Improved Rellich inequalities for the polyharmonic operator

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Abstract

We prove two improved versions of the Hardy-Rellich inequality for the polyharmonic operator $(-\Delta)^m$ involving the distance to the boundary. The first involves an infinite series improvement using logarithmic functions, while the second contains L^2 norms and involves as a coefficient the volume of the domain. We find explicit constants for these inequalities, and we prove their optimality in the first case.

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1 Introduction

Let Ω be a convex domain in $\mathbf{R}^N$ and let $d(x) = \mathrm{dist}(x, \partial\Omega)$. The classical Hardy's inequality asserts that

$$\int_\Omega |\nabla u|^2 dx \geq \frac{1}{4} \int_\Omega \frac{u^2}{d^2} dx \ , \qquad u \in C_c^\infty(\Omega). \tag{1}$$

There has recently been an increased interest in so-called inproved Hardy's inequalities, where additional non-negative terms appear in the right-hand side of (1). Such inequalities were first established by Maz'ya [M] in the case where Ω is a half-space. Renewed interest in such inequalities followed the work of Brezis and Marcus [BM] where (1) was improved in two ways. More precisely, let $X_1(s) = (1 - \log s)^{-1}$, $s \in (0, 1]$, a function that vanishes at logarithmic speed at $s = 0$. It is shown in [BM] that if Ω is bounded with diameter D then there holds

$$\int_\Omega |\nabla u|^2 dx \geq \frac{1}{4} \int_\Omega \frac{u^2}{d^2} dx + \frac{1}{4} \int_\Omega \frac{u^2}{d^2} X_1^2(d/D) dx \ , \qquad u \in C_c^\infty(\Omega), \tag{2}$$

and also

$$\int_\Omega |\nabla u|^2 dx \geq \frac{1}{4} \int_\Omega \frac{u^2}{d^2} dx + \frac{1}{4D^2} \int_\Omega u^2 dx \ , \qquad u \in C_c^\infty(\Omega). \tag{3}$$

Inequalities (2) and (3) subsequently led to additional improvements and generalizations, which broadly can be termed logarithmic and non-logarithmic respectively.

Let us define recursively $X_i(s) = X_1(X_{i-1}(s))$, $i \geq 2$, $s \in (0, 1]$. Hence the X_i's are iterated logarithmic functions that vanish at an increasingly slow rate at $s = 0$

and satisfy $X_i(1) = 1$. In was proved in [BFT1] that for any $p > 1$ there exists $D \geq \sup_\Omega d(x)$ such that

$$\int_\Omega |\nabla u|^p dx \geq \Big(\frac{p-1}{p}\Big)^p \int_\Omega \frac{|u|^p}{d^p} dx + \frac{1}{2}\Big(\frac{p-1}{p}\Big)^{p-1} \sum_{i=1}^\infty \int_\Omega \frac{|u|^p}{d^p} X_1^2(d/D) \dots X_i^2(d/D) dx, \tag{4}$$

for all $u \in C_c^\infty(\Omega)$. Each new term in this series is optimal, with respect to both the exponent two of X_i and the constant $(1/2)((p-1)/p)^{p-1}$. An analogous result for the bilaplacian is obtained in [BT] where it is shown that

$$\int_\Omega (\Delta u)^2 dx \geq \frac{9}{16} \int_\Omega \frac{u^2}{d^4} dx + \frac{5}{8} \sum_{i=1}^\infty \int_\Omega \frac{u^2}{d^2} X_1^2(d/D) \dots X_i^2(d/D) dx, \tag{5}$$

which is, again, sharp.

Concerning non-logarithmic inequalities and answering a question of [BM], Hoffmann-Ostenhof et al. [HHL] proved that $\mathrm{diam}(\Omega)^{-2}$ in (3) can be replaced by $c|\Omega|^{-2/N}$, where $|\Omega|$ stands for the volume of Ω; more precisely, they showed that

$$\int_\Omega |\nabla u|^2 dx \geq \frac{1}{4} \int_\Omega \frac{u^2}{d^2} dx + \frac{N}{4} \Big(\frac{|\Omega|}{a_N}\Big)^{-2/N} \int_\Omega u^2 dx, \tag{6}$$

where, here and below, a_N stands for the volume of the unit ball in $\mathbf{R}^N$. This was generlized to $p \neq 2$ by Tidblom [T1] who obtained

$$\int_\Omega |\nabla u|^p dx \geq \Big(\frac{p-1}{p}\Big)^p \int_\Omega \frac{|u|^p}{d^p} dx + (p-1)\Big(\frac{p-1}{p}\Big)^p \frac{\sqrt{\pi}\,\Gamma(\frac{N+p}{2})}{\Gamma(\frac{p+1}{2})\Gamma(\frac{N}{2})} \Big(\frac{a_N}{|\Omega|}\Big)^{\frac{p}{N}} \int_\Omega |u|^p dx. \tag{7}$$

Such inequalities, where the volume of Ω appears in the right-hand side, have also been called *geometric*, and we follow this terminology. In the case of geometric improvements the identification of best constants is significantly more complex, since the problem has a global character as opposed to local in the logarithmic case. Results in this direction where obtained in [BFT2] in the linear case and when Ω is the unit ball B; in particular, the best constant was identified in dimension $N = 3$. The constants appearing in (6) and (7) are not sharp. A different type of non-logarithmic L^p improvemnts, rather in the spirit of [M], is obtained in [T2]. See also [FMT1, FMT2] for recent results on improved L^p Hardy-Sobolev inequalities, where an L^q norm, $q > p$, is added to the right-hand side of Hardy's inequality.

The Hardy-Rellich inequalities have various applications in the study of elliptic and parabolic PDE's. Improved Rellich inequalities are useful if critical potentials are additionally present and they also serve to identify such potentials. As the simplest example, one obtains information on the existence of solution and asymptotic behavior for the equation $u_t = \Delta + V$ (or $u_t = -\Delta^2 + V$) for critical potentials V. We refer to [D, BM, O, MMP, BT] and references therein for more on applications.

Our aim in this article is the study of analogous problems for the polyharmonic operator $(-\Delta)^m$. The Hardy-Rellich inequality for $(-\Delta)^m$ was established by Owen [O] who showed that if Ω is convex then

$$\int_\Omega (\Delta^{m/2} u)^2 dx \geq A(m) \int_\Omega \frac{u^2}{d^{2m}} dx\ , \qquad u \in C_c^\infty(\Omega), \tag{8}$$

where

$$A(m) = \frac{1^2 \cdot 3^2 \cdot \ldots \cdot (2m-1)^2}{4^m}$$

is sharp. Here and below we abuse the notation and write $\int (\Delta^{m/2} u)^2 dx$ to stand for $\int |\nabla \Delta^{(m-1)/2} u|^2 dx$ when m is odd. In the main theorems of this paper we obtain two improvements of (8), a logarithmic and a geometric improvement. To state our results, let us define the constants

$$B(m) = \frac{1}{4^m} \sum_{i=1}^{m} \prod_{\substack{k=1 \\ k \neq i}}^{m} (2k-1)^2,$$

$$\Gamma(m) = \frac{N(N+2)\ldots(N+2m-2)}{1 \cdot 3 \cdots (2m-1)} \Big(\sum_{i=1}^{m} \frac{1}{2^{m+i}} \prod_{k=1}^{i} (2k-1)^2 \Big) a_N^{2m/N}.$$

Our first theorem yields a logarithmic series improvement:

Theorem 1 *Let Ω be convex and such that $d(x)$ is bounded in Ω. Then there exists $D \geq \sup_\Omega d(x)$ such that*

$$\int_\Omega (\Delta^{m/2} u)^2 dx \geq A(m) \int_\Omega \frac{u^2}{d^{2m}} dx + B(m) \sum_{i=1}^{\infty} \int_\Omega \frac{u^2}{d^{2m}} X_1^2(d/D) \ldots X_i^2(d/D) dx\,,$$

for all functions $u \in C_c^\infty(\Omega)$.

