On a class of Rellich inequalities

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Dedicated to Professor E.B. Davies on the occasion of his 60th birthday

Abstract

We prove Rellich and improved Rellich inequalities that involve the distance function from a hypersurface of codimension k, under a certain geometric assumption. In case the distance is taken from the boundary, that assumption is the convexity of the domain. We also discuss the best constant of these inequalities.
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1 Introduction

The classical Rellich inequality states that for $p > 1$

$$\int_0^\infty |u''|^p dt \geq \frac{(p-1)^p(2p-1)^p}{p^{2p}} \int_0^\infty \frac{|u|^p}{t^{2p}} dt \,, \tag{1.1}$$

for all $u \in C_c^\infty(0,\infty)$. A multi-dimensional version of (1.1) for $p = 2$ is also classical and states that for any $\Omega \subset \mathbf{R}^N$, $N \geq 5$, there holds

$$\int_\Omega (\Delta u)^2 dx \geq \frac{N^2(N-4)^2}{16} \int_\Omega \frac{u^2}{|x|^4} dx \,, \tag{1.2}$$

for all $u \in C_c^\infty(\Omega)$.

Davies and Hinz [DH] generalized (1.2) and showed that for any $p \in (1, N/2)$ there holds

$$\int_\Omega |\Delta u|^p dx \geq \left(\frac{(p-1)N|N-2p|}{p^2} \right)^p \int_\Omega \frac{|u|^p}{|x|^{2p}} dx \,, \qquad u \in C_c^\infty(\Omega \setminus \{0\}) \,. \tag{1.3}$$

Inequality (1.1) has also been generalized to higher dimensions in another direction, where the singularity involves the distance $d(x) = \mathrm{dist}(x, \partial\Omega)$. Owen [O] proved among other results that if Ω is bounded and convex then

$$\int_\Omega (\Delta u)^2 dx \geq \frac{9}{16} \int_\Omega \frac{u^2}{d(x)^4} dx \,, \qquad u \in C_c^\infty(\Omega) \,. \tag{1.4}$$

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Recently, an improved version of (1.2) has been established in [TZ]. Among several other results they showed that for a bounded domain Ω in $\mathbf{R}^N$, $N \geq 5$, there holds

$$\int_\Omega (\Delta u)^2 dx \geq \frac{N^2(N-4)^2}{16} \int_\Omega \frac{u^2}{|x|^4} dx + \Big(1 + \frac{N(N-4)}{8}\Big) \sum_{i=1}^{\infty} \int_\Omega \frac{u^2}{|x|^4} X_1^2 X_2^2 \dots X_i^2 dx \,, \tag{1.5}$$

as well as

$$\int_\Omega (\Delta u)^2 dx \geq \frac{N^2}{4} \int_\Omega \frac{|\nabla u|^2}{|x|^2} dx + \frac{1}{4} \sum_{i=1}^{\infty} \int_\Omega \frac{|\nabla u|^2}{|x|^2} X_1^2 X_2^2 \dots X_i^2 dx \,,$$

for all $u \in C_c^\infty(\Omega \backslash \{0\})$. Here X_k are iterated logarithmic functions; see (1.6) for precise the definition.

Rellich inequalities have various applications in the study of fourth-order elliptic and parabolic PDE's; see e.g. [DH, O, B]. Improved Rellich inequalities are useful if critical potentials are additionally present. As a simplest example, one obtains information on the existence of solution and asymptotic behavior for the equation $u_t = -\Delta^2 + V$ for critical potentials V. Corresponding problems for improved Hardy's inequalities have recently attracted considerable attention: see [BV, BM, BFT1] and references therein.

Our aim in this paper is to obtain sharp improved versions of inequalities (1.3) and (1.4), where additional non-negative terms are present in the respective right-hand sides. At the same time we obtain some new improved Rellich inequalities which are new even at the level of plain Rellich inequalities; these involve the distance to a surface K of intermediate codimension.

Statement of results

Before stating our theorems let us first introduce some notation. We denote by Ω a domain in $\mathbf{R}^N$, $N \geq 2$. For the sake of simplicity all functions considered below are assumed to be real-valued; in relation to this we note however that minor modifications of the proofs or a suitable application of [D, Lemma 7.5] can yield the validity of Theorems 1-3 below for complex-valued functions u. We let K be a closed, piecewise smooth surface of codimension k, $k = 1, \dots, N$. We do not assume that K is connected but only that it has finitely many connected components. In the case $k = N$ we assume that K is a finite union of points while in the case $k = 1$ we assume that $K = \partial\Omega$. We then set

$$d(x) = \mathrm{dist}(x, K) \,,$$

and assume that $d(x)$ is bounded in Ω.

We define recursively

$$\begin{aligned} X_1(t) &= (1 - \log t)^{-1}, \quad t \in (0,1], \\ X_i(t) &= X_1(X_{i-1}(t)), \qquad i = 2, 3, \dots, t \in (0,1]. \end{aligned} \tag{1.6}$$

These are iterated logarithmic functions that vanish at an increasingly slow rate at $t = 0$ and satisfy $X_i(1) = 1$. Given an integer $m \geq 1$ we define

$$\eta_m(t) = \sum_{i=1}^{m} X_1(t) \dots X_i(t), \qquad \zeta_m(t) = \sum_{i=1}^{m} X_1^2(t) \dots X_i^2(t). \tag{1.7}$$

We note that $\lim_{t\to 0}\eta_m(t) = \lim_{t\to 0}\zeta_m(t) = 0$. Now, it has been shown [BFT2] that both series in (1.7) converge for any $t \in (0,1)$. This allows us to also introduce the functions η_∞ and ζ_∞ as the infinite series.

We fix a parameter $s \in \mathbf{R}$ and we assume that the following inequality holds in the distributional sense:

$$p \neq k+s, \qquad (k+s-p)(d\Delta d - k + 1) \geq 0 \quad \text{in } \Omega \setminus K.$$

For a detailed discussion of this condition we refer to [BFT1]. Here we simply note that it is satisfied in the following two important cases: (i) it is satisfied as an equality if $k = N$ and K consists of single point and (ii) it is also satisfied if $K = \partial\Omega$ (so $k = 1$), $s + 1 - p < 0$ and Ω is convex.

