



Sharp Hardy and Hardy–Sobolev inequalities with point singularities on the boundary



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ABSTRACT

We study the Hardy inequality when the singularity is placed on the boundary of a bounded domain in \mathbb{R}^n that satisfies both an interior and exterior ball condition at the singularity. We obtain the sharp Hardy constant $n^2/4$ in case the exterior ball is large enough and show the necessity of the large exterior ball condition. We improve Hardy inequality with the best constant by adding a sharp Sobolev term. We next produce criteria that lead to characterizing maximal potentials that improve Hardy inequality. Breaking the criteria one produces successive improvements with sharp constants. Our approach goes through in less regular domains, like cones. In the case of a cone, contrary to the smooth case, the Sobolev constant does depend on the opening of the cone.

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R É S U M É

Nous étudions l'inégalité de Hardy dans le cas où la singularité se trouve sur la frontière d'un domaine borné sur \mathbb{R}^n qui satisfait à la fois une condition de boule intérieure et extérieure sur la singularité. Nous présentons la constante explicite de Hardy $n^2/4$ obtenue dans le cas où la boule extérieure est suffisamment large et montrons la nécessité de la condition de la boule extérieure. Nous présentons une amélioration de l'inégalité de Hardy avec la meilleure constante en ajoutant un terme explicite de Sobolev. Par la suite, nous présentons certains critères capables de caractériser les potentiels maximaux qui améliorent l'inégalité de Hardy. En bafouant les critères nous produisons des améliorations successives avec des constantes explicites. Notre approche peut être appliquée dans des domaines moins réguliers, comme des cônes. Dans le cas d'une cône, contrairement au cas régulier, la constante de Sobolev dépend de l'ouverture de la cône.

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1. Introduction and main results

For $n \geq 3$ Hardy inequality states, that for any $u \in C_c^\infty(\mathbb{R}^n)$ there holds

$$\int_{\mathbb{R}^n} |\nabla u|^2 dx \geq \left(\frac{n-2}{2}\right)^2 \int_{\mathbb{R}^n} \frac{u^2}{|x|^2} dx ,$$

where $\frac{(n-2)^2}{4}$ is the best constant. On the other hand Sobolev inequality reads as follows

$$\int_{\Omega} |\nabla u|^2 dx \geq S_n \left(\int_{\Omega} |u|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}}, \quad u \in C_c^\infty(\Omega)$$

where $S_n = \pi n(n-2) \left(\Gamma(\frac{n}{2})/\Gamma(n)\right)^{\frac{2}{n}}$ is the best Sobolev constant for any domain $\Omega \subset \mathbb{R}^n$.

There are various improved versions of either Hardy or Sobolev inequalities in the case of a bounded domain Ω containing the origin see e.g. [7,24,23,1,17,3,4,16,6,5]. We mention in particular the following sharp Hardy–Sobolev inequality from [17,2] that combines both inequalities

$$\int_{\Omega} |\nabla u|^2 dx \geq \left(\frac{n-2}{2}\right)^2 \int_{\Omega} \frac{u^2}{|x|^2} dx + (n-2)^{-\frac{2(n-1)}{n}} S_n \left(\int_{\Omega} X_1^{\frac{2n-2}{n-2}} |u|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}}$$

for all $u \in C_c^\infty(\Omega)$. Here $X_1 = X_1(|x|/D)$, with

$$X_1(t) = \frac{1}{1 - \ln t}, \quad t \in (0, 1), \quad D := \sup_{x \in \Omega} |x| .$$

A natural question is what are the analogues of Hardy and Hardy–Sobolev inequalities in case the origin is on the boundary of Ω instead of being in the interior. As we shall see, contrary to the previous case, the geometry of Ω plays an important role. In the simplest case of the half space $\mathbb{R}_+^n = \{(x', x_n) : x_n > 0\}$, Hardy inequality with best constant reads (cf. [22,18])

$$\int_{\mathbb{R}_+^n} |\nabla u|^2 dx \geq \frac{n^2}{4} \int_{\mathbb{R}_+^n} \frac{u^2}{|x|^2} dx, \quad \forall u \in C_c^\infty(\mathbb{R}_+^n) . \quad (1)$$

In the more general case where the domain is a cone \mathcal{C} with its vertex at the origin the sharp Hardy inequality reads (cf. [22])

$$\int_{\mathcal{C}} |\nabla u|^2 dx \geq \left(\left(\frac{n-2}{2}\right)^2 + \mu_1(\Sigma) \right) \int_{\mathcal{C}} \frac{u^2}{|x|^2} dx, \quad \forall u \in C_c^\infty(\mathcal{C}) , \quad (2)$$

where $\Sigma = \mathcal{C} \cap S^{n-1}$ and $\mu_1(\Sigma)$ is the first Dirichlet eigenvalue of the Dirichlet Laplace–Beltrami operator on Σ .

If on the other hand, the origin is on the boundary of a smooth near zero domain, then, related types of problems have been studied in [19,20,12,21]. More precisely the following minimization problem has been considered for $0 < s < 2$ and $n \geq 4$,

$$\mu_s(\Omega) = \inf_{u \in H_0^1(\Omega)} \frac{\int_{\Omega} |\nabla u|^2 dx}{\left(\int_{\Omega} \frac{|u|^{\frac{2(n-s)}{n-2}}}{|x|^s} dx \right)^{\frac{n-2}{n-s}}} , \quad (3)$$

and it was established that the geometry of Ω around zero plays an important role. In particular if the mean curvature at zero is negative then $\mu_s(\Omega) < \mu_s(\mathbb{R}_+^n)$ and there exists a minimizer for (3). In the limit case $s = 2$ the infimum $\mu_2(\Omega)$ is the best Hardy constant and under certain geometric assumptions on Ω has been studied in [8–11,15].

In [14] it was realized that the geometry plays no role for the local best Hardy constant. That is, for $r > 0$ small enough if we denote by B_r the ball of radius r centered at the origin, then for a smooth near zero domain Ω one has

$$\int_{\Omega \cap B_r} |\nabla u|^2 dx \geq \frac{n^2}{4} \int_{\Omega \cap B_r} \frac{u^2}{|x|^2} dx, \quad u \in C_c^\infty(\Omega \cap B_r), \quad (4)$$

which in particular implies the existence of a constant $\lambda \geq 0$ such that

$$\lambda \int_{\Omega} u^2 dx + \int_{\Omega} |\nabla u|^2 dx \geq \frac{n^2}{4} \int_{\Omega} \frac{u^2}{|x|^2} dx, \quad u \in C_c^\infty(\Omega).$$

The first question we raise in this work is to find a more quantitative result that connects the local inequality (4) to the global inequality in the half space (1). To state our first result we denote by $B_\rho(x_0)$ the ball of radius ρ centered at x_0 and simply by B_ρ in case the ball is centered at the origin; we also denote by $\mathcal{C}A$ the complement of a set $A \subset \mathbb{R}^n$.

Throughout this work $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded domain with $0 \in \partial\Omega$ satisfying an exterior ball condition at zero, that is there exists a ball

$$B_\rho(-\rho e_n) \subset \mathcal{C}\overline{\Omega}.$$

We also denote

$$D := \sup_{\Omega} |x|.$$

Theorem 1. *There exists a positive constant τ_n depending only on n such that if the radius of the exterior ball satisfies $\rho \geq D/\tau_n$ then*

$$\int_{\Omega} |\nabla u|^2 dx \geq \frac{n^2}{4} \int_{\Omega} \frac{u^2}{|x|^2} dx, \quad (5)$$

for all $u \in C_c^\infty(\Omega)$. If in addition Ω satisfies an interior ball condition at 0 then the constant $n^2/4$ is sharp.

Thus in the case of a smooth (near zero) domain Ω , if the exterior ball at zero is large enough compared to the size of Ω then the Hardy constant is $n^2/4$. If however the (largest) exterior ball is not large enough, at the end of Section 3 we present an example where the Hardy constant is smaller than $n^2/4$.

We next improve Hardy inequality by adding a Sobolev term:

Theorem 2. *Let $n \geq 3$. There exist positive constants σ_n and C_n that depend only on n such that, if the radius of the exterior ball satisfies $\rho \geq D/\sigma_n$ the following holds true:*

$$\int_{\Omega} |\nabla u|^2 dx \geq \frac{n^2}{4} \int_{\Omega} \frac{u^2}{|x|^2} dx + C_n \left(\int_{\Omega} X_1^{\frac{2n-2}{n-2}} |u|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}},$$

for all $u \in C_c^\infty(\Omega)$. Here $X_1 = X_1(|x|/3D)$. If in addition Ω satisfies an interior ball condition at 0 then the exponent $(2n-2)/(n-2)$ of X_1 is sharp.

If the radius of the exterior ball is small then there exists a non negative constant λ (that depends on Ω) so that we have

$$\lambda \int_{\Omega} u^2 dx + \int_{\Omega} |\nabla u|^2 dx \geq \frac{n^2}{4} \int_{\Omega} \frac{u^2}{|x|^2} dx + C_n \left(\int_{\Omega} X_1^{\frac{2n-2}{n-2}} |u|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}}, \quad u \in C_c^\infty(\Omega);$$

for the precise statement see Theorem 9.

Under the assumptions of Theorem 2, a simple application of Holder's inequality yields that for any $\alpha > 2$ there exists a positive constant $c(\alpha, \Omega)$ such that

$$\int_{\Omega} |\nabla u|^2 dx \geq \frac{n^2}{4} \int_{\Omega} \frac{u^2}{|x|^2} dx + c(\alpha, \Omega) \int_{\Omega} \frac{X_1^\alpha}{|x|^2} u^2 dx, \quad u \in C_c^\infty(\Omega).$$

If $c(\alpha, \Omega)$ is the best constant then this inequality cannot be further improved, see Theorem 11. On the other hand, as we shall see, in the limiting case $\alpha = 2$ the inequality is also true, that is

$$\int_{\Omega} |\nabla u|^2 dx \geq \frac{n^2}{4} \int_{\Omega} \frac{u^2}{|x|^2} dx + \frac{1}{4} \int_{\Omega} \frac{X_1^2}{|x|^2} u^2 dx, \quad u \in C_c^\infty(\Omega),$$

and the constant $1/4$ is sharp. In contrast with the case $\alpha > 2$ this inequality can be further improved.

This is a particular case of a more general situation where one has a non negative potential V that for some λ non negative and some sharp positive constant C the following inequality is true:

$$\lambda \int_{\Omega} u^2 dx + \int_{\Omega} |\nabla u|^2 dx \geq \frac{n^2}{4} \int_{\Omega} \frac{u^2}{|x|^2} dx + C \int_{\Omega} V u^2 dx, \quad u \in C_c^\infty(\Omega). \quad (6)$$

In Section 5 we characterize maximal potentials, that is potentials V such that (6) cannot be improved, with C being the best constant for (6); such examples are the subcritical potentials, see Definition 2. The main result of Section 5 is Theorem 11. We note that this description of maximal potentials is analogous to the description in [23,17] for the interior point singularity case.

In Section 6 we consider the problem of successively improving Hardy inequality by critical potentials. Before stating our result we first define the iterated logarithms (cf. [17])

$$X_{k+1}(t) = X_k(X_1(t)), \quad t \in (0, 1], \quad k = 1, 2, \dots$$

One can check that for $t \in (0, 1)$ the series $\sum_{i=1}^{\infty} X_1(t) X_2(t) \dots X_i(t)$ converges (see the proof of Lemma 6.3 in [17] or the Appendix in [13]) and that it is a strictly increasing function of t . We denote by κ the unique $\kappa > 1$ for which

$$\sum_{i=1}^{\infty} X_1(1/\kappa) \dots X_i(1/\kappa) = \frac{1}{4}. \quad (7)$$

We then have

Theorem 3. *There exists $\sigma_n > 0$ that depends only on n such that if the radius of the exterior ball satisfies $\rho \geq D/\sigma_n$ the following holds true:*

$$\int_{\Omega} |\nabla u|^2 dx \geq \frac{n^2}{4} \int_{\Omega} \frac{u^2}{|x|^2} dx + \frac{1}{4} \sum_{i=1}^{\infty} \int_{\Omega} \frac{u^2}{|x|^2} X_1^2 \dots X_i^2 dx,$$

for all $u \in C_c^\infty(\Omega)$; here $X_i = X_i(|x|/(3\kappa D))$. If in addition Ω satisfies an interior ball condition at 0 then the constants $1/4$ are sharp at each step, that is

$$\inf_{u \in C_c^\infty(\Omega)} \frac{\int_{\Omega} |\nabla u|^2 dx - \frac{n^2}{4} \int_{\Omega} \frac{u^2}{|x|^2} dx}{\int_{\Omega} \frac{u^2}{|x|^2} X_1^2 dx} = \frac{1}{4},$$

and for each $m = 2, 3, \dots$

$$\inf_{u \in C_c^\infty(\Omega)} \frac{\int_{\Omega} |\nabla u|^2 dx - \frac{n^2}{4} \int_{\Omega} \frac{u^2}{|x|^2} dx - \frac{1}{4} \sum_{i=1}^{m-1} \int_{\Omega} \frac{u^2}{|x|^2} X_1^2 \dots X_i^2 dx}{\int_{\Omega} \frac{u^2}{|x|^2} X_1^2 \dots X_m^2 dx} = \frac{1}{4}.$$

We also have the Hardy–Sobolev analogue:

Theorem 4. Let $n \geq 3$. There exist positive constants σ_n and C_n that depend only on n such that, if the radius of the exterior ball satisfies $\rho \geq D/\sigma_n$ then for any $m \in \mathbb{N}$ the following holds true:

$$\int_{\Omega} |\nabla u|^2 dx \geq \frac{n^2}{4} \int_{\Omega} \frac{u^2}{|x|^2} dx + \frac{1}{4} \sum_{i=1}^m \int_{\Omega} \frac{u^2}{|x|^2} X_1^2 \dots X_i^2 dx + C_n \left(\int_{\Omega} (X_1 \dots X_{m+1})^{\frac{2n-2}{n-2}} |u|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}},$$

for all $u \in C_c^\infty(\Omega)$; here $X_i = X_i(|x|/(3\kappa D))$. If in addition Ω satisfies an interior ball condition at 0 then the exponents $(2n-2)/(n-2)$ of X_i are also sharp.

