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Homogenization of Maxwell's equations in dissipative bianisotropic media

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SUMMARY

We study the periodic homogenization of Maxwell's equations for dissipative bianisotropic media in the time domain, both in $\mathbb{R}^3$ and in a bounded domain with the perfect conductor boundary condition. We consider both local with respect to time (optical response region) and non-local in time (allowing dispersive effects) constitutive laws; in the non-local case the explicit description of the homogenized coefficients is given in terms of the Laplace transform. The principal result of this work is the description of the asymptotic behaviour of the solutions of the considered problems as the period of the electromagnetic parameters tends to zero. Copyright © 2003 John Wiley & Sons, Ltd.

KEY WORDS: Maxwell's equations; bianisotropic media; homogenization

1. INTRODUCTION

In Mechanics, Physics, Chemistry and Engineering, in the study of composite materials, one is often led to the study of boundary value problems in media with periodic structure. If the period of the structure is small compared to the size of the region in which the system is to be studied, then asymptotic analysis is called for in order to obtain an asymptotic expansion of the solution in terms of a small parameter which is the ratio of the two length-scales. The aim of homogenization theory is to establish the macroscopic behaviour of such a system. This means that the non-homogeneous material is replaced by a homogeneous fictitious one (the 'homogenized' material) whose global characteristics are a good approximation of the initial ones. From the mathematical point of view this signifies mainly that the solutions of a boundary value problem depending on a small parameter, converge to the solution of a limit boundary value problem which is explicitly described. Representative of the mathematical

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work on homogenization are the monographs [1–4]; see also [5] for optimization problems leading to homogenization questions.

The concept of bianisotropic medium, which is actually synonymous to 'general linear medium', was introduced in 1968 by Cheng and Kong. The bianisotropic description of materials has fundamental importance from the point of view of relativity [6]. In recent years, the study of linear complex media (of which bianisotropic media constitute the more general form) is very intensive in the electromagnetic community both at theoretical level and in relation to experimental work related to important new technologies [7,8].

Within the electromagnetic community, homogenization of composites has a huge literature —see [9,10] and references therein—the biggest part of which is devoted to dielectrics. The literature on bianisotropic composites is much less. Among the recent developments are the work on Maxwell Garnett and Bruggeman formalisms for different classes of bianisotropic inclusions (see References [11,12] and the references therein) and the work on the strong property fluctuation theory for bianisotropic composites (see References [11,13] and the references therein).

The mathematical literature on electromagnetics in complex media is not, as yet, very extended. The bigger part deals with the study of time-harmonic waves in chiral media, which leads to frequency domain studies; the references in Reference [14] give a comprehensive account of research activity in this direction. The literature in the time domain is even more restricted; we refer to [15–17] and references therein.

We work in the time domain and consider dissipative bianisotropic media. This is a large class and besides isotropic media it also contains chiral and bi-isotropic media, uniaxial dielectric/magnetic media, uniaxial bianisotropic media, gyrotropic media, biaxial anisotropic media, biaxial bianisotropic media and general anisotropic media. To the best of our knowledge, our results are knew for these special cases too. For the corresponding problem for isotropic media see References [4,18–21] and the references therein.

In Section 2 we formulate the problem to be studied, introduce some notation and state a compensated compactness result which will be used in the sequel. In Section 3 we consider the problem in the optical response region, i.e. assuming local with respect to time constitutive relations, establish its unique solvability and describe the asymptotic behaviour of its solution as the period of the electromagnetic parameters tends to zero. The proof of the main result makes use of an auxiliary elliptic system with the aid of which the homogenized coefficients are expressed. We note that the latter can also be obtained formally if one postulates a double-scale expansion for the periodic problem. Finally, in Section 4 we extend the results of Section 3 to general (not necessarily in the optical response region) bianisotropic media.

2. FORMULATION

Let Ω be a domain in $\mathbb{R}^3$. We consider Maxwell's equations

$$\partial_t \mathbf{D} = \operatorname{curl} \mathbf{H} + \mathbf{F}(\mathbf{x}, t) \tag{1}$$

$$\partial_t \mathbf{B} = -\operatorname{curl} \mathbf{E} + \mathbf{G}(\mathbf{x}, t), \quad \mathbf{x} \in \Omega, \quad t > 0 \tag{2}$$

with initial conditions

$$\mathbf{E}(\mathbf{x},0)=\mathbf{0}, \quad \mathbf{H}(\mathbf{x},0)=\mathbf{0}, \quad \mathbf{x}\in\Omega \tag{3}$$

and the perfect conductor boundary condition

$$\mathbf{n}\times\mathbf{E}=\mathbf{0}, \quad \mathbf{x}\in\partial\Omega,\ t>0 \tag{4}$$

where $\mathbf{n}$ is the outward unit normal on $\partial\Omega$. This boundary condition is, of course, only considered if $\Omega\neq\mathbb{R}^3$, in which case we further assume that the boundary $\partial\Omega$ is C^1.

In this paper, we shall investigate the homogenization of the above system when the material involved is bianisotropic. The constitutive relations for a bianisotropic medium have the following general form [16]:

$$\begin{aligned}\mathbf{D}&=\eta\mathbf{E}+\xi\mathbf{H}+\eta_d*\mathbf{E}+\xi_d*\mathbf{H}\\ \mathbf{B}&=\zeta\mathbf{E}+\mu\mathbf{H}+\zeta_d*\mathbf{E}+\mu_d*\mathbf{H}\end{aligned} \tag{5}$$

where $*$ stands for temporal convolution, i.e. $u*v=\int_{-\infty}^{t}u(t-s)v(s)\,\mathrm{d}s$. The functions η,ξ,ζ and μ take values in the space $M_3(\mathbb{R})$ of 3×3 real matrices and describe the optical (instantaneous) response of the material. The susceptibility functions η_d,ξ_d,ζ_d and μ_d have an additional explicit time dependence and also take values in $M_3(\mathbb{R})$; they vanish for $t<0$ due to causality. The symbols ε and ε_d are usually used instead of η and η_d, but, as is typical in homogenization problems, we reserve the letter ε to stand for a typical length at the microscopic scale. We do not include electric and magnetic current densities in Maxwell's equations, since in view of [22], such terms can be incorporated in the dispersion terms by a suitable gauge transformation.