Our first theorem involves the functions $\eta_m = \eta_m(d(x)/D)$ and $\zeta_m = \zeta_m(d(x)/D)$, $x \in \Omega$, for a large enough parameter $D > 0$. In any case D will be large enough so that the quantity $1 + \alpha\eta_m + \beta\eta_m^2 + \gamma\zeta_m$ is positive in Ω. We also set

$$H = \frac{k+s-p}{p}. \tag{1.8}$$

Theorem 1 (weighted improved Hardy inequality) *Let $p > 1$ and $m \in \mathbf{N}\cup\{\infty\}$. Let Ω be a domain in $\mathbf{R}^N$ and K a piecewise smooth surface of codimension k, $k = 1,\ldots,N$. Suppose that $p \neq k+s$, that $\sup_{x\in\Omega} d(x) < \infty$ and that*

$$(k+s-p)(d\Delta d - k + 1) \geq 0 \qquad in\ \Omega \setminus K. \tag{1.9}$$

Also, let $\alpha, \beta, \gamma \in \mathbf{R}$ be fixed. Then there exists a positive constant $D_0 \geq \sup_{x\in\Omega} d(x)$ such that for any $D \geq D_0$ and all $u \in C_c^\infty(\Omega \setminus K)$ there holds

$$\begin{aligned}\int_\Omega d^s(1+\alpha\eta_m+\beta\eta_m^2+\gamma\zeta_m)|\nabla u|^p dx &\geq |H|^p \int_\Omega d^{s-p}|u|^p dx + \\ &+|H|^p\alpha\int_\Omega d^{s-p}\eta_m|u|^p dx + \Big(|H|^p\beta + \frac{|H|^{p-2}H\alpha}{2}\Big)\int_\Omega d^{s-p}\eta_m^2|u|^p dx \\ &+\Big(\frac{p-1}{2p}|H|^{p-2} + \frac{|H|^{p-2}H\alpha}{2} + |H|^p\gamma\Big)\int_\Omega d^{s-p}\zeta_m|u|^p dx\,,\end{aligned}$$

where $\eta_m = \eta_m(d(x)/D) = \sum_{i=1}^m X_1(d(x)/D)\ldots X_i(d(x)/D)$ and $\zeta_m = \zeta_m(d(x)/D) = \sum_{i=1}^m X_1^2(d(x)/D)\ldots X_i^2(d(x)/D)$.

We note that the special case $s = \alpha = \beta = \gamma = 0$ has been proved in [BFT2].

To state our next theorem we define the constant

$$Q = \frac{(p-1)k(k-2p)}{p^2}. \tag{1.10}$$

Theorem 2 (improved Rellich inequality I) *Let $p > 1$. Let Ω be a domain in $\mathbf{R}^N$ and K a piecewise smooth surface of codimension k, $k = 1,\ldots,N$. Suppose that $\sup_{x\in\Omega} d(x) < \infty$. Suppose also that $k > 2p$ and that*

$$d\Delta d - k + 1 \geq 0 \quad , \quad in\ \Omega \setminus K$$

in the distributional sense. Then there exists a positive constant $D_0 \geq \sup_{x\in\Omega} d(x)$ *such that for any* $D \geq D_0$ *and all* $u \in C_c^\infty(\Omega\setminus K)$ *there holds*

$$\int_\Omega |\Delta u|^p dx \geq Q^p \int_\Omega \frac{|u|^p}{d^{2p}}dx + \tag{1.11}$$
$$+\frac{p-1}{2p^3}|Q|^{p-2}\Big\{k^2(p-1)^2+(k-2p)^2\Big\}\sum_{i=1}^\infty \int_\Omega \frac{|u|^p}{d^{2p}}X_1^2X_2^2\dots X_i^2dx,$$

where $X_j = X_j(d(x)/D)$.

It is remarkable that the geometric assumption of this Theorem 2 only involves Δd, as in the case of Theorem 1, and not higher-order derivatives of d as one might expect. The above theorem does not cover the important case $k = 1$ which corresponds to $d(x) = \mathrm{dist}(x, \partial\Omega)$. This is done in the following theorem for the case $p = 2$.

Theorem 3 (improved Rellich inequality II) *Let* Ω *be convex and such that* $d(x):=$ $\mathrm{dist}(x,\partial\Omega)$ *is bounded in* Ω. *Then there exists a positive constant* $D_0 \geq \sup_{x\in\Omega} d(x)$ *such that for any* $D \geq D_0$ *and all* $u \in C_c^\infty(\Omega)$ *there holds*

$$\text{(i)} \qquad \int_\Omega (\Delta u)^2 dx \geq \frac{1}{4}\int_\Omega \frac{|\nabla u|^2}{d^2}dx + \frac{1}{4}\sum_{i=1}^\infty \int_\Omega \frac{|\nabla u|^2}{d^2}X_1^2X_2^2\dots X_i^2dx\ , \tag{1.12}$$

$$\text{(ii)} \qquad \int_\Omega (\Delta u)^2 dx \geq \frac{9}{16}\int_\Omega \frac{u^2}{d^4}dx + \frac{5}{8}\sum_{i=1}^\infty \int_\Omega \frac{u^2}{d^4}X_1^2X_2^2\dots X_i^2dx, \tag{1.13}$$

where $X_j = X_j(d(x)/D)$.

In our last theorem we prove the optimality of the constants appearing in Theorems 2 and 3 above. In a similar manner one can prove the optimality of the constants in Theorem 1; we omit the proof since it follows very closely the proof of [BFT2, Proposition 3.1]. Anyway, we note that in some particular cases the optimality of Theorem 1 follows indirectly from the optimality of Theorems 2 and 3, which we do prove. In relation to Theorem 4 see also the remark at the end of the paper.

We define

$$J_0[u] = \int_\Omega |\Delta u|^p dx - |Q|^p \int_\Omega \frac{|u|^p}{d^{2p}}dx$$

and for $m \in \mathbf{N}$,

$$\begin{aligned} J_m[u] \;\; &= \;\; \int_\Omega |\Delta u|^p dx - |Q|^p \int_\Omega \frac{|u|^p}{d^{2p}}dx - \\ &\quad -\frac{p-1}{2p^3}|Q|^{p-2}\Big\{k^2(p-1)^2+(k-2p)^2\Big\}\sum_{i=1}^m \int_\Omega \frac{|u|^p}{d^{2p}}X_1^2X_2^2\dots X_i^2dx. \end{aligned}$$

Our next theorem reads:

Theorem 4 *Let* $p > 1$. *Let* Ω *be a domain in* $\mathbf{R}^N$. (i) *If* $2 \leq k \leq N-1$ *then we take* K *to be a piecewise smooth surface of codimension* k *and assume* $K \cap \Omega \neq \emptyset$; (ii) *if*

$k = N$ *then we take* $K = \{0\} \subset \Omega$; (iii) *if* $k = 1$ *then we assume* $K = \partial\Omega$. *For any* $D \geq \sup_\Omega d(x)$ *we have*

$$\text{(i)} \qquad \inf_{C_c^\infty(\Omega\setminus K)} \frac{\int_\Omega |\Delta u|^p dx}{\int_\Omega \frac{|u|^p}{d^{2p}} dx} \leq |Q|^p;$$

$$\text{(ii)} \qquad \inf_{C_c^\infty(\Omega\setminus K)} \frac{J_{m-1}[u]}{\int_\Omega \frac{|u|^p}{d^{2p}} X_1^2 X_2^2 \dots X_m^2} \leq \frac{p-1}{2p^3}|Q|^{p-2}\Big\{k^2(p-1)^2 + (k-2p)^2\Big\}, \ m \geq 1.$$

where $X_j = X_j(d(x)/D)$.