We then proceed to obtain a characterization for maximal potentials in the context of logarithmic improvements; see Theorem 15.

Analogues of these theorems hold true if the domain Ω is a cone with vertex at zero and Section 2 is entirely devoted to this. What is interesting in this case is that the Sobolev constant depends on the cone. As a typical result we mention here the following theorem that refers to a bounded cone $\mathcal{C}_1 := \mathcal{C} \cap B_1$, the intersection of an infinite cone \mathcal{C} with vertex at the origin with the unit ball B_1 .

Theorem 5. Let $n \geq 3$. There exists a positive constant C that depends only on Σ such that

$$\int_{\mathcal{C}_1} |\nabla u|^2 dx \geq \left(\frac{(n-2)^2}{4} + \mu_1(\Sigma) \right) \int_{\mathcal{C}_1} \frac{u^2}{|x|^2} dx + C \left(\int_{\mathcal{C}_1} X_1^{\frac{2n-2}{n-2}} |u|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}}, \quad (8)$$

for all $u \in C_c^\infty(\mathcal{C}_1)$; here $X_1 = X_1(|x|)$. The exponent $(2n-2)/(n-2)$ of X_1 is the best possible. Moreover the best constant C for inequality (8) satisfies the estimate

$$C \leq C_n |\Sigma|^{\frac{2}{n}} \quad (9)$$

for some positive constant C_n that depends only on n . In particular the best constant C of inequality (8) cannot be taken to be independent of Σ .

Finally, a similar analysis goes through if one has potentials with multiple singularities on the boundary, see Theorem 20 for one such result.

Our results about point singularities on the boundary, are analogous to the case of interior point singularities see [17,23,2]. We note however that whereas in the interior singularity case the geometry of Ω is irrelevant, in this work the curvature of the boundary introduces several technical difficulties even in the case of the plain Hardy inequality (5) as already noted in several recent works see e.g. [8–11,14,15,20,21]. To overcome these difficulties we produce new improved inequalities in the flat case, see Lemmas 1, 2, 3 and then we use suitable conformal transformations thus obtaining sharp inequalities under the exterior ball assumption.

2. Distance from the vertex of a cone

In this section we consider the case of a finite cone and we obtain both homogeneous and nonhomogeneous improvements of the Hardy inequality (2). We pay particular attention to the special case where the cone is the half ball B_R^+ . In this case the estimates we obtain are stronger than in the case of a general cone and play a crucial role in our subsequent analysis.

Let $\Sigma \subset S^{n-1}$ be a domain in S^{n-1} (that is a set that is open and connected in the relative topology) with Lipschitz boundary. Let $\mu_k = \mu_k(\Sigma)$ be the k th Dirichlet eigenvalue of the Laplace–Beltrami operator on Σ and let ϕ_k be a corresponding eigenfunction that is,

$$\begin{cases} -\Delta_{S^{n-1}} \phi_k(\omega) = \mu_k \phi_k(\omega), & \omega \in \Sigma, \\ \phi_k|_{\partial\Sigma} = 0. \end{cases}$$

We may assume that $\{\phi_k\}$ is a complete orthonormal system in $L^2(\Sigma)$. We note that μ_1 is a simple eigenvalue and we take ϕ_1 to be positive.

We define

$$\mathcal{C} = \{x \in \mathbb{R}^n \setminus \{0\} : \frac{x}{|x|} \in \Sigma\}, \quad \mathcal{C}_1 = \mathcal{C} \cap B_1 = \{x \in \mathbb{R}^n \setminus \{0\} : \frac{x}{|x|} \in \Sigma, |x| < 1\}.$$

Proof of Theorem 5. Let $u \in C_c^\infty(\mathcal{C}_1)$ be given and let

$$u(x) = \sum_{k=1}^{\infty} u_k(r) \phi_k(\omega)$$

be its decomposition into the spherical harmonics of Σ . We then have

$$u_k(r) = \int_{\Sigma} u(x) \phi_k(\omega) dS(\omega).$$

Let ω_{n-1} denote the surface measure of the unit sphere S^{n-1} . Throughout this proof for any radial function G (which sometimes shall be written as $G(x)$ and sometimes as $G(r)$) we shall use the notation

$$\int_{B_1} G(x) dx = \omega_{n-1} \int_0^1 G(r) r^{n-1} dr.$$

It then easily follows that

$$\int_{\mathcal{C}_1} |\nabla u|^2 dx = \frac{1}{\omega_{n-1}} \int_{B_1} (|\nabla u_1|^2 + \mu_1 \frac{u_1^2}{|x|^2}) dx + \frac{1}{\omega_{n-1}} \sum_{k=2}^{\infty} \int_{B_1} (|\nabla u_k|^2 + \mu_k \frac{u_k^2}{|x|^2}) dx$$

$$= \frac{1}{\omega_{n-1}} \int_{B_1} \left(|\nabla u_1|^2 + \mu_1 \frac{u_1^2}{|x|^2} \right) dx + \int_{\mathcal{C}_1} |\nabla(u - u_1 \phi_1)|^2 dx.$$

Moreover for any bounded radial function G we have

$$\int_{\mathcal{C}_1} G u^2 dx = \frac{|\Sigma|}{\omega_{n-1}} \int_{B_1} G u_1^2 dx + \int_{\mathcal{C}_1} G (u - u_1 \phi_1)^2 dx.$$

Therefore

$$\begin{aligned} & \int_{\mathcal{C}_1} |\nabla u|^2 dx - \left(\frac{(n-2)^2}{4} + \mu_1 \right) \int_{\mathcal{C}_1} \frac{u^2}{|x|^2} dx \\ &= \frac{1}{\omega_{n-1}} \left[\int_{B_1} |\nabla u_1|^2 dx - \left(\frac{n-2}{2} \right)^2 \int_{B_1} \frac{u_1^2}{|x|^2} dx \right] \\ & \quad + \frac{1}{\omega_{n-1}} \sum_{k=2}^{\infty} \int_{B_1} \left(|\nabla u_k|^2 + \mu_k \frac{u_k^2}{|x|^2} - \left(\frac{(n-2)^2}{4} + \mu_1 \right) \frac{u_k^2}{|x|^2} \right) dx \\ &\geq \frac{1}{\omega_{n-1}} \left[\int_{B_1} |\nabla u_1|^2 dx - \left(\frac{n-2}{2} \right)^2 \int_{B_1} \frac{u_1^2}{|x|^2} dx \right] \\ & \quad + \frac{1}{\omega_{n-1}} \cdot \frac{\mu_2 - \mu_1}{\left(\frac{n-2}{2} \right)^2 + \mu_2} \sum_{k=2}^{\infty} \int_{B_1} \left(|\nabla u_k|^2 + \mu_k \frac{u_k^2}{|x|^2} \right) dx \\ &\geq C_n \left(\int_{B_1} X_1^{\frac{2(n-1)}{n-2}} |u_1|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{2}} + \frac{\mu_2 - \mu_1}{\left(\frac{n-2}{2} \right)^2 + \mu_2} \int_{\mathcal{C}_1} |\nabla(u - u_1 \phi_1)|^2 dx \\ &\geq \frac{C_n}{1 + \mu_1} \left(\int_{\mathcal{C}_1} X_1^{\frac{2(n-1)}{n-2}} |u_1 \phi_1|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{2}} + \frac{(\mu_2 - \mu_1) S_n}{\left(\frac{n-2}{2} \right)^2 + \mu_2} \left(\int_{\mathcal{C}_1} |u - u_1 \phi_1|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{2}} \\ &\geq C(n, \Sigma) \left(\int_{\mathcal{C}_1} X_1^{\frac{2(n-1)}{n-2}} |u|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{2}}. \end{aligned}$$

For the optimality of the exponent, suppose to the contrary that there exists $p < (2n-2)/(n-2)$ such that

$$\int_{\mathcal{C}_1} |\nabla u|^2 dx \geq \left(\frac{(n-2)^2}{4} + \mu_1(\Sigma) \right) \int_{\mathcal{C}_1} \frac{u^2}{|x|^2} dx + C \left(\int_{\mathcal{C}_1} X_1^p |u|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}},$$

for all $u \in C_c^\infty(\mathcal{C}_1)$. Considering functions u of the form $u(x) = v(r)\phi_1(\omega)$ with $v(1) = 0$ we obtain that any such v satisfies

$$\int_{B_1} |\nabla v|^2 dx \geq \left(\frac{n-2}{2} \right)^2 \int_{B_1} \frac{v^2}{|x|^2} dx + C \left(\int_{B_1} X_1^p |v|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}}. \quad (10)$$

This is a contradiction since the best exponent of X_1 in (10) is $2(n-1)/(n-2)$; see [17].

To prove estimate (9) we test inequality (8) with a function of the form $u(x) = v(r)\phi_1(\omega)$. Then an easy calculation gives

$$\begin{aligned}
C &\leq \frac{\int_{\mathcal{C}_1} |\nabla u|^2 dx - \left(\frac{(n-2)^2}{4} + \mu_1 \right) \int_{\mathcal{C}_1} \frac{u^2}{|x|^2} dx}{\left(\int_{\mathcal{C}_1} |u|^{\frac{2n}{n-2}} X_1^{\frac{2n-2}{n-2}} dx \right)^{\frac{n-2}{n}}} \\
&= \frac{\frac{1}{\omega_{n-1}} \int_{B_1} \left[|\nabla v|^2 dx - \left(\frac{n-2}{2} \right)^2 \frac{v^2}{|x|^2} \right] dx}{\left(\frac{1}{\omega_{n-1}} \int_{B_1} |v|^{\frac{2n}{n-2}} X_1^{\frac{2n-2}{n-2}} dx \right)^{\frac{n-2}{n}} \left(\int_{\Sigma} |\phi_1|^{\frac{2n}{n-2}} dS \right)^{\frac{n-2}{n}}}.
\end{aligned}$$

Minimizing with respect to v (see [2, Theorem B]) we conclude that

$$C \leq \frac{\omega_{n-1}^{-\frac{2}{n}} (n-2)^{-\frac{2}{n}} S_n}{\left(\int_{\Sigma} |\phi_1|^{\frac{2n}{n-2}} dS \right)^{\frac{n-2}{n}}}.$$

By the normalization of ϕ_1 and Hölder inequality we conclude that

$$C \leq \omega_{n-1}^{-\frac{2(n-1)}{n}} (n-2)^{-\frac{2}{n}} S_n |\Sigma|^{\frac{2}{n}},$$

which concludes the proof. \square

In a similar fashion we obtain

Theorem 6. *Let $n \geq 3$. There exists a constant C that depends only on Σ such that for any $m \in \mathbb{N}$*

$$\begin{aligned}
\int_{\mathcal{C}_1} |\nabla u|^2 dx &\geq \left(\frac{(n-2)^2}{4} + \mu_1(\Sigma) \right) \int_{\mathcal{C}_1} \frac{u^2}{|x|^2} dx + \frac{1}{4} \sum_{i=1}^m \int_{\mathcal{C}_1} \frac{u^2}{|x|^2} X_1^2 \dots X_i^2 dx \\
&\quad + C \left(\int_{\mathcal{C}_1} (X_1 \dots X_{m+1})^{\frac{2n-2}{n-2}} |u|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}}, \tag{11}
\end{aligned}$$

for all $u \in C_c^\infty(\mathcal{C}_1)$; here $X_i = X_i(|x|)$. Each constant $1/4$ is the best possible, that is,

$$\inf_{u \in C_c^\infty(\mathcal{C}_1)} \frac{\int_{\mathcal{C}_1} |\nabla u|^2 dx - \left(\left(\frac{n-2}{2} \right)^2 + \mu_1(\Sigma) \right) \int_{\mathcal{C}_1} \frac{u^2}{|x|^2} dx}{\int_{\mathcal{C}_1} \frac{u^2}{|x|^2} X_1^2 dx} = \frac{1}{4},$$

and for each $m = 2, 3, \dots$

$$\inf_{u \in C_c^\infty(\mathcal{C}_1)} \frac{\int_{\mathcal{C}_1} |\nabla u|^2 dx - \left(\left(\frac{n-2}{2} \right)^2 + \mu_1(\Sigma) \right) \int_{\mathcal{C}_1} \frac{u^2}{|x|^2} dx - \frac{1}{4} \sum_{i=1}^{m-1} \int_{\mathcal{C}_1} \frac{u^2}{|x|^2} X_1^2 \dots X_i^2 dx}{\int_{\mathcal{C}_1} \frac{u^2}{|x|^2} X_1^2 \dots X_m^2 dx} = \frac{1}{4}.$$

The exponent $(2n-2)/(n-2)$ is the best possible. Moreover the best constant C for inequality (11) satisfies the estimate

$$C \leq C_n |\Sigma|^{\frac{2}{n}} \tag{12}$$

for some positive constant C_n that depends only on n . In particular the best constant C of inequality (11) cannot be taken to be independent of Σ .

