(n^2-4)(n^2-4n+8)}{16n(n-1)}X^{\frac{4}{n}} \\
& - \frac{n(n-2)(15n^3-104n^2+256n-152)}{32(n-1)^3\beta^2}X^2 \\
& - \frac{5n(n-2)(n-4)(3n-2)}{16(n-1)^3\beta}X^3 + \frac{45(n-2)^2(3n-2)^2}{n(n-1)^3}X^4.
\end{aligned}
$$

Proof. Let r_1, r_2 be real numbers to be fixed later. We have

$$
\begin{aligned}
0 & \leq \int_0^1 t^{\frac{\beta+n-4}{\beta}} G^{\#}(t)\Big(w_t + \frac{r_1 + r_2 X(t)}{t} w\Big)^2 dt \\
& = \int_0^1 t^{\frac{\beta+n-4}{\beta}} G^{\#}(t) w_t^2 dt + \int_0^1 \Big\{ t^{\frac{-\beta+n-4}{\beta}} G^{\#}(t)(r_1^2 + 2r_1 r_2 X + r_2^2 X^2) \\
& \qquad - \Big(t^{\frac{n-4}{\beta}} G^{\#}(t)\big(r_1 + r_2 X(t)\big)\Big)_t \Big\} w^2 dt
\end{aligned}
$$

Substituting from (29) and carrying out the computations we arrive at

$$
\begin{aligned}
0 \leq & \int_0^1 t^{\frac{\beta+n-4}{\beta}} G^{\#}(t) w_t^2 dt + \\
& \int_0^1 t^{\frac{-\beta+n-4}{\beta}} \Big\{ p_1 r_1 (r_1 - \frac{n-4}{\beta}) X^{\frac{4-2n}{n}} + p_1 (2r_1 r_2 - r_2 \frac{n-4}{\beta} + \frac{2n-4}{n} r_1) X^{\frac{4-n}{n}} \\
& \quad + p_2 r_1 (r_1 - \frac{n-4}{\beta}) + p_1 r_2 (r_2 + \frac{n-4}{n}) X^{\frac{4}{n}} + p_2 r_2 (2r_1 - \frac{n-4}{\beta}) X \\
& \quad + \big(p_2 r_2^2 - p_2 r_2 + p_3 r_1^2 - p_3 r_1 \frac{n-4}{\beta}\big) X^2 + \big(2p_3 r_1 r_2 - 2p_3 r_1 - p_3 r_2 \frac{n-4}{\beta}\big) X^3 \\
& \quad + (p_3 r_2^2 - 3p_3 r_2) X^4 \Big\} w^2 dt \, .
\end{aligned}
$$

We now choose

$$
r_1 = \frac{n-4}{2\beta}\,, \qquad r_2 = -\frac{3(n-2)}{2n}.
$$

The choice for r_1 minimizes the coefficient of the leading term in the last integral; the parameter r_2 is less important and the choice is made for convenience.

Substituting we obtain

$$0 \leq \int_0^1 t^{\frac{\beta+n-4}{\beta}} G^{\#}(t) w_t^2 dt +$$

$$\int_0^1 t^{\frac{-\beta+n-4}{\beta}} \Big\{ -\frac{n(n-4)^2(n^2-4n+8)}{16(n-1)\beta^2} X^{\frac{4-2n}{n}} + \frac{(n-2)(n-4)(n^2-4n+8)}{4(n-1)\beta} X^{\frac{4-n}{n}}$$

$$+ \frac{n^3(n-4)^2(n^2-4n+8)}{64(n-1)^3\beta^4} + \frac{3(n^2-4)(n^2-4n+8)}{16n(n-1)} X^{\frac{4}{n}}$$

$$- \frac{n(n-2)(15n^3-104n^2+256n-152)}{32(n-1)^3\beta^2} X^2 - \frac{5n(n-2)(n-4)(3n-2)}{16(n-1)^3\beta} X^3$$

$$+ \frac{45(n-2)^2(3n-2)^2}{n(n-1)^3} X^4 \Big\} w^2 dt\,.$$

which is the stated inequality. □

We next define the positive constants

$$\gamma_1 = \frac{n^6(n-4)^2}{256(n-1)^4}, \qquad \gamma_2 = \frac{3n^2(n-2)(5n-6)(n^2-4n+8)}{128(n-1)^4},$$