It follows in particular that all constants in Theorem 2 and Theorem 3 (ii) are sharp. The sharpness of Theorem 3 (i) follows implicitly from the sharpness of 3 (ii).

2 Series expansion for weighted Hardy inequality

In this section we give the proof of Theorem 1. We note that in the special case $s = \alpha = \beta = \gamma = 0$ the theorem has already been proved in [BFT2]. In the sequel we shall repeatedly use the differentiation rule

$$\frac{d}{dt} X_i^\beta(t) = \frac{\beta}{t} X_1 X_2 \dots X_{i-1} X_i^{1+\beta}, \qquad \beta \neq 0\,, \tag{2.1}$$

which is easily proved by induction.

Proof of Theorem 1. We set for simplicity $\psi = (1 + \alpha\eta_m + \beta\eta_m^2 + \gamma\zeta_m)$. If T is a vector field in Ω, then, for any $u \in C_c^\infty(\Omega \setminus K)$ we first integrate by parts and then use Young's inequality to obtain

$$\begin{aligned}\int_\Omega \operatorname{div} T \ |u|^p dx &\leq p\int_\Omega |T||\nabla u||u|^{p-1} dx \\ &\leq \int_\Omega d^s\psi|\nabla u|^p dx + (p-1)\int_\Omega d^{-\frac{s}{p-1}}|T|^{\frac{p}{p-1}}\psi^{-\frac{1}{p-1}}|u|^p dx\,,\end{aligned}$$

and thus conclude that

$$\int_\Omega d^s\psi|\nabla u|^p dx \geq \int_\Omega (\operatorname{div}T - (p-1)d^{-\frac{s}{p-1}}|T|^{\frac{p}{p-1}}\psi^{-\frac{1}{p-1}})|u|^p dx. \tag{2.2}$$

We recall that $H = (k+s-p)/p$ and define

$$T(x) = H|H|^{p-2} d^{s+1-p}(x)\nabla d(x)\left(1 + (\alpha + \frac{p-1}{pH})\eta_m(d(x)/D) + B\eta_m^2(d(x)/D)\right).$$

where $D \geq \sup_\Omega d(x)$ and $B \in \mathbf{R}$ is a free parameter to be chosen later. In any case, once B is chosen, D will be large enough so that the quantity $1 + (\alpha + \frac{p-1}{pH})\eta_m(d/D) + B\eta_m^2(d/D)$ is positive on Ω. Note that T is singular on K, but since $u \in C_c^\infty(\Omega \setminus K)$ all previous calculations are legitimate. In view of (2.2), to prove the theorem it is enough to show that there exists $D_0 \geq \sup_\Omega d(x)$ such that for $D \geq D_0$

$$\begin{aligned}&\operatorname{div}T - (p-1)d^{-\frac{s}{p-1}}|T|^{\frac{p}{p-1}}\psi^{-\frac{1}{p-1}} - d^{s-p}\Bigg\{|H|^p + |H|^p\alpha\eta_m \\ &+\left(|H|^p\beta + \frac{|H|^{p-2}H\alpha}{2}\right)\eta_m^2 + \left(\frac{p-1}{2p}|H|^{p-2} + \frac{|H|^{p-2}H\alpha}{2} + |H|^p\gamma\right)\zeta_m\Bigg\} \geq 0\end{aligned} \tag{2.3}$$

for all $x \in \Omega$.

To compute $\mathrm{div}T$ we shall need to differentiate $\eta_m(d/D)$. For this we note that (2.1) easily implies

$$\eta_m'(t) = \frac{1}{t}\left(X_1^2 + (X_1^2X_2 + X_1^2X_2^2) + \cdots + (X_1^2X_2\ldots X_m + \cdots + X_1^2\ldots X_m^2)\right),$$

from which follows that

$$t\eta_m^{'}(t) = \frac{1}{2}\zeta_m(t) + \frac{1}{2}\eta_m^2(t). \tag{2.4}$$

We also define θ_m on $(0,1)$ by

$$\zeta_m'(t) = \frac{\theta_m(t)}{t},$$

and, for simplicity, we set $A = \alpha + (p-1)/(pH)$ so that $T = |H|^{p-2}Hd^{s+1-p}\nabla d(1 + A\eta_m + B\eta_m^2)$. We think of η_m as an independent variable, which we may assume to be small by taking D large enough. Simple computations together with assumption (1.9) and the fact that $|\nabla d| = 1$ give

$$\begin{aligned}
&\mathrm{div}T \qquad\qquad (2.5)\\
=\ & d^{s-p}\Big\{p|H|^p + p|H|^pA\eta_m + (p|H|^pB + |H|^{p-2}H\frac{A}{2})\eta_m^2 + |H|^{p-2}H\frac{A}{2}\zeta_m + \\
&\quad + |H|^{p-2}HB(\eta_m^3 + \eta_m\zeta_m)\Big\} + |H|^{p-2}Hd^{s-p}(d\Delta d + s - p + 1)(1 + A\eta_m + B\eta_m^2)\\
\geq\ & d^{s-p}\Big\{p|H|^p + p|H|^pA\eta_m + (p|H|^pB + |H|^{p-2}H\frac{A}{2})\eta_m^2 + |H|^{p-2}H\frac{A}{2}\zeta_m + \\
&\quad + |H|^{p-2}HB(\eta_m^3 + \eta_m\zeta_m)\Big\} + |H|^{p-2}Hd^{s-p}(d\Delta d - k + 1)(1 + A\eta_m + B\eta_m^2)\\
\geq\ & d^{s-p}\Big\{p|H|^p + p|H|^pA\eta_m + (p|H|^pB + |H|^{p-2}H\frac{A}{2})\eta_m^2 + |H|^{p-2}H\frac{A}{2}\zeta_m + \\
&\quad + |H|^{p-2}HB(\eta_m^3 + \eta_m\zeta_m)\Big\} \qquad\qquad (2.6)
\end{aligned}$$

