Series expansion for L^p Hardy inequalities

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Abstract

We consider a general class of sharp L^p Hardy inequalities in $I\!R^N$ involving distance from a surface of general codimension $1 \le k \le N$. We show that we can succesively improve them by adding to the right hand side a lower order term with optimal weight and best constant. This leads to an infinite series improvement of L^p Hardy inequalities.

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1 Introduction

Let Ω be a bounded domain in $I\!R^N$ containing the origin. Hardy inequality asserts that for any $p > 1$

$$\int_\Omega |\nabla u|^p dx \ge \left|\frac{N-p}{p}\right|^p \int_\Omega \frac{|u|^p}{|x|^p} dx, \qquad u \in C_c^\infty(\Omega \setminus \{0\}), \tag{1.1}$$

with $|\frac{N-p}{p}|^p$ being the best constant, see for example [HLP], [OK], [DH]. An analogous result asserts that for a convex domain $\Omega \subset I\!R^N$ with smooth boundary, and $d(x) = {\rm dist}(x, \partial\Omega)$, there holds

$$\int_\Omega |\nabla u|^p dx \ge \Big(\frac{p-1}{p}\Big)^p \int_\Omega \frac{|u|^p}{d^p} dx, \qquad u \in C_c^\infty(\Omega), \tag{1.2}$$

with $(\frac{p-1}{p})^p$ being the best constant, cf [MS], [MMP]. See [OK] for a comprehensive account of Hardy inequalities and [D] for a review of recent results.

In the last few years improved versions of the above inequalities have been obtained, in the sense that nonnegative terms are added in the right hand side of (1.1) or (1.2). Improved Hardy inequalities are useful in the study of critical phenomena in elliptic and parabolic PDE's; see, e.g., [BM, BV, MMP, VZ]. In this work we obtain an infinite series improvement for general Hardy inequalities, that include (1.1) or (1.2) as special cases.

Before stating our main theorems let us first introduce some notation. Let Ω be a domain in $I\!R^N$, $N \geq 2$, and K a piecewise smooth surface of codimension k, $k = 1, \dots, N$. In case $k = N$, we adopt the convention that K is a point, say, the origin. We also set

$$d(x) = \mathrm{dist}(x, K),$$

and we assume that the following inequality holds in the weak sense:

$$p \neq k, \qquad \Delta_p d^{\frac{p-k}{p-1}} \leq 0, \qquad \text{in } \Omega \setminus K. \tag{C}$$

Here Δ_p denotes the usual p-Laplace operator, $\Delta_p w = \mathrm{div}(|\nabla w|^{p-2}\nabla w)$. When $k = N$ (C) is satisfied as equality, since $d^{\frac{p-k}{p-1}} = |x|^{\frac{p-N}{p-1}}$ is the fundamental solution of the p-Laplacian. Also, if $k = 1$, Ω is convex and $K = \partial\Omega$ condition (C) is satisfied. For a detailed analysis of this condition, as well as for examples in the intermediate cases $1 < k < N$, we refer to [BFT].

We next define the function:

$$X_1(t) = (1 - \log t)^{-1}, \quad t \in (0, 1), \tag{1.3}$$

and recursively

$$X_k(t) = X_1(X_{k-1}(t)), \qquad k = 2, 3, \dots;$$

these are iterrated logarithmic functions suitably normalized. We also set

$$I_m[u] := \int_\Omega |\nabla u|^p dx - |H|^p \int_\Omega \frac{|u|^p}{d^p} dx - \frac{p-1}{2p}|H|^{p-2} \sum_{i=1}^m \int_\Omega \frac{|u|^p}{d^p} X_1^2 X_2^2 \dots X_i^2 dx. \tag{1.4}$$

where

$$H = \frac{k-p}{p}.$$

Our main result reads

Theorem A *Let Ω be a domain in $I\!R^N$ and K a piecewise smooth surface of codimension k, $k = 1, \dots, N$. Suppose that $\sup_{x\in\Omega} d(x) < \infty$ and condition (C) is satisfied. Then:*
(1) There exists a positive constant $D_0 = D_0(k, p) \geq \sup_{x\in\Omega} d(x)$ such that for any $D \geq D_0$ and all $u \in W_0^{1,p}(\Omega \setminus K)$ there holds

$$\int_\Omega |\nabla u|^p dx - |H|^p \int_\Omega \frac{|u|^p}{d^p} dx \geq \frac{p-1}{2p}|H|^{p-2} \left(\sum_{i=1}^\infty \int_\Omega \frac{|u|^p}{d^p} X_1^2(d/D) \dots X_i^2(d/D) dx \right). \tag{1.5}$$

If in addition $2 \le p < k$, *then we can take* $D_0 = \sup_{x\in\Omega} d(x)$.
(2) Moreover, for each $m = 1, 2, \ldots$ *the constant* $\frac{p-1}{2p}|H|^{p-2}$ *is the best constant for the corresponding* m*-Improved Hardy inequality, that is,*

$$\frac{p-1}{2p}|H|^{p-2} = \inf_{u\in W_0^{1,p}(\Omega\setminus K)} \frac{I_{m-1}[u]}{\int_\Omega \frac{|u|^p}{d^p} X_1^2 X_2^2 \ldots X_m^2 dx},$$

in either of the following cases: (a) $k = N$ *and* $K = \{0\} \subset \Omega$, *(b)* $k = 1$ *and* $K = \partial\Omega$, *(c)* $2 \le k \le N-1$ *and* $\Omega \cap K \neq \emptyset$.

We also note that the exponent two of the logarithmic corrections in (1.5) are optimal; see Proposition 3.1 for the precise statement.

For $p = 2$, Ω convex and $K = \partial\Omega$ the first term in the infinite series of (1.5) was obtained in [BM]. In the more general framework of Theorem A, the first term in the above series was obtained in [BFT]. On the other hand, when $p = 2$ and $K = \{0\}$ the full series was obtained in [FT] by a different method. For other types of improved Hardy inequalities we refer to [BV, GGM, M, VZ]; in all these works one correction term is added in the right hand side of the plain Hardy inequality.