Proof. Arguing as in the proof of Theorem 5 we arrive at

$$\begin{aligned}
& \int_{\mathcal{C}_1} |\nabla u|^2 dx - \left(\frac{(n-2)^2}{4} + \mu_1 \right) \int_{\mathcal{C}_1} \frac{u^2}{|x|^2} dx - \frac{1}{4} \sum_{i=1}^m \int_{\mathcal{C}_1} \frac{u^2}{|x|^2} X_1^2 \dots X_i^2 dx \\
&= \frac{1}{\omega_{n-1}} \left[\int_{B_1} |\nabla u_1|^2 dx - \left(\frac{(n-2)^2}{4} + \mu_1 \right) \int_{B_1} \frac{u_1^2}{|x|^2} dx - \frac{1}{4} \sum_{i=1}^m \int_{B_1} \frac{u_1^2}{|x|^2} X_1^2 \dots X_i^2 dx \right] \\
&\quad + \frac{1}{\omega_{n-1}} \sum_{k=2}^{\infty} \int_{B_1} \left(|\nabla u_k|^2 + \mu_k \frac{u_k^2}{|x|^2} - \left(\frac{(n-2)^2}{4} + \mu_1 \right) \frac{u_k^2}{|x|^2} - \frac{1}{4} \sum_{i=1}^m \frac{u_k^2}{|x|^2} X_1^2 \dots X_i^2 \right) dx \\
&\geq \frac{1}{\omega_{n-1}} \left[\int_{B_1} |\nabla u_1|^2 dx - \left(\frac{(n-2)^2}{4} + \mu_1 \right) \int_{B_1} \frac{u_1^2}{|x|^2} dx - \frac{1}{4} \sum_{i=1}^m \int_{B_1} \frac{u_1^2}{|x|^2} X_1^2 \dots X_i^2 dx \right] \\
&\quad + \frac{\mu_2 - \mu_1}{2 \left[\left(\frac{n-2}{2} \right)^2 + \mu_2 \right]} \frac{1}{\omega_{n-1}} \sum_{k=2}^{\infty} \int_{B_1} \left(|\nabla u_k|^2 + \mu_k \frac{u_k^2}{|x|^2} \right) dx \\
&\geq C_n \left(\int_{B_1} (X_1 \dots X_{m+1})^{\frac{2(n-1)}{n-2}} |u_1|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{2}} + \frac{\mu_2 - \mu_1}{2 \left[\left(\frac{n-2}{2} \right)^2 + \mu_2 \right]} \int_{\mathcal{C}_1} |\nabla(u - u_1 \phi_1)|^2 dx \\
&\geq \frac{C_n}{1 + \mu_1} \left(\int_{\mathcal{C}_1} (X_1 \dots X_{m+1})^{\frac{2(n-1)}{n-2}} |u_1 \phi_1|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{2}} + \frac{(\mu_2 - \mu_1) S_n}{2 \left[\left(\frac{n-2}{2} \right)^2 + \mu_2 \right]} \left(\int_{\mathcal{C}_1} |u - u_1 \phi_1|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{2}} \\
&\geq C(n, \Sigma) \left(\int_{\mathcal{C}_1} (X_1 \dots X_{m+1})^{\frac{2(n-1)}{n-2}} |u|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{2}}.
\end{aligned}$$

For the optimality of the constants $1/4$ we make once again the choice $u(x) = v(r)\phi_1(\omega)$ to conclude that

$$\begin{aligned}
& \inf_{C_c^\infty(\mathcal{C}_1)} \frac{\int_{\mathcal{C}_1} |\nabla u|^2 dx - \left(\left(\frac{n-2}{2} \right)^2 + \mu_1(\Sigma) \right) \int_{\mathcal{C}_1} \frac{u^2}{|x|^2} dx - \frac{1}{4} \sum_{i=1}^{m-1} \int_{\mathcal{C}_1} \frac{u^2}{|x|^2} X_1^2 \dots X_i^2 dx}{\int_{\mathcal{C}_1} \frac{u^2}{|x|^2} X_1^2 \dots X_m^2 dx} \\
&\leq \inf_{C_c^\infty(B_1)} \frac{\int_{B_1} |\nabla v|^2 dx - \left(\frac{n-2}{2} \right)^2 \int_{B_1} \frac{v^2}{|x|^2} dx - \frac{1}{4} \sum_{i=1}^{m-1} \int_{B_1} \frac{v^2}{|x|^2} X_1^2 \dots X_i^2 dx}{\int_{B_1} \frac{v^2}{|x|^2} X_1^2 \dots X_m^2 dx} \\
&= \frac{1}{4},
\end{aligned}$$

by [17, Theorem 6.1]. The optimality of the exponent in the Sobolev term follows as before from the optimality of the corresponding exponent of the Hardy–Sobolev inequality for an interior point, [17, Theorem A].

Finally to prove estimate (12) we once again test inequality (11) with a function of the form $u(x) = v(r)\phi_1(\omega)$. We then obtain

$$\begin{aligned}
C &\leq \frac{\int_{\mathcal{C}_1} |\nabla u|^2 dx - \left(\frac{(n-2)^2}{4} + \mu_1 \right) \int_{\mathcal{C}_1} \frac{u^2}{|x|^2} dx - \frac{1}{4} \sum_{i=1}^m \int_{\mathcal{C}_1} \frac{u^2}{|x|^2} X_1^2 \dots X_i^2 dx}{\left(\int_{\mathcal{C}_1} |u|^{\frac{2n}{n-2}} (X_1 \dots X_{m+1})^{\frac{2n-2}{n-2}} dx \right)^{\frac{n-2}{n}}} \\
&= \frac{\frac{1}{\omega_{n-1}} \int_{B_1} \left[|\nabla v|^2 dx - \left(\frac{n-2}{2} \right)^2 \frac{v^2}{|x|^2} - \frac{1}{4} \sum_{i=1}^m \int_{B_1} \frac{v^2}{|x|^2} X_1^2 \dots X_i^2 \right] dx}{\left(\frac{1}{\omega_{n-1}} \int_{B_1} |v|^{\frac{2n}{n-2}} (X_1 \dots X_{m+1})^{\frac{2n-2}{n-2}} dx \right)^{\frac{n-2}{n}} \left(\int_{\Sigma} |\phi_1|^{\frac{2n}{n-2}} dS \right)^{\frac{n-2}{n}}}.
\end{aligned}$$

Minimizing with respect to v (see [2, Theorem B]) and using Hölder inequality we conclude that

$$C \leq \frac{\omega_{n-1}^{-\frac{2}{n}}(n-2)^{-\frac{2(n-1)}{n}}S_n}{\left(\int_{\Sigma}|\phi_1|^{\frac{2n}{n-2}}dS\right)^{\frac{n-2}{n}}} \leq \omega_{n-1}^{-\frac{2}{n}}(n-2)^{-\frac{2}{n}}S_n|\Sigma|^{\frac{2}{n}},$$

which concludes the proof. \square

Theorem 7. *Let $n \geq 2$. There holds*

$$\int_{\mathcal{C}_1} |\nabla u|^2 dx \geq \left(\left(\frac{n-2}{2} \right)^2 + \mu_1(\Sigma) \right) \int_{\mathcal{C}_1} \frac{u^2}{|x|^2} dx + \frac{1}{4} \sum_{i=1}^{\infty} \int_{\mathcal{C}_1} \frac{u^2}{|x|^2} X_1^2 \dots X_i^2 dx$$

for all $u \in C_c^\infty(\mathcal{C}_1)$; here $X_i = X_i(|x|)$. Each constant $1/4$ is sharp.

Proof. This follows from Theorem 6 by letting $m \rightarrow +\infty$. The optimality of the constants $1/4$ has been established in Theorem 6. \square

2.1. Improved Hardy inequalities in half balls

The case of half ball where $\Sigma = S_+^{n-1}$ is of particular importance for our approach. In this case the Hardy constant becomes

$$\left(\frac{n-2}{2} \right)^2 + n - 1 = \frac{n^2}{4},$$

and the Sobolev constants of Theorems 5 and 6 depend only on n . As a special case of the previous results we have the following sharp inequalities for all functions $u \in C_c^\infty(B_R^+)$:

$$\int_{B_R^+} |\nabla u|^2 dx \geq \frac{n^2}{4} \int_{B_R^+} \frac{u^2}{|x|^2} dx + C_n \left(\int_{B_R^+} X_1^{\frac{2n-2}{n-2}} |u|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}}, \quad (13)$$

$$\begin{aligned} \int_{B_R^+} |\nabla u|^2 dx &\geq \frac{n^2}{4} \int_{B_R^+} \frac{u^2}{|x|^2} dx + \frac{1}{4} \sum_{i=1}^m \int_{B_R^+} \frac{u^2}{|x|^2} X_1^2 \dots X_i^2 dx \\ &\quad + C_n \left(\int_{B_R^+} (X_1 \dots X_{m+1})^{\frac{2n-2}{n-2}} |u|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}}, \end{aligned} \quad (14)$$

$$\int_{B_R^+} |\nabla u|^2 dx \geq \frac{n^2}{4} \int_{B_R^+} \frac{u^2}{|x|^2} dx + \frac{1}{4} \sum_{i=1}^{\infty} \int_{B_R^+} \frac{u^2}{|x|^2} X_1^2 \dots X_i^2 dx, \quad (15)$$

where $X_i = X_i(|x|/R)$.

In these inequalities the singularity lies on a flat part of the boundary. However if the boundary is not flat near the singularity, then curvature plays a role. To overcome these difficulties, in the next three lemmas we establish stronger versions of (13), (14) and (15) that will be used to prove Theorems 2, 3 and 4.

We recall (cf. (7)) that κ is the unique $\kappa > 1$ for which $\sum_{i=1}^{\infty} X_1(1/\kappa) \dots X_i(1/\kappa) = \frac{1}{4}$. We also denote for $t \in (0, 1)$,

$$\eta(t) := \sum_{i=1}^{\infty} X_1(t)X_2(t)\dots X_i(t), \quad B(t) := \sum_{i=1}^{\infty} X_1^2(t)X_2^2(t)\dots X_i^2(t).$$

Using the identity

$$\frac{d}{dt}X_k(t) = \frac{1}{t}X_1(t)\dots X_{k-1}(t)X_k^2(t)$$

we easily obtain cf. [4]

$$\frac{d}{dt}\eta(t) = \frac{1}{2t}(\eta(t)^2 + B(t)), \quad t \in (0, 1).$$

We next have.

Lemma 1. *For any $R > 0$ there holds*

$$\int_{B_R^+} |\nabla u|^2 dx \geq \frac{n^2}{4} \int_{B_R^+} \frac{u^2}{|x|^2} dx + \frac{1}{4} \sum_{i=1}^{\infty} \int_{B_R^+} \frac{u^2}{|x|^2} X_1^2 \dots X_i^2 dx + \frac{1}{8R^{1/2}} \int_{B_R^+} \frac{u^2}{|x|^{3/2}} dx, \quad (16)$$

for all $u \in C_c^\infty(B_R^+)$; here $X_i = X_i(|x|/\kappa R)$.