Moreover, since $|\nabla d| = 1$, Taylor's expansion gives

$$\begin{aligned}
|T|^{\frac{p}{p-1}} &= |H|^p d^{\frac{(s+1-p)p}{p-1}}(1 + A\eta_m + B\eta_m^2)^{\frac{p}{p-1}}\\
&= |H|^p d^{\frac{(s+1-p)p}{p-1}}\Big\{1 + \frac{pA}{p-1}\eta_m + \Big(\frac{pB}{p-1} + \frac{pA^2}{2(p-1)^2}\Big)\eta_m^2\\
&\quad + \Big(\frac{pAB}{(p-1)^2} - \frac{p(p-2)A^3}{6(p-1)^3}\Big)\eta_m^3 + O(\eta_m^4)\Big\} \qquad (2.7)
\end{aligned}$$

and also

$$\begin{aligned}
\psi^{-\frac{1}{p-1}} &= 1 - \frac{\alpha}{p-1}\eta_m + \Big(-\frac{\beta}{p-1} + \frac{p\alpha^2}{2(p-1)^2}\Big)\eta_m^2 - \frac{\gamma}{p-1}\zeta_m\\
&\quad + \Big(\frac{p\alpha\beta}{(p-1)^2} - \frac{p(2p-1)\alpha^3}{6(p-1)^3}\Big)\eta_m^3 + \frac{p\alpha\gamma}{(p-1)^2}\eta_m\zeta_m + O(\eta_m^4). \qquad (2.8)
\end{aligned}$$

Using (2.6), (2.7) and (2.8) we see that the LHS of (2.3) is greater than or equal to d^{s-p} times a linear combination of powers of η_m, ζ_m and θ_m plus $O(\eta_m^4)$. Recalling that $A = \alpha + (p-1)/(pH)$, we easily see that the constant term and the coefficients of η_m, η_m^2 and ζ_m vanish, independently of the choice of the parameter B. The remaining two coefficients, that is the coefficients of η_m^3 and $\eta_m\zeta_m$ are, respectively,

$$\frac{(p-1)\alpha}{2pH^2} + \frac{\beta}{H} + \frac{(p-2)(p-1)}{6p^2H^3} \qquad , \qquad \frac{B+\gamma}{H}.$$

Since $\zeta_m \leq \eta_m^2 \leq m\zeta_m$, we conclude that taking B to be large and positive (if $H > 0$) or large and negative (if $H < 0$), inequality (2.3) is satisfied provided η_m is small enough, which amounts to D being large enough. This completes the proof of the theorem. //

In the proof of Theorem 2 we are going to use the last theorem in the following special case which corresponds to taking $p = 2$ and $s = -2q + 2$:

Special case. Assume that $k \neq 2q$ and that $(k-2q)(d\Delta d - k + 1) \geq 0$ on $\Omega \setminus K$. Then for D large enough there holds

$$\begin{aligned}\int_\Omega d^{-2q+2}(1 + \alpha\eta_m + \beta\eta_m^2 + \gamma\zeta_m)|\nabla u|^2 dx \geq \frac{(k-2q)^2}{4}\int_\Omega d^{-2q}u^2 dx + \\ + \frac{(k-2q)^2\alpha}{4}\int_\Omega d^{-2q}\eta_m u^2 dx + \left(\frac{(k-2q)^2\beta}{4} + \frac{(k-2q)\alpha}{4}\right)\int_\Omega d^{-2q}\eta_m^2 u^2 dx \\ + \left(\frac{1}{4} + \frac{(k-2q)\alpha}{4} + \frac{(k-2q)^2\gamma}{4}\right)\int_\Omega d^{-2q}\zeta_m u^2 dx \end{aligned} \tag{2.9}$$

for all $u \in C_c^\infty(\Omega \setminus K)$.

3 The improved Rellich inequality

In this section we are going to prove Theorems 2 and 3 as well as the corresponding optimality theorem. We begin with the following lemma where, we note, $\Delta\phi$ is to be understood in the distributional sense.

Lemma 5 *For any locally bounded function ϕ with $|\nabla u| \in L^2_{loc}(\Omega \setminus K)$ we have*

$$\int_\Omega |\Delta u|^p dx \geq p(p-1)\int_\Omega \phi|u|^{p-2}|\nabla u|^2 dx - \int_\Omega \Big(\Delta\phi + (p-1)|\phi|^{\frac{p}{p-1}}\Big)|u|^p dx\,, \tag{3.10}$$

for all $u \in C_c^\infty(\Omega \setminus K)$.

Proof. Given $u \in C_c^\infty(\Omega \setminus K)$ we have

$$\begin{aligned} -\int_\Omega \Delta\phi|u|^p dx &= p\int_\Omega \nabla\phi \cdot (|u|^{p-2}u\nabla u)dx \\ &= -p\int_\Omega \phi|u|^{p-2}u\Delta u dx - p(p-1)\int_\Omega \phi|u|^{p-2}|\nabla u|^2 dx \\ &\leq p\left(\frac{p-1}{p}\int_\Omega |\phi|^{\frac{p}{p-1}}|u|^p dx + \frac{1}{p}\int_\Omega |\Delta u|^p dx\right) - \\ &\qquad -p(p-1)\int_\Omega \phi|u|^{p-2}|\nabla u|^2 dx. \end{aligned}$$

which is (3.10).