We next consider the degenerate case $p = k$ for which we do not have the usual Hardy inequality. In [BFT] a substitute for Hardy inequality was given in that case. The analogue of condition (C) is now:

$$p = k, \qquad \Delta_p(-\ln d) \le 0, \qquad \text{in } \Omega\setminus K. \qquad (\mathbf{C'})$$

If (C') is satisfied then for any $D \ge \sup_\Omega d(x)$ there holds (cf [BFT], Theorems 4.2 and 5.4):

$$\int_\Omega |\nabla u|^k dx \ge \left(\frac{k-1}{k}\right)^k \int_\Omega \frac{|u|^k}{d^k} X_1^k(d/D)dx, \quad u \in W_0^{1,p}(\Omega\setminus K), \tag{1.6}$$

with $\left(\frac{k-1}{k}\right)^k$ being the best constant. In our next result we obtain a series improvement for inequality (1.6). We set

$$\tilde{I}_m[u] =: \int_\Omega |\nabla u|^k dx - \left(\frac{k-1}{k}\right)^k \int_\Omega \frac{|u|^k}{d^k} X_1^k(d/D)dx - \\ -\frac{1}{2}\left(\frac{k-1}{k}\right)^{k-1} \sum_{i=2}^m \int_\Omega \frac{|u|^k}{d^k} X_1^k(d/D)X_2^2(d/D)\ldots X_m^2(d/D)dx.$$

We then have

Theorem B *Let* Ω *be a domain in* $I\!R^N$ *and* K *a piecewise smooth surface of codimension* k, $k = 2, \ldots, N$. *Suppose that* $\sup_{x\in\Omega} d(x) < \infty$ *and condition (C') is satisfied. Then,*

(1) for any $D \ge \sup_\Omega d(x)$ *and all* $u \in W_0^{1,k}(\Omega\setminus K)$ *there holds*

$$\int_\Omega |\nabla u|^k dx - \left(\frac{k-1}{k}\right)^k \int_\Omega \frac{|u|^k}{d^k} X_1^k(d/D)dx \ge \\ \frac{1}{2}\left(\frac{k-1}{k}\right)^{k-1} \sum_{i=2}^\infty \int_\Omega \frac{|u|^k}{d^k} X_1^k(d/D)X_2^2(d/D)\ldots X_i^2(d/D)dx. \tag{1.7}$$

(2) Moreover, for each $m = 2, 3, \dots$ the constant $\frac{1}{2}\left(\frac{k-1}{k}\right)^{k-1}$ is the best constant for the corresponding m-Improved inequality. That is

$$\frac{1}{2}\left(\frac{k-1}{k}\right)^{k-1} = \inf_{u \in W_0^{1,p}(\Omega \setminus K)} \frac{\tilde{I}_{m-1}[u]}{\int_\Omega \frac{|u|^k}{d^k} X_1^k X_2^2 \dots X_m^2 dx}.$$

in either of the following cases: (a) $k = N$ and $K = \{0\} \subset \Omega$, (b) $2 \le k \le N-1$ and $\Omega \cap K \neq \emptyset$.

To prove parts (1) of the above Theorems, we make use of suitable vector fields and elementary inequalities; this is carried out in Section 2. To prove the second parts, we use a local argument and appropriate test functions; this is done in Section 3.

2 The series expansion

In this Section we will derive the series improvement that appear in part (1) of Theorems A and B. We shall repeatedly use the differentiation rule

$$\frac{d}{dt} X_i^\beta(t) = \frac{\beta}{t} X_1(t) X_2(t) \dots X_{i-1}(t) X_i^{1+\beta}(t), \qquad i = 1, 2, \dots, \quad \beta \neq -1, \tag{2.1}$$

which is proved by induction: for $i = 1$ (2.1) follows immediately from the definition of $X_1(t)$, cf. (1.3):

$$\frac{d}{dt} X_1^\beta(t) = \frac{\beta}{t}(1 - \log t)^{-\beta-1} = \frac{\beta}{r} X_1^{\beta+1}(t).$$

Moreover assuming (2.1) for a fixed $i \ge 1$ we have

$$\begin{aligned} \frac{d}{dt} X_{i+1}^\beta(t) &= \frac{d}{dt}[X_1^\beta(X_i(t))] \\ &= \frac{\beta}{X_i(t)} X_1^{\beta+1}(X_i(t)) \frac{dX_i(t)}{dt} \\ &= \frac{\beta}{X_i(t)} X_{i+1}^{\beta+1}(t) \frac{1}{t} X_1(t) \dots X_{i-1}(t) X_i^2(t) \\ &= \frac{\beta}{t} X_1(t) \dots X_i(t) X_{i+1}^{\beta+1}(t); \end{aligned}$$

hence (2.1) is proved.

Proof of Theorem A(1): We will make use of a suitable vectot field as in [BFT]. If T is a C^1 vector field in Ω, then, for any $u \in C_c^\infty(\Omega \setminus K)$ we first integrate by parts and then use Hölder's inequality to obtain

$$\begin{aligned} \int_\Omega \operatorname{div} T \; |u|^p dx &= -p \int_\Omega (T \cdot \nabla u)|u|^{p-2} u dx \\ &\le p \left(\int_\Omega |\nabla u|^p dx\right)^{\frac{1}{p}} \left(\int_\Omega |T|^{\frac{p}{p-1}} |u|^p dx\right)^{\frac{p-1}{p}} \\ &\le \int_\Omega |\nabla u|^p dx + (p-1) \int_\Omega |T|^{\frac{p}{p-1}} |u|^p dx. \end{aligned}$$

We therefore arrive at

$$\int_\Omega |\nabla u|^p dx \geq \int_\Omega (\mathrm{div}T - (p-1)|T|^{\frac{p}{p-1}})|u|^p dx. \tag{2.2}$$

For $m \geq 1$ we introduce the notation

$$\eta(t) = \sum_{i=1}^m X_1(t)\dots X_i(t),$$
$$B(t) = \sum_{i=1}^m X_1^2(t)\dots X_i^2(t), \qquad t \in (0,1).$$

In view of (2.2) in order to prove (1.5) it is enough to establish the following pointwise estimate:

$$\mathrm{div}\, T - (p-1)|T|^{\frac{p}{p-1}} \geq \frac{|H|^p}{d^p}\left(1 + \frac{p-1}{2pH^2}B(d(x)/D)\right). \tag{2.3}$$

To proceed we now make a specific choice of T. We take

$$T(x) = H|H|^{p-2}\frac{\nabla d(x)}{d^{p-1}(x)}\left(1 + \frac{p-1}{pH}\eta(d(x)/D) + a\eta^2(d(x)/D)\right).$$

where a is a free parameter to be chosen later. In any case a will be such that the quantity $1 + \frac{p-1}{pH}\eta(d/D) + a\eta^2(d/D)$ is positive on Ω. Note that $T(x)$ is singular at $x \in K$, but since $u \in C_c^\infty(\Omega \setminus K)$ all previous calculations are legitimate.