In what follows we will use boldface capital letters to denote three-vectors and calligraphic capital letters to denote six-vectors.

Using the electromagnetic six-vector field $\mathscr{E}$ and the six-vector flux density $\mathscr{D}$, given, respectively, by

$$\mathscr{E}=\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix}, \quad \mathscr{D}=\begin{pmatrix}\mathbf{D}\\ \mathbf{B}\end{pmatrix}$$

the constitutive relations (5) are written as a single six-vector equation

$$\mathscr{D}=\mathbf{A}\mathscr{E}+\mathbf{K}*\mathscr{E} \tag{6}$$

where

$$\mathbf{A}(\mathbf{x})=\begin{pmatrix}\eta & \xi\\ \zeta & \mu\end{pmatrix}, \quad \mathbf{K}(\mathbf{x},t)=\begin{pmatrix}\eta_d & \xi_d\\ \zeta_d & \mu_d\end{pmatrix}$$

are, respectively, the six-dyadic of the optical response and the susceptibility kernel six-dyadic which models the dispersive effects. It is known [23] that in certain frequency ranges one can ignore the dispersive component and work in the optical response region ($\mathbf{K}=\mathbf{0}$). A study of the error in the optical response approximation for chiral media is performed in Reference [24]. We will first treat the optical response approximation for dissipative media and then the general case.

To complete this section, let us introduce some notation. Given a domain $V \subset \mathbb{R}^3$ we denote by $H(V, \mathrm{div})$ (resp. $H(V, \mathrm{curl})$) the closure of $C_0^\infty(V)$ (infinitely differentiable functions of compact support) in the norm $\{\|\mathbf{u}\|_2^2 + \|\mathrm{div}\,\mathbf{u}\|_2^2\}^{1/2}$ (resp. $\{\|\mathbf{u}\|_2^2 + \|\mathrm{curl}\,\mathbf{u}\|_2^2\}^{1/2}$). We recall the following compensated compactness result of Tartar [1, Chapter 1, Section 11.4]:

Theorem 1
Let $V \subset \mathbb{R}^3$ be bounded and let $(\mathbf{T}^n)$ and $(\mathbf{S}^n)$ be two sequences of vector fields in $H(V, \mathrm{div})$ and $H(V, \mathrm{curl})$, respectively. Suppose that

$$\mathbf{T}^n \to \mathbf{T} \quad \text{weakly in } H(V, \mathrm{div})$$

$$\mathbf{S}^n \to \mathbf{S} \quad \text{weakly in } H(V, \mathrm{curl})$$

Then

$$\mathbf{T}^n \cdot \mathbf{S}^n \to \mathbf{T} \cdot \mathbf{S} \quad \text{in } \mathscr{D}'(V)$$

3. DISSIPATIVE MEDIA IN THE OPTICAL RESPONSE REGION

We consider dissipative bianisotropic media in the optical response region. By Fridén *et al.* [22] the matrix $\mathbf{A}$ in (6) is symmetric and uniformly coercive. Hence, denoting by ξ^{T} the transpose of a matrix ξ, we have

$$\begin{aligned} \mathbf{D} &= \eta \mathbf{E} + \xi \mathbf{H} \\ \mathbf{B} &= \xi^{\mathrm{T}} \mathbf{E} + \mu \mathbf{H} \end{aligned} \tag{7}$$

where η, ξ and μ are 3×3 real matrices with entries in $L^\infty(\Omega)$ and there exists $c > 0$ such that

$$\langle \mathbf{A}(\mathbf{x})\mathscr{U}, \mathscr{U} \rangle \geqslant c\|\mathscr{U}\|^2, \quad \mathbf{x} \in \Omega, \ \ \mathscr{U} \in \mathbb{R}^6 \tag{8}$$

Of course the submatrices η and μ are also symmetric and uniformly coercive. We then have

Theorem 2
Assume that $\mathbf{F}, \mathbf{G} : (0, \infty) \to L^2(\Omega)$ are locally Hölder continuous and that $\int_0^\infty (\|\mathbf{F}\|_2 + \|\mathbf{G}\|_2)\,\mathrm{d}t < +\infty$. Then the Maxwell system (1)–(3) subject to the constitutive relations (7) has a unique solution $(\mathbf{E}, \mathbf{H})$ in $C((0, \infty), L^2(\Omega))$.

Proof
The operator

$$\mathbf{Q} := -i \begin{pmatrix} \mathbf{0} & \mathrm{curl} \\ -\mathrm{curl} & \mathbf{0} \end{pmatrix}$$

with domain $H(\Omega, \mathrm{curl}) \oplus H(\Omega, \mathrm{curl})$ is self-adjoint on $L^2(\Omega)$ [25, Lemma VII 4.4]. Writing

$$\mathscr{E} = \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}, \quad \mathscr{F} = \begin{pmatrix} \mathbf{F} \\ \mathbf{G} \end{pmatrix} \tag{9}$$

the Maxwell system takes the form

$$\mathbf{A}\mathscr{E}' = i\mathbf{Q}\mathscr{E} + \mathscr{F}, \quad \mathscr{E}(0) = \mathbf{0} \tag{10}$$

and has a unique solution in $L^2(\Omega)$ by standard semigroup theory [26, Theorem 1.16]. □