Proof of Theorem 2. Let $m \in \mathbf{N}$ be fixed and let η_m and ζ_m be as in (1.7). We apply (3.10) with $\phi(x) = \lambda d(x)^{-2p+2}(1 + \alpha\eta_m + \beta\eta_m^2)$, $\lambda > 0$, where, as always, $\eta_m = \eta_m(d(x)/D)$ and D is yet to be determined. We thus obtain

$$\int_\Omega |\Delta u|^p dx \geq T_1 + T_2 + T_3 \tag{3.11}$$

where

$$\begin{aligned}
T_1 &= p(p-1)\int_\Omega \phi|u|^{p-2}|\nabla u|^2 dx\,,\\
T_2 &= -\int_\Omega \Delta\phi|u|^p dx\,,\\
T_3 &= -(p-1)\int_\Omega |\phi|^{\frac{p}{p-1}}|u|^p dx\,.
\end{aligned}$$

To estimate T_1 we set $v = |u|^{p/2}$ and apply (2.9) for $q = p$,

$$\begin{aligned}
T_1 &= \frac{4(p-1)\lambda}{p}\int_\Omega d^{-2p+2}(1 + \alpha\eta_m + \beta\eta_m^2)|\nabla v|^2 dx\\
&\geq \frac{4(p-1)\lambda}{p}\int_\Omega d^{-2p}\Big\{\frac{(k-2p)^2}{4} + \frac{(k-2p)^2\alpha}{4}\eta_m +\\
&\quad \Big(\frac{1}{4} + \frac{(k-2p)\alpha}{4}\Big)\zeta_m + \Big(\frac{(k-2p)\alpha}{4} + \frac{(k-2p)^2\beta}{4}\Big)\eta_m^2\Big\}|u|^p dx
\end{aligned} \tag{3.12}$$

To estimate T_2 we first note that

$$\nabla\phi = \lambda d^{-2p+1}\Big\{-2(p-1)(1 + \alpha\eta_m + \beta\eta_m^2) + \frac{\alpha}{2}(\eta_m^2 + \zeta_m^2) + \beta(\eta_m^3 + \eta_m\zeta_m)\Big\}\nabla d$$

and hence compute

$$\begin{aligned}
-\Delta\phi &= \lambda d^{-2p}(-2p + 1 + d\Delta d)\Big\{2(p-1)(1 + \alpha\eta_m + \beta\eta_m^2) - \frac{\alpha}{2}(\eta_m^2 + \zeta_m) -\\
&\qquad -\beta(\eta_m^3 + \eta_m\zeta_m)\Big\} +\\
&\quad +\lambda d^{-2p}\Big\{(p-1)\alpha(\eta_m^2 + \zeta_m) - \beta(\eta_m^3 + \eta_m\zeta_m) -\\
&\qquad -\frac{\alpha}{2}(\eta_m^3 + \eta_m\zeta_m + \theta_m) + O(\eta_m^4)\Big\}
\end{aligned}$$

Using the geometric assumption $d\Delta d - k + 1 \geq 0$ and collecting similar terms we conclude that

$$\begin{aligned}
T_2 &\geq \lambda\int_\Omega d^{-2p}\Big\{2(p-1)(k-2p) + 2(p-1)(k-2p)\alpha\eta_m +\\
&\quad +\Big(2(p-1)(k-2p)\beta + \frac{-k+4p-2}{2}\alpha\Big)\eta_m^2\\
&\quad +\frac{-k+4p-2}{2}\alpha\zeta_m - \Big((k-2p+1)\beta + \frac{\alpha}{2}\Big)(\eta_m^3 + \eta_m\zeta_m) - \frac{\alpha}{2}\theta_m + O(\eta_m^4)\Big\}dx.
\end{aligned} \tag{3.13}$$

From Taylor's theorem we have

$$\begin{aligned}(1+\alpha\eta_m+\beta\eta_m^2)^{\frac{p}{p-1}} &= 1+\frac{p\alpha}{p-1}\eta_m+\Big(\frac{p\beta}{p-1}+\frac{p\alpha^2}{2(p-1)^2}\Big)\eta_m^2+\\ &\quad +\Big(\frac{p\alpha\beta}{(p-1)^2}+\frac{p(2-p)\alpha^3}{6(p-1)^3}\Big)\eta_m^3+O(\eta_m^4)\end{aligned}$$

from which follows that

$$\begin{aligned}T_3 &= -(p-1)|\lambda|^{\frac{p}{p-1}}\int_\Omega d^{-2p}\Big\{1+\frac{p\alpha}{p-1}\eta_m+\Big(\frac{p\beta}{p-1}+\frac{p\alpha^2}{2(p-1)^2}\Big)\eta_m^2+\\ &\quad +\Big(\frac{p\alpha\beta}{(p-1)^2}+\frac{p(2-p)\alpha^3}{6(p-1)^3}\Big)\eta_m^3+O(\eta_m^4)\Big\}dx. \qquad (3.14)\end{aligned}$$

Using the above estimates on T_1, T_2 and T_3 and going back to (3.11) we obtain the inequality

$$\int_\Omega |\Delta u|^p dx \geq \int_\Omega d^{-2p}V|u|^p dx \qquad (3.15)$$

where the potential V has the form

$$V(x)=r_0+r_1\eta_m+r_2\eta_m^2+r_2'\zeta_m+r_3\eta_m^3+r_3'\eta_m\zeta_m+r_3''\theta_m.$$

We compute the coefficients r_i, r_i' by adding the corresponding coefficients from (3.12), (3.13) and (3.14). We ignore for now the coefficients of the third-order terms. For the others we find

$$\begin{aligned}r_0 &= (p-1)\Big(\frac{k(k-2p)}{p}\lambda-|\lambda|^{\frac{p}{p-1}}\Big)\\ r_1 &= \frac{(p-1)k(k-2p)}{p}\alpha\lambda-p\alpha|\lambda|^{\frac{p}{p-1}}\\ r_2 &= \frac{pk-2k+2p}{2p}\alpha\lambda+\frac{(p-1)k(k-2p)}{p}\beta\lambda-(p-1)\Big(\frac{p\beta}{p-1}+\frac{p\alpha^2}{2(p-1)^2}\Big)|\lambda|^{\frac{p}{p-1}}\\ r_2' &= \Big(\frac{p-1}{p}+\frac{pk-2k+2p}{2p}\alpha\Big)\lambda\end{aligned}$$

We now make a specific choice for α and λ. We recall that $Q=(p-1)k(k-2p)/p^2$, and choose

$$\lambda = Q^{p-1} \qquad , \qquad \alpha=\frac{(p-1)(pk-2k+2p)}{p^2Q}.$$

We then have $r_0=Q^p$, $r_1=r_2=0$, irrespective of the value of β. We also have

$$r_2'=\frac{p-1}{p}Q^{p-2}\Big(Q+\frac{(pk-2k+2p)^2}{2p^2}\Big).$$

Substituting these values in (3.15) we thus obtain

$$\begin{aligned}\int_\Omega|\Delta u|^p dx \geq\ & Q^p\int_\Omega d^{-2p}|u|^p dx+\frac{p-1}{p}Q^{p-2}\Big(Q+\frac{(pk-2k+2p)^2}{2p^2}\Big)\int_\Omega d^{-2p}\zeta_m|u|^p dx\\ &+\int_\Omega d^{-2p}(r_3\eta_m^3+r_3'\eta_m\zeta_m+r_3''\theta_m+O(\eta_m^4))|u|^p dx\end{aligned}$$