When computing $\mathrm{div}T$ we need to diffferentiate $\eta(d(x)/D)$. Recalling (2.1) a straightforward calculation gives

$$\eta^{'}(t) = \frac{1}{t}\left(X_1^2 + (X_1^2X_2 + X_1^2X_2^2) + \dots + (X_1^2X_2\dots X_m + \dots + X_1^2\dots X_m^2)\right),$$

from which follows that

$$t\eta^{'}(t) = \frac{1}{2}B(t) + \frac{1}{2}\eta^2(t). \tag{2.4}$$

On the other hand, observing that

$$\Delta_p d^{\frac{p-k}{p-1}} = \frac{p-k}{p-1}\left|\frac{p-k}{p-1}\right|^{p-2} d^{-k}(d\Delta d + (1-k)|\nabla d|^2),$$

condition (C) implies

$$(p-k)(d\Delta d + 1 - k) \leq 0. \tag{2.5}$$

For the sake of simplicity we henceforth omit the argument $d(x)/D$ from $\eta(d(x)/D)$ and $B(d(x)/D)$. Using (2.4) and (2.5) a straightforward calculation shows that

$$\mathrm{div}\, T \geq \frac{|H|^p}{d^p}\left(p + \frac{p-1}{H}\eta + pa\eta^2 + \frac{p-1}{2pH^2}(B+\eta^2) + \frac{a}{H}(B+\eta^2)\eta\right). \tag{2.6}$$

It then follows that (2.3) will be established once we prove the following inequality

$$(p-1) + \frac{p-1}{H}\eta + (pa + \frac{p-1}{2pH^2})\eta^2 + \frac{a}{H}B\eta + \frac{a}{H}\eta^3 - (p-1)\left(1 + \frac{p-1}{pH}\eta + a\eta^2\right)^{\frac{p}{p-1}} \geq 0,$$

for all $x \in \Omega$. We set for convenience

$$\begin{aligned} f(B,\eta) &= (p-1) + \frac{p-1}{H}\eta + (pa + \frac{p-1}{2pH^2})\eta^2 + \frac{a}{H}B\eta + \frac{a}{H}\eta^3, \\ g(\eta) &= \left(1 + \frac{p-1}{pH}\eta + a\eta^2\right)^{\frac{p}{p-1}}, \end{aligned}$$

and the required inequality is written as

$$f(B,\eta) - (p-1)g(\eta) \geq 0. \tag{2.7}$$

When $\eta = \eta(d(x)/D) > 0$ is small, the Taylor expansion of $g(\eta)$ about $\eta = 0$, gives

$$g(\eta) = 1 + \frac{1}{H}\eta + \frac{1}{2}\left(\frac{2ap}{p-1} + \frac{1}{pH^2}\right)\eta^2 + \frac{1}{6}\left(\frac{6a}{(p-1)H} + \frac{2-p}{p^2H^3}\right)\eta^3 + O(\eta^4). \tag{2.8}$$

Let us also note, that in the special case $a = 0$, there holds

$$g(\eta) = 1 + \frac{1}{H}\eta + \frac{1}{2pH^2}\eta^2 + \frac{2-p}{6p^2H^3}\left(1 + \frac{p-1}{pH}\xi_\eta\right)^{\frac{3-2p}{p-1}}\eta^3, \qquad (a=0), \tag{2.9}$$

for some $\xi_\eta \in (0,\eta)$, without any smallness assumption on η.

In view of (2.8), if η is small, inequality (2.7) will be proved once we show:

$$\frac{a}{H} \geq \left(\frac{(2-p)(p-1)}{6p^2H^3} + O(\eta)\right)\frac{\eta^2}{B}. \tag{2.10}$$

From the definition of η and B it follows easily that

$$m \geq \frac{\eta^2}{B} \geq 1. \tag{2.11}$$

We will show that for any choice of H and $p > 1$, there exists an $a \in I\!R$, such that (2.7) holds true. We distinguish various cases:

(a) $H > 0$, $1 < p < 2$. We assume that η is small, which amounts to taking D big. It is enough to show that we can choose a such that (2.10) holds. In view of (2.11) we see that for (2.10) to be valid, it is enough to take a to be big and positive.

(b) $H > 0$, $p \geq 2$. In this case we choose $a = 0$ and we use (2.9). Notice that under our current assumptions on H, p the last term in (2.9) is negative and therefore

$$g(\eta) \leq 1 + \frac{1}{H}\eta + \frac{1}{2pH^2}\eta^2, \qquad (a=0). \tag{2.12}$$

On the other hand

$$f(B,\eta) = (p-1) + \frac{p-1}{H}\eta + \frac{p-1}{2pH^2}\eta^2, \qquad (a=0),$$

and therefore (2.7) is satisfied, without any smallness assumption on η. In particular, we can take $D_0 = \sup_{x\in\Omega} d(x)$ in this case.

(c) $H < 0$, $1 < p < 2$. We assume that η is small. In this case, the right hand side of (2.10) is negative. Hence, we can choose $a = 0$ and (2.10) holds true.

(d) $H<0$, $p\geq 2$. Arguing as in case (a) we take a to be big and negative, and (2.10) holds true. //

We next consider the degenerate case $p=k$.

Proof of Theorem B(1): We assume that $p=k\geq 2$ and that condition (C') is satisfied. The proof is quite similar to the previous one.