In this section, we will consider a homogenization problem associated to the Maxwell system (2). More precisely for $\varepsilon > 0$ we consider the system:

$$\begin{aligned} \partial_t \mathbf{D}^\varepsilon &= \operatorname{curl} \mathbf{H}^\varepsilon + \mathbf{F}(\mathbf{x}, t) \\ \partial_t \mathbf{B}^\varepsilon &= -\operatorname{curl} \mathbf{E}^\varepsilon + \mathbf{G}(\mathbf{x}, t), \quad \mathbf{x} \in \Omega, \ t > 0 \\ \mathbf{E}^\varepsilon(\mathbf{x}, 0) &= \mathbf{0}, \quad \mathbf{H}^\varepsilon(\mathbf{x}, 0) = \mathbf{0} \\ \mathbf{n} \times \mathbf{E}^\varepsilon &= \mathbf{0}, \quad \mathbf{x} \in \partial\Omega, \ t > 0 \end{aligned} \tag{11}$$

subject to the constitutive laws

$$\begin{aligned} \mathbf{D}^\varepsilon(\mathbf{x}, t) &= \eta^\varepsilon(\mathbf{x})\mathbf{E}^\varepsilon(\mathbf{x}, t) + \xi^\varepsilon(\mathbf{x})\mathbf{H}^\varepsilon(\mathbf{x}, t) \\ \mathbf{B}^\varepsilon(\mathbf{x}, t) &= \xi^{\varepsilon\mathrm{T}}(\mathbf{x})\mathbf{E}^\varepsilon(\mathbf{x}, t) + \mu^\varepsilon(\mathbf{x})\mathbf{H}^\varepsilon(\mathbf{x}, t) \end{aligned} \tag{12}$$

In addition to the assumptions of Section 2, we assume that η^ε, μ^ε and ξ^ε are periodic with period of small scale $\varepsilon > 0$; more precisely we assume that

$$\eta^\varepsilon(\mathbf{x}) = \eta(\mathbf{x}/\varepsilon), \quad \mu^\varepsilon(\mathbf{x}) = \mu(\mathbf{x}/\varepsilon), \quad \xi^\varepsilon(\mathbf{x}) = \xi(\mathbf{x}/\varepsilon)$$

where η, ξ and μ are periodic matrix-valued functions on $\mathbb{R}^3$ of common period Y, say (so Y is a parallelepiped). Our aim is to describe the asymptotic behaviour of the solution $(\mathbf{E}^\varepsilon, \mathbf{H}^\varepsilon)$ of the above system in the limit $\varepsilon \to 0$.

We let $H^1_{\mathrm{per}}(Y)$ denote the closed subspace of $H^1(Y)$ that consists of periodic functions and define the operator $L_{\mathrm{per}} : H^1_{\mathrm{per}}(Y) \to (H^1_{\mathrm{per}}(Y))^*$ by

$$L_{\mathrm{per}} = \begin{pmatrix} -\operatorname{div}(\eta \operatorname{grad}) & -\operatorname{div}(\xi \operatorname{grad}) \\ -\operatorname{div}(\xi^{\mathrm{T}} \operatorname{grad}) & -\operatorname{div}(\mu \operatorname{grad}) \end{pmatrix}$$

The coercivity assumption (8) implies that L_{per} is invertible modulo constants. In particular, we can define (modulo constants) the functions u_1^j, u_2^j, v_1^j and v_2^j, $j = 1, 2, 3$, by the relations

$$L_{\mathrm{per}} \begin{pmatrix} u_1^j \\ u_2^j \end{pmatrix} = \begin{pmatrix} \partial \eta_{ij}/\partial y_i \\ \partial \xi_{ji}/\partial y_i \end{pmatrix}, \quad L_{\mathrm{per}} \begin{pmatrix} v_1^j \\ v_2^j \end{pmatrix} = \begin{pmatrix} \partial \xi_{ij}/\partial y_i \\ \partial \mu_{ij}/\partial y_i \end{pmatrix}$$

We define the *homogenized* constant coefficient matrices η^h, ξ^h and μ^h by

$$\begin{aligned} \eta^h_{ij} &= \langle \eta_{ij} + \eta_{ik}\partial y_k u_1^j + \xi_{ik}\partial y_k u_2^j \rangle \\ \xi^h_{ij} &= \langle \xi_{ij} + \xi_{ik}\partial y_k v_2^j + \eta_{ik}\partial y_k v_1^j \rangle \\ \mu^h_{ij} &= \langle \mu_{ij} + \mu_{ik}\partial y_k v_2^j + \xi_{ki}\partial y_k v_1^j \rangle \end{aligned} \tag{13}$$

where $\langle g \rangle := |Y|^{-1} \int_Y g$. Note that definition (13) is independent of the additive constants modulo which the functions u_1^j, u_2^j, v_1^j and v_2^j, $j = 1, 2, 3$, are defined. It is not obvious but it is easy to prove that the block matrix

$$\mathbf{A}^h = \begin{pmatrix} \eta^h & \xi^h \\ \xi^{h\mathrm{T}} & \mu^h \end{pmatrix}$$

is symmetric and positive definite. We note here that one can also deduce relations (13) formally by postulating a double-scale expansion for $\mathbf{E}^\varepsilon$ and $\mathbf{H}^\varepsilon$. We shall prove the following:

Theorem 3

Let $\mathbf{F}, \mathbf{G}$ satisfy the assumptions of Theorem 2 and let $(\mathbf{E}^\varepsilon, \mathbf{H}^\varepsilon)$ be the solution of the Maxwell system (1)–(4) subject to the constitutive laws (12). Then

$$\mathbf{E}^\varepsilon \to \mathbf{E}^*, \quad \mathbf{H}^\varepsilon \to \mathbf{H}^* \quad * \text{-weakly in } L^\infty((0,\infty), L^2(\Omega))$$

where $(\mathbf{E}^*, \mathbf{H}^*)$ is the unique solution of the Maxwell system

$$\begin{aligned} \partial_t \mathbf{D}^* &= \operatorname{curl} \mathbf{H}^* + \mathbf{F} \\ \partial_t \mathbf{B}^* &= -\operatorname{curl} \mathbf{E}^* + \mathbf{G}, \quad \mathbf{x} \in \Omega, \ t > 0 \\ \mathbf{E}^*(\mathbf{x}, 0) = \mathbf{0}, &\quad \mathbf{H}^*(\mathbf{x}, 0) = \mathbf{0}, \quad \mathbf{x} \in \Omega \\ \mathbf{n} \times \mathbf{E}^* &= \mathbf{0}, \quad \mathbf{x} \in \partial\Omega, \ t > 0 \end{aligned} \tag{14}$$

subject to the homogeneous constitutive laws

$$\begin{aligned} \mathbf{D}^*(\mathbf{x}, t) &= \eta^h \mathbf{E}^*(\mathbf{x}, t) + \xi^h \mathbf{H}^*(\mathbf{x}, t) \\ \mathbf{B}^*(\mathbf{x}, t) &= \xi^{h\mathrm{T}} \mathbf{E}^*(\mathbf{x}, t) + \mu^h \mathbf{H}^*(\mathbf{x}, t) \end{aligned} \tag{15}$$

Proof

We take the inner product of the first and second Maxwell equation (11) with $\mathbf{E}^\varepsilon$ and $\mathbf{H}^\varepsilon$ correspondingly and then add the resulting relations. Using the identity

$$\int_\Omega \operatorname{curl} \mathbf{H}^\varepsilon \cdot \mathbf{E}^\varepsilon = \int_\Omega \operatorname{curl} \mathbf{E}^\varepsilon \cdot \mathbf{H}^\varepsilon + \int_{\partial\Omega} \operatorname{curl} \mathbf{H}^\varepsilon \cdot (\mathbf{E}^\varepsilon \times \mathbf{n})$$

and the boundary condition of (11) we obtain

$$\langle \partial_t \mathbf{D}^\varepsilon, \mathbf{E}^\varepsilon \rangle + \langle \partial_t \mathbf{B}^\varepsilon, \mathbf{H}^\varepsilon \rangle = \langle \mathbf{F}, \mathbf{E}^\varepsilon \rangle + \langle \mathbf{G}, \mathbf{H}^\varepsilon \rangle \tag{16}$$

Using the constitutive laws (12) and recalling the six-vector notation (9) we write (16) as

$$\langle \mathbf{A}^\varepsilon \partial_t \mathscr{E}^\varepsilon, \mathscr{E}^\varepsilon \rangle = \langle \mathscr{F}, \mathscr{E}^\varepsilon \rangle \tag{17}$$

Letting $f(t) = \frac{1}{2} \langle \mathbf{A}^\varepsilon \mathscr{E}^\varepsilon, \mathscr{E}^\varepsilon \rangle$ we have

$$\begin{aligned} f'(t) &= \langle \mathbf{A}^\varepsilon \partial_t \mathscr{E}^\varepsilon, \mathscr{E}^\varepsilon \rangle \\ &= \langle \mathscr{E}^\varepsilon, \mathscr{F} \rangle \\ &\leqslant \|\mathscr{E}^\varepsilon\|_2 \|\mathscr{F}\|_2 \\ &\leqslant c \|\mathscr{F}\|_2 f^{1/2}(t) \end{aligned}$$

where the last inequality follows from (8). Hence, using the fact that $\int \|\mathscr{F}\|_2 \, \mathrm{d}t < \infty$ and using (8) once more, we conclude that there exists $c>0$ such that

$$\|\mathbf{E}^\varepsilon\|_2 \leqslant c, \quad \|\mathbf{H}^\varepsilon\|_2 \leqslant c, \quad \varepsilon>0, \ t>0 \tag{18}$$

The boundedness of $\mathbf{A}^\varepsilon$ together with (18) imply that $\mathbf{D}^\varepsilon$ and $\mathbf{B}^\varepsilon$ are also bounded in $L^2(\Omega)$ uniformly in $\varepsilon, t>0$. It is then standard [2, Theorem 1.26] that there exists $\mathbf{E}^*, \mathbf{H}^*, \mathbf{D}^*, \mathbf{B}^* \in L^\infty((0,\infty), L^2(\Omega))$ such that, up to taking a subsequence $\varepsilon \to 0$, there holds

$$\left.\begin{aligned} &\mathbf{E}^\varepsilon \to \mathbf{E}^*, \quad \mathbf{H}^\varepsilon \to \mathbf{H}^* \\ &\mathbf{D}^\varepsilon \to \mathbf{D}^*, \quad \mathbf{B}^\varepsilon \to \mathbf{B}^* \end{aligned}\right\} * \text{-weakly in } L^\infty((0,\infty), L^2(\Omega)) \tag{19}$$

The ensuing arguments will identify $(\mathbf{E}^*, \mathbf{H}^*)$ and will show that *any* $*$-weakly convergent subsequence of $(\mathbf{E}^\varepsilon, \mathbf{H}^\varepsilon)$ has $(\mathbf{E}^*, \mathbf{H}^*)$ as its limit. This implies the convergence of the full sequence $(\mathbf{E}^\varepsilon, \mathbf{H}^\varepsilon)$; see Reference [2, Theorem 1.26].