$$\gamma_3 = \frac{9(n-2)(3n-2)(5n-6)(7n-6)}{256(n-1)^4}. \tag{30}$$

Lemma 10. *Let $n \geq 5$ and $\beta \geq \beta_n$. Let $v \in C_c^\infty(B_1 \setminus \{0\})$ and let w be defined by* (22)*. We then have*

$$J_{\rm R}[v] + \int_0^\infty \int_{{\rm S}^{n-1}} v^2 r^{-1} \Big(\frac{\gamma_1}{\beta^4} X(r)^{\frac{2(n-2)}{n-1}} - \frac{\gamma_2}{\beta^2} X(r)^{\frac{3n-4}{n-1}} + \gamma_3 X(r)^4 \Big) dS\, dt$$

$$\geq \Big(\frac{n}{2(n-1)}\Big)^3 \mathcal{A}_{\rm R}[w].$$

Proof. From Lemmas 7 and 9 we have

$$J_{\rm R}[v] \geq \Big(\frac{n}{2(n-1)}\Big)^3 \mathcal{A}_{\rm R}[w] + \int_0^1 \int_{{\rm S}^{n-1}} t^{\frac{n-\beta-4}{\beta}} w^2 \big(H(t) - H^{\#}(t)\big) dS\, dt.$$

But we easily see that

$$\frac{n}{2(n-1)}(H(t) - H^{\#}(t)) = -\frac{\gamma_1}{\beta^4} + \frac{\gamma_2}{\beta^2} X(t)^2 - \gamma_3 X(t)^4,$$

hence

$$J_{\rm R}[v] + \frac{2(n-1)}{n} \int_0^1 \int_{{\rm S}^{n-1}} t^{\frac{n-\beta-4}{\beta}} w^2 \big(\frac{\gamma_1}{\beta^4} - \frac{\gamma_2}{\beta^2} X(t)^2 + \gamma_3 X(t)^4 \big) dS\, dt$$

$$\geq \Big(\frac{n}{2(n-1)}\Big)^3 \mathcal{A}_{\rm R}[w].$$

We now express the double integral above in terms of the function v using once again (22). We note that for any $\sigma \geq 0$ we have

$$\int_0^1 t^{\frac{n-\beta-4}{\beta}} w^2 X(t)^\sigma dt = \frac{n}{2(n-1)} \int_0^1 r^{-1} v^2 X(r)^{\frac{\sigma n + 4(n-2)}{2(n-1)}} dr\,.$$

Applying this for $\sigma = 0, 2, 4$ we obtain the required inequality. □

Proof of Theorem 2. Let $u\in C_c^\infty(\Omega)$. Without loss of generality we may assume that $\Omega=B_1$ and that $u\in C_c^\infty(B_1\setminus\{0\})$. Let $v=|x|^{\frac{n-4}{2}}u$. By the discussion following Lemma 5, the required inequality is written

$$\frac{J_{\rm R}[v]+\frac{n^2(n-4)^2}{16}\int_0^1\int_{{\rm S}^{n-1}}r^{-1}v^2X(r)^{\frac{2(n-2)}{n-1}}dS\,dr+J_{\rm NR}[v]}{\Big(\int_0^1\int_{{\rm S}^{n-1}}r^{-1}X(r)^{\frac{2n-4}{n-4}}|v|^{\frac{2n}{n-4}}dS\,dr\Big)^{\frac{n-4}{n}}}\geq S_{2,n}.$$

We make the choice

$$\beta=\frac{n}{2(n-1)}.$$

We shall prove the following two inequalities where v and w are related by the change of variables (22):

$$J_{\rm R}[v]+\frac{n^2(n-4)^2}{16}\int_0^1\int_{{\rm S}^{n-1}}r^{-1}v^2X(r)^{\frac{2(n-2)}{n-1}}dS\,dr\geq\Big(\frac{n}{2(n-1)}\Big)^3\mathcal{A}_{\rm R}[w]\quad(31)$$

$$J_{\rm NR}[v]\geq\Big(\frac{n}{2(n-1)}\Big)^3\mathcal{A}_{\rm NR}[w].\quad(32)$$

We claim that if these are proved then the result will follow. Indeed, by Lemma 6 (iii) the Sobolev terms are related by

$$\int_0^1\int_{{\rm S}^{n-1}}r^{-1}X(r)^{\frac{2n-4}{n-4}}|v|^{\frac{2n}{n-4}}dS\,dr=\frac{2(n-1)}{n}\int_0^1\int_{{\rm S}^{n-1}}|w|^{\frac{2n}{n-4}}t^{\frac{n-\beta}{\beta}}dS\,dt\,.$$