An easy calculation shows that condition (C') implies that

$$d\Delta d+1-k\geq 0. \tag{2.13}$$

We now choose the vector field

$$T(x)=\Big(\frac{k-1}{k}\Big)^{k-1}\frac{\nabla d}{d^{k-1}}\Big(X_1^{k-1}+\sum_{i=2}^{m}X_1^{k-1}X_2\dots X_i\Big). \tag{2.14}$$

where, here and below, $X_j=X_j(d(x)/D)$. Taking into account (2.13) a straightforward calculation yields that

$$\begin{aligned}\mathrm{div}T-(p-1)|T|^{\frac{p}{p-1}}\geq\ &\frac{(k-1)^k}{k^{k-1}}\frac{X_1^k}{d^k}\Bigg(1+\sum_{i=2}^{m}X_2\dots X_i\\&+\frac{1}{k-1}\sum_{i=2}^{m}\sum_{j=2}^{i}X_2^2\dots X_j^2X_{j+1}\dots X_i-\frac{k-1}{k}\Big(1+\sum_{i=2}^{m}X_2\dots X_i\Big)^{\frac{k}{k-1}}\Bigg).\end{aligned} \tag{2.15}$$

To estimate the last term in the right hand side of (2.15) we use Taylor's expansion to obtain the inequality

$$\Big(1+\sum_{i=2}^{m}X_2\dots X_i\Big)^{\frac{k}{k-1}}\leq 1+\frac{k}{k-1}\sum_{i=2}^{m}X_2\dots X_i+\frac{k}{2(k-1)^2}\Big(\sum_{i=2}^{m}X_2\dots X_i\Big)^2.$$

It then follows that

$$\begin{aligned}&\mathrm{div}T-(p-1)|T|^{\frac{p}{p-1}}\geq\\&\frac{(k-1)^{k-1}}{k^{k-1}}\frac{X_1^k}{d^k}\left(\frac{k-1}{k}+\sum_{i=2}^{m}\sum_{j=2}^{i}X_2^2\dots X_j^2X_{j+1}\dots X_i-\frac{1}{2}\left(\sum_{i=2}^{m}X_2\dots X_i\right)^2\right).\end{aligned} \tag{2.16}$$

Expanding the square in the last term in (2.16) we conclude that

$$\mathrm{div}T-(p-1)|T|^{\frac{p}{p-1}}\geq\left(\frac{k-1}{k}\right)^{k-1}\frac{X_1^k}{d^k}\left(\frac{k-1}{k}+\frac{1}{2}\sum_{i=2}^{m}X_2^2\dots X_i^2\right),$$

and the result follows. //

3 Best constants

In this section we are going to prove the optimality of the Improved Hardy Inequality of Section 2. More precisely, for any $m\geq 1$ let us recall that

$$\begin{aligned}I_m[u]\ =\ &\int_\Omega|\nabla u|^pdx-|H|^p\int_\Omega\frac{|u|^p}{d^p}dx-\\&-\frac{p-1}{2p}|H|^{p-2}\int_\Omega\frac{|u|^p}{d^p}\left(X_1^2+X_1^2X_2^2+\dots+X_1^2\dots X_m^2\right)dx,\end{aligned}$$

where $X_i=X_i(d(x)/D)$. We have the following

Proposition 3.1 *Let Ω be a domain in $\mathbb{R}^N$.* (i) *If $2 \le k \le N-1$ then we take K to be a piecewise smooth surface of codimension k and assume $K \cap \Omega \neq \emptyset$;* (ii) *if $k = N$ then we take $K = \{0\} \subset \Omega$;* (iii) *if $k = 1$ then we assume $K = \partial\Omega$. Let $D \ge \sup_\Omega d(x)$ be fixed and suppose that for some constants $B > 0$ and $\gamma \in \mathbb{R}$ the following inequality holds true for all $u \in W_0^{1,p}(\Omega \setminus K)$*

$$I_{m-1}[u] \ge B \int_\Omega \frac{|u|^p}{d^p} X_1^2(d/D) \dots X_{m-1}^2(d/D) X_m^\gamma(d/D) dx. \tag{3.1}$$

Then

(i) $\gamma \ge 2$

(ii) *If $\gamma = 2$ then $B \le \frac{p-1}{2p}|H|^{p-2}$.*

Proof. All our analysis will be local, say, in a fixed ball of radius δ (denoted by B_δ) centered at the origin, for some fixed small δ. The proof we present works for any $k = 1, 2, \dots, N$. We note however that for $k = N$ (distance from a point) the subsequent calculations are substantially simplified, whereas for $k = 1$ (distance from the boundary) one should replace B_δ by $B_\delta \cap \Omega$. This last change entails some minor modifications, the arguments otherwise being the same. Without any loss of generality we may assume that $0 \in K \cap \Omega$ ($k \neq 1$), or $0 \in \partial\Omega$ if $k = 1$. We divide the proof into several steps.

Step 1. Let $\phi \in C_c^\infty(B_\delta)$ be such that $0 \le \phi \le 1$ in B_δ and $\phi = 1$ in $B_{\delta/2}$. We fix small parameters $\alpha_0, \alpha_1, \dots, \alpha_m > 0$ and define the functions

$$w(x) = d^{-H+\frac{\alpha_0}{p}} X_1^{\frac{-1+\alpha_1}{p}}(d/D) \dots X_m^{\frac{-1+\alpha_m}{p}}(d/D)$$

and

$$u(x) = \phi(x) w(x).$$

It is an immediate consequence of (3.17) below that $u \in W^{1,p}(\Omega)$. Moreover, if $k < p$ then $H < 0$ and therefore $u|_K = 0$. On the other hand, if $k > p$ then a standard approximation argument – using cut-off functions – shows that $W_0^{1,p}(\Omega \setminus K) = W^{1,p}(\Omega \setminus K)$. Hence $u \in W_0^{1,p}(\Omega \setminus K)$. To prove the proposition we shall estimate the corresponding Rayleigh quotient of u in the limit $\alpha_0 \to 0$, $\alpha_1 \to 0$, …, $\alpha_m \to 0$ in this order.

It is easily seen that

$$\nabla w = -d^{\frac{-k+\alpha_0}{p}} X_1^{\frac{-1+\alpha_1}{p}} \dots X_m^{\frac{-1+\alpha_m}{p}} \Big(H + \frac{\zeta(x)}{p}\Big) \nabla d. \tag{3.2}$$

where

$$\zeta(x) = -\alpha_0 + (1-\alpha_1)X_1 + \dots + (1-\alpha_m)X_1 X_2 \dots X_m, \tag{3.3}$$

where, here and below, we omit the argument $d(x)/D$ from $X_i(d/D)$. Since δ is small the X_i's are also small. Hence $\zeta(x)$ can be thought as a small parameter in the rest of the proof.