In the direction of geometric improvement we have

Theorem 2 *Let Ω be bounded and convex. Then there holds*

$$\int_\Omega (\Delta^{m/2} u)^2 dx \geq A(m) \int_\Omega \frac{u^2}{d^{2m}} dx + \Gamma(m) |\Omega|^{-2m/N} \int_\Omega u^2 dx\,,$$

for all functions $u \in C_c^\infty(\Omega)$.

For $m = 2$ Theorem 1 recovers inequality (5), while for $m = 1$ Theorem 2 recovers (6). The constant $B(m)$ of Theorem 1 is sharp; this is contained in the next theorem: we set

$$I_r[u] = \int_\Omega (\Delta^{m/2} u)^2 dx - A(m) \int_\Omega \frac{u^2}{d^{2m}} dx - B(m) \sum_{i=1}^{r} \int_\Omega \frac{u^2}{d^{2m}} X_1^2(d/D) \ldots X_i^2(d/D) dx\,.$$

Theorem 3 *Let $r \geq 1$ and suppose that for some constants $C > 0$, $\theta \in \mathbf{R}$ and $D \geq \sup_\Omega d(x)$ the following inequality holds true,*

$$I_{r-1}[u] \geq C \int_\Omega \frac{u^2}{d^{2m}} X_1^2(d/D) \ldots X_{r-1}^2(d/D) X_r^\theta(d/D) dx. \tag{9}$$

for all $u \in C_c^\infty(\Omega)$. Then (i) *$\theta \geq 2$.* (ii) *If $\theta = 2$ then $C \leq B(m)$.*

We point out that the value of D does not affect the optimality of Theorem 2 in the sense that for any $D_1, D_2 \geq \sup_\Omega d(x)$ there holds $\lim(X_i(d/D_1))/(X_i(d/D_2)) = 1$ as $x \to \partial\Omega$.

Our proofs of Theorems 1 and 2 are surprisingly simple once some one-dimensional inequalities are available. These inequalities are obtained in Section 2. With these in hand the proof is completed using the mean-distance function introduced by Davies [D], as adapted by Owen in [O]; this is carried out in Section 3. What is significantly more involved is the proof of the optimality of the constant $B(m)$ in Theorem 3. This is established in Section 4.

2 One dimensional estimates

For $\gamma > -1$ we define the constants

$$A(m,\gamma) = \frac{(\gamma+1)^2(\gamma+3)^2\dots(\gamma+2m-1)^2}{4^m},$$

$$B(m,\gamma) = \frac{1}{4^m}\sum_{i=1}^m \prod_{\substack{k=1\\k\neq i}}^m (\gamma+2k-1)^2\,,$$

$$\Gamma(m,\gamma) = \frac{N(N+2)\dots(N+2m-2)}{(\gamma+1)(\gamma+3)\dots(\gamma+2m-1)}\Big(\sum_{i=1}^m \frac{1}{2^{m+i}}\prod_{k=1}^i(\gamma+2k-1)^2\Big)a_N^{2m/N}.$$

Note that when $\gamma = 0$ these reduce to the constants $A(m)$, $B(m)$ and $\Gamma(m)$ defined in the introduction. In relation to the case $m = 1$ of this definition, throughout the paper we adopt the convention that empty sums equal zero and empty products equal one.

To simplify the notation we define

$$\zeta(s) = \sum_{i=1}^\infty X_1^2(s)\dots X_i^2(s)\,, \quad s \in (0,1) \tag{10}$$

We claim that that the series converges for all $s \in (0,1)$. Indeed, let us define the functions

$$Y_1(s) = (2-\log s)^{-1} \quad , \quad Y_1(s) = Y_1(Y_{i-1}(s))\,, \quad i \geq 2\,, \quad s \in (0,1)\,.$$

Let s^* be the unique fixed point of Y_1 in $(0,1)$, that is $s^* \in (0,1)$ satisfies $2s^* - s^*\log s^* = 1$, and let $s \in (0,e)$ be given. Then a simple argument shows that $Y_i(s)$ is monotone convergent (incresing, decreasing or constant, according to whether s is smaller, larger or equal to s^*, respectively) and its limit is precisely s^*. Hence it follows from the ratio test that the series $\sum_{i=1}^\infty Y_1^2(s)\dots Y_i^2(s)$ converges for all $s \in (0,e)$. Since $Y_i(es) = X_i(s)$, we conclude that the series (10) converges for all $s \in (0,1)$, as claimed.

Throughout this section we fix an open interval $(0,2b)$ and let $\rho(t) = \min\{t, 2b-t\}$, the distance of t to the boundary of $[0,2b]$. We have

Proposition 4 *Let $m \geq 1$ be fixed. Then there exists $D \geq b$ such that for any $\gamma > -1$ and $\lambda \geq 0$ there holds*

$$\int_0^{2b}(1+\lambda\zeta(\rho/D))\frac{(u^{(m)})^2}{\rho^\gamma(t)}dt \geq A(m,\gamma)\int_0^{2b}\frac{u^2}{\rho^{\gamma+2m}}dt +$$

$$+\Big[B(m,\gamma)+\lambda A(m,\gamma)\Big]\int_0^{2b}\frac{u^2}{\rho^{\gamma+2m}}\zeta(\rho/D)\,dt\,, \tag{11}$$

for all $u\in C_c^\infty(0,2b)$.

Proof. We use induction. For $m=1$ the result is contained in [BT, Theorem 1]; crucially, the constant D does *not* depend on γ. We assume that (11) is valid for $m-1$ (for the same D and for *any* $\gamma>-1$) and writing for simplicity ζ for $\zeta(\rho(t)/D)$, we have

$$\begin{aligned}
&\int_0^{2b}(1+\lambda\zeta)\frac{(u^{(m)})^2}{\rho^\gamma}dt\\
\geq\;& A(m-1,\gamma)\int_0^{2b}\frac{(u')^2}{\rho^{\gamma+2m-2}}dt+\\
&+\Big[B(m-1,\gamma)+\lambda A(m-1,\gamma)\Big]\int_0^{2b}\frac{(u')^2}{\rho^{\gamma+2m-2}}\zeta\,dt\\
=\;& A(m-1,\gamma)\left\{\int_0^{2b}\left(1+\Big[\lambda+\frac{B(m-1,\gamma)}{A(m-1,\gamma)}\Big]\zeta\right)\frac{(u')^2}{\rho^{\gamma+2m-2}}dt\right\}\\
\geq\;& A(m-1,\gamma)\left\{A(1,\gamma+2m-2)\int_0^{2b}\frac{u^2}{\rho^{\gamma+2m}}dt+\right.\\
&\left.+\Big[B(1,\gamma+2m-2)+\Big[\lambda+\frac{B(m-1,\gamma)}{A(m-1,\gamma)}\Big]A(1,\gamma+2m-2)\Big]\int_0^{2b}\frac{u^2}{\rho^{\gamma+2m}}\zeta\,dt\right\}\\
=\;& A(m-1,\gamma)A(1,\gamma+2m-2)\int_0^{2b}\frac{u^2}{\rho^{\gamma+2m}}dt+\\
&+\Big\{\Big[A(m-1,\gamma)B(1,\gamma+2m-2)+B(m-1,\gamma)A(1,\gamma+2m-2)\Big]+\\
&+\lambda A(m-1,\gamma)A(1,\gamma+2m-2)\int_0^{2b}\frac{u^2}{\rho^{\gamma+2m}}\zeta\,dt\,.
\end{aligned}$$