Let us take the Laplace transform $g(t) \mapsto \hat{g}(p)$, $p \in \mathbb{C}_+ := \{\mathrm{Re}\, p>0\}$, of Maxwell's equations (11); we obtain

$$\begin{aligned} p\hat{\mathbf{D}}^\varepsilon &= \operatorname{curl} \hat{\mathbf{H}}^\varepsilon + \hat{\mathbf{F}} \\ p\hat{\mathbf{B}}^\varepsilon &= -\operatorname{curl} \hat{\mathbf{E}}^\varepsilon + \hat{\mathbf{G}}, \quad p \in \mathbb{C}_+, \ \mathbf{x} \in \Omega \end{aligned} \tag{20}$$

Moreover (19) implies that

$$\left.\begin{aligned} &\hat{\mathbf{E}}^\varepsilon \to \hat{\mathbf{E}}^* \quad \hat{\mathbf{H}}^\varepsilon \to \hat{\mathbf{H}}^* \\ &\hat{\mathbf{D}}^\varepsilon \to \hat{\mathbf{D}}^* \quad \hat{\mathbf{B}}^\varepsilon \to \hat{\mathbf{B}}^* \end{aligned}\right\} \quad \text{weakly in } L^2(\Omega) \quad (\text{fixed } p \in \mathbb{C}_+.) \tag{21}$$

Combining (20) and (21) implies that for fixed $p \in \mathbb{C}_+$ the vector fields $\operatorname{curl} \hat{\mathbf{E}}^\varepsilon$ and $\operatorname{curl} \hat{\mathbf{H}}^\varepsilon$ have L^2 norms that remain bounded as $\varepsilon \to 0$. Hence they have weak limits in $L^2(\Omega)$. It then follows from (21) that $\hat{\mathbf{E}}^*$ and $\hat{\mathbf{H}}^*$ belong to $H(\Omega, \operatorname{curl})$ and moreover

$$\hat{\mathbf{E}}^\varepsilon \to \hat{\mathbf{E}}^*, \quad \hat{\mathbf{H}}^\varepsilon \to \hat{\mathbf{H}}^* \quad \text{weakly in } H(\Omega, \operatorname{curl}) \tag{22}$$

Letting $\varepsilon \to 0$ in (20) then yields

$$\begin{aligned} p\hat{\mathbf{D}}^* &= \operatorname{curl} \hat{\mathbf{H}}^* + \hat{\mathbf{F}} \\ p\hat{\mathbf{B}}^* &= -\operatorname{curl} \hat{\mathbf{E}}^* + \hat{\mathbf{G}}, \quad p \in \mathbb{C}_+, \ \mathbf{x} \in \Omega \end{aligned} \tag{23}$$

which implies that $\mathbf{E}^*$, $\mathbf{H}^*$, $\mathbf{D}^*$ and $\mathbf{B}^*$ satisfy the Maxwell system:

$$\partial_t \mathbf{D}^* = \operatorname{curl} \mathbf{H}^* + \mathbf{F}$$

$$\partial_t \mathbf{B}^* = -\operatorname{curl} \mathbf{E}^* + \mathbf{G}, \quad \mathbf{x} \in \Omega, \ t>0 \tag{24}$$

$$\mathbf{E}^*(\mathbf{x}, 0) = \mathbf{0}, \quad \mathbf{H}^*(\mathbf{x}, 0) = \mathbf{0}, \quad \mathbf{x} \in \Omega \tag{25}$$

Hence it remains to establish that the boundary condition $\mathbf{n} \times \mathbf{E}^* = \mathbf{0}$ is also satisfied and that the vector fields $\mathbf{E}^*$, $\mathbf{H}^*$, $\mathbf{D}^*$ and $\mathbf{B}^*$ are related by the constitutive laws (15).

Validity of the boundary condition: We first note that the boundary condition is understood in the sense of the trace operator $H(\mathrm{curl},\Omega)\to H^{-1/2}(\partial\Omega)$, $\mathbf{U}\mapsto\mathbf{n}\times\mathbf{U}|_{\partial\Omega}$. Let us fix a function $\phi\in H^{1/2}(\partial\Omega)$. There exists [25, p. 341] $\mathbf{\Phi}\in H^1(\Omega)$ such that $\mathbf{\Phi}|_{\partial\Omega}=\phi$. Now, for $\varepsilon>0$ there holds

$$\int_\Omega \mathrm{curl}\,\mathbf{\Phi}\cdot\mathscr{E}^\varepsilon=\int_\Omega \mathrm{curl}\,\mathbf{E}^\varepsilon\cdot\mathbf{\Phi}+\int_{\partial\Omega}\mathbf{\Phi}(\mathbf{n}\times\mathbf{E}^\varepsilon)$$

$$\int_\Omega \mathrm{curl}\,\mathbf{\Phi}\cdot\mathscr{E}^*=\int_\Omega \mathrm{curl}\,\mathbf{E}^*\cdot\mathbf{\Phi}+\int_{\partial\Omega}\mathbf{\Phi}(\mathbf{n}\times\mathbf{E}^*)$$

Combining these with the fact that $\mathbf{n}\times\mathbf{E}^\varepsilon|_{\partial\Omega}=\mathbf{0}$ and the relations

$$\int_\Omega \mathrm{curl}\,\mathbf{\Phi}\cdot\mathbf{E}^\varepsilon\to\int_\Omega \mathrm{curl}\,\mathbf{\Phi}\cdot\mathbf{E}^*$$

$$\int_\Omega \mathrm{curl}\,\mathbf{E}^\varepsilon\cdot\mathbf{\Phi}\to\int_\Omega \mathrm{curl}\,\mathbf{E}^*\cdot\mathbf{\Phi}\quad(\varepsilon\to 0)$$

we obtain

$$\int_{\partial\Omega}\phi(\mathbf{n}\times\mathbf{E}^*)=\int_{\partial\Omega}\mathbf{\Phi}(\mathbf{n}\times\mathbf{E}^*)=0$$

Since $\phi\in H^{1/2}(\partial\Omega)$ was arbitrary, we conclude that $\mathbf{n}\times\mathbf{E}^*=\mathbf{0}$ on $\partial\Omega$.