Now $\nabla u = \phi \nabla w + \nabla\phi\, w$ and hence, using the elementary inequality

$$|a+b|^p \le |a|^p + c_p(|a|^{p-1}|b| + |b|^p), \quad a, b \in \mathbb{R}^N, \qquad p > 1, \tag{3.4}$$

we obtain

$$\begin{aligned}\int_\Omega |\nabla u|^p dx &\le \int_{B_\delta} \phi^p |\nabla w|^p dx + c_p \int_{B_\delta} |\nabla \phi||\phi|^{p-1}|\nabla w|^{p-1}|w|\, dx + \\ &\quad + c_p \int_{B_\delta} |\nabla \phi|^p |w|^p\, dx \\ &=: I_1 + I_2 + I_3. \end{aligned} \tag{3.5}$$

We claim that

$$I_2,\ I_3 = O(1) \quad \text{uniformly as } \alpha_0, \alpha_1, \ldots, \alpha_m \text{ tend to zero.} \tag{3.6}$$

Let us give the proof for I_2. Using the definition of $w(x)$ and the regularity of ϕ we obtain

$$I_2 \le c \int_{B_\delta} d^{1-k+\alpha_0} X_1^{-1+\alpha_1} \ldots X_m^{-1+\alpha_m} \Big| H + \frac{\zeta(x)}{p} \Big|^{p-1} dx.$$

The appearance of d^{-k+1} together with the fact that ζ is small compared to H implies that I_2 is uniformly bounded (see step 2). The integral I_3 is treated similarly.

Step 2. We shall repeatedly deal with integrals of the form

$$Q = \int_\Omega \phi^p d^{-k+\beta_0} X_1^{1+\beta_1}(d/D) \ldots X_m^{1+\beta_m}(d/D) dx, \quad \beta_i \in I\!R, \tag{3.7}$$

we therefore provide precise conditions under which $Q < \infty$. From our assumptions on ϕ we have

$$\int_{B_{\delta/2}} d^{-k+\beta_0} X_1^{1+\beta_1} \ldots X_m^{1+\beta_m} dx \le Q \le \int_{B_\delta} d^{-k+\beta_0} X_1^{1+\beta_1} \ldots X_m^{1+\beta_m} dx.$$

Using the coarea formula and the fact that

$$c_1 r^{k-1} \le \int_{\{d=r\}\cap B_\delta} dS < c_2 r^{k-1}$$

we conclude that

$$c_1 \int_0^{\delta/2} r^{-1+\beta_0} X_1^{1+\beta_1} \ldots X_m^{1+\beta_m} dr \le Q \le c_2 \int_0^{\delta} r^{-1+\beta_0} X_1^{1+\beta_1} \ldots X_m^{1+\beta_m} dr.$$

where $X_i = X_i(r/D)$. Hence, recalling (2.1) we conclude that

$$Q < \infty \Longleftrightarrow \left\{ \begin{array}{ll} & \beta_0 > 0 \\ \text{or} & \beta_0 = 0 \text{ and } \beta_1 > 0 \\ \text{or} & \beta_0 = \beta_1 = 0 \text{ and } \beta_2 > 0 \\ & \cdots \\ \text{or} & \beta_0 = \beta_1 = \ldots = \beta_{m-1} = 0 \text{ and } \beta_m > 0. \end{array} \right. \tag{3.8}$$

Step 3. We introduce some auxiliary quantities and prove some simple relations about them. For $0 \le i \le j \le m$ we define

$$\begin{aligned} A_0 &= \int_\Omega \phi^p d^{-k+\alpha_0} X_1^{-1+\alpha_1} \ldots X_m^{-1+\alpha_m} dx \\ A_i &= \int_\Omega \phi^p d^{-k+\alpha_0} X_1^{1+\alpha_1} \ldots X_i^{1+\alpha_i} X_{i+1}^{-1+\alpha_{i+1}} \ldots X_m^{-1+\alpha_m} dx \\ \Gamma_{0j} &= \int_\Omega \phi^p d^{-k+\alpha_0} X_1^{\alpha_1} \ldots X_i^{\alpha_i} X_{i+1}^{-1+\alpha_{i+1}} \ldots X_m^{-1+\alpha_m} dx \\ \Gamma_{ij} &= \int_\Omega \phi^p d^{-k+\alpha_0} X_1^{1+\alpha_1} \ldots X_i^{1+\alpha_i} X_{i+1}^{\alpha_{i+1}} \ldots X_j^{\alpha_j} X_{j+1}^{-1+\alpha_{j+1}} \ldots X_m^{-1+\alpha_m} dx, \end{aligned}$$

with $\Gamma_{ii} = A_i$. We have the following

Two identities: Let $0 \le i \le m-1$ be given and assume that $\alpha_0 = \alpha_1 = \ldots = \alpha_{i-1} = 0$. Then

$$\alpha_i A_i = \sum_{j=i+1}^{m} (1-\alpha_j)\Gamma_{ij} + O(1) \tag{3.9}$$

$$\alpha_i \Gamma_{ij} = -\sum_{k=i+1}^{j} \alpha_k \Gamma_{kj} + \sum_{k=j+1}^{m} (1-\alpha_k)\Gamma_{jk} + O(1) \tag{3.10}$$

where the $O(1)$ is uniform as the α_i's tend to zero. Let us give the proof for (3.9). We assume that $i > 0$, the case $i = 0$ being a straight-forward adaptation. A direct computation gives

$$\alpha_i d^{-k} X_1 \ldots X_{i-1} X_i^{1+\alpha_i} = \mathrm{div}(d^{-k+1} X_i^{\alpha_i} \nabla d) - d^{-k}(d\Delta d + 1 - k) X_i^{\alpha_i}. \tag{3.11}$$

hence

$$\begin{aligned}
\alpha_i A_i &= \int_\Omega \phi^p \mathrm{div}(d^{-k+1} X_i^{\alpha_i} \nabla d) X_{i+1}^{-1+\alpha_{i+1}} \ldots X_m^{-1+\alpha_m} dx - \\
&\quad - \int_\Omega \phi^p d^{-k}(d\Delta d + 1 - k) X_i^{\alpha_i} X_{i+1}^{-1+\alpha_{i+1}} \ldots X_m^{-1+\alpha_m} dx \\
&=: E_1 - E_2.
\end{aligned}$$

It is a direct consequence of [AS, Theorem 3.2] that

$$d\Delta d + 1 - k = O(d), \quad \text{as } d \to 0, \tag{3.12}$$

hence E_2 is estimated by a constant times $\int_\Omega \phi^p d^{-k+1} X_i^{\alpha_i} X_{i+1}^{-1+\alpha_{i+1}} \ldots X_m^{-1+\alpha_m} dx$ and therefore is bounded uniformly in $\alpha_0, \alpha_1, \ldots, \alpha_m$. To handle E_1 we integrate by parts obtaining

$$\begin{aligned}
E_1 &= -\int_\Omega \nabla\phi^p \cdot \nabla d\, d^{-k+1} X_i^{\alpha_i} X_{i+1}^{-1+\alpha_{i+1}} \ldots X_m^{-1+\alpha_m} dx \\
&\quad - \int_\Omega \phi^p d^{-k+1} X_i^{\alpha_i} \nabla d \cdot \nabla\Big(X_1^{-1+\alpha_1} \ldots X_m^{-1+\alpha_m} \Big) dx
\end{aligned}$$

The first integral is of order $O(1)$ (similarly to I_2, I_3 above) while the second is equal to $\sum_{j=i+1}^m (1-\alpha_j)\Gamma_{ij}$. Hence (3.9) has been proved. To prove (3.10) we use (3.11) once more and proceed similarly; we omit the details.