Now, simple calculations together with the relations $A(1,\gamma)=(\gamma+1)^2/4$ and $B(1,\gamma)=1/4$ show that

$$\begin{aligned}
A(m,\gamma)&=A(m-1,\gamma)A(1,\gamma+2m-2)\,,\\
B(m,\gamma)&=A(m-1,\gamma)B(1,\gamma+2m-2)+B(m-1,\gamma)A(1,\gamma+2m-2)\,.
\end{aligned}$$

This concludes the proof. //

Lemma 5 *Let* $\gamma>-1$ *be fixed. Then*

$$\int_0^{2b}\frac{(u')^2}{\rho^\gamma}dt\geq\frac{(\gamma+1)^2}{4}\int_0^{2b}\frac{u^2}{\rho^{\gamma+2}}dt+\frac{(\gamma+1)^2}{4}\frac{1}{b^{\gamma+2}}\int_0^{2b}u^2dt, \tag{12}$$

for all functions $u\in C_c^\infty(0,2b)$.

Proof. Let $u\in C_c^\infty(0,2b)$ be given and let g be a differentiable function on $(0,b]$. There holds

$$\begin{aligned}
\int_0^b g'(\rho(t))u^2dt&=g(b)u^2(b)-2\int_0^b g(\rho(t))uu'dt\\
&\leq g(b)u^2(b)+\int_0^b g^2(\rho(t))\rho^\gamma u^2dt+\int_0^b\frac{(u')^2}{\rho^\gamma}dt,
\end{aligned}$$

that is

$$\int_0^b \frac{(u')^2}{\rho^\gamma}dt \geq \int_0^b \Big(g'(\rho(t)) - g^2(\rho(t))\rho^\gamma\Big)u^2dt - g(b)u^2(b).$$

Similarly,

$$\int_b^{2b} \frac{(u')^2}{\rho^\gamma}dt \geq \int_b^{2b} \Big(g'(\rho(t)) - g^2(\rho(t))\rho^\gamma\Big)u^2dt - g(b)u^2(b).$$

Adding up we obtain

$$\int_0^{2b} \frac{(u')^2}{\rho^\gamma}dt \geq \int_0^{2b} \Big(g'(\rho(t)) - g^2(\rho(t))\rho^\gamma\Big)u^2dt - 2g(b)u^2(b).$$

Replacing $g(\cdot)$ by $g(\cdot) - g(b)$ we conclude that

$$\int_0^{2b} \frac{(u')^2}{\rho^\gamma}dt \geq \int_0^{2b} \Big(g'(\rho(t)) - [g(\rho(t)) - g(b)]^2\rho^\gamma\Big)u^2dt. \tag{13}$$

Choosing

$$g(s) = -\frac{\gamma+1}{2}s^{-\gamma-1},$$

yields after some simple calculations

$$\begin{aligned}
\int_0^{2b} \frac{(u')^2}{\rho^\gamma}dt &\geq \frac{(\gamma+1)^2}{4}\int_0^{2b}\frac{u^2}{\rho^{\gamma+2}}dt + \frac{(\gamma+1)^2}{2}\int_0^{2b}\frac{u^2}{b^{\gamma+1}\rho}dt - \frac{(\gamma+1)^2}{4}\int_0^{2b}\frac{\rho^\gamma u^2}{b^{2\gamma+2}}dt \\
&\geq \frac{(\gamma+1)^2}{4}\int_0^{2b}\frac{u^2}{\rho^{\gamma+2}}dt + \frac{(\gamma+1)^2}{4}\int_0^{2b}\frac{u^2}{b^{\gamma+2}}dt, \qquad (14)
\end{aligned}$$

as required. //

For $\gamma > -1$ we define

$$E(m,\gamma) = \sum_{i=1}^m \frac{1}{2^{m+i}}\prod_{k=1}^i(\gamma+2k-1)^2.$$

Proposition 6 *For any $\gamma > -1$ there holds*

$$\int_0^{2b} \frac{(u^{(m)})^2}{\rho^\gamma}dt \geq A(m,\gamma)\int_0^{2b}\frac{u^2}{\rho^{\gamma+2m}}dt + E(m,\gamma)\frac{1}{b^{\gamma+2m}}\int_0^{2b}u^2dt, \tag{15}$$

for all functions $u \in C_c^\infty(0,2b)$.

Proof. For $m = 1$ this has been proved in the last lemma. Assuming (15) to be true for $m-1$ we compute

$$\begin{aligned}
\int_0^{2b} \frac{(u^{(m)})^2}{\rho^\gamma}dt &\geq A(m-1,\gamma)\int_0^{2b}\frac{(u')^2}{\rho^{\gamma+2m-2}}dt + E(m-1,\gamma)\int_0^{2b}\frac{(u')^2}{b^{\gamma+2m-2}}dt \\
&\geq A(m-1,\gamma)\frac{(2m-1+\gamma)^2}{4}\int_0^{2b}\frac{u^2}{\rho^{\gamma+2m}}dt + \\
&\quad + \Big(A(m-1,\gamma)\frac{(2m-1+\gamma)^2}{4} + \frac{1}{2}\frac{E(m-1,\gamma)}{2}\Big)\frac{1}{b^{\gamma+2m}}\int_0^{2b}u^2dt\,.
\end{aligned}$$

The result follows if we note that

$$A(m,\gamma) = A(m-1,\gamma)\frac{(2m-1+\gamma)^2}{4} \quad , \quad E(m,\gamma) = A(m,\gamma) + \frac{1}{2}E(m-1,\gamma). \ //$$

Remark. We could use the intermediate inequality in (14), hence obtaining $b^{-\gamma-1}\rho^{-1}$ instead of $b^{-\gamma-2}$ in (12). This would lead to a better constant $\hat{E}(m,\gamma)$, defined inductively by

$$\hat{E}(1,\gamma) = \frac{(\gamma+1)^2}{4} \, , \qquad \hat{E}(m,\gamma) = \frac{(\gamma+1)^2}{4}\Big[\hat{E}(m-1,\gamma+2)+\hat{E}(m-1,1)+A(m-1,1)\Big].$$

3 Higher dimensions

Let Ω be a convex domain in $\mathbf{R}^N$. We introduce some additional notation (see [D, HHL]). For $\omega \in S^{N-1}$ and $x \in \Omega$ we define the following functions with values in $(0,+\infty]$:

$$\begin{aligned} \tau_\omega(x) &= \inf\{s>0 \mid x+s\omega \notin \Omega\} \\ \rho_\omega(x) &= \min\{\tau_\omega(x), \tau_{-\omega}(x)\} \\ b_\omega(x) &= \frac{1}{2}(\tau_\omega(x)+\tau_{-\omega}(x)). \end{aligned} \tag{16}$$

We can now prove Theorems 1 and 2.