Validity of the constitutive laws: Let us fix a bounded domain V with $\bar{V}\subset\Omega$. Since $\mathrm{div}\,\mathrm{curl}=0$, (20) and (23) imply that $\mathrm{div}\,\hat{\mathbf{D}}^\varepsilon=\mathrm{div}\,\hat{\mathbf{D}}^*$ and $\mathrm{div}\,\hat{\mathbf{B}}^\varepsilon=\mathrm{div}\,\hat{\mathbf{B}}^*$, and (21) then yields

$$\hat{\mathbf{D}}^\varepsilon\to\hat{\mathbf{D}}^*,\quad \hat{\mathbf{B}}^\varepsilon\to\hat{\mathbf{B}}^*\quad\text{weakly in } H(V,\mathrm{div})\tag{26}$$

Let L^ε denote the elliptic operator $H_0^1(V)\to H^{-1}(V)$ given in block form by

$$L^\varepsilon=\begin{pmatrix}-\mathrm{div}(\eta^\varepsilon\ \mathrm{grad}) & -\mathrm{div}(\xi^\varepsilon\ \mathrm{grad})\\ -\mathrm{div}(\xi^{\varepsilon\mathrm{T}}\ \mathrm{grad}) & -\mathrm{div}(\mu^\varepsilon\ \mathrm{grad})\end{pmatrix}$$

Then L^ε is invertible for all $\varepsilon>0$. Now, let $g_1,g_2\in H^{-1}(V)$ be fixed and let $u^\varepsilon,v^\varepsilon\in H_0^1(V)$ solve the system

$$L^\varepsilon\begin{pmatrix}u^\varepsilon\\ v^\varepsilon\end{pmatrix}=\begin{pmatrix}g_1\\ g_2\end{pmatrix}$$

Moreover, let $L^h:H_0^1(V)\to H^{-1}(V)$ be the constant coefficient operator

$$L^h=\begin{pmatrix}-\mathrm{div}(\eta^h\,\mathrm{grad}) & -\mathrm{div}(\xi^h\,\mathrm{grad})\\ -\mathrm{div}(\xi^{h\mathrm{T}}\,\mathrm{grad}) & -\mathrm{div}(\mu^h\,\mathrm{grad})\end{pmatrix}$$

By standard homogenization theory, [1], L^h is the limit as $\varepsilon \to 0$ of L^ε in the following sense: if (u, v) is the unique solution of

$$L^h \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$$

then

$$\left.\begin{aligned} \operatorname{grad} u^\varepsilon &\to \operatorname{grad} u, \\ \operatorname{grad} v^\varepsilon &\to \operatorname{grad} v, \end{aligned}\right\} \quad \text{weakly in } L^2(V) \tag{27}$$

and moreover

$$\left.\begin{aligned} \eta^\varepsilon \operatorname{grad} u^\varepsilon + \xi^\varepsilon \operatorname{grad} v^\varepsilon &\to \eta^h \operatorname{grad} u + \xi^h \operatorname{grad} v \\ \xi^{\varepsilon\mathrm{T}} \operatorname{grad} u^\varepsilon + \mu^\varepsilon \operatorname{grad} v^\varepsilon &\to \xi^{h\mathrm{T}} \operatorname{grad} u + \mu^h \operatorname{grad} v \end{aligned}\right\} \quad \text{weakly in } L^2(V) \tag{28}$$

Relations (27) together with the fact that $\operatorname{curl} \operatorname{grad} = 0$ imply that

$$\operatorname{grad} u^\varepsilon \to \operatorname{grad} u, \quad \operatorname{grad} v^\varepsilon \to \operatorname{grad} v \quad \text{weakly in } H(V, \operatorname{curl}) \tag{29}$$

Combining (26) and (29) and applying Theorem 1 we obtain

$$\hat{\mathbf{D}}^\varepsilon \cdot \operatorname{grad} u^\varepsilon \to \hat{\mathbf{D}}^* \cdot \operatorname{grad} u \tag{30}$$

$$\hat{\mathbf{B}}^\varepsilon \cdot \operatorname{grad} v^\varepsilon \to \hat{\mathbf{B}}^* \cdot \operatorname{grad} v \tag{31}$$

in $\mathscr{D}'(V)$. Moreover, we have

$$\begin{aligned} -\operatorname{div}(\eta^\varepsilon \operatorname{grad} u^\varepsilon + \xi^\varepsilon \operatorname{grad} v^\varepsilon) &= g_1 = -\operatorname{div}(\eta^h \operatorname{grad} u + \xi^h \operatorname{grad} v) \\ -\operatorname{div}(\xi^{\varepsilon\mathrm{T}} \operatorname{grad} u^\varepsilon + \mu^\varepsilon \operatorname{grad} v^\varepsilon) &= g_2 = -\operatorname{div}(\xi^{h\mathrm{T}} \operatorname{grad} u + \mu^h \operatorname{grad} v) \end{aligned}$$

and these together with (28) imply

$$\left.\begin{aligned} \eta^\varepsilon \operatorname{grad} u^\varepsilon + \xi^\varepsilon \operatorname{grad} v^\varepsilon &\to \eta^h \operatorname{grad} u + \xi^h \operatorname{grad} v \\ \xi^{\varepsilon\mathrm{T}} \operatorname{grad} u^\varepsilon + \mu^\varepsilon \operatorname{grad} v^\varepsilon &\to \xi^{h\mathrm{T}} \operatorname{grad} u + \mu^h \operatorname{grad} v \end{aligned}\right\} \quad \text{weakly in } H(V, \operatorname{div})$$