We still have not imposed any restriction on β. We now observe that r_3' and r_3'' are independent of β, while $r_3 = c_1\beta + c_2$ with $c_1 = Q^{p-1}((2k)/p - 2k + 2p - 3) < 0$. Hence, since the functions η_m^3, $\eta_m\zeta_m$ and θ_m are comparable in size to each other, the integral is made positive by choosing β to be large and negative and η_m small enough, which amounts to D being large enough. Hence we have proved that for $D \geq D_0$ there holds

$$\int_\Omega |\Delta u|^p dx \geq Q^p \int_\Omega \frac{|u|^p}{d^{2p}} dx + \\ + \Big(\frac{p-1}{p} Q^{p-1} + \frac{p-1}{2p} Q^{p-2} R^2\Big) \sum_{i=1}^m \int_\Omega \frac{|u|^p}{d^{2p}} X_1^2 X_2^2 \dots X_i^2 dx,$$

where $X_j = X_j(d(x)/D)$. This concludes the proof of the theorem. //

Remark. Let us mention here that in the proofs of Theorems 1 and 2 we did not use at any point the assumption that k is the codimension of the set K. Indeed, a careful look at the two proofs shows that K can be any closed set such that $\mathrm{dist}(x, K)$ is bounded in Ω and for which the condition $d\Delta d - k + 1 \geq 0$ or ≤ 0 is satisfied; the proof does not even require k to be an integer. Of course, the natural realizations of these conditions are that K is smooth and $k = \mathrm{codim}(K)$. However, the argument also applies in the case where K is a union of sets of different codimensions; see [BFT1].

We next prove Theorem 3.

Proof of Theorem 3. We note that the convexity of Ω implies that $\Delta d \leq 0$ on Ω in the distributional sense [EG, Theorem 6.3.2]. Now, let $u \in C_c^\infty(\Omega)$ be given. Applying Theorem 1 (with $k = 1$, $p = 2$, $s = 0$ and $\alpha = \beta = \gamma = 0$) to the partial derivatives u_{x_i} we have

$$\begin{aligned} \int_\Omega (\Delta u)^2 dx &= \sum_{i=1}^n \int_\Omega |\nabla u_{x_i}|^2 dx \\ &\geq \sum_{i=1}^n \Big\{ \frac{1}{4} \int_\Omega \frac{u_{x_i}^2}{d^2} dx + \frac{1}{4} \int_\Omega \frac{u_{x_i}^2}{d^2} \zeta_m dx \\ &= \frac{1}{4} \int_\Omega \frac{|\nabla u|^2}{d^2} (1 + \zeta_m) dx \Big\}, \end{aligned} \tag{3.16}$$

for D large enough, where $\zeta_m = \zeta_m(d(x)/D)$. Applying Theorem 1 once more (this time with $k = 1$, $p = 2$, $s = -2$, $\alpha = \beta = 0$ and $\gamma = 1$) we obtain

$$\int_\Omega \frac{|\nabla u|^2}{d^2} (1 + \zeta_m) dx \geq \frac{9}{4} \int_\Omega \frac{u^2}{d^4} dx + \frac{5}{2} \int_\Omega \frac{u^2}{d^4} \zeta_m dx\,. \tag{3.17}$$

Combining (3.16) and (3.17) we obtain

$$\int_\Omega (\Delta u)^2 dx \geq \frac{9}{16} \int_\Omega \frac{u^2}{d^4} dx + \frac{5}{8} \int_\Omega \frac{u^2}{d^4} \zeta_m dx\,,$$

which is the stated inequality. //

We next give the proof of Theorem 4. We recall that Ω is a domain in $\mathbf{R}^N$ and that K is a piecewise smooth surface of codimension k such that $K \cap \Omega \neq \emptyset$, unless $k = 1$ in which case $K = \partial\Omega$. All the calculations below are local, in a small ball of radius

δ, and indeed, it would be enough to assume that K has a smooth part. We also note that for $k = N$ (distance from a point) the subsequent calculations are substantially simplified, whereas for $k = 1$ (distance from the boundary) one should replace B_δ by $B_\delta \cap \Omega$. This last change entails some minor modifications, the arguments otherwise being the same.

Proof. We shall only give the proof of (ii) since the proof of (i) is much simpler. For the proof we shall use some of the ideas and tools developed in [BFT2]. All our analysis will be local, say, in a fixed ball of $B(x_0, \delta)$ where $x_0 \in K$ and δ is small, but fixed throughout th proof. We therefore fix a smooth, non-negative function ϕ such that $\phi(x) = 1$ on $\{|x - x_0| < \delta/2\}$ and $\phi(x) = 1$ on $\{|x - x_0| > \delta\}$. For given $\epsilon_0, \epsilon_1 \dots, \epsilon_m > 0$ we then define the function

$$
\begin{aligned}
u &= \phi d^{\frac{-k+2p+\epsilon_0}{p}} X_1^{\frac{-1+\epsilon_1}{p}} X_2^{\frac{-1+\epsilon_2}{p}} \dots X_m^{\frac{-1+\epsilon_m}{p}} \\
&=: \phi v.
\end{aligned}
$$

A standard argument using cut-off functions shows that u belongs in $W_0^{2,p}(\Omega \setminus K)$ and therefore is a legitimate test-function for the infimum above. We intend to see how $J_{m-1}[u]$ behaves as the ϵ_i's tend to zero. We shall not be interested in terms that remain bounded for small values of the ϵ_i's. To distinguish such terms we shall need the following fact, cf. [BFT2, (3.8)]: we have

$$
\begin{aligned}
&\int_\Omega \phi^p d^{-k+\beta_0} X_1^{1+\beta_1}(d/D) \dots X_m^{1+\beta_m}(d/D) dx < \infty \Longleftrightarrow \\
&\Longleftrightarrow \begin{cases} & \beta_0 > 0 \\ \text{or} & \beta_0 = 0 \text{ and } \beta_1 > 0 \\ \text{or} & \beta_0 = \beta_1 = 0 \text{ and } \beta_2 > 0 \\ & \dots \\ \text{or} & \beta_0 = \beta_1 = \dots = \beta_{m-1} = 0 \text{ and } \beta_m > 0. \end{cases}
\end{aligned}
\tag{3.18}
$$