Proof of Theorem 1. Let $u \in C_c^\infty(\Omega)$ be given. Let us fix a direction $\omega \in S^{N-1}$ and let Ω_ω be the orthogonal projection of Ω on the hyperplane perpendicular to ω. For each $z \in \Omega_\omega$ we apply Proposition 4 (with $\gamma = 0$) on the segment defined by z and ω. By continuity and compactness, D can be chosen to be independent of ω. We then integrate over $z \in \Omega_\omega$ and using the convexity of Ω we conclude that

$$\int_\Omega (\partial_\omega^m u)^2 dx \geq A(m)\int_\Omega \frac{u^2}{\rho_\omega^{2m}}dx + B(m)\int_\Omega \frac{u^2}{\rho_\omega^{2m}}\zeta(\rho_\omega(x)/D)dx \, .$$

Since ζ is an increasing function, this implies

$$\int_\Omega (\partial_\omega^m u)^2 dx \geq A(m)\int_\Omega \frac{u^2}{\rho_\omega^{2m}}dx + B(m)\int_\Omega \frac{u^2}{\rho_\omega^{2m}}\zeta(d(x)/D)dx \, . \tag{17}$$

We now integrate over $\omega \in S^{N-1}$. It is shown in [O] that

$$\int_{S^{N-1}}\int_\Omega (\partial_\omega^m u)^2 dx \, dS(\omega) = C(m,N)\int_\Omega (\Delta^{m/2}u)^2 dx \, , \tag{18}$$

where

$$C(m,N) = \frac{1\cdot 3\ldots(2m-1)}{N(N+2)\ldots(N+2m-2)}.$$

In the same article it was shown that the convexity of Ω implies

$$\int_{S^{N-1}} \frac{dS(\omega)}{\rho_\omega^{2m}(x)} \geq C(m,N)\frac{1}{d^{2m}(x)}. \tag{19}$$

Combining (17), (18) and (19) we obtain the stated inequality. //

Proof of Theorem 2. Let $u \in C_c^\infty(\Omega)$ be given. Arguing as before, but using now Proposition 6 instead of Proposition 4, we have

$$\int_\Omega (\partial_\omega^m u)^2 dx \geq A(m) \int_\Omega \frac{u^2}{\rho_\omega^{2m}} dx + E(m) \int_\Omega \frac{u^2}{b_\omega^{2m}} dx\,, \qquad \omega \in S^{N-1}.$$

Integrating over $\omega \in S^{N-1}$ and using (18) and (19) yields

$$\int_\Omega (\Delta^{m/2} u)^2 dx \geq A(m) \int_\Omega \frac{u^2}{d^{2m}} dx + \frac{E(m)}{C(m,N)} \int_\Omega \int_{S^{N-1}} \frac{u^2}{b_\omega^{2m}} dS(\omega) dx\,. \tag{20}$$

But [T1, Lemma 2.1] the convexity of Ω implies that

$$\int_{S^{N-1}} \frac{1}{b_\omega(x)^{2m}} dS(\omega) \geq \Big(\frac{|\Omega|}{a_N}\Big)^{-2m/N}. \tag{21}$$

Combining (20) and (21) and observing that

$$\Gamma(m) = \frac{E(m)}{C(m,N)} a_N^{2m/N},$$

concludes the proof of the theorem. //

4 Optimality of the constants

This section is more technical than the previous ones. Our main purpose will be the computation of $I_{r-1}[u]$ for an appropriate test function u. Throughout the section we shall repeatedly use the differentiation rule

$$\frac{d}{dt} X_i^\beta(t) = \frac{\beta}{t} X_1(t) X_2(t) \dots X_{i-1}(t) X_i^{1+\beta}(t), \qquad i = 1, 2, \dots, \quad \beta \in \mathbf{R}, \tag{22}$$

which is easily proved by induction.

Let $m \in \mathbf{N}$. We recall our convention about empty sums or products and define the functions

$$\sigma_0^{(m)}(x) = x(x-1)\dots(x-m+1)\ \ ,\ \ \sigma_1^{(m)}(x) = \sum_{i=1}^{m} \prod_{k \neq i} (x-k+1)$$

$$\sigma_2^{(m)}(x) = \sum_{1 \leq i < j \leq r}^{m} \prod_{k \neq i,j} (x-k+1).$$

Lemma 7 *Let $s_0, s_1, \dots, s_r \in \mathbf{R}$ and $u(t) = t^{s_0} X_1^{s_1} \dots X_r^{s_r}$. Let*

$$Y_{ij} = X_1^2 \dots X_i^2 X_{i+1} \dots X_j \qquad 0 \leq i \leq j \leq r,$$

with the conventions $Y_{00} = 1$, $Y_{ii} = X_1^2 \dots X_i^2$, $Y_{0j} = X_1 \dots X_j$. Then there holds

$$u^{(m)}(t) = t^{s_0 - m} X_1^{s_1} \dots X_r^{s_r} \sum_{0 \leq i \leq j \leq r} c_{ij}^{(m)} Y_{ij}(t) + t^{s_0 - m} O(X_1^{s_1+3} X_2^{s_2} \dots X_r^{s_r}), \tag{23}$$

where:

$$
\begin{array}{llll}
c_{00}^{(m)} = \sigma_0^{(m)}(s_0), & & c_{0j}^{(m)} = s_j\sigma_1^{(m)}(s_0), & j \geq 1, \\
c_{ii}^{(m)} = s_i(s_i+1)\sigma_2^{(m)}(s_0)\,, & 1 \leq i \leq r, & c_{ij}^{(m)} = (2s_i+1)s_j\sigma_2^{(m)}(s_0), & 1 \leq i < j \leq r\,.
\end{array}
$$

Proof. We use induction. When $m = 1$ (23) follows directly from (22). Let us assume that

$$u^{(m-1)}(t) = t^{s_0-m+1}X_1^{s_1}\dots X_r^{s_r}\sum_{0\leq i\leq j\leq r} c_{ij}^{(m-1)}Y_{ij}(t) + t^{s_0-m+1}O(X_1^{s_1+3}X_2^{s_2}\dots X_r^{s_r}).$$

We differentiate and again use (22). The $t^{s_0-m+1}O(X_1^{s_1+3}X_2^{s_2}\dots X_r^{s_r})$ will give a term $t^{s_0-m}O(X_1^{s_1+3}X_2^{s_2}\dots X_r^{s_r})$. After some simple calculations we obtain modulo $O(X_1^{s_1+3}X_2^{s_2}\dots X_r^{s_r})$,

$$
\begin{aligned}
u^{(m)}(t) &= t^{s_0-m}X_1^{s_1}\dots X_r^{s_r}\Bigg\{\sum_{0\leq i\leq j\leq r} c_{ij}^{(m-1)}(s_0-m+1)Y_{ij} + \sum_{j=i}^{r} c_{00}^{(m-1)}s_jY_{0j} \\
&\quad + \sum_{j=1}^{r}\sum_{k=1}^{r} c_{0j}^{(m-1)}Y_{kj} + \sum_{j=1}^{r}\sum_{k=j+1}^{r} c_{0j}^{(m-1)}s_kY_{jk}\Bigg\} \\
&= t^{s_0-m}X_1^{s_1}\dots X_r^{s_r}\Bigg\{(s_0-m+1)c_{00}^{(m-1)}Y_{00} \\
&\quad + \sum_{j=1}^{r}\Big[(s_0-m+1)c_{0j}^{(m-1)} + s_jc_{00}^{(m-1)}\Big]Y_{0j} \\
&\quad + \sum_{i=1}^{r}\Big[(s_0-m+1)c_{ii}^{(m-1)} + (s_i+1)c_{0i}^{(m-1)}\Big]Y_{ii} \\
&\quad + \sum_{1\leq i<j\leq r}\Big[(s_0-m+1)c_{ij}^{(m-1)} + (s_i+1)c_{0j}^{(m-1)} + s_jc_{0i}^{(m-1)}\Big]Y_{ij}\Bigg\}.
\end{aligned}
$$