Combining these with (22) and applying Theorem 1 we obtain

$$(\eta^\varepsilon \operatorname{grad} u^\varepsilon + \xi^\varepsilon \operatorname{grad} v^\varepsilon) \cdot \hat{\mathbf{E}}^\varepsilon \to (\eta^h \operatorname{grad} u + \xi^h \operatorname{grad} v) \cdot \hat{\mathbf{E}}^* \tag{32}$$

$$(\xi^{\varepsilon\mathrm{T}} \operatorname{grad} u^\varepsilon + \mu^\varepsilon \operatorname{grad} v^\varepsilon) \cdot \hat{\mathbf{H}}^\varepsilon \to (\xi^{h\mathrm{T}} \operatorname{grad} u + \mu^h \operatorname{grad} v) \cdot \hat{\mathbf{H}}^* \tag{33}$$

in $\mathscr{D}'(V)$.

Now we observe that the left-hand side of the sum of (30) and (31) coincides with the left-hand side of the sum of (32) and (33). Hence the corresponding right-hand sides are equal, that is

$$\hat{\mathbf{D}}^* \cdot \operatorname{grad} u + \hat{\mathbf{B}}^* \cdot \operatorname{grad} v = (\eta^h \operatorname{grad} u + \xi^h \operatorname{grad} v) \cdot \hat{\mathbf{E}}^* + (\xi^{h\mathrm{T}} \operatorname{grad} u + \mu^h \operatorname{grad} v) \cdot \hat{\mathbf{H}}^*$$

The fact that g_1 and g_2 were arbitrary, together with the symmetry of η^h and μ^h imply that

$$\hat{\mathbf{D}}^* = \eta^h \hat{\mathbf{E}}^* + \xi^h \hat{\mathbf{H}}^*$$
$$\hat{\mathbf{B}}^* = \xi^{h\mathrm{T}} \hat{\mathbf{E}}^* + \mu^h \hat{\mathbf{H}}^*, \quad \mathbf{x} \in V, \ \ p \in \mathbb{C}_+$$

Since V is arbitrary, we obtain the Laplace transforms of the stated constitutive laws; this completes the proof. □

4. GENERAL BIANISOTROPIC MEDIA

If we observe carefully the proof of Theorem 3 we see that the special form (12) of the constitutive laws was used at two points and in order to guarantee (i) existence and uniqueness for Maxwell's equations; and (ii) the validity of the finite energy condition (18) on the solution $(\mathbf{E}^\varepsilon, \mathbf{H}^\varepsilon)$. If one assumes *a priori* that properties (i) and (ii) are valid then the arguments of the proof go through without essential modifications for the wider class of constitutive laws (5) that take into account dispersive effects.

Consider the initial boundary value problem for Maxwell's equations:

$$\begin{aligned} \partial_t \mathbf{D}^\varepsilon &= \operatorname{curl} \mathbf{H}^\varepsilon + \mathbf{F}(\mathbf{x}, t) \\ \partial_t \mathbf{B}^\varepsilon &= -\operatorname{curl} \mathbf{E}^\varepsilon + \mathbf{G}(\mathbf{x}, t), \quad \mathbf{x} \in \Omega, \ \ t > 0 \\ \mathbf{E}^\varepsilon(\mathbf{x}, 0) = \mathbf{0}, &\quad \mathbf{H}^\varepsilon(\mathbf{x}, 0) = \mathbf{0}, \quad \mathbf{x} \in \Omega \\ \mathbf{n} \times \mathbf{E}^\varepsilon = \mathbf{0}, &\quad \mathbf{x} \in \partial\Omega, \ \ t > 0 \end{aligned} \tag{34}$$

subject to the constitutive laws

$$\begin{aligned} \mathbf{D}^\varepsilon &= \eta^\varepsilon \mathbf{E}^\varepsilon + \xi^\varepsilon \mathbf{H}^\varepsilon + \eta_d^\varepsilon * \mathbf{E}^\varepsilon + \xi_d^\varepsilon * \mathbf{H}^\varepsilon \\ \mathbf{B}^\varepsilon &= \zeta^\varepsilon \mathbf{E}^\varepsilon + \mu^\varepsilon \mathbf{H}^\varepsilon + \zeta_d^\varepsilon * \mathbf{E}^\varepsilon + \mu_d^\varepsilon * \mathbf{H}^\varepsilon \end{aligned} \tag{35}$$

The functions $\eta^\varepsilon(\mathbf{x}), \zeta^\varepsilon(\mathbf{x}), \xi^\varepsilon(\mathbf{x}), \mu^\varepsilon(\mathbf{x})$ as well as the functions $\eta_d^\varepsilon(\mathbf{x}, t)$, $\zeta_d^\varepsilon(\mathbf{x}, t)$, $\xi_d^\varepsilon(\mathbf{x}, t)$, $\mu_d^\varepsilon(\mathbf{x}, t)$ are periodic in $\mathbf{x}$ of period εY. As in Section 3 we denote by $\hat{\alpha}(p)$ the Laplace transform of a function $\alpha(t)$. We assume that there exists $c > 0$ such that the block matrix

$$\begin{pmatrix} \eta + \hat{\eta}_d & \xi + \hat{\xi}_d \\ \zeta + \hat{\zeta}_d & \mu + \hat{\mu}_d \end{pmatrix} =: \mathbf{A}(\mathbf{x}, p) \tag{36}$$

satisfies

$$\langle \mathbf{A}(\mathbf{x}, p)\mathcal{U}, \mathcal{U} \rangle \geqslant c \|\mathcal{U}\|^2, \quad \mathbf{x} \in \Omega, \ \ p \in \mathbb{C}_+, \ \ \mathcal{U} \in \mathbb{R}^6$$