Proof. Let \mathbf{T} be a C^1 vector field in B_R^+ and $u \in C_c^\infty(B_R^+)$. We have

$$\int_{B_R^+} \operatorname{div} \mathbf{T} u^2 dx = -2 \int_{B_R^+} u \nabla u \cdot \mathbf{T} dx \leq \int_{B_R^+} (|\nabla u|^2 + |\mathbf{T}|^2 u^2) dx$$

and therefore

$$\int_{B_R^+} |\nabla u|^2 dx \geq \int_{B_R^+} (\operatorname{div} \mathbf{T} - |\mathbf{T}|^2) u^2 dx.$$

We shall apply this for the vector field

$$\mathbf{T} = \frac{n}{2} \frac{x}{|x|^2} - \frac{e_n}{x_n} + \frac{\eta}{2} \frac{x}{|x|^2} + \frac{1}{2(R^{1/2} - |x|^{1/2})} \frac{x}{|x|^{3/2}}, \quad (\eta = \eta(|x|/\kappa R)).$$

We have

$$\begin{aligned} \operatorname{div} \mathbf{T} &= \frac{n(n-2)}{2|x|^2} + \frac{1}{x_n^2} + \frac{n-2}{2|x|^2} \eta + \frac{\eta^2 + B}{4|x|^2} \\ &\quad + \frac{n - \frac{3}{2}}{2(R^{1/2} - |x|^{1/2})|x|^{3/2}} + \frac{1}{4(R^{1/2} - |x|^{1/2})^2|x|}, \end{aligned}$$

hence

$$\begin{aligned} \operatorname{div} \mathbf{T} - |\mathbf{T}|^2 &= \frac{n^2}{4|x|^2} + \frac{1}{4|x|^2} \sum_{i=1}^{\infty} X_1^2 \dots X_i^2 + \frac{\frac{1}{2} - \eta}{2(R^{1/2} - |x|^{1/2})|x|^{3/2}} \\ &\geq \frac{n^2}{4|x|^2} + \frac{1}{4|x|^2} \sum_{i=1}^{\infty} X_1^2 \dots X_i^2 + \frac{1}{8R^{1/2}|x|^{3/2}}, \end{aligned}$$

where in the last inequality we used that $\eta \leq \frac{1}{4}$, because of the choice of κ , and the result follows. \square

Lemma 2. Let $n \geq 3$. There exists a constant C_n that depends only on n such that for any $R > 0$ there holds

$$\int_{B_R^+} |\nabla u|^2 dx \geq \frac{n^2}{4} \int_{B_R^+} \frac{u^2}{|x|^2} dx + \frac{1}{16R^{1/2}} \int_{B_R^+} \frac{u^2}{|x|^{3/2}} dx + C_n \left(\int_{B_R^+} X_1^{\frac{2n-2}{n-2}} |u|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}},$$

for all $u \in C_c^\infty(B_R^+)$; here $X_1 = X_1(|x|/R)$.

Proof. The result follows by taking a convex combination of (13) and (16) and discarding the logarithmic terms that do not come with the sharp constant; see also the next lemma. \square

Lemma 3. Let $n \geq 3$ and $m \in \mathbb{N}$. There exists a constant C_n that depends only on n such that for all $R > 0$ there holds

$$\begin{aligned} \int_{B_R^+} |\nabla u|^2 dx &\geq \frac{n^2}{4} \int_{B_R^+} \frac{u^2}{|x|^2} dx + \frac{1}{4} \sum_{i=1}^m \int_{B_R^+} \frac{u^2}{|x|^2} X_1^2 \dots X_i^2 dx \\ &\quad + \frac{1}{16R^{1/2}} \int_{B_R^+} \frac{u^2}{|x|^{3/2}} dx + C_n \left(\int_{B_R^+} (X_1 \dots X_{m+1})^{\frac{2n-2}{n-2}} |u|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}}, \end{aligned}$$

for all $u \in C_c^\infty(B_R^+)$; here $X_i = X_i(|x|/\kappa R)$.

Proof. This follows by taking a convex combination of (14) and (16). \square

3. Hardy inequality in bounded domains

In this section we provide the proof of Theorem 1 and use an example to establish the necessity of a relatively large exterior ball assumption. We also analyse the Hardy constant in the case of annuli (see Theorem 8).

We initially establish that $n^2/4$ is an upper bound for the Hardy constant under an interior ball condition.

Lemma 4. If Ω satisfies an interior ball condition at 0 then for any $r > 0$ we have

$$\inf_{u \in C_c^\infty(\Omega \cap B_r)} \frac{\int_{\Omega \cap B_r} |\nabla u|^2 dx}{\int_{\Omega \cap B_r} \frac{u^2}{|x|^2} dx} \leq \frac{n^2}{4}.$$

Proof. Without loss of generality we may assume that the interior ball is $B_\rho(\rho e_n)$ and satisfies $B_\rho(\rho e_n) \subset \Omega \cap B_r$, therefore it is enough to establish that

$$\inf_{u \in C_c^\infty(B_\rho(\rho e_n))} \frac{\int_{B_\rho(\rho e_n)} |\nabla u|^2 dx}{\int_{B_\rho(\rho e_n)} \frac{u^2}{|x|^2} dx} \leq \frac{n^2}{4}.$$

Using a scaling argument we find that this infimum is equal to

$$\inf_{u \in C_c^\infty(\mathbb{R}_+^n)} \frac{\int_{\mathbb{R}_+^n} |\nabla u|^2 dx}{\int_{\mathbb{R}_+^n} \frac{u^2}{|x|^2} dx},$$

which is equal to $n^2/4$. \square

We shall next prove a result about annuli. We use the notation

$$\mathcal{D}(x_0; r_1, r_2) := \{x \in \mathbb{R}^n : r_1 < |x - x_0| < r_2\},$$

or simply $\mathcal{D}(r_1, r_2)$ in case $x_0 = 0$. Also, e_n shall denote the unit vector in the x_n direction.

Theorem 8. *Let $n \geq 2$ and let λ_τ denote the best constant for the Hardy inequality*

$$\int_{\mathcal{D}(\rho, \rho(1+\tau))} |\nabla u|^2 dx \geq \lambda_\tau \int_{\mathcal{D}(\rho, \rho(1+\tau))} \frac{u^2}{|x - \rho e_n|^2} dx, \quad u \in C_c^\infty(\mathcal{D}(\rho, \rho(1+\tau))). \quad (17)$$

There exists a constant $\tau_n > 0$ which depends only on n such that

- (i) *For all $0 < \tau \leq \tau_n$ there holds $\lambda_\tau = n^2/4$*
- (ii) *For all $\tau > \tau_n$ there holds $\lambda_\tau < n^2/4$.*

Moreover λ_τ is strictly decreasing in $(\tau_n, +\infty)$ and $\lim_{\tau \rightarrow +\infty} \lambda_\tau = (n-2)^2/4$.

Proof. It is enough to establish the result for $\rho = 1$, the general case then follows by scaling. To prove (i) it is enough to establish that for small enough $\tau > 0$ we have inequality (17). We apply (15) with $R = 2$ where we place the singularity at e_n and we obtain the inequality

$$\int_{B_1} |\nabla u|^2 dx \geq \frac{n^2}{4} \int_{B_1} \frac{u^2}{|x - e_n|^2} dx + \frac{1}{4} \int_{B_1} \frac{u^2}{|x - e_n|^2} X_1^2 dx, \quad \forall u \in C_0^\infty(B_1), \quad (18)$$

where $X_1 = X_1(|x - e_n|/2)$. Next we apply the Kelvin transform

$$u(x) = |y|^{n-2} v(y), \quad y = \frac{x}{|x|^2}.$$

Then by standard calculations using the conformality of the Kelvin transform we have

$$\int_{B_1} |\nabla u(x)|^2 dx = \int_{\mathcal{C}B_1} |\nabla v(y)|^2 dy,$$

and since

$$|x - e_n| = \frac{|y - e_n|}{|y|},$$

inequality (18) takes the equivalent form

$$\int_{\mathcal{C}B_1} |\nabla v|^2 dy \geq \frac{n^2}{4} \int_{\mathcal{C}B_1} \frac{v^2}{|y|^2 |y - e_n|^2} dy + \frac{1}{4} \int_{\mathcal{C}B_1} \frac{X_1^2}{|y|^2 |y - e_n|^2} v^2 dy \quad (19)$$

for all $v \in C_c^\infty(\mathcal{C}B_1)$, where $X_1 = X_1(|y - e_n|/2|y|)$.

It follows from (19) that for any $\tau > 0$ and any $v \in C_c^\infty(B_{1+\tau} \setminus B_1)$ there holds

$$\int_{B_{1+\tau} \setminus B_1} |\nabla v|^2 dy \geq \frac{n^2}{4} \int_{B_{1+\tau} \setminus B_1} \frac{v^2}{|y - e_n|^2} dy + \frac{1}{4} \int_{B_{1+\tau} \setminus B_1} \frac{n^2 - n^2|y|^2 + X_1^2 \left(\frac{|y - e_n|}{2|y|} \right)}{|y|^2 |y - e_n|^2} v^2 dy.$$

To conclude the proof it suffices to show that the last term above is nonnegative for small enough $\tau > 0$. For this it is enough to have the inequality

$$X_1^2\left(\frac{|y - e_n|}{2|y|}\right) \geq n^2(|y|^2 - 1), \quad 1 < |y| < 1 + \tau.$$

Writing $|y| = 1 + t$, $0 < t < \tau$, we have that $|y - e_n| \geq t$. Hence

$$X_1^2\left(\frac{|y - e_n|}{2|y|}\right) \geq X_1^2\left(\frac{t}{2(t+1)}\right),$$

and therefore it is enough to have

$$X_1^2\left(\frac{t}{2(t+1)}\right) \geq n^2 t(t+2), \quad 0 < t < \tau.$$

Since $\lim_{t \rightarrow 0^+} X_1^2(t)/t = +\infty$, the result follows.

We shall next establish that the set of all $\tau > 0$ for which inequality (17) holds true is bounded and therefore we may define

$$\tau_n = \sup\{\tau > 0 : \text{inequality (17) holds true}\}.$$

For this we first note that for $\tau > 2$ we have the inclusion

$$B_\tau \setminus B_2 \subset B_{1+\tau}(-e_n) \setminus B_1(-e_n),$$

and therefore

$$\inf_{C_c^\infty(\mathcal{D}(-e_n; 1, 1+\tau))} \frac{\int_{\mathcal{D}(-e_n; 1, 1+\tau)} |\nabla u|^2 dx}{\int_{\mathcal{D}(-e_n; 1, 1+\tau)} \frac{u^2}{|x|^2} dx} \leq \inf_{C_c^\infty(B_\tau \setminus B_2)} \frac{\int_{B_\tau \setminus B_2} |\nabla u|^2 dx}{\int_{B_\tau \setminus B_2} \frac{u^2}{|x|^2} dx}.$$

Using the radial function

$$u(r) = r^{-\frac{n-2}{2}} \sin\left(\frac{\ln(r/2)\pi}{\ln(\tau/2)}\right), \quad 2 < r < \tau,$$

we easily see that the last infimum is equal to $\left(\frac{n-2}{2}\right)^2 + \left(\frac{\pi}{\ln \frac{\tau}{2}}\right)^2$ and in particular it is smaller than $n^2/4$ if

$$\tau > 2e^{\frac{\pi}{\sqrt{n-1}}}.$$

This implies the existence of an H_0^1 minimizer (see e.g. [20], Theorem 4.2) and therefore the strict monotonicity of λ_τ for $\tau > \tau_n$. The above computation also gives that $\lim_{\tau \rightarrow +\infty} \lambda_\tau \leq (n-2)^2/4$; this combined with the standard Hardy inequality gives $\lim_{\tau \rightarrow +\infty} \lambda_\tau = (n-2)^2/4$ thus concluding the proof of the theorem. \square

We next have

Proof of Theorem 1. As we shall see, the constant τ_n of Theorem 1 is the same as that of Theorem 8 above. Since $\Omega \cap B_{\rho\tau_n} \subset \mathcal{D}(-\rho e_n; \rho, \rho(1+\tau_n))$, it follows from Theorem 8 that

$$\int_{\Omega \cap B_{\rho\tau_n}} |\nabla u|^2 dx \geq \frac{n^2}{4} \int_{\Omega \cap B_{\rho\tau_n}} \frac{u^2}{|x|^2} dx, \quad u \in C_c^\infty(\Omega \cap B_{\rho\tau_n}). \quad (20)$$

The assumption $\rho\tau_n \geq D$ implies $\Omega \subset B_{\rho\tau_n}$ and therefore (5) follows from (20). The sharpness of the constant $n^2/4$ follows directly from Lemma 4. \square

It is natural to ask whether the assumption of having a large exterior ball at zero is necessary in order to have the Hardy inequality with constant $n^2/4$. In the following example we will see that for small exterior balls inequality (5) fails.