Now, we have $\Delta u = \phi \Delta v + 2\nabla\phi \cdot \nabla v + v\Delta\phi$ and hence, using the inequality

$$
|a + b|^p \le |a|^p + c(|a|^{p-1}|b| + |b|^p), \tag{3.19}
$$

we have

$$
\begin{aligned}
&\int_\Omega |\Delta u|^p dx \\
\le\ & \int_\Omega \phi^p |\Delta v|^p dx + c \int_\Omega \Big\{ (\phi|\Delta v|)^{p-1} (|\nabla\phi||\nabla v| + |v||\Delta\phi|) + (|\nabla\phi||\nabla v| + |v||\Delta\phi|)^p \Big\} \\
\le\ & \int_\Omega \phi^p |\Delta v|^p dx + \int_\Omega \Big(|\Delta v|^{p-1} (|\nabla v| + |v|) + (|\nabla v| + |v|)^p \Big).
\end{aligned}
$$

The first integral involves d to the power $-k + \epsilon_0/p$ (see below) and is therefore important. On the other hand, all terms in the second integral involve d to the power that is larger than $-k$ and in fact bounded away from $-k$, independently of ϵ_0; hence

$$
\int_\Omega |\Delta u|^p dx = \int_\Omega \phi^p |\Delta v|^p dx + O(1), \tag{3.20}
$$

where the $O(1)$ is uniform in all the ϵ_i's.

We next define the function

$$g(t) = \epsilon_0 + (-1+\epsilon_1)X_1 + (-1+\epsilon_2)X_1X_2 + \cdots (-1+\epsilon_m)X_1X_2\ldots X_m, \quad t>0,$$

where $X_i = X_i(t/D)$. We shall always think of $g(d(x))$ as a small quantity. Recalling (2.1) one easily sees that

$$\begin{aligned} \frac{d\eta_m}{dt} &= \frac{1}{t}\sum_{1\le i\le j\le m}(-1+\epsilon_j)X_1^2\ldots X_i^2 X_{i+1}\ldots X_j \\ &=: \frac{h(t)}{t}. \end{aligned} \tag{3.21}$$

Also, for any β there holds

$$\frac{d}{dt}\big(t^{\frac{\beta+\epsilon_0}{p}}X_1^{\frac{-1+\epsilon_1}{p}}X_2^{\frac{-1+\epsilon_2}{p}}\ldots X_m^{\frac{-1+\epsilon_m}{p}}\big) = t^{\frac{\beta-p+\epsilon_0}{p}}X_1^{\frac{-1+\epsilon_1}{p}}X_2^{\frac{-1+\epsilon_2}{p}}\ldots X_m^{\frac{-1+\epsilon_m}{p}}\big[\frac{\beta}{p}+\frac{g(t)}{p}\big]. \tag{3.22}$$

Applying (3.22) first for $\beta = k-2p$, then for $\beta = k-p$ and using (3.21) we obtain

$$\Delta v = d^{\frac{-k+\epsilon_0}{p}}X_1^{\frac{-1+\epsilon_1}{p}}X_2^{\frac{-1+\epsilon_2}{p}}\ldots X_m^{\frac{-1+\epsilon_m}{p}}\left\{\Big(\frac{k-p}{p} - d\Delta d - \frac{g}{p}\Big)\Big(\frac{k-2p}{p}-\frac{g}{p}\Big)+\frac{h}{p}\right\},$$

where, here and below, we use g, h and X_i to denote $g(d(x))$, $h(d(x))$ and $X_i(d(x)/D)$. Now, by [AS, Theorem 3.2] we have $d\Delta d = k-1+O(d)$ as $d(x)\to 0$. Hence, the expression in the braces equals

$$Q + \frac{R}{p}g - \frac{1}{p^2}g^2 - \frac{1}{p}h + O(d) \text{ as } d(x)\to 0,$$

where $R = (2k-pk-2p)/p$. The $O(d)$ gives a bounded contribution by an application of (3.19) – as was done earlier. Hence (3.20) gives

$$\int_\Omega |\Delta u|^p dx = \int_\Omega \phi^p d^{-k+\epsilon_0}X_1^{-1+\epsilon_1}\ldots X_m^{-1+\epsilon_m}\left|Q+\frac{R}{p}g-\frac{1}{p^2}g^2-\frac{1}{p}h\right|^p dx + O(1). \tag{3.23}$$

To estimate this we take the Taylor's expansion of $|Q+t|^p$ about $t=0$. We obtain

$$\begin{aligned} \int_\Omega |\Delta u|^p dx &= \int_\Omega \phi^p d^{-k+\epsilon_0}X_1^{-1+\epsilon_1}\ldots X_m^{-1+\epsilon_m}\Big\{|Q|^p + |Q|^{p-2}QRg + \\ &\quad + \Big(-\frac{1}{p}|Q|^{p-2}Q + \frac{p-1}{2p}|Q|^{p-2}R^2\Big)g^2 - \\ &\quad - |Q|^{p-2}Q\zeta_m + O(g^3) + O(gh) + O(h^2)\Big\}dx + O(1). \end{aligned} \tag{3.24}$$

Using (3.19) once again, it is not difficult to see that the terms $O(g^3)$, $O(gh)$ and $O(h^2)$ give a contribution that is bounded uniformly in the ϵ_i's and can therefore be dropped.

At this point, and in order to simplify the notation, we introduce some auxiliary quantities. For $0\le i\le j\le m$ we define

$$A_0 = \int_\Omega \phi^p d^{-k+\epsilon_0}X_1^{-1+\epsilon_1}\ldots X_m^{-1+\epsilon_m}dx$$

$$
\begin{aligned}
A_i &= \int_\Omega \phi^p d^{-k+\epsilon_0} X_1^{1+\epsilon_1} \dots X_i^{1+\epsilon_i} X_{i+1}^{-1+\epsilon_{i+1}} \dots X_m^{-1+\epsilon_m} dx \\
\Gamma_{0j} &= \int_\Omega \phi^p d^{-k+\epsilon_0} X_1^{\epsilon_1} \dots X_i^{\epsilon_i} X_{i+1}^{-1+\epsilon_{i+1}} \dots X_m^{-1+\epsilon_m} dx \\
\Gamma_{ij} &= \int_\Omega \phi^p d^{-k+\epsilon_0} X_1^{1+\epsilon_1} \dots X_i^{1+\epsilon_i} X_{i+1}^{\epsilon_{i+1}} \dots X_j^{\epsilon_j} X_{j+1}^{-1+\epsilon_{j+1}} \dots X_m^{-1+\epsilon_m} dx,
\end{aligned}
$$