The proof is concluded by observing that the constants $c_{ij}^{(k)}$, $0 \leq i \leq j \leq r$, satisfy the induction relations

$$
\begin{array}{ll}
c_{00}^{(m)} = (s_0-m+1)c_{00}^{(m-1)}, & \\
c_{0j}^{(m)} = (s_0-m+1)c_{0j}^{(m-1)} + s_jc_{00}^{(m-1)}, & 1 \leq j \leq r, \\
c_{ii}^{(m)} = (s_0-m+1)c_{ii}^{(m-1)} + (s_i+1)c_{0i}^{(m-1)}, & 1 \leq i \leq r, \\
c_{ij}^{(m)} = (s_0-m+1)c_{ij}^{(m-1)} + (s_i+1)c_{0j}^{(m-1)} + s_jc_{0i}^{(m-1)}, & 1 \leq i < j \leq r.
\end{array}
$$

//

In the sequel we shall denote the constants $c_{ij}^{(m)}$ simply by c_{ij}, since only the mth order derivative of u will appear. Similarly, we shall write $\sigma_i(x)$ instead of $\sigma_i^{(m)}(x)$, $i = 0, 1, 2$. Let $s_0 > (2m-1)/2$, $s_1, \dots, s_r \in \mathbf{R}$ be fixed. For $0 \leq i \leq j \leq r$ we define

$$
\begin{aligned}
\Gamma_{ij} &= \int_0^1 t^{2s_0-2m}X_1^{2s_1}\dots X_r^{2s_r}Y_{ij}dt \\
&= \int_0^1 t^{2s_0-2m}X_1^{2s_1+2}\dots X_i^{2s_i+2}X_{i+1}^{2s_{i+1}+1}\dots X_j^{2s_j+1}X_{j+1}^{2s_{j+1}}\dots X_r^{2s_r}dt.
\end{aligned}
$$

Lemma 8 *Let $u(t) = t^{s_0} X_1^{s_1} \dots X_r^{s_r}$. There holds*

$$I_{r-1}[u] = \sum_{0 \le i \le j \le r} a_{ij}\Gamma_{ij} + \int_0^1 t^{2s_0-2m} O(X_1^{s_1+3} X_2^{s_2} \dots X_r^{s_r}) dt. \tag{24}$$

where

$$\begin{aligned} a_{00} &= c_{00}^2 - A(m), & & a_{0j} = 2c_{00}c_{0j}, & 1 \le j \le r, \\ a_{ii} &= c_{0i}^2 + 2c_{00}c_{ii} - B(m), & 1 \le i \le r-1, \quad & a_{rr} = c_{0r}^2 + 2c_{00}c_{rr}, \\ a_{ij} &= 2c_{00}c_{ij} + 2c_{0i}c_{0j}, & 1 \le i < j \le r. \end{aligned} \tag{25}$$

Proof. From Lemma 7 we have modulo $\int_0^1 t^{2s_0-2m} O(X_1^{s_1+3} X_2^{s_2} \dots X_r^{s_r}) dt$,

$$\int_0^1 (u^{(m)})^2 dt = \int_0^1 t^{2s_0-2m} X_1^{2s_1} \dots X_r^{2s_r} \Big(\sum_{0 \le i \le j \le r} c_{ij} Y_{ij} \Big)^2 dt\,.$$

We expand the square and hence obtain a linear combination of terms of the form $\int_0^1 t^{2s_0-2m} X_1^{2s_1} \dots X_r^{2s_r} Y_{ij} Y_{kl} dt$, where $0 \le i \le j \le r$, $0 \le k \le l \le r$. Now, we observe that $Y_{ij}Y_{kl} = O(X_1^3)$ unless (1) $i = j = 0$ or (2) $k = l = 0$ or (3) $i = k = 0$. Hence, denoting by S the last parenthesis above we have

$$\begin{aligned} S &= c_{00}^2 + 2 \sum_{\substack{0 \le i \le j \le r \\ (i,j) \ne (0,0)}} c_{00} c_{ij} Y_{ij} + \sum_{j,l=1}^{r} c_{0j} c_{0l} Y_{0j} Y_{0l} + O(X_1^3) \\ &= c_{00}^2 + 2 \sum_{j=1}^{r} c_{00} c_{0j} Y_{0j} + 2 \sum_{1 \le i \le j \le r} c_{00} c_{ij} Y_{ij} + \sum_{i=1}^{r} c_{0i}^2 Y_{0i}^2 + 2 \sum_{1 \le i < j \le r} c_{0i} c_{0j} Y_{0i} Y_{0j} + O(X_1^3). \end{aligned}$$

Using the fact that $Y_{0i}Y_{0j} = Y_{ij}$, $i \le j$, we thus conclude that

$$S = c_{00}^2 + 2\sum_{j=1}^{r} c_{00}c_{0j}Y_{0j} + \sum_{i=1}^{r} (c_{0i}^2 + 2c_{00}c_{ii}) Y_{ii} + 2 \sum_{1 \le i < j \le r} (c_{00}c_{ij} + 2c_{0i}c_{0j}) Y_{ij}.$$

The proof is complete if we recall that

$$\int_0^1 \frac{u^2}{t^{2m}} dt = \Gamma_{00} \text{ and } \int_0^1 \frac{u^2}{t^{2m}} X_1^2 \dots X_i^2 dt = \Gamma_{ii}\,, \quad 1 \le i \le r-1\,. \qquad //$$

Up to this point the parameters $s_0, s_1, \dots, s_r$ where arbitrary subject only to $s_0 > (2m-1)/2$. We now make a more specific choice, taking

$$s_0 = \frac{2m - 1 + \epsilon_0}{2}\,, \qquad s_j = \frac{-1 + \epsilon_j}{2}\,, \quad 1 \le j \le r\,,$$

where $\epsilon_0, \dots, \epsilon_r$ are small parameters. We consider the functional $I_{r-1}[u]$ as a function of these parameters and intend to take succesively the limits $\epsilon_0 \searrow 0, \dots, \epsilon_r \searrow 0$. In taking these limits we shall ignore terms that are bounded uniformly in the ϵ_i's. In order to distinguish such terms we shall make use of the following fact: we have [BFT1, (3.8)]:

$$\int_0^1 t^{-1+\epsilon_0} X_1^{1+\epsilon_1} \dots X_r^{1+\epsilon_r} dt < \infty \Longleftrightarrow \begin{cases} & \epsilon_0 > 0 \\ \text{or} & \epsilon_0 = 0 \text{ and } \epsilon_1 > 0 \\ \text{or} & \epsilon_0 = \epsilon_1 = 0 \text{ and } \epsilon_2 > 0 \\ & \dots \\ \text{or} & \epsilon_0 = \epsilon_1 = \dots = \epsilon_{r-1} = 0 \text{ and } \epsilon_r > 0. \end{cases} \tag{26}$$

For the terms that diverge as the ϵ_i's tend to zero, we shall need some quantitive information on the rate of divergence. This is contained in the following

Lemma 9 *For any $\beta < 1$ there exists $c_\beta > 0$ such that*

$$\begin{aligned} &\text{(i)} \quad \int_0^1 t^{-1+\epsilon_0} X_1^\beta dt \le c_\beta \epsilon_0^{-1+\beta} \,, \\ &\text{(ii)} \quad \int_0^1 t^{-1} X_1 \dots X_{i-1} X_i^{1+\epsilon_i} X_{i+1}^\beta dt \le c_\beta \epsilon_i^{-1+\beta} \,, \qquad 1 \le i \le r-1 \,. \end{aligned}$$