Step 4. We proceed to estimate I_1. It follows from (3.2) that

$$I_1 = \int_\Omega \phi^p d^{-k+\alpha_0} X_1^{-1+\alpha_1} \ldots X_m^{-1+\alpha_m} \Big| H + \frac{\zeta}{p} \Big|^p dx.$$

Since ζ is small compared to H we may use Taylor's expansion to obtain

$$\Big| H + \frac{\zeta}{p} \Big|^p \le |H|^p + |H|^{p-2} H\zeta + \frac{p-1}{2p} |H|^{p-2}\zeta^2 + c|\zeta|^3. \tag{3.13}$$

Using this inequality we can bound I_1 by

$$I_1 \le I_{10} + I_{11} + I_{12} + I_{13}, \tag{3.14}$$

where

$$I_{10} = |H|^p \int_{B_\delta} \phi^p d^{-k+\alpha_0} X_1^{-1+\alpha_1} \dots X_m^{-1+\alpha_m} dx = |H|^p \int_\Omega \frac{|u|^p}{d^p} dx, \qquad (3.15)$$

$$I_{11} = |H|^{p-2} H \int_{B_\delta} \phi^p d^{-k+\alpha_0} X_1^{-1+\alpha_1} \dots X_m^{-1+\alpha_m} \zeta(x) dx,$$

$$I_{12} = \frac{p-1}{2p} |H|^{p-2} \int_{B_\delta} \phi^p d^{-k+\alpha_0} X_1^{-1+\alpha_1} \dots X_m^{-1+\alpha_m} \zeta^2(x) dx,$$

$$I_{13} = c \int_{B_\delta} \phi^p d^{-k+\alpha_0} X_1^{-1+\alpha_1} \dots X_m^{-1+\alpha_m} |\zeta(x)|^3 dx.$$

Step 5. We shall prove that

$$I_{11},\ I_{13} = O(1) \quad \text{uniformly in } \alpha_0, \alpha_1, \dots, \alpha_m. \qquad (3.16)$$

Indeed, substituting for ζ in I_{11} we see by a direct application of (3.9) (for $i = 0$) that $I_{11} = O(1)$. To estimate I_{13} we observe that $X_1 \dots X_i \le cX_1$ for some $c > 0$ and thus obtain

$$I_{13} \le c_1 \alpha_0^3 \int_\Omega \phi^p d^{-k+\alpha_0} X_1^{-1+\alpha_1} \dots X_m^{-1+\alpha_m} dx + $$
$$+ c_2 \int_\Omega \phi^p d^{-k+\alpha_0} X_1^{2+\alpha_1} X_2^{-1+\alpha_2} \dots X_m^{-1+\alpha_m} dx.$$

The second integral is bounded uniformly in the α_i's due to the factor X_1^2. Moreover, using the fact $0 \le \phi \le 1$ and $\int_{\{d=r\} \cap B_\delta} dS < cr^{k-1}$ we obtain

$$\alpha_0^3 \int_\Omega \phi^p d^{-k+\alpha_0} X_1^{-1+\alpha_1} \dots X_m^{-1+\alpha_m} dx$$
$$\le c\alpha_0^3 \int_0^\delta r^{-1+\alpha_0} X_1^{-1+\alpha_1}(r/D) \dots X_m^{-1+\alpha_m}(r/D) dr$$
$$\le c\alpha_0^3 \int_0^\delta r^{-1+\alpha_0} X_1^{-2}(r/D) dr$$
$$(r = Ds^{1/\alpha_0}) \qquad = cD^{\alpha_0} \alpha_0^2 \int_0^{(\delta/D)^{\alpha_0}} \Big(1 - \frac{1}{\alpha_0} \log s\Big)^2 ds$$
$$= O(1)$$

as $\alpha_0 \to 0$, uniformly in $\alpha_1, \dots \alpha_m$. Hence (3.16) has been proved. Combining (3.5), (3.6), (3.14), (3.15) and (3.16) we conclude that

$$\int_\Omega |\nabla u|^p dx - |H|^p \int_\Omega \frac{|u|^p}{d^p} dx \le I_{12} + O(1), \qquad (3.17)$$

uniformly in the α_i's.

Step 6. Recalling the definition of $I_{m-1}[\cdot]$ we obtain from (3.17)

$$I_{m-1}[u] \le \frac{p-1}{2p} |H|^{p-2} \int_\Omega \phi^p d^{-k+\alpha_0} X_1^{-1+\alpha_1} \dots X_m^{-1+\alpha_m} \times$$
$$\times \Big(\zeta^2(x) - \sum_{i=1}^{m-1} X_1^2 \dots X_i^2\Big) dx + O(1)$$
$$=: \frac{p-1}{2p} |H|^{p-2} J + O(1). \qquad (3.18)$$

Expanding $\zeta^2(x)$ (cf (3.3)) and collecting similar terms we obtain

$$\begin{aligned} J &= \int_\Omega \phi^p d^{-k+\alpha_0} X_1^{-1+\alpha_1}\dots X_m^{-1+\alpha_m}\Big\{\alpha_0^2+\sum_{i=1}^m (1-\alpha_i)^2 X_1^2\dots X_i^2 - \\ &\qquad -\sum_{i=1}^{m-1} X_1^2\dots X_i^2 - 2\alpha_0\sum_{j=1}^m (1-\alpha_j)X_1\dots X_j + \\ &\qquad +2\sum_{i=1}^{m-1}\sum_{j=i+1}^m (1-\alpha_i)(1-\alpha_j)X_1^2\dots X_i^2 X_{i+1}\dots X_j\Big\}\,dx \\ &= \alpha_0^2 A_0 + A_m + \sum_{i=1}^m(\alpha_i^2-2\alpha_i)A_i - 2\alpha_0\sum_{j=1}^m(1-\alpha_j)\Gamma_{0j} + \\ &\qquad + \sum_{i=1}^{m-1}\sum_{j=i+1}^m 2(1-\alpha_i)(1-\alpha_j)\Gamma_{ij}. \end{aligned} \tag{3.19}$$