with the convention that $\Gamma_{ii} = A_i$. It is then easily seen that

$$
\begin{aligned}
&\int_\Omega \phi^p d^{-k+\epsilon_0} X_1^{-1+\epsilon_1} \dots X_m^{-1+\epsilon_m} g\, dx = \epsilon_0 A_0 - \sum_{i=1}^m (1-\epsilon_i)\Gamma_{0i} \\
&\int_\Omega \phi^p d^{-k+\epsilon_0} X_1^{-1+\epsilon_1} \dots X_m^{-1+\epsilon_m} g^2 dx = \epsilon_0^2 A_0 + \sum_{i=1}^m (1-\epsilon_i)^2 A_i - 2\epsilon_0 \sum_{i=1}^m (1-\epsilon_i)\Gamma_{0i} + \\
&\qquad\qquad\qquad\qquad +2\sum_{i<j}(1-\epsilon_i)(1-\epsilon_j)\Gamma_{ij} \\
&\int_\Omega \phi^p d^{-k+\epsilon_0} X_1^{-1+\epsilon_1} \dots X_m^{-1+\epsilon_m} h\, dx = -\sum_{i=1}^m (1-\epsilon_i) A_i - \sum_{i<j}(1-\epsilon_j)\Gamma_{ij}.
\end{aligned}
$$

(Here and below $\sum_{i<j}$ means $\sum_{1\le i<j\le m}$.) Let us also define the constant

$$
P = -\frac{1}{p}|Q|^{p-2}Q + \frac{p-1}{2p}|Q|^{p-2}R^2.
$$

Going back to (3.24) and noting that

$$
\begin{aligned}
\int_\Omega \frac{|u|^p}{d^{2p}} dx &= A_0 \\
\sum_{i=1}^{m-1} \int_\Omega \frac{|u|^p}{d^{2p}} X_1^2 \dots X_i^2 dx &= \sum_{i=1}^{m-1} A_i.
\end{aligned}
$$

we obtain

$$
\begin{aligned}
J_{m-1}[u] &= |Q|^{p-2}QR\Big(\epsilon_0 A_0 - \sum_{i=1}^m (1-\epsilon_i)\Gamma_{0i}\Big) + \\
&\quad +P\Big(\epsilon_0^2 A_0 + \sum_{i=1}^m (1-\epsilon_i)^2 A_i - 2\epsilon_0 \sum_{i=1}^m (1-\epsilon_i)\Gamma_{0i} + 2\sum_{i<j}(1-\epsilon_i)(1-\epsilon_j)\Gamma_{ij}\Big) \\
&\quad +|Q|^{p-2}Q\Big(\sum_{i=1}^m (1-\epsilon_i)A_i + \sum_{i<j}(1-\epsilon_j)\Gamma_{ij}\Big) - G\sum_{i=1}^{m-1} A_i + O(1). \qquad (3.25)
\end{aligned}
$$

Now, by [BFT2, p184],

$$
\epsilon_0^2 - 2\epsilon_0 \sum_{i=1}^m (1-\epsilon_i)\Gamma_{0i} = \sum_{i=1}^m (\epsilon_i - \epsilon_i^2)A_i + \sum_{i<j}(2\epsilon_i - 1)(1-\epsilon_j)\Gamma_{ij} + O(1).
$$

For the sake of simplicity, we set

$$
G = \frac{p-1}{2p^3}|Q|^{p-2}\Big\{k^2(p-1)^2 + (k-2p)^2\Big\}.
$$

One then easily sees that $P+|Q|^{p-2}Q=G$. Hence, collecting similar terms,

$$\begin{aligned} J_{m-1}[u] &= |Q|^{p-2}QR(\epsilon_0 A_0-\sum_{j=1}^m(1-\epsilon_j)\Gamma_{0j}-G\Big(\sum_{i=1}^m\epsilon_iA_i-\sum_{i<j}(1-\epsilon_j)\Gamma_{ij}\Big)+ \\ & \quad +GA_m+O(1), \end{aligned}$$

where the $O(1)$ is uniform for small ϵ_i's.

Up to this point the parameters $\epsilon_0,\epsilon_1,\ldots,\epsilon_m$ where positive. We intend to take limits as they tend to zero in that order. Due to (3.18), as $\epsilon_0\to 0$ all terms have finite limits except those involving A_0 and Γ_{0j} which, when viewed separately, diverge. However a simple argument involving an integration by parts (see [BFT2, (3.9)] shows that

$$\epsilon_0A_0-\sum_{j=1}^m(1-\epsilon_j)\Gamma_{0j}=O(1) \tag{3.26}$$

uniformly in $\epsilon_0,\ldots,\epsilon_m$. Hence, letting $\epsilon_0\to 0$ we conclude that

$$J_{m-1}[u]=-G\Big(\sum_{i=1}^m\epsilon_iA_i-\sum_{i<j}(1-\epsilon_j)\Gamma_{ij}\Big)+GA_m+O(1) \qquad (\epsilon_0=0)$$

Now – as was the case with (3.26) – an integration by parts shows that (see [BFT2, (3.9)]) if $\epsilon_0=\epsilon_1=\ldots=\epsilon_{i-1}=0$, then

$$\epsilon_iA_i-\sum_{j=1}^m(1-\epsilon_j)\Gamma_{ij}=O(1). \tag{3.27}$$

We now let $\epsilon_1\to 0$. Again, all terms have finite limits except those involving A_1 and Γ_{1j} which diverge. Using (3.27) we see that when combined these terms stay bounded in the limit $\alpha_1\to 0$. We proceed in this way and after letting $\epsilon_{m-1}\to 0$ we are left with

$$J_{m-1}[u]=G(1-\epsilon_m)A_m+O(1) \qquad (\epsilon_0=\ldots=\epsilon_{m-1}=0).$$

Let us denote by G' the infimum in the left-hand side of part (ii) of Theorem 4. We have thus proved that

$$G'\le\frac{G(1-\epsilon_m)A_m+O(1)}{A_m} \qquad (\epsilon_0=\ldots=\epsilon_{m-1}=0).$$

Letting now $\epsilon_m\to 0$ we have $A_m\to+\infty$ (by 3.18)), and thus conclude that $G'\le G$, as required. //

Remark. Slightly modifying the above argument one can also prove the optimality of the power X_m^2 of the improved Rellich inequalities (1.11) and (1.13). Namely, for any $\epsilon>0$ there holds

$$\inf_{u\in C_c^\infty(\Omega\setminus K)}\frac{J_{m-1}[u]}{\int_\Omega\frac{|u|^p}{d^{2p}}X_1^2X_2^2\ldots X_m^{2-\epsilon}dx}=0\,.$$

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