Example. Given $\rho \in (0, 1/2)$ and $\theta \in (0, \pi/2)$ we define the domain

$$\mathcal{A}_{\rho,\theta} = \{x = (x', x_n) \in B_1 : x_n < \cot \theta |x'| \text{ and } |x - \rho e_n| > \rho\}.$$

Let Ω be a domain containing $\mathcal{A}_{\rho,\theta}$ and having the same largest exterior ball at zero, namely $B(\rho e_n, \rho)$.

We denote by $\lambda_1(n, \theta)$ the first Dirichlet eigenvalue of the Laplace operator on the spherical cap

$$\Sigma_\theta = \{(x', x_n) \in S^{n-1} : x_n < \cot \theta |x'|\}.$$

By monotonicity it follows that for $\theta < \pi/2$ we have $\lambda_1(n, \theta) < \lambda_1(n, \pi/2) = n - 1$. We shall prove that if

$$\rho < \frac{1}{2 \cos \theta} e^{-\frac{\pi}{\sqrt{n-1-\lambda_1(n,\theta)}}}, \quad (21)$$

then

$$\inf_{C_c^\infty(\Omega)} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega \frac{u^2}{|x|^2} dx} \leq \inf_{C_c^\infty(\mathcal{A}_{\rho,\theta})} \frac{\int_{\mathcal{A}_{\rho,\theta}} |\nabla u|^2 dx}{\int_{\mathcal{A}_{\rho,\theta}} \frac{u^2}{|x|^2} dx} < \frac{n^2}{4}, \quad (22)$$

that is the Hardy inequality with constant $n^2/4$ fails in Ω if the exterior ball at zero is small enough.

Proof of (22). We first note that

$$(B_1 \setminus B_{2\rho \cos \theta}) \cap \{(x', x_n) : x_n < \cot \theta |x'|\} \subset \mathcal{A}_{\rho,\theta}.$$

Separating variables we then conclude that

$$\begin{aligned} \inf_{C_c^\infty(\mathcal{A}_{\rho,\theta})} \frac{\int_{\mathcal{A}_{\rho,\theta}} |\nabla u|^2 dx}{\int_{\mathcal{A}_{\rho,\theta}} \frac{u^2}{|x|^2} dx} &\leq \inf_{f(2\rho \cos \theta)=f(1)=0} \frac{\int_{2\rho \cos \theta}^1 f'(r)^2 r^{n-1} dr}{\int_{2\rho \cos \theta}^1 f(r)^2 r^{n-3} dr} + \inf_{g(\theta)=g'(\pi)=0} \frac{\int_\theta^\pi \sin^{n-2} t g'(t)^2 dt}{\int_\theta^\pi \sin^{n-2} t g(t)^2 dt} \\ &= \left(\frac{n-2}{2}\right)^2 + \left(\frac{\pi}{\ln(2\rho \cos \theta)}\right)^2 + \lambda_1(n, \theta) < \frac{n^2}{4}, \end{aligned}$$

by assumption (21). \square

4. Improved Hardy–Sobolev inequalities for bounded domains

In this section we shall establish improved Hardy and Hardy–Sobolev inequalities and in particular we will provide the proof of Theorem 2. We start with the following lemma.

Lemma 5. *Let $n \geq 3$. There exist $\sigma_n \in (0, 1)$ and a constant $C_n > 0$, both depending only on n , such that for all $\rho > 0$ and all $r \leq \sigma_n \rho$ we have*

$$\int_{\mathcal{CB}(\rho) \cap B(\rho e_n, r)} |\nabla u|^2 dx \geq \frac{n^2}{4} \int_{\mathcal{CB}(\rho) \cap B(\rho e_n, r)} \frac{u^2}{|x - \rho e_n|^2} dx + C_n \left(\int_{\mathcal{CB}(\rho) \cap B(\rho e_n, r)} X_1^{\frac{2n-2}{n-2}} |u|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}}, \quad (23)$$

for all $u \in C_c^\infty(\mathcal{CB}(\rho) \cap B(\rho e_n, r))$; here $X_1 = X_1(|x - \rho e_n|/3r)$.

Proof. We establish (23) for $\rho = 1$, the general case will then follow by scaling. The map

$$S(v) = \frac{1}{|v + e_n|^2} (2v', 1 - |v|^2) \quad (24)$$

maps conformally \mathbb{R}_+^n onto the unit ball B_1 . We note that

$$|S(v)| = \frac{|v - e_n|}{|v + e_n|}. \quad (25)$$

Composing S with the Kelvin transform K we obtain that the map

$$T(v) = (KS)(v) = \frac{1}{|v - e_n|^2} (2v', 1 - |v|^2) \quad (26)$$

maps conformally \mathbb{R}_+^n onto \mathcal{CB}_1 . The Jacobian determinant $JS(v)$ of S can be computed explicitly and one finds

$$|JS(v)| = \frac{2^n}{|v + e_n|^{2n}}.$$

The Jacobian of the Kelvin map $K(y)$ is $|y|^{-2n}$ hence, using also (25), the Jacobian of T is

$$|JT(v)| = |JK(Sv)| |JS(v)| = \frac{2^n |Sv|^{-2n}}{|v + e_n|^{2n}} = \frac{2^n}{|v - e_n|^{2n}}. \quad (27)$$

Now, simple computations give that $S^{-1} = S$ and therefore $T^{-1} = S^{-1}K^{-1} = SK$. From this we find

$$T^{-1}(x) = \frac{1}{|x'|^2 + (x_n + 1)^2} (2x', |x|^2 - 1)$$

and therefore

$$|T^{-1}(x)| = \frac{|x - e_n|}{|x + e_n|}. \quad (28)$$

Now let $r < 1$ be fixed (this will be chosen later on) and let $F \in C_c^\infty(T(B_r^+))$ be given. We define the function G on B_r^+ by

$$G(v) = F(T(v)) |(JT)(v)|^{\frac{n-2}{2n}} = F(T(v)) \left(\frac{2}{|v - e_n|^2} \right)^{\frac{n-2}{2}}.$$

We then have by Lemma 2,

$$\int_{B_r^+} |\nabla G|^2 dv \geq \frac{n^2}{4} \int_{B_r^+} \frac{G^2}{|v|^2} dv + \frac{1}{16r^{1/2}} \int_{B_r^+} \frac{G^2}{|v|^{3/2}} dv + C_n \left(\int_{B_r^+} X_1^{\frac{2n-2}{n-2}} |G|^{\frac{2n}{n-2}} dv \right)^{\frac{n-2}{n}}, \quad (29)$$

where $X_1 = X_1(|v|/r)$. We next change variables in (29).

We have

$$G(v) = 2^{\frac{n-2}{2}} F(T(v)) |v - e_n|^{2-n}$$

and therefore

$$\begin{aligned} |\nabla_v G(v)|^2 &= 2^{n-2} \left(|\nabla F(T(v))|^2 |v - e_n|^{2(2-n)} + 2 |v - e_n|^{2-n} F(T(v)) \nabla(F(T(v))) \cdot \nabla |v - e_n|^{2-n} \right. \\ &\quad \left. + F(T(v))^2 |\nabla |v - e_n|^{2-n}|^2 \right). \end{aligned}$$

After integration over B_r^+ and a change of variables the first term turns out to be equal to $\int_{T(B_r^+)} |\nabla F|^2 dx$. Integrating the other two terms yields

$$\begin{aligned} &\int_{B_r^+} \left(2 |v - e_n|^{2-n} F(T(v)) \nabla(F(T(v))) \cdot \nabla |v - e_n|^{2-n} + F(T(v))^2 |\nabla |v - e_n|^{2-n}|^2 \right) dv \\ &= \int_{B_r^+} \left(|v - e_n|^{2-n} \nabla(F(T(v)))^2 \cdot \nabla |v - e_n|^{2-n} + F(T(v))^2 |\nabla |v - e_n|^{2-n}|^2 \right) dv \\ &= \int_{B_r^+} (F(T(v)))^2 \left(-\operatorname{div}(|v - e_n|^{2-n} \nabla |v - e_n|^{2-n}) + |\nabla |v - e_n|^{2-n}|^2 \right) dv \\ &= 0. \end{aligned}$$

We thus conclude that

$$\int_{B_r^+} |\nabla G|^2 dv = \int_{T(B_r^+)} |\nabla F|^2 dx.$$

Using (27) and (28) we also find that

$$\int_{B_r^+} \frac{G^2}{|v|^2} dv = \int_{T(B_r^+)} \frac{4F^2}{|x - e_n|^2 |x + e_n|^2} dx.$$

The other two integrals in (29) can similarly be transformed and we conclude that (29) takes the form

$$\begin{aligned} \int_{T(B_r^+)} |\nabla F|^2 dx &\geq \frac{n^2}{4} \int_{T(B_r^+)} \frac{4F^2}{|x - e_n|^2 |x + e_n|^2} dx + \frac{1}{16r^{1/2}} \int_{T(B_r^+)} \frac{4F^2}{|x - e_n|^{3/2} |x + e_n|^{5/2}} dx \\ &\quad + C_n \left(\int_{T(B_r^+)} X_1^{\frac{2n-2}{n-2}} |F|^{\frac{2n}{n-2}} dv \right)^{\frac{n-2}{n}}, \end{aligned}$$

where $X_1 = X_1(|x - e_n|/r |x + e_n|)$. Now, it follows from (28) and some simple geometry that for any $r < 1$

$$T(B(r)) = \{x \in \mathbb{R}^n : |x - e_n| < r |x + e_n|\} = B\left(\frac{1+r^2}{1-r^2} e_n, \frac{2r}{1-r^2}\right) \supset B(e_n, r),$$

therefore

$$T(B_r^+) \supset B_1^c \cap B(e_n, r). \quad (30)$$

We will choose $\sigma_n \in (0, 1)$ such that for all $r \leq \sigma_n$ and for all $x \in B_1^c \cap B(e_n, r) \subset T(B_r^+)$ there holds

$$\frac{n^2}{4} \frac{4}{|x - e_n|^2 |x + e_n|^2} + \frac{1}{16r^{1/2}} \frac{4}{|x - e_n|^{3/2} |x + e_n|^{5/2}} \geq \frac{n^2}{4|x - e_n|^2},$$

or equivalently,

$$|x - e_n|^{1/2} \geq n^2 r^{1/2} |x + e_n|^{5/2} \left(1 - \frac{4}{|x + e_n|^2}\right). \quad (31)$$

Indeed, this is immediate for $|x + e_n| \leq 2$. Assuming that $|x + e_n| > 2$ we set $t = |x - e_n|$. We then have $|x + e_n| \leq t + 2$ and therefore (31) will follow provided

$$n^2 r^{1/2} t^{1/2} (t + 4) (t + 2)^{1/2} \leq 1,$$

for all $t \leq r$. Simple computations give that the last inequality holds true provided $t \leq 1/(75n^4r)$. This will be true for all $t \leq r$ and all $r \leq \sigma_n$ if σ_n is chosen as

$$\sigma_n = \frac{1}{\sqrt{75}n^2}.$$

Finally, the inequality $|x + e_n| \leq 3$ implies $X_1(|x - e_n|/r|x + e_n|) \geq X_1(|x - e_n|/3r)$. This completes the proof. \square

Proof of Theorem 2. We shall actually prove that the constant σ_n in the statement of the theorem is the same as the constant σ_n in the statement of Lemma 5. Without loss of generality we may assume that $\rho = 1$, the general case then follows by scaling.

We first note that from Lemma 5 and the inclusions

$$\Omega \cap B_\rho \subset \mathcal{CB}(-e_n, 1) \cap B_\rho \subset \mathcal{CB}(-e_n, 1) \cap B_{\sigma_n},$$

we obtain that for all $r \leq \sigma_n \rho$ there holds

$$\int_{\Omega \cap B_r} |\nabla u|^2 dx \geq \frac{n^2}{4} \int_{\Omega \cap B_r} \frac{u^2}{|x|^2} dx + C_n \left(\int_{\Omega \cap B_r} X_1^{\frac{2n-2}{n-2}} |u|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}}, \quad (32)$$

for all $u \in C_c^\infty(\Omega \cap B_r)$, where $X_1 = X_1(|x|/3r)$.

We apply (32) for $r = D$ (which is allowed since $D \leq \sigma_n \rho$) and the result follows immediately from the inclusion $\Omega \subset B_D$. To establish the optimality of the exponent $(2n - 2)/(n - 2)$, it is enough to show the following

Claim. *If $p < (2n - 2)/(n - 2)$ then there is no $\sigma > 1$ such that the inequality*

$$\int_{B_1 \cap B_\rho(e_n)} |\nabla u|^2 dx \geq \frac{n^2}{4} \int_{B_1 \cap B_\rho(e_n)} \frac{u^2}{|x - e_n|^2} dx + C \left(\int_{B_1 \cap B_\rho(e_n)} X_1^p |u|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}} \quad (33)$$

with $X_1 = X_1(|x - e_n|/\sigma\rho)$ holds true for some small $\rho > 0$ and some $C > 0$ and all $u \in C_c^\infty(\Omega \cap B_\rho)$.