Step 7. We intend to take the limit $\alpha_0 \to 0$ in (3.19). All terms have finite limits except those containing A_0 and Γ_{0j} which, when viewed separately, diverge. When combined however, they give

$$\begin{aligned} & \alpha_0^2 A_0 - 2\alpha_0 \sum_{j=1}^m (1-\alpha_j)\Gamma_{0j} \\ \text{(by (3.9))} \quad &= -\alpha_0\sum_{j=1}^m(1-\alpha_j)\Gamma_{0j} + O(1) \\ \text{(by (3.10))} \quad &= -\sum_{j=1}^m (1-\alpha_j)\Big(-\sum_{i=1}^j \alpha_i\Gamma_{ij} + \sum_{i=j+1}^m (1-\alpha_i)\Gamma_{ji}\Big) + O(1) \\ &= \sum_{i=1}^m(\alpha_i-\alpha_i^2)A_i + \sum_{i=1}^{m-1}\sum_{j=i+1}^m (2\alpha_i-1)(1-\alpha_j)\Gamma_{ij} + O(1). \end{aligned}$$

All the terms in the last expression remain bounded as $\alpha_0 \to 0$; hence taking the limit in (3.19) we obtain

$$J = A_m - \sum_{i=1}^m \alpha_i A_i + \sum_{i=1}^{m-1}\sum_{j=i+1}^m (1-\alpha_j)\Gamma_{ij} + O(1) \qquad (\alpha_0 = 0) \tag{3.20}$$

where the $O(1)$ is uniform with respect to $\alpha_1, \dots, \alpha_m$.

Step 8. We next let $\alpha_1 \to 0$ in (3.20). All terms have finite limits except those involving A_1 and Γ_{1j} which diverge. Using (3.9) once more – this time for $i = 1$ – we see that when combined these terms stay bounded in the limit $\alpha_1 \to 0$. Hence

$$J = A_m - \sum_{i=2}^m \alpha_i A_i + \sum_{i=2}^{m-1}\sum_{j=i+1}^m (1-\alpha_j)\Gamma_{ij} + O(1) \qquad (\alpha_0 = \alpha_1 = 0) \tag{3.21}$$

We proceed in this way and after letting $\alpha_{m-1} \to 0$ we are left with

$$J = (1-\alpha_m)A_m + O(1), \qquad (\alpha_0 = \alpha_1 = \dots \alpha_{m-1} = 0), \tag{3.22}$$

uniformly in α_m.

Combining (3.1), (3.18) and (3.22) we conclude that

$$B \leq \frac{p-1}{2p}|H|^{p-2}\frac{(1-\alpha_m)A_m + O(1)}{\int_\Omega \phi^p d^{-k}X_1 \dots X_{m-1}X_m^{\gamma-1+\alpha_m}dx}. \qquad (3.23)$$

Suppose now that $\gamma < 2$. Then letting $\alpha_m \to 2-\gamma > 0$ we observe that the denominator in (3.23) tends to infinity while the numerator stays bounded. This implies $B = 0$ proving part (i) of the Proposition.

Now, if $\gamma = 2$ then the denominator in (3.23) is equal to A_m. Hence letting $\alpha_m \to 0$ we have $A_m \to \infty$ (by (3.8)) and hence $B \leq \frac{p-1}{2p}|H|^{p-2}$. This concludes the proof. //

We next consider the degenerate case $p = k$. We have the following

Proposition 3.2 *Let Ω be a domain in $I\!R^N$.* (i) *If $2 \leq k \leq N-1$ then we take K to be a piecewise smooth surface of codimension k and assume $K \cap \Omega \neq \emptyset$;* (ii) *if $k = N$ then we take $K = \{0\} \subset \Omega$. Let $D \geq \sup_{x\in\Omega} d(x)$ be fixed and suppose that for some constants $B > 0$ and $\gamma \in I\!R$ the following inequality holds true for all $u \in C_c^\infty(\Omega \setminus K)$*

$$\tilde{I}_{m-1}[u] \geq B\int_\Omega \frac{|u|^k}{d^k}X_1^k(d/D)X_2^2(d/D)\dots X_m^\gamma(d/D)dx. \qquad (3.24)$$

Then:

(i) $\gamma \geq 2$

(ii) *If $\gamma = 2$ then $B \leq \frac{1}{2}(\frac{k-1}{k})^{k-1}$.*

Proof. The proof is similar to that of Proposition 3.1. Without any loss of generality we assume that $0 \in K \cap \Omega$. As in the previous theorem we let ϕ be a non-negative, smooth cut-off function supported in $B_\delta = \{|x| < \delta\}$, equal to one on $B_{\delta/2}$ and taking values in $[0, 1]$.

Given small parameters $\alpha_1, \dots, \alpha_m > 0$ we define

$$w(x) = X_1^{\frac{-k+1+\alpha_1}{k}}(d/D)X_2^{\frac{-1+\alpha_2}{k}}(d/D)\dots X_m^{\frac{-1+\alpha_m}{k}}(d/D).$$

and

$$u(x) = \phi(x)w(x).$$

Subsequent calculations will establish that $u \in W^{1,k}(\Omega)$ (see (3.29)). We will prove that $u \in W_0^{1,k}(\Omega \setminus K)$ by showing that

$$d^{\frac{\alpha_0}{k}}u \to u \quad \text{in } W^{1,k}(\Omega) \text{ as } \alpha_0 \to 0. \qquad (3.25)$$

We have

$$\int_\Omega |\nabla(d^{\frac{\alpha_0}{k}}u) - \nabla u|^k dx \leq c\alpha_0^k \int_\Omega d^{-k+\alpha_0}u^k dx + \int_\Omega |d^{\frac{\alpha_0}{k}} - 1|^k|\nabla u|^k dx. \qquad (3.26)$$

The second term in the right hand side of (3.26) tends to zero as $\alpha_0 \to 0$ by the dominated convergence theorem. Moreover, there exists a constant c_{α_1} such that

$X_1^{\alpha_1/2}X_2^{-1}\dots X_m^{-1} \le c_{\alpha_1}$. Hence the first term in the right hand side of (3.26) is estimated by

$$c_{\alpha_1}\alpha_0^k\int_\Omega d^{-k+\alpha_0}X_1^{-k+1+\frac{\alpha_1}{2}}\,dx.$$

A direct application of [BFT, Lemma 5.2] shows that this tends to zero as $\alpha_0\to 0$. Hence $u\in W_0^{1,p}(\Omega\setminus K)$.