Proof. (i) Setting $s = \epsilon_0^{-1} X_1(t)$ we have $t = \exp(1 - \epsilon_0^{-1} s^{-1})$, $ds = \epsilon_0^{-1} t^{-1} X_1^2 dt$, and therefore

$$\begin{aligned} \int_0^1 t^{-1+\epsilon_0} X_1^\beta dt &= e^{\epsilon_0} \epsilon_0^{-1+\beta} \int_0^{\frac{1}{\epsilon_0}} e^{-\frac{1}{s}} s^{-2+\beta} ds \\ &\le e^{\epsilon_0} \epsilon_0^{-1+\beta} \int_0^\infty e^{-\frac{1}{s}} s^{-2+\beta} ds. \end{aligned}$$

(ii) Similarly, we set $s = \epsilon_i^{-1} X_{i+1}(t)$. Then

$$X_i(t) = \exp(1 - \epsilon_i^{-1} s^{-1}) \ , \qquad ds = \epsilon_i^{-1} t^{-1} X_1 \dots X_i X_{i+1}^2 dt.$$

Hence (22) gives

$$\int_0^1 t^{-1} X_1 \dots X_{i-1} X_i^{1+\epsilon_i} X_{i+1}^\beta dt = e^{\epsilon_i} \epsilon_i^{-1+\beta} \int_0^{\frac{1}{\epsilon_i}} e^{-\frac{1}{s}} s^{-2+\beta} ds,$$

yielding the stated estimate. //

We shall also need the following

Lemma 10 (i) *There holds*

$$\epsilon_0^2 \Gamma_{00} - 2\epsilon_0 \sum_{j=i+1}^r (1-\epsilon_j)\Gamma_{0j} = \sum_{i=1}^r (\epsilon_i - \epsilon_i^2)\Gamma_{ii} - \sum_{1 \le i < j \le r} (1-\epsilon_j)(1-2\epsilon_i)\Gamma_{ij} + O(1),$$

where the $O(1)$ is uniform in $\epsilon_0, \dots, \epsilon_r$.
(ii) *Let $i \ge 0$ and (if $i \ge 1$) assume that $\epsilon_0 = \dots = \epsilon_{i-1} = 0$. Then*

$$\epsilon_i \Gamma_{ii} = \sum_{j=i+1}^r (1-\epsilon_j)\Gamma_{ij} + O(1),$$

where the O(1) is uniform in $\epsilon_i, \dots, \epsilon_r$.

Proof. The two parts of the lemma have been proved in [BFT1, p184] and [BFT1, p181] respectively. //

Remark. We are now in position to prove Theorem 3, but before proceeding some comments are necessary. The proof of the theorem is local: we fix a point $x_0 \in \partial\Omega$ and work entirely in a small ball $B(x_0, \delta)$ using a cut-off function ϕ. The sequence of functions that is used is then given by

$$u(x) = \phi(x) d(x)^{\frac{-1+2m+\epsilon_0}{2}} X_1(d(x)/D)^{\frac{-1+\epsilon_1}{2}} \dots X_r(d(x)/D)^{\frac{-1+\epsilon_r}{2}}, \qquad (\epsilon_0, \dots, \epsilon_r > 0)$$

and, as already mentioned, we take the successive limits $\epsilon_0 \searrow 0, \dots, \epsilon_r \searrow 0$; in taking this limits, we work modulo terms that are bounded uniformly in the remaining ϵ_i's. Such are any terms that contain derivatives of ϕ; such are also any terms that contain derivatives of $d(x)$ of order higher than one since such derivatives are bounded near $\partial\Omega$; see, e.g. [LN, Section 1.3]. These considerations are to a large extent the justification of the fact that, for the proof of Theorem 3 we can, without any loss of generality, restrict ourselves to the one-dimensional case. We shall thus take $\Omega = (0, 1)$, and consider the sequence

$$u(t) = t^{\frac{-1+2m+\epsilon_0}{2}} X_1(t)^{\frac{-1+\epsilon_1}{2}} \dots X_r(t)^{\frac{-1+\epsilon_r}{2}}.$$

discussed earlier; multiplication by an appropriate cut-off function shows that u lies in the appropriate Sobolev space. Note that u does not vanish at $t = 1$, but the cut-off function ϕ would take care of that. For a complete picture of what the full proof would look like, we refer to [BT] where the case $m = 2$ has been carried out in every detail.

Proof of Theorem 3 (see also the remark above) We define

$$u(t) = t^{\frac{-1+2m+\epsilon_0}{2}} X_1(t)^{\frac{-1+\epsilon_1}{2}} \dots X_r(t)^{\frac{-1+\epsilon_r}{2}}, \tag{27}$$

where $\epsilon_0, \dots, \epsilon_r$ are small positive parameters. For the reader's convenience we recall from Lemma 8 that

$$I_{r-1}[u] = \sum_{0 \le i \le j \le r} a_{ij} \Gamma_{ij} + O(1), \tag{28}$$

where the $O(1)$ is uniform in $\epsilon_0, \dots \epsilon_r$ (by (26)) and the constants a_{ij} are given by

$$\begin{aligned} &a_{00} = c_{00}^2 - A(m), && a_{0j} = 2c_{00}c_{0j}, \quad && 1 \le j \le r, \\ &a_{ii} = c_{0i}^2 + 2c_{00}c_{ii} - B(m), \quad 1 \le i \le r-1, && a_{rr} = c_{0r}^2 + 2c_{00}c_{rr}, && \\ &a_{ij} = 2c_{00}c_{ij} + 2c_{0i}c_{0j}, \qquad 1 \le i < j \le r. && && \end{aligned} \tag{29}$$

The c_{ij}'s are given by

$$\begin{array}{llll} c_{00} = \sigma_0(s_0), & & c_{0j} = s_j \sigma_1(s_0), & j \ge 1, \\ c_{ii} = s_i(s_i + 1)\sigma_2(s_0)\,, & 1 \le i \le r, & c_{ij} = (2s_i + 1)s_j \sigma_2(s_0), & 1 \le i < j \le r\,. \end{array}$$

where, in turn,

$$s_0 = \frac{2m - 1 + \epsilon_0}{2}\,, \qquad s_j = \frac{\epsilon_j - 1}{2}\,, \quad 1 \le j \le r,$$

and

$$\sigma_0(x) = x(x-1)\dots(x-m+1) \;\;,\;\; \sigma_1(x) = \sum_{i=1}^{m}\prod_{k\neq i}(x-k+1)$$
$$\sigma_2(x) = \sum_{1\leq i<j\leq m}\prod_{k\neq i,j}(x-k+1).$$