Hence, applying Lemma 5 we shall obtain

$$\begin{aligned}&\frac{J_{\rm R}[v]+\frac{n^2(n-4)^2}{16}\int_0^1\int_{{\rm S}^{n-1}}r^{-1}v^2X(r)^{\frac{2(n-2)}{n-1}}dS\,dr+J_{\rm NR}[v]}{\Big(\int_0^1\int_{{\rm S}^{n-1}}r^{-1}X(r)^{\frac{2n-4}{n-4}}|v|^{\frac{2n}{n-4}}dS\,dr\Big)^{\frac{n-4}{n}}}\\
&\geq\Big(\frac{n}{2(n-1)}\Big)^{\frac{4(n-1)}{n}}\frac{\mathcal{A}_{\rm R}[w]+\mathcal{A}_{\rm NR}[w]}{\Big(\int_0^1\int_{{\rm S}^{n-1}}|w|^{\frac{2n}{n-4}}t^{\frac{n-\beta}{\beta}}dS\,dt\Big)^{\frac{n-4}{n}}}\\
&\geq\Big(\frac{n}{2(n-1)\beta}\Big)^{\frac{4(n-1)}{n}}S_{2,n}\\
&=S_{2,n},\end{aligned}$$

and the proof will be complete.

Proof of (31). For the specific choice of β we have

$$\begin{aligned}&\frac{\gamma_1}{\beta^4}X(r)^{\frac{2(n-2)}{n-1}}-\frac{\gamma_2}{\beta^2}X(r)^{\frac{3n-4}{n-1}}+\gamma_3X(r)^4\\
=\ &\frac{\gamma_1}{\beta^4}X(r)^{\frac{2(n-2)}{n-1}}\Big(1-\frac{\gamma_2}{\gamma_1}\beta^2X(r)^{\frac{n}{n-1}}+\frac{\gamma_3}{\gamma_1}\beta^4X(r)^{\frac{2n}{n-1}}\Big)\\
=\ &\frac{n^2(n-4)^2}{16}X(r)^{\frac{2(n-2)}{n-1}}\Big(1-\frac{3(n-2)(5n-6)(n^2-4n+8)}{2n^2(n-1)^2(n-4)^2}X(r)^{\frac{n}{n-1}}\\
&+\frac{9(n-2)(3n-2)(5n-6)(7n-6)}{16n^2(n-1)^4(n-4)^2}X(r)^{\frac{2n}{n-1}}\Big)\end{aligned}$$

The function

$$y \mapsto 1 - \frac{3(n-2)(5n-6)(n^2-4n+8)}{2n^2(n-1)^2(n-4)^2}y + \frac{9(n-2)(3n-2)(5n-6)(7n-6)}{16n^2(n-1)^4(n-4)^2}y^2$$

is convex and its values at the endpoints $y=0$ and $y=1$ do not exceed one. Noting that $n/(2n-2) > \beta_n$ the result follows by Lemma 10.

Proof of (32). We recall that the functional $\mathcal{A}_{\rm NR}[w]$ has been defined in Lemma 5 and the functional $J_{\rm NR}[v]$ is expressed in terms of the function w in Lemma 6.

We observe that the coefficients of the terms involving $(\Delta_\omega w)^2$ in the two sides of (32) are equal. The same is true for the coefficients of the terms involving $|\nabla_\omega w_t|^2$. Hence the result will follow if we establish that

$$K(t) \geq \Big(\frac{n}{2(n-1)}\Big)^3 \cdot \frac{2(n-4)}{\beta^4} = \frac{4(n-1)(n-4)}{n}.$$

Indeed, the first two terms of $K(t)$ are enough for this, that is there holds

$$(n-1)(n-4)X(t)^{\frac{8-4n}{n}} - \frac{(n-1)(n-4)^2}{n}X(t)^{\frac{4-2n}{n}} - \frac{4(n-1)(n-4)}{n} \geq 0$$

for all $t \in (0,1)$. This completes the proof of the Rellich-Sobolev inequality of Theorem 2.

The sharpness of the constant $S_{2,n}$ in the Rellich-Sobolev inequality follows easily by concentrating near a point $x_0 \in \partial\Omega$ with $|x_0| = D$. □

Acknowledgement. We wish to thank the anonymous referee for a very careful reading of the initial manuscript and for bringing to our attention the relevance of article [8].

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