To proceed we use (3.4) obtaining

$$\begin{aligned}\int_\Omega|\nabla u|^k dx &\le \int_\Omega \phi^k|\nabla w|^k dx + c_k\int_\Omega|\nabla\phi||\phi|^{k-1}|\nabla w|^{k-1}|w|\,dx+\\ &\quad +c_k\int_\Omega|\nabla\phi|^k|w|^k\,dx\\ &=: I_1+I_2+I_3. \end{aligned}\tag{3.27}$$

Arguing as in the proof of the previous proposition (cf step 1) we see that I_2 and I_3 are bounded uniformly with respect to the α_i's. Hence

$$\int_\Omega|\nabla u|^k dx\le\int_\Omega\phi^k|\nabla w|^k dx+O(1)\tag{3.28}$$

uniformly as $\alpha_1,\dots,\alpha_m\to 0$.

Now, a direct computation yields

$$\nabla w=-d^{-1}X_1^{\frac{1+\alpha_1}{k}}X_2^{\frac{-1+\alpha_2}{k}}\dots X_m^{\frac{-1+\alpha_m}{k}}\left(\frac{k-1}{k}+\frac{\zeta(x)}{k}\right)\nabla d$$

where

$$\zeta(x)=-\alpha_1+\sum_{i=2}^m(1-\alpha_i)X_2(d/D)\dots X_i(d/D).$$

From (3.28) and recalling (3.13) we have

$$\begin{aligned}\int_\Omega|\nabla u|^k dx\le\int_\Omega &\phi^k d^{-k}X_1^{1+\alpha_1}X_2^{-1+\alpha_2}\dots X_m^{-1+\alpha_m}\times\\ &\times\left\{\left(\frac{k-1}{k}\right)^k+\left(\frac{k-1}{k}\right)^{k-1}\zeta+\frac12\left(\frac{k-1}{k}\right)^{k-1}\zeta^2+c|\zeta|^3\right\}dx.\end{aligned}\tag{3.29}$$

The term containing $|\zeta|^3$ is bounded uniformly with respect to $\alpha_1,\dots,\alpha_m$ (cf Step 4 in the previous proposition). Moreover it is immediately seen that

$$\phi^k d^{-k}X_1^{1+\alpha_1}X_2^{-1+\alpha_2}\dots X_m^{-1+\alpha_m}=\frac{|u|^k}{d^k}X_1^k,\tag{3.30}$$

Hence

$$\begin{aligned}\tilde I_{m-1}[u] &\le \int_\Omega\phi^k d^{-k}X_1^{1+\alpha_1}X_2^{-1+\alpha_2}\dots X_m^{-1+\alpha_m}\times\\ &\quad\left\{\left(\frac{k-1}{k}\right)^{k-1}(-\alpha_1+\sum_{i=2}^m(1-\alpha_i)X_2\dots X_i)+\right.\\ &\quad+\frac12\left(\frac{k-1}{k}\right)^{k-1}\Big(-\alpha_1+\sum_{i=2}^m(1-\alpha_i)X_2\dots X_i\Big)^2-\\ &\quad\left.-\frac12\left(\frac{k-1}{k}\right)^{k-1}\sum_{i=2}^{m-1}X_2^2\dots X_i^2\right\}dx+O(1)\end{aligned}\tag{3.31}$$

where the $O(1)$ is uniform with respect to all the α_i's. Expanding the square and collecting similar terms we conclude that

$$\tilde{I}_{m-1}[u] \le \frac{1}{2}\left(\frac{k-1}{k}\right)^{k-1} \tilde{J} + O(1), \quad \text{uniformly in } \alpha_1, \ldots, \alpha_m, \tag{3.32}$$

where

$$\tilde{J} = A_m + \sum_{i=1}^{m}(\alpha_i^2 - 2\alpha_i)A_i + 2\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}(1-\alpha_i)(1-\alpha_j)\Gamma_{ij}. \tag{3.33}$$

We intend to take the limit $\alpha_1 \to 0$ in (3.33). All terms have a finite limit except A_1 and Γ_{1j} which do not contain the factor $X_2^{1+\alpha_2}$. When combined they give

$$\begin{aligned}
& (\alpha_1^2 - 2\alpha_1)A_1 + 2\sum_{j=2}^{m}(1-\alpha_1)(1-\alpha_j)\Gamma_{1j} \\
\text{(by (3.9))} \quad = \quad & \alpha_1^2 A_1 - 2\alpha_1 \sum_{j=2}^{m}(1-\alpha_j)\Gamma_{1j} + O(1) \\
\text{(by (3.9))} \quad = \quad & -\sum_{j=2}^{m}(1-\alpha_j)\alpha_1\Gamma_{1j} + O(1) \\
\text{(by (3.10))} \quad = \quad & \sum_{j=2}^{m}(1-\alpha_j)\Big(\sum_{i=2}^{j}\alpha_i\Gamma_{ij} - \sum_{i=j+1}^{m}(1-\alpha_i)\Gamma_{ji}\Big) + O(1) \\
= \quad & \sum_{i=2}^{m}(\alpha_i - \alpha_i^2)A_i + \sum_{i=2}^{m-1}\sum_{j=i+1}^{m}(2\alpha_i - 1)(1-\alpha_j)\Gamma_{ij} + O(1).
\end{aligned}$$

In this expression we can let $\alpha_1 \to 0$. Hence (3.33) becomes

$$\tilde{J} = A_m - \sum_{i=2}^{m}\alpha_i A_i + \sum_{i=2}^{m-1}\sum_{j=i+1}^{m}(1-\alpha_j)\Gamma_{ij} + O(1), \qquad (\alpha_1 = 0). \tag{3.34}$$

This relation is completely analogous to (3.20). For the rest of the proof we argue as in the proof of Proposition 3.1; we omit the details. //

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