We observe that

$$\sigma_0'(x) = \sigma_1(x) \quad , \qquad \sigma_1'(x) = 2\sigma_2(x).$$

We now let $\epsilon_0 \searrow 0$ in (28). It follows from (26) that all Γ_{ij}'s with $i \geq 1$ have finite limits. As for the remaining terms Γ_{0j} , applying Lemma 9 with $\beta = -3/2$ (for $j = 0$) and with $\beta = -1/2$ (for $j \geq 1$) we obtain respectively

$$\begin{aligned} \Gamma_{00} &= \int_0^1 t^{-1+\epsilon_0} X_1^{-1+\epsilon_1} \dots\dots X_r^{-1+\epsilon_r} dt \\ &\leq c\int_0^1 t^{-1+\epsilon_0} X_1^{-\frac{3}{2}} dt \\ &\leq c\epsilon_0^{-\frac{5}{2}} \end{aligned} \tag{30}$$

and

$$\begin{aligned} \Gamma_{0j} &= \int_0^1 t^{-1+\epsilon_0} X_1^{\epsilon_1} \dots X_j^{\epsilon_j} X_{j+1}^{-1+\epsilon_{j+1}} \dots X_r^{-1+\epsilon_r} dt \\ &\leq c\int_0^1 t^{-1+\epsilon_0} X_1^{-\frac{1}{2}} dt \\ &\leq c\epsilon_0^{-\frac{3}{2}}, \end{aligned} \tag{31}$$

where in both cases $c > 0$ is independent of $\epsilon_1, \dots, \epsilon_r$. Now, we think of the contants a_{0j} and c_{0j} as functions of ϵ_0, writting $a_{0j} = a_{0j}(\epsilon_0)$, $c_{0j} = c_{0j}(\epsilon_0)$ and considering $\epsilon_1, \dots, \epsilon_r$ as small positive parameters. Using Taylor's theorem we shall expand the coefficient a_{0j} of Γ_{0j}, $j = 0$ (resp. $j \geq 1$) in powers of ϵ_0, and relation (30) (resp. (31)) shows that we can discard powers with exponent ≥ 3 (resp. ≥ 2). We compute the remaining ones. Denoting by $A_{k,0j}$ the coefficient of ϵ_0^k in a_{0j} we have:

- *Constant term in a_{00}:* We have $A_{0,00} = a_{00}(0) = c_{00}^2(0) - A(m) = 0$.

- *Coefficient of ϵ_0 in a_{00}:* We have $c_{00}(\epsilon_0) = \sigma_0(\frac{2m-1+\epsilon_0}{2})$ and therefore $c_{00}'(0) = \frac{1}{2}\sigma_1(\frac{2m-1}{2})$. Hence $a_{00}'(\epsilon_0) = 2c_{00}(\epsilon_0)c_{00}'(\epsilon_0) = \sigma_0(\frac{2m-1+\epsilon_0}{2})\sigma_1(\frac{2m-1+\epsilon_0}{2})$ and the coefficient is

$$A_{1,00} = a_{00}'(0) = \sigma_0(\frac{2m-1}{2})\sigma_1(\frac{2m-1}{2}).$$

We henceforth write σ_i for $\sigma_i((2m-1)/2)$, $i = 0, 1, 2$.

- *Coefficient of ϵ_0^2 in a_{00}:* The coefficient is

$$A_{2,00} = \frac{1}{2}a_{00}''(0) = [c_{00}'(0)]^2 + c_{00}(0)c_{00}''(0) = \frac{1}{4}\sigma_1^2 + \frac{1}{2}\sigma_0\sigma_2.$$

- *Constant term in a_{0j}, $j \geq 1$:* This is

$$A_{0,0j} = a_{0j}(0) = 2c_{00}(0)c_{0j}(0) = -(1-\epsilon_j)\sigma_0\sigma_1\,.$$

- *Coefficient of* ϵ_0 *in* a_{0j}*:* This is

$$
\begin{aligned}
A_{1,0j} &= a'_{0j}(0) = 2c'_{00}(0)c_{0j}(0) + 2c_{00}(0)c'_{0j}(0) \\
&= -\frac{1}{2}(1-\epsilon_j)\sigma_1^2 - (1-\epsilon_j)\sigma_0\sigma_2.
\end{aligned}
$$

Now, we observe that $A_{0,0j} = -(1-\epsilon_j)A_{1,00}$. Hence (ii) of Lemma 10 implies that

$$
A_{1,00}\epsilon_0\Gamma_{00} + \sum_{j=1}^{r} A_{0,0j}\Gamma_{0j} = O(1) \tag{32}
$$

uniformly in $\epsilon_1, \ldots, \epsilon_r$. Similarly, we observe that $A_{1,0j} = -2(1-\epsilon_j)A_{2,00}$. Hence, by (i) of Lemma 10, the remaining 'bad' terms when combined give

$$
\begin{aligned}
&A_{2,00}\epsilon_0^2\Gamma_{00} + \epsilon_0 \sum_{j=1}^{r} A_{1,0j}\Gamma_{0j} = \\
&\quad = A_{2,00}\Big(\epsilon_0^2\Gamma_{00} - 2\epsilon_0\sum_{j=1}^{r}(1-\epsilon_j)\Gamma_{0j}\Big) \qquad (33) \\
&\quad = A_{2,00}\Big(\sum_{i=1}^{r}(\epsilon_i - \epsilon_i^2)\Gamma_{ii} - \sum_{1\le i<j\le r}(1-\epsilon_j)(1-2\epsilon_i)\Gamma_{ij}\Big) + O(1),
\end{aligned}
$$

uniformly in $\epsilon_1, \ldots, \epsilon_r$. Note that the right-hand side of (33) has a finite limit as $\epsilon_0 \searrow 0$. Combining (29) , (32) and (33) we conclude that, after letting $\epsilon_0 \searrow 0$, we are left with

$$
\begin{aligned}
&I_{r-1}[u] \\
&= \sum_{i=1}^{r}\Big(a_{ii} + A_{2,00}(\epsilon_i - \epsilon_i^2)\Big)\Gamma_{ii} + \sum_{1\le i<j\le r}\Big(a_{ij} - A_{2,00}(1-\epsilon_j)(1-2\epsilon_i)\Big)\Gamma_{ij} + O(1) \\
&=: \sum_{i=1}^{r} b_{ii}\Gamma_{ii} + \sum_{1\le i<j\le r} b_{ij}\Gamma_{ij} + O(1)\ , \qquad (\epsilon_0 = 0), \qquad (34)
\end{aligned}
$$

where the $O(1)$ is uniform in $\epsilon_1, \ldots, \epsilon_r$.

We next let $\epsilon_1 \searrow 0$ in (34). It follows from (26) that all the Γ_{ij}'s have a finite limit, except those with $i = 1$ which diverge to $+\infty$. The latter terms are again estimated with the aid of Lemma 9, this time with $i = 1$. Part (i) of the lemma (with $\beta = -3/2$) yields

$$
\begin{aligned}
\Gamma_{11} &= \int_0^1 t^{-1} X_1^{1+\epsilon_1} X_2^{-1+\epsilon_2} \ldots X_r^{-1+\epsilon_r} dt \\
&\le c\int_0^1 t^{-1} X_1^{1+\epsilon_1} X_2^{-\frac{3}{2}} dt \\
&\le c\epsilon_1^{-\frac{5}{2}}\ ,
\end{aligned} \tag{35}
$$

uniformly in $\epsilon_2, \ldots, \epsilon_r$. For $j \ge 2$ it also yields (now with $\beta = -1/2$)

$$
\begin{aligned}
\Gamma_{1j} &= \int_0^1 t^{-1} X_1^{1+\epsilon_1} X_2^{\epsilon_2} \ldots X_j^{\epsilon_j} X_{j+1}^{-1+\epsilon_{j+1}} \ldots X_r^{-1+\epsilon_r} dt \\
&\le c\int_0^1 t^{-1} X_1^{1+\epsilon_1} X_2^{-\frac{1}{2}} dt \\
&\le c\epsilon_1^{-\frac{3}{2}},
\end{aligned} \tag{36}
$$

again, uniformly in $\epsilon_2, \dots, \epsilon_r$. We think of the coefficients b_{1j} and a_{1j} as functions of ϵ_1 and we expand these in powers of ϵ_1. Estimate (35) (resp. (36)) implies that only the terms $1, \epsilon_1$ and ϵ_1^2 (resp. 1 and ϵ_1) give contributions for Γ_{11} (resp. Γ_{1j}, $j \geq 2$) that do not vanish as $\epsilon_1 \searrow 0$. We shall compute the coefficients of these terms; note that c_{00} is now treated simply as a constant. Denoting by $B_{k,1j}$ the coefficient of ϵ_1^k in b_{1j}, $j \geq 1$, we have:

- *Constant term in b_{11}:* For $\epsilon_1 = 0$ we have $s_1 = -1/2$. Hence

$$\begin{aligned}
B_{0,11} &= b_{11}(0) \\
&= c_{01}^2(0) + 2c_{00}c_{11}(0) - B(m) \\
&= \frac{1}{4}\sigma_1^2 - \frac{1}{2}\sigma_0\sigma_2 - B(m) \\
&= \frac{1}{4}\Big(\sum_{i=1}^{m}\prod_{k\neq i}\frac{2k-1}{2}\Big)^2 - \frac{1}{2}\Big(\prod_{k=1}^{m}\frac{2k-1}{2}\Big)\Big(\sum_{1\leq i<j\leq m}\prod_{k\neq i,j}\frac{2k-1}{2}\Big) - \\
&\quad -\frac{1}{4}\sum_{i=1}^{m}\prod_{k\neq i}\Big(\frac{2k-1}{2}\Big)^2.
\end{aligned}$$

This is zero as is seen by expanding the square:

$$\Big(\sum_{i=1}^{m}\prod_{k\neq i}\frac{2k-1}{2}\Big)^2 = \sum_{i=1}^{m}\prod_{k\neq i}\Big(\frac{2k-1}{2}\Big)^2 + 2\sum_{i<j}\Big(\prod_{k\neq i}\frac{2k-1}{2}\Big)\Big(\prod_{k\neq j}\frac{2k-1}{2}\Big).$$

- *Coefficient of ϵ_1 in b_{11}:* We have

$$\begin{aligned}
b_{11}'(\epsilon_1) &= a_{11}'(\epsilon_1) + A_{2,00} - 2A_{2,00}\epsilon_1 \\
&= 2c_{01}(\epsilon_1)c_{01}'(\epsilon_1) + 2c_{00}c_{11}'(\epsilon_1) + A_{2,00}(1-2\epsilon_1) \\
&= \frac{\epsilon_1 - 1}{2}\sigma_1^2 + \epsilon_1\sigma_0\sigma_2 + \Big(\frac{1}{4}\sigma_1^2 + \frac{1}{2}\sigma_0\sigma_2\Big)(1-2\epsilon_1),
\end{aligned}$$

and therefore the coefficient is

$$B_{1,11} = b_{11}'(0) = -\frac{1}{4}\sigma_1^2 + \frac{1}{2}\sigma_0\sigma_2 = -B(m).$$

- *Coefficient of ϵ_1^2 in b_{11}:* The coefficient is

$$B_{2,11} = \frac{1}{2}b_{11}''(0) = \frac{1}{2}a_{11}''(0) - A_{2,00} = \frac{1}{4}\sigma_1^2 + \frac{1}{2}\sigma_0\sigma_2 - A_{2,00} = 0.$$

- *Constant term in b_{1j}, $j \geq 2$:* We have

$$\begin{aligned}
b_{1j}(\epsilon_1) &= 2c_{00}c_{1j}(\epsilon_1) + 2c_{01}(\epsilon_1)c_{0j}(\epsilon_1) - A_{2,00}(1-\epsilon_j)(1-2\epsilon_1) \\
&= \epsilon_1(\epsilon_j - 1)\sigma_0\sigma_2 + \frac{(\epsilon_1-1)(\epsilon_j-1)}{2}\sigma_1^2 - A_{2,00}(1-\epsilon_j)(1-2\epsilon_1),
\end{aligned}$$

and therefore the constant term is

$$B_{0,1j} = b_{1j}(0) = (1-\epsilon_j)\left(\frac{\sigma_1^2}{2} - A_{2,00}\right) = (1-\epsilon_j)B(m).$$

- *Coefficient of ϵ_1 in b_{1j}, $j \geq 2$:* The coefficient is

$$B_{1,1j} = b'_{1j}(0) = (\epsilon_j - 1)\sigma_0\sigma_2 + \frac{\epsilon_j - 1}{2}\sigma_1^2 + 2A_{2,00}(1 - \epsilon_j) = 0.$$

We obsrerve that $B_{0,1j} = -(1-\epsilon_j)B_{1,11}$, $j \geq 2$. Hence part (ii)of Lemma 10 gives

$$\epsilon_1 B_{1,11}\Gamma_{11} + \sum_{j=2}^{r} B_{0,1j}\Gamma_{1j} = O(1), \tag{37}$$

uniformly in $\epsilon_2, \ldots, \epsilon_r$. Combining (34) and (37) we conclude that after letting $\epsilon_1 \searrow 0$ we are left with

$$I_{r-1}[u] = \sum_{2\leq i\leq j\leq r} b_{ij}\Gamma_{ij} + O(1) \ , \qquad (\epsilon_0 = \epsilon_1 = 0), \tag{38}$$

uniformly in $\epsilon_2, \ldots, \epsilon_r$. Note that we have the same coefficients b_{ij} as in (34), unlike the case where the limit $\epsilon_0 \searrow 0$ was taken, in which case we passed from the original coefficients a_{ij} to the coefficients b_{ij}.

We proceed in this way. At the ith step we denote by $B_{k,ij}$ the coefficient of ϵ_i^k in b_{ij}, $j \geq i$, and observe that (exactly as in the case $i = 1$) there holds

$$B_{0,ij} = -(1-\epsilon_j)B_{1,ii} \quad , \qquad B_{2,ii} = B_{1,ij} = 0 \ , \qquad j \geq i+1.$$

Hence (ii) of Lemma 10 implies the cancelation (modulo uniformly bounded terms) of all terms that, individually, diverge as $\epsilon_i \searrow 0$. Eventually, after letting $\epsilon_{r-1} \searrow 0$, we arrive at

$$I_{r-1}[u] = b_{rr}\Gamma_{rr} + O(1) \ , \quad (\epsilon_0 = \epsilon_1 = \ldots = \epsilon_{r-1} = 0), \tag{39}$$

where b_{rr} has been defined in (34). We observe now that

$$\int_0^1 \frac{u^2}{t^2} X_1^2 \ldots X_r^2 dt = \Gamma_{rr}.$$

Hence, using the fact that $\Gamma_{rr} \to +\infty$ as $\epsilon_r \searrow 0$ (cf (26)) we obtain

$$\begin{aligned}
\inf_{C_c^\infty(0,1)} \frac{I_{r-1}[v]}{\int_0^1 \frac{v^2}{t^2} X_1^2 \ldots X_r^2 dt} &\leq \lim_{\epsilon_r \to 0+} \frac{b_{rr}\Gamma_{rr} + O(1)}{\Gamma_{rr}} \\
&= \lim_{\epsilon_r \to 0+} a_{rr} \\
&= \lim_{\epsilon_r \to 0} (c_{0r}^2 + 2c_{00}c_{rr}) \\
&= \frac{1}{4}\sigma_1^2 - \frac{1}{2}\sigma_0\sigma_2 \\
&= B(m).
\end{aligned}$$

This proves part (ii) of the theorem. Part (i) follows from (39) by slightly varying the above argument. //

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