Suppose to the contrary that (33) is true for all $u \in C_c^\infty(B_1 \cap B_\rho(e_n))$. We use the conformal map S defined by (24) to pull-back (33) to $S^{-1}(B_1 \cap B_\rho(e_n))$. We write $x = Sv$ and define

$$w(v) = u(S(v)) \left(\frac{2}{|v + e_n|^2} \right)^{\frac{n-2}{n}}.$$

Noting that $|x - e_n| = 2|v|/|v + e_n|$ we obtain that there exists $R > 0$ such that the following inequality holds true for all $w \in C_c^\infty(B_R^+)$

$$\int_{B_R^+} |\nabla w|^2 dv \geq \frac{n^2}{4} \int_{B_R^+} \frac{w^2}{|v|^2 |v + e_n|^2} dv + C \left(\int_{B_R^+} X_1^p |w|^{\frac{2n}{n-2}} dv \right)^{\frac{n-2}{n}},$$

or equivalently

$$\frac{n^2}{4} \int_{B_R^+} \frac{|v|^2 + 2v_n}{|v|^2 |v + e_n|^2} w^2 dv + \int_{B_R^+} |\nabla w|^2 dv \geq \frac{n^2}{4} \int_{B_R^+} \frac{w^2}{|v|^2} dv + C \left(\int_{B_R^+} X_1^p |w|^{\frac{2n}{n-2}} dv \right)^{\frac{n-2}{n}}, \quad (34)$$

where $X_1 = X_1(2|v|/\sigma\rho|v + e_n|)$. Now, from Lemma 1 we have the inequality

$$\int_{B_R^+} \frac{w^2}{|x|^{3/2}} dx \leq 4R^{1/2} \left(\int_{B_R^+} |\nabla w|^2 dx - \frac{n^2}{4} \int_{B_R^+} \frac{w^2}{|x|^2} dx \right). \quad (35)$$

By taking R small enough we obtain from (34) and (35) that

$$\int_{B_R^+} |\nabla w|^2 dv \geq \frac{n^2}{4} \int_{B_R^+} \frac{w^2}{|v|^2} dv + C \left(\int_{B_R^+} X_1^p |w|^{\frac{2n}{n-2}} dv \right)^{\frac{n-2}{n}}, \quad w \in C_c^\infty(B_R^+).$$

This violates the optimality of the exponent $2(n-1)/(n-2)$ of Theorem 5, concluding the proof. \square

If the radius of the exterior ball is small we then have

Theorem 9. *Let $n \geq 3$. There exist positive constants λ_n and C_n that depend only on n such that, if the radius of the exterior ball satisfies $\rho < D/\sigma_n$ the following holds true:*

$$\int_{\Omega} |\nabla u|^2 dx + \frac{\lambda_n}{\rho^2} \int_{\Omega} u^2 dx \geq \frac{n^2}{4} \int_{\Omega} \frac{u^2}{|x|^2} dx + C_n \left(\int_{\Omega} X_1^{\frac{2n-2}{n-2}} |u|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}},$$

for all $u \in C_c^\infty(\Omega)$; here $X_1 = X_1(|x|/3D)$. If in addition Ω satisfies an interior ball condition at 0 then the constant $n^2/4$ and the exponent $(2n-2)/(n-2)$ of X_1 are sharp in both inequalities.

Proof. Without loss of generality we may assume that $\rho = 1$, the general case following by scaling. We consider a C^∞ cutoff function $\phi(r)$ such that $\phi(r) = 1$ for $0 \leq r \leq \sigma_n/2$ and $\phi(r) = 0$ for $r \geq \sigma_n$, and $\phi(|x|)$ we have $0 \leq \phi \leq 1$, $|\nabla \phi| \leq C_1$, $|\Delta \phi| \leq C_2$ for some constants depending only on n . We then compute

$$\begin{aligned}
\int_{\Omega} |\nabla u|^2 dx &= \int_{\Omega} |\nabla(\phi u) + \nabla((1-\phi)u)|^2 dx \\
&= \int_{\Omega} |\nabla(\phi u)|^2 dx + \int_{\Omega} |\nabla((1-\phi)u)|^2 dx + 2 \int_{\Omega} \phi(1-\phi) |\nabla u|^2 dx \\
&\quad + \int_{\Omega} (2\phi - 1) \Delta \phi u^2 dx \\
&\geq \int_{\Omega} |\nabla(\phi u)|^2 dx + \int_{\Omega} |\nabla((1-\phi)u)|^2 dx - c_n \int_{\Omega} u^2 dx \\
(\text{by (32)}) &\geq \frac{n^2}{4} \int_{\Omega \cap B(\sigma_n)} \frac{\phi^2 u^2}{|x|^2} dx + C_n \left(\int_{\Omega \cap B(\sigma_n)} X_1^{\frac{2n-2}{n-2}} (|x|/3\sigma_n) |\phi u|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}} \\
&\quad + S_n \left(\int_{\Omega} |(1-\phi)u|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}} - c_n \int_{\Omega} u^2 dx \\
&\geq \frac{n^2}{4} \int_{\Omega} \frac{u^2}{|x|^2} dx + C_n \left(\int_{\Omega} X_1^{\frac{2n-2}{n-2}} (|x|/3\sigma_n) |\phi u|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}} \\
&\quad + S_n \left(\int_{\Omega} X_1^{\frac{2n-2}{n-2}} (|x|/3\sigma_n) |(1-\phi)u|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}} - c'_n \int_{\Omega} u^2 dx \\
&\geq \frac{n^2}{4} \int_{\Omega} \frac{u^2}{|x|^2} dx + C'_n \left(\int_{\Omega} X_1^{\frac{2n-2}{n-2}} (|x|/3D) |u|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}} - c'_n \int_{\Omega} u^2 dx,
\end{aligned}$$

where for the last inequality we used the fact that $D > \sigma_n$.

The sharpness of the constant $n^2/4$ and of the exponent $(2n-2)/(n-2)$ follow as in the proof of Theorems 1 and 2. \square

5. Characterizing maximal potentials

Throughout this section we assume that Ω satisfies both an interior and exterior ball condition at 0. Without loss of generality we may assume that the exterior ball at 0 is $B(-2\rho e_n, 2\rho)$ for some $\rho > 0$.

Our starting point is the following improved Hardy inequality contained in Theorem 9,

$$\lambda \int_{\Omega} u^2 dx + \int_{\Omega} |\nabla u|^2 dx \geq \frac{n^2}{4} \int_{\Omega} \frac{u^2}{|x|^2} dx \quad (36)$$

for all $u \in C_c^\infty(\Omega)$. We shall be interested in the problem of improvements of (36) and whether corresponding best constants are attained. In connection with this we make the following definition

Definition 1. A non-negative potential $V \in L_{\text{loc}}^{n/2}(\overline{\Omega} \setminus \{0\})$ is called *admissible* if there exist $\lambda \geq 0$ and $C > 0$ such that

$$\lambda \int_{\Omega} u^2 dx + \int_{\Omega} |\nabla u|^2 dx \geq \frac{n^2}{4} \int_{\Omega} \frac{u^2}{|x|^2} dx + C \int_{\Omega} V u^2 dx, \quad u \in C_c^\infty(\Omega). \quad (37)$$

The class of all admissible potentials for the domain Ω is denoted by $\mathcal{A}(\Omega)$.

For a given $V \in \mathcal{A}(\Omega)$ we denote by $b(\lambda) > 0$ the best constant C of inequality (37). We next address the question whether there exists non-negative potentials $W \in \mathcal{A}(\Omega)$ and a positive constant C such that

$$\lambda \int_{\Omega} u^2 dx + \int_{\Omega} |\nabla u|^2 dx \geq \frac{n^2}{4} \int_{\Omega} \frac{u^2}{|x|^2} dx + b(\lambda) \int_{\Omega} V u^2 dx + C \int_{\Omega} W u^2 dx, \quad u \in C_c^\infty(\Omega).$$

In case there does not exist such a potential W we say that the potential

$$\frac{n^2}{4} \frac{1}{|x|^2} + b(\lambda) V(x),$$

is a *maximal potential*. Our next goal is to characterize maximal potentials. In this direction for $V \in \mathcal{A}(\Omega)$ and small $r > 0$ we define

$$C_r(V) = \inf_{u \in C_c^\infty(\Omega \cap B_r)} \frac{\lambda \int_{\Omega \cap B_r} u^2 dx + \int_{\Omega \cap B_r} |\nabla u|^2 dx - \frac{n^2}{4} \int_{\Omega \cap B_r} \frac{u^2}{|x|^2} dx}{\int_{\Omega \cap B_r} V u^2 dx}. \quad (38)$$

Since $C_r(V)$ is a non-increasing function we can define

$$\mathcal{C}^0(V) = \lim_{r \rightarrow 0+} C_r(V),$$

which may also be equal to $+\infty$. This definition gives the impression that $\mathcal{C}^0(V)$ might depend on the choice of λ . We will now establish that $\mathcal{C}^0(V)$ is independent of λ . Let us denote at the moment the infimum in (38) by $\mathcal{C}^0(V, \lambda)$ to express the dependence on λ . We have seen (cf. (32)) that for small $r > 0$ there exists a positive constant C_n that depends only on n so that

$$\int_{\Omega \cap B_r} |\nabla u|^2 dx \geq \frac{n^2}{4} \int_{\Omega \cap B_r} \frac{u^2}{|x|^2} dx + C_n \left(\int_{\Omega \cap B_r} X_1^{\frac{2n-2}{n-2}} |u|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}}, \quad u \in C_c^\infty(\Omega \cap B_r).$$

Using Holder's inequality we conclude the existence of a positive constant c independent of r , so that for small r we have

$$\int_{\Omega \cap B_r} |\nabla u|^2 dx \geq \frac{n^2}{4} \int_{\Omega \cap B_r} \frac{u^2}{|x|^2} dx + \frac{c}{r^2} \int_{\Omega \cap B_r} u^2 dx, \quad u \in C_c^\infty(\Omega \cap B_r).$$

This implies

$$\begin{aligned} 0 &\leq \frac{\int_{\Omega \cap B_r} |\nabla u|^2 dx - \frac{n^2}{4} \int_{\Omega \cap B_r} \frac{u^2}{|x|^2} dx}{\int_{\Omega \cap B_r} V u^2 dx} \leq \frac{\lambda \int_{\Omega \cap B_r} u^2 dx + \int_{\Omega \cap B_r} |\nabla u|^2 dx - \frac{n^2}{4} \int_{\Omega \cap B_r} \frac{u^2}{|x|^2} dx}{\int_{\Omega \cap B_r} V u^2 dx} \\ &\leq (1 + \lambda c r^2) \frac{\int_{\Omega \cap B_r} |\nabla u|^2 dx - \frac{n^2}{4} \int_{\Omega \cap B_r} \frac{u^2}{|x|^2} dx}{\int_{\Omega \cap B_r} V u^2 dx}. \end{aligned}$$

Hence $C_r(V, 0) \leq C_r(V, \lambda) \leq (1 + \lambda c r^2) C_r(V, 0)$. Letting $r \rightarrow 0$ we conclude that $\mathcal{C}^0(V)$ is indeed independent of the choice of $\lambda \geq 0$.

Definition 2. We say that the potential $V \in \mathcal{A}(\Omega)$ is subcritical if $\mathcal{C}^0(V) = +\infty$.

Lemma 6. *Let V be a non-negative potential satisfying*

$$\int_{\Omega} V^{n/2} X_1^{1-n} dx < +\infty, \quad (39)$$

where $X_1 = X_1(|x|/D)$. Then V is a subcritical potential.