We fix a domain V with $\bar{V} \subset \Omega$ and consider the operator

$$L^\varepsilon = \begin{pmatrix} -\operatorname{div}((\eta + \hat{\eta}_d)\operatorname{grad}) & -\operatorname{div}((\xi + \hat{\xi}_d)\operatorname{grad}) \\ -\operatorname{div}((\zeta + \hat{\zeta}_d)\operatorname{grad}) & -\operatorname{div}((\mu + \hat{\mu}_d)\operatorname{grad}) \end{pmatrix} : H_0^1(V) \to H^{-1}(V)$$

and the corresponding homogenization limit

$$L^h =: \begin{pmatrix} -\operatorname{div}(\tilde{\eta}^h \operatorname{grad}) & -\operatorname{div}(\tilde{\xi}^h \operatorname{grad}) \\ -\operatorname{div}(\tilde{\zeta}^h \operatorname{grad}) & -\operatorname{div}(\tilde{\mu}^h \operatorname{grad}) \end{pmatrix}$$

Note that while the coefficients of L^h are spatially constant, they do depend on $p \in \mathbb{C}_+$. We assume that for fixed $x \in \Omega$ the functions $\tilde{\eta}^h, \tilde{\xi}^h, \tilde{\zeta}^h$ and $\tilde{\mu}^h$ are the Laplace transforms of functions $\eta^h, \xi^h, \zeta^h, \mu^h$ on $(0, \infty)$. We then have

Theorem 4
Assume that the Maxwell system (34) and (35) is uniquely solvable for all $\varepsilon > 0$ and that $\|\mathbf{E}^\varepsilon\|_2, \|\mathbf{H}^\varepsilon\|_2 \leqslant c$ for all $\varepsilon, t > 0$. Then the solution $(\mathbf{E}^\varepsilon, \mathbf{H}^\varepsilon)$ of the above system satisfies

$$\mathbf{E}^\varepsilon \to \mathbf{E}^*, \quad \mathbf{H}^\varepsilon \to \mathbf{H}^* \quad * \text{-weakly in } L^\infty((0,\infty), L^2(\Omega))$$

where $(\mathbf{E}^*, \mathbf{H}^*)$ is the unique solution of the Maxwell system

$$\begin{aligned} \partial_t \mathbf{D}^* &= \operatorname{curl} \mathbf{H}^* + \mathbf{F} \\ \partial_t \mathbf{B}^* &= -\operatorname{curl} \mathbf{E}^* + \mathbf{G}, \quad \mathbf{x} \in \Omega, \ t > 0 \\ \mathbf{E}^*(\mathbf{x}, 0) &= \mathbf{0}, \quad \mathbf{H}^*(\mathbf{x}, 0) = \mathbf{0} \\ \mathbf{n} \times \mathbf{E}^* &= \mathbf{0}, \quad \mathbf{x} \in \partial\Omega, \ t > 0 \end{aligned} \tag{37}$$

subject to the constitutive laws

$$\begin{aligned} \mathbf{D}^* &= \eta^h * \mathbf{E}^* + \xi^h * \mathbf{H}^* \\ \mathbf{B}^* &= \zeta^h * \mathbf{E}^* + \mu^h * \mathbf{H}^* \end{aligned} \tag{38}$$

Proof
Arguing as in the proof of Theorem 3 we first prove that there exist vector fields $\mathbf{E}^*, \mathbf{H}^*, \mathbf{D}^*$ and $\mathbf{B}^*$ that are limits as $\varepsilon \to 0$ of $\mathbf{E}^\varepsilon, \mathbf{H}^\varepsilon, \mathbf{D}^\varepsilon$ and $\mathbf{B}^\varepsilon$ and that $\mathbf{E}^*, \mathbf{H}^*$ satisfy the stated initial condition. It then remains to establish the boundary condition and the constitutive laws (38). For the boundary condition, one works in the space of Laplace transforms and argues as in the proof of Theorem 3.

To prove (38) we take the Laplace transform of (35). Recalling definition (36) we obtain

$$\begin{pmatrix} \hat{\mathbf{D}}^\varepsilon \\ \hat{\mathbf{B}}^\varepsilon \end{pmatrix} = \mathbf{A}(\mathbf{x}, p) \begin{pmatrix} \hat{\mathbf{E}}^\varepsilon \\ \hat{\mathbf{H}}^\varepsilon \end{pmatrix}, \quad \mathbf{x} \in \Omega, \ p \in \mathbb{C}_+$$

The argument of Theorem 3 goes through, $p \in \mathbb{C}_+$ being carried along as a parameter. We conclude that

$$\begin{aligned} \hat{\mathbf{D}}^* &= \tilde{\eta}^h \hat{\mathbf{E}}^* + \tilde{\xi}^h \hat{\mathbf{H}}^* \\ \hat{\mathbf{B}}^* &= \tilde{\zeta}^h \hat{\mathbf{E}}^* + \tilde{\mu}^h \hat{\mathbf{H}}^*, \quad \mathbf{x} \in \Omega, \ p \in \mathbb{C}_+ \end{aligned} \tag{39}$$

from which (38) follows. □

Remark

(1) The above theorem gives the homogenized coefficients as inverse Laplace transforms of certain functions. In concrete cases one can use numerical schemes to obtain precise approximations of η^h, ξ^h, ζ^h, μ^h.
(2) Clearly both Theorems 3 and 4 have, additionally, versions non-global in time, where $(0,+\infty)$ is everywhere replaced by $(0,T)$.
(3) It is clear that the functions $\mathbf{F}$ and $\mathbf{G}$ can also depend on $\varepsilon>0$, provided one makes suitable assumptions on their behaviour as $\varepsilon\to 0$.

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