Proof. Applying Theorem 2 we obtain that for $r > 0$ small enough we have

$$\int_{\Omega \cap B_r} |\nabla u|^2 dx \geq \frac{n^2}{4} \int_{\Omega \cap B_r} \frac{u^2}{|x|^2} dx + C_n \left(\int_{\Omega \cap B_r} X_1^{\frac{2n-2}{n-2}} |u|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}}$$

for all $u \in C_c^\infty(\Omega \cap B_r)$, where $X_1 = X_1(|x|/D)$. Applying Hölder inequality we then easily obtain that

$$\frac{\int_{\Omega \cap B_r} |\nabla u|^2 dx - \frac{n^2}{4} \int_{\Omega \cap B_r} \frac{u^2}{|x|^2} dx}{\int_{\Omega \cap B_r} V u^2 dx} \geq \frac{1}{C_n} \left(\int_{\Omega \cap B_r} V^{n/2} X_1^{1-n} dx \right)^{-\frac{2}{n}}.$$

Letting $r \rightarrow 0+$ we conclude that $\mathcal{C}^0(V) = +\infty$. \square

We shall also consider the following more general situation. Assume that V, W_1, W_2 are non-negative potentials in $\mathcal{A}(\Omega)$ and assume that there exist $c > 0$ and a radius $R > 0$ so that

$$\int_{\Omega \cap B_R} W_1 u^2 dx + \int_{\Omega \cap B_R} |\nabla u|^2 dx \geq \frac{n^2}{4} \int_{\Omega \cap B_R} \frac{u^2}{|x|^2} dx + \int_{\Omega \cap B_R} W_2 u^2 dx + c \int_{\Omega \cap B_R} V u^2 dx \quad (40)$$

for all $u \in C_c^\infty(\Omega \cap B_R)$. For $0 < r < R$ we define

$$C_r(W_1, W_2; V) = \inf_{C_c^\infty(\Omega \cap B_r)} \frac{\int_{\Omega \cap B_r} W_1 u^2 dx + \int_{\Omega \cap B_r} |\nabla u|^2 dx - \frac{n^2}{4} \int_{\Omega \cap B_r} \frac{u^2}{|x|^2} dx - \int_{\Omega \cap B_r} W_2 u^2 dx}{\int_{\Omega \cap B_r} V u^2 dx}$$

and we denote

$$\mathcal{C}^0(W_1, W_2; V) = \lim_{r \rightarrow 0} C_r(W_1, W_2; V).$$

We next show that subcritical potentials do not affect the concentration level $\mathcal{C}^0(V)$. More precisely we have

Lemma 7. *Let V, W_1, W_2 be non-negative potentials in $\mathcal{A}(\Omega)$ and assume that for some $R > 0$ there exists $c > 0$ such that (40) holds true. If in addition W_1, W_2 are subcritical then $\mathcal{C}^0(W_1, W_2; V) = \mathcal{C}^0(V)$.*

Proof. The subcriticality of W_i implies that for small $r > 0$ there holds

$$\int_{\Omega \cap B_r} |\nabla u|^2 dx \geq \frac{n^2}{4} \int_{\Omega \cap B_r} \frac{u^2}{|x|^2} dx + c_r(W_i) \int_{\Omega \cap B_r} W_i u^2 dx, \quad u \in C_c^\infty(\Omega \cap B_r), \quad (41)$$

with $\lim_{r \rightarrow 0} c_r(W_i) = +\infty$. From inequalities (40) and (41) follows that for $r > 0$ small enough we have

$$\begin{aligned}
& \left(1 - \frac{1}{c_r(W_2)}\right) \frac{\int_{\Omega \cap B_r} |\nabla u|^2 dx - \frac{n^2}{4} \int_{\Omega \cap B_r} \frac{u^2}{|x|^2} dx}{\int_{\Omega \cap B_r} V u^2 dx} \\
& \leq \frac{\int_{\Omega \cap B_r} |\nabla u|^2 dx - \frac{n^2}{4} \int_{\Omega \cap B_r} \frac{u^2}{|x|^2} dx + \int_{\Omega \cap B_r} (W_1 - W_2) u^2 dx}{\int_{\Omega \cap B_r} V u^2 dx} \\
& \leq \left(1 + \frac{1}{c_r(W_1)}\right) \frac{\int_{\Omega \cap B_r} |\nabla u|^2 dx - \frac{n^2}{4} \int_{\Omega \cap B_r} \frac{u^2}{|x|^2} dx}{\int_{\Omega \cap B_r} V u^2 dx}.
\end{aligned}$$

This implies that

$$\left(1 - \frac{1}{c_r(W_2)}\right) c_r(V) \leq C_r(W_1, W_2; V) \leq \left(1 + \frac{1}{c_r(W_1)}\right) c_r(V),$$

and the result follows by letting $r \rightarrow 0+$. \square

Given $u \in C_c^\infty(\Omega)$ we define the function w by

$$u(x) = \frac{|x + \rho e_n|^2 - \rho^2}{|x|^{\frac{n}{2}} |x + 2\rho e_n|^{\frac{n}{2}}} w(x). \quad (42)$$

Then $w \in C_c^\infty(\Omega)$ by our assumption that the exterior ball at zero is $B(-2\rho e_n, 2\rho)$. After some computations and using integration by parts we arrive at

$$\int_{\Omega} |\nabla u|^2 dx = \int_{\Omega} \frac{(|x + \rho e_n|^2 - \rho^2)^2}{|x|^n |x + 2\rho e_n|^n} |\nabla w|^2 dx + n^2 \rho^2 \int_{\Omega} \frac{(|x + \rho e_n|^2 - \rho^2)^2}{|x|^{n+2} |x + 2\rho e_n|^{n+2}} w^2 dx,$$

so inequality (37) is written

$$\begin{aligned}
& \lambda \int_{\Omega} \frac{(|x + \rho e_n|^2 - \rho^2)^2}{|x|^n |x + 2\rho e_n|^n} w^2 dx + \int_{\Omega} \frac{(|x + \rho e_n|^2 - \rho^2)^2}{|x|^n |x + 2\rho e_n|^n} |\nabla w|^2 dx + n^2 \rho^2 \int_{\Omega} \frac{(|x + \rho e_n|^2 - \rho^2)^2}{|x|^{n+2} |x + 2\rho e_n|^{n+2}} w^2 dx \\
& \geq \frac{n^2}{4} \int_{\Omega} \frac{(|x + \rho e_n|^2 - \rho^2)^2}{|x|^{n+2} |x + 2\rho e_n|^n} w^2 dx + c \int_{\Omega} \frac{(|x + \rho e_n|^2 - \rho^2)^2}{|x|^n |x + 2\rho e_n|^n} V w^2 dx
\end{aligned}$$

which can also take the equivalent form

$$\begin{aligned}
& \lambda \int_{\Omega} \frac{(|x + \rho e_n|^2 - \rho^2)^2}{|x|^n |x + 2\rho e_n|^n} w^2 dx + \int_{\Omega} \frac{(|x + \rho e_n|^2 - \rho^2)^2}{|x|^n |x + 2\rho e_n|^n} |\nabla w|^2 dx \\
& \geq \frac{n^2}{4} \int_{\Omega} \frac{(|x + \rho e_n|^2 - \rho^2)^2 (|x|^2 + 4\rho x_n)}{|x|^{n+2} |x + 2\rho e_n|^{n+2}} w^2 dx + c \int_{\Omega} \frac{(|x + \rho e_n|^2 - \rho^2)^2}{|x|^n |x + 2\rho e_n|^n} V w^2 dx.
\end{aligned} \quad (43)$$

It is clear from the above that (43) is valid for all functions $w \in C_c^\infty(\Omega)$ and moreover, for a fixed $\lambda \geq 0$ the best constants c for inequalities (37) and (43) coincide. That common best constant shall be denoted by $b = b(\lambda)$.

Denoting

$$\phi(x) = \frac{|x + \rho e_n|^2 - \rho^2}{|x|^{\frac{n}{2}} |x + 2\rho e_n|^{\frac{n}{2}}}$$

it then follows that (cf. (38))

$$C_r(V) = \inf_{W_0^{1,2}(\Omega \cap B_r; \phi^2)} \frac{\lambda \int_{\Omega \cap B_r} \phi^2 w^2 dx + \int_{\Omega \cap B_r} \phi^2 |\nabla w|^2 dx - \frac{n^2}{4} \int_{\Omega \cap B_r} \frac{|x|^2 + 4\rho x_n}{|x|^2 |x + 2\rho e_n|^2} \phi^2 w^2 dx}{\int_{\Omega \cap B_r} V \phi^2 w^2 dx}$$

where $W_0^{1,2}(\Omega \cap B_r; \phi^2)$ denotes the closure of $C_c^\infty(\Omega \cap B_r)$ under the norm

$$\int_{\Omega \cap B_r} \phi^2 |\nabla w|^2 dx + \int_{\Omega \cap B_r} \phi^2 w^2 dx.$$

We shall now see a simpler way for expressing $\mathcal{C}^0(V) = \lim_{r \rightarrow 0+} C_r(V)$ in terms of the weight ϕ^2 . For this we define

$$C_r(V; \phi^2) = \inf_{W_0^{1,2}(\Omega \cap B_r; \phi^2)} \frac{\int_{\Omega \cap B_r} \phi^2 |\nabla w|^2 dx}{\int_{\Omega \cap B_r} V \phi^2 w^2 dx},$$

and

$$\mathcal{C}^0(V; \phi^2) = \lim_{r \rightarrow 0+} C_r(V; \phi^2).$$

Lemma 8. *Let V be a non-negative potential in $\mathcal{A}(\Omega)$. Then $\mathcal{C}^0(V) = \mathcal{C}^0(V; \phi^2)$.*

Proof. For the sake of simplicity we assume that $\rho = 1$; the general case then follows by scaling. On the one hand we have for small $r > 0$ that

$$cr^{-2} \int_{\Omega \cap B_r} \phi^2 w^2 dx \leq \int_{\Omega \cap B_r} \phi^2 |\nabla w|^2 dx - \frac{n^2}{4} \int_{\Omega \cap B_r} \frac{|x|^2 + 4x_n}{|x|^2 |x + 2e_n|^2} \phi^2 w^2 dx, \quad u \in C_c^\infty(\Omega \cap B_r)$$

for some universal constant $c > 0$, which implies the inequality

$$\frac{\lambda \int_{\Omega \cap B_r} \phi^2 w^2 dx + \int_{\Omega \cap B_r} \phi^2 |\nabla w|^2 dx - \frac{n^2}{4} \int_{\Omega \cap B_r} \frac{|x|^2 + 4x_n}{|x|^2 |x + 2e_n|^2} \phi^2 w^2 dx}{\int_{\Omega \cap B_r} \phi^2 V w^2 dx} \leq \left(1 + \frac{\lambda}{c} r^2\right) \frac{\int_{\Omega \cap B_r} \phi^2 |\nabla w|^2 dx}{\int_{\Omega \cap B_r} \phi^2 V w^2 dx}. \quad (44)$$

On the other hand, since

$$0 \leq \frac{|x|^2 + 4x_n}{|x|^2 |x + 2e_n|^2} \leq \frac{c}{|x|},$$

we have the inequality

$$cr^{-1} \int_{\Omega \cap B_r} \frac{|x|^2 + 4x_n}{|x|^2 |x + 2e_n|^2} \phi^2 w^2 dx \leq \int_{\Omega \cap B_r} \phi^2 |\nabla w|^2 dx, \quad w \in C_c^\infty(\Omega \cap B_r)$$

which in turn implies

$$\frac{\int_{\Omega \cap B_r} \phi^2 |\nabla w|^2 dx}{\int_{\Omega \cap B_r} \phi^2 V w^2 dx} (1 - cr) \leq \frac{\lambda \int_{\Omega \cap B_r} \phi^2 w^2 dx + \int_{\Omega \cap B_r} \phi^2 |\nabla w|^2 dx - \frac{n^2}{4} \int_{\Omega \cap B_r} \frac{|x|^2 + 4x_n}{|x|^2 |x + 2e_n|^2} \phi^2 w^2 dx}{\int_{\Omega \cap B_r} \phi^2 V w^2 dx}. \quad (45)$$

The result follows by combining inequalities (44) and (45) and letting $r \rightarrow 0+$. \square

One important consequence of subcriticality is the following compactness property.

Lemma 9. *Assume that the positive potential $V \in \mathcal{A}(\Omega)$ is subcritical. Then for any sequence (w_k) which is bounded in $W_0^{1,2}(\Omega; \phi^2)$ there exists a subsequence, also denoted by (w_k) , and a function $w_0 \in W_0^{1,2}(\Omega; \phi^2)$ so that*

$$\begin{aligned} \text{(i)} \quad & w_k \rightharpoonup w_0 \text{ in } W_0^{1,2}(\Omega; \phi^2) \\ \text{(ii)} \quad & \int_{\Omega} \phi^2 V (w_k - w_0)^2 dx \longrightarrow 0. \end{aligned}$$

Proof. Part (i) is standard. To prove (ii) we may assume without loss of generality that $w_0 = 0$. We consider a small $r > 0$ and a smooth cut-off function ψ such that $\psi = 1$ in $B_{r/2}$ and $\psi = 0$ outside B_r . We then have

$$\begin{aligned} \int_{\Omega} \phi^2 |\nabla w_k|^2 dx &= \int_{\Omega} \phi^2 |\nabla(\psi w_k) + \nabla((1 - \psi)w_k)|^2 dx \\ &= \int_{\Omega} \phi^2 |\nabla(\psi w_k)|^2 dx + \int_{\Omega} \phi^2 |\nabla((1 - \psi)w_k)|^2 dx + 2 \int_{\Omega} \phi^2 \psi(1 - \psi) |\nabla w_k|^2 dx \\ &\quad + \int_{\Omega} \phi^2 (1 - 2\psi) w_k \nabla \psi \cdot \nabla w_k dx - \int_{\Omega} \phi^2 |\nabla \psi|^2 w_k^2 dx \\ &\geq \int_{\Omega} \phi^2 |\nabla(\psi w_k)|^2 dx + o(1) \\ &\geq C_r(V; \phi^2) \int_{\Omega} \phi^2 V \psi^2 w_k^2 dx + o(1) \\ &= C_r(V; \phi^2) \int_{\Omega} \phi^2 V w_k^2 dx + o(1), \end{aligned}$$

that is

$$\int_{\Omega} V \phi^2 w_k^2 dx \leq \frac{1}{C_r(V; \phi^2)} \int_{\Omega} \phi^2 |\nabla w_k|^2 dx + o(1). \quad (46)$$

The result follows by noting that the RHS of (46) can be made arbitrarily small by choosing $r > 0$ small enough. \square

We can now state and prove the main result of this section.

Theorem 10. *Let $V \in \mathcal{A}(\Omega)$ and $\lambda \geq 0$ be given and let $b(\lambda)$ be the best constant for the inequality*

$$\lambda \int_{\Omega} \phi^2 w^2 dx + \int_{\Omega} \phi^2 |\nabla w|^2 dx \geq \frac{n^2}{4} \int_{\Omega} \frac{|x|^2 + 4\rho x_n}{|x|^2 |x + 2\rho e_n|^2} \phi^2 w^2 dx + b(\lambda) \int_{\Omega} V \phi^2 w^2 dx, \quad w \in W_0^{1,2}(\Omega; \phi^2). \quad (47)$$

If in addition

$$b(\lambda) < C^0(V)$$

then the best constant $b(\lambda)$ in (47) is realized by a function $w_0 \in W_0^{1,2}(\Omega; \phi^2)$. In particular the best constant $b(\lambda)$ is realized if the potential V is subcritical.

Proof. We denote

$$Q(x) = \frac{n^2}{4} \frac{|x|^2 + 4\rho x_n}{|x|^2|x + 2\rho e_n|^2}.$$

Then it is easily seen that

$$0 \leq Q \leq \frac{c}{|x|}, \quad x \in \Omega,$$

which implies that Q is a subcritical potential by Lemma 6. We consider a minimizing sequence (w_k) for (47) and without loss of generality we assume that

$$\int_{\Omega} Q \phi^2 w_k^2 dx + b(\lambda) \int_{\Omega} V \phi^2 w_k^2 dx = 1, \quad \lambda \int_{\Omega} \phi^2 w_k^2 dx + \int_{\Omega} \phi^2 |\nabla w_k|^2 dx \longrightarrow 1, \text{ as } k \rightarrow \infty. \quad (48)$$

Since (w_k) is bounded in $W_0^{1,2}(\Omega; \phi^2)$, it has a subsequence, which we assume is (w_k) itself, which converges weakly to some $w_0 \in W_0^{1,2}(\Omega; \phi^2)$. We define $v_k = w_k - w_0$.

We consider a small enough $r > 0$ so that $C_r(V; \phi^2) > b(\lambda)$ and a smooth cut-off function ψ such that $\psi = 1$ in $B_{r/2}$ and $\psi = 0$ outside B_r . Arguing as in the proof of Lemma 9 we have

$$\begin{aligned} \int_{\Omega} \phi^2 |\nabla v_k|^2 dx &= \int_{\Omega} \phi^2 |\nabla [\psi v_k + (1 - \psi)v_k]|^2 dx \\ &\geq \int_{\Omega} \phi^2 |\nabla (\psi v_k)|^2 dx + o(1) \\ &\geq C_r(V; \phi^2) \int_{\Omega} \phi^2 V \psi^2 v_k^2 dx + o(1) \\ &= C_r(V; \phi^2) \int_{\Omega} \phi^2 V v_k^2 dx + o(1). \end{aligned} \quad (49)$$

Now, substituting $w_k = v_k + w_0$ in the normalization relations (48) and using Lemma 9 we obtain

$$\int_{\Omega} \phi^2 Q w_0^2 dx + b(\lambda) \int_{\Omega} \phi^2 V w_0^2 dx + b(\lambda) \int_{\Omega} \phi^2 V v_k^2 dx = 1 + o(1) \quad (50)$$

and

$$\lambda \int_{\Omega} \phi^2 w_0^2 dx + \int_{\Omega} \phi^2 |\nabla w_0|^2 dx + \int_{\Omega} \phi^2 |\nabla v_k|^2 dx = 1 + o(1). \quad (51)$$

From (49) and (50) we obtain

$$\int_{\Omega} \phi^2 |\nabla v_k|^2 dx \geq \frac{C_r(V; \phi^2)}{b(\lambda)} \left(1 - \int_{\Omega} \phi^2 Q w_0^2 dx + b(\lambda) \int_{\Omega} V \phi^2 w_0^2 dx \right) + o(1). \quad (52)$$

Moreover using (47) for $w = w_0$ we obtain from (51) that

$$\int_{\Omega} \phi^2 |\nabla v_k|^2 dx \leq 1 - \int_{\Omega} \phi^2 Q w_0^2 dx - b(\lambda) \int_{\Omega} \phi^2 V w_0^2 dx + o(1). \quad (53)$$

From (52) and (53) we conclude that

$$\left(1 - \frac{C_r(V; \phi^2)}{b(\lambda)}\right) \left(1 - \int_{\Omega} \phi^2 Q w_0^2 dx + b(\lambda) \int_{\Omega} V \phi^2 w_0^2 dx\right) \geq 0.$$

Since $C_r(V; \phi^2) > b(\lambda)$, this implies that

$$\int_{\Omega} \phi^2 Q w_0^2 dx + b(\lambda) \int_{\Omega} \phi^2 V w_0^2 dx \geq 1.$$

But by lower semicontinuity,

$$\int_{\Omega} \phi^2 Q w_0^2 dx + b(\lambda) \int_{\Omega} \phi^2 V w_0^2 dx \leq 1.$$

Hence w_0 is a minimizer. \square

The next theorem is an immediate consequence of Theorem 10

Theorem 11. *Let V be a non-negative potential in $\mathcal{A}(\Omega)$.*

(a) *Let $\lambda \geq 0$ and $b(\lambda) > 0$ be such that*

$$\lambda \int_{\Omega} u^2 dx + \int_{\Omega} |\nabla u|^2 dx \geq \frac{n^2}{4} \int_{\Omega} \frac{u^2}{|x|^2} dx + b(\lambda) \int_{\Omega} V u^2 dx, \quad u \in C_c^\infty(\Omega), \quad (54)$$

where $b(\lambda)$ is the best constant. If in addition

$$b(\lambda) < \mathcal{C}^0(V),$$

then the potential $n^2/4|x|^2 + b(\lambda)V(x)$ is a maximal potential, that is inequality (54) cannot be improved by adding a non-negative potential W in the RHS.

(b) *If V is a subcritical potential then there exist $\lambda \geq 0$ and a best constant $b(\lambda) > 0$ such that (54) is true. Moreover the potential $n^2/4|x|^2 + b(\lambda)V(x)$ is a maximal potential.*

Proof. (a) Suppose that $n^2/4|x|^2 + b(\lambda)V(x)$ is not a maximal potential, that is there exists a non-trivial potential $W \geq 0$ in $\mathcal{A}(\Omega)$ such that

$$\lambda \int_{\Omega} u^2 dx + \int_{\Omega} |\nabla u|^2 dx \geq \frac{n^2}{4} \int_{\Omega} \frac{u^2}{|x|^2} dx + b(\lambda) \int_{\Omega} V u^2 dx + \int_{\Omega} W u^2 dx$$

holds true for all $u \in H_0^1(\Omega)$. Using the transformation (42) this is equivalently written as

$$\lambda \int_{\Omega} \phi^2 w^2 dx + \int_{\Omega} \phi^2 |\nabla w|^2 dx \geq \int_{\Omega} Q \phi^2 w^2 dx + b(\lambda) \int_{\Omega} \phi^2 V w^2 dx + \int_{\Omega} W \phi^2 w^2 dx, \quad w \in W_0^{1,2}(\Omega; \phi^2).$$

Using $w = w_0$ where w_0 is the minimizer from Theorem 10 we conclude that $\int_{\Omega} W \phi^2 w_0^2 dx \leq 0$, which is a contradiction.

Part (b) is an immediate consequence of part (a) since any subcritical potential V is in $\mathcal{A}(\Omega)$ and satisfies $\mathcal{C}^0(V) = +\infty$. \square

6. Logarithmic improvements and maximal potentials

Throughout this section we continue to assume that Ω satisfies both an interior and exterior ball condition at 0. We also continue to assume that the exterior ball at 0 is $B(-2\rho e_n, 2\rho)$ for some $\rho > 0$.

In this section we will provide the proofs of Theorems 3, 4 and also study maximal potentials in the context of logarithmic improvements of Hardy inequality.

6.1. Logarithmic improvements

To prove Theorems 3 and 4 we first establish the following lemmas:

Lemma 10. *Let $n \geq 2$. There exists a positive constant σ_n depending only on n such that for all $\rho > 0$ and all $r \leq \sigma_n \rho$ we have*

$$\begin{aligned}
 \text{(i)} \quad & \int_{CB(\rho) \cap B(\rho e_n, r)} |\nabla u|^2 dx \geq \frac{n^2}{4} \int_{CB(\rho) \cap B(\rho e_n, r)} \frac{u^2}{|x - \rho e_n|^2} dx \\
 & \quad + \frac{1}{4} \sum_{i=1}^{\infty} \int_{CB(\rho) \cap B(\rho e_n, r)} \frac{u^2}{|x - \rho e_n|^2} X_1^2 \dots X_i^2 dx \\
 \text{(ii)} \quad & \text{If in addition } n \geq 3 \text{ there exists a constant } C_n \text{ depending only on } n \text{ such that for all } m \in \mathbb{N} \\
 & \int_{CB(\rho) \cap B(\rho e_n, r)} |\nabla u|^2 dx \geq \frac{n^2}{4} \int_{CB(\rho) \cap B(\rho e_n, r)} \frac{u^2}{|x - \rho e_n|^2} dx \\
 & \quad + \frac{1}{4} \sum_{i=1}^m \int_{CB(\rho) \cap B(\rho e_n, r)} \frac{u^2}{|x - \rho e_n|^2} X_1^2 \dots X_i^2 dx \\
 & \quad + C_n \left(\int_{CB(\rho) \cap B(\rho e_n, r)} (X_1 \dots X_{m+1})^{\frac{2n-2}{n-2}} |u|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}}. \tag{55}
 \end{aligned}$$

Both inequalities are valid for all $u \in C_c^\infty(CB(\rho) \cap B(\rho e_n, r))$ and in both cases $X_i = X_i(|x - \rho e_n|/(3\kappa r))$.

Proof. To prove (i) it is enough to consider the case $\rho = 1$. We fix $r < 1$ and we apply Lemma 1. Changing variables via T (cf. (26)) we obtain

$$\begin{aligned}
 \int_{T(B_r^+)} |\nabla u|^2 dx & \geq \frac{n^2}{4} \int_{T(B_r^+)} \frac{4u^2}{|x - e_n|^2 |x + e_n|^2} dx \\
 & \quad + \frac{1}{4} \sum_{i=1}^{\infty} \int_{T(B_r^+)} \frac{4u^2 X_1^2 \dots X_i^2}{|x - e_n|^2 |x + e_n|^2} dx + \frac{1}{8r^{1/2}} \int_{T(B_r^+)} \frac{4u^2}{|x - e_n|^{3/2} |x + e_n|^{5/2}} dx \tag{56}
 \end{aligned}$$

for all $u \in C_c^\infty(T(B_r^+))$; here $X_k = X_k(|x - e_n|/(\kappa r|x + e_n|))$. As already noted (cf. (30)) we have

$$\mathcal{CB}_1 \cap B(e_n, r) \subset T(B_r^+).$$

So integrals in (56) can be taken over $\mathcal{CB}_1 \cap B(e_n, r)$.

Once again it is enough to find $\sigma_n < 1$ such that for all $r \leq \sigma_n$ and all $x \in \mathcal{CB}_1 \cap B(e_n, r)$ there holds

$$\begin{aligned} & \frac{n^2}{4} \frac{4}{|x - e_n|^2 |x + e_n|^2} + \frac{1}{4} \sum_{i=1}^{\infty} \frac{4}{|x - e_n|^2 |x + e_n|^2} X_1^2 \dots X_i^2 \\ & + \frac{4}{8r^{1/2} |x - e_n|^{3/2} |x + e_n|^{5/2}} \geq \frac{n^2}{4|x - e_n|^2} + \frac{1}{4|x - e_n|^2} \sum_{i=1}^{\infty} X_1^2 \dots X_i^2, \end{aligned}$$

or equivalently,

$$|x - e_n|^{1/2} \geq \frac{1}{2} r^{1/2} |x + e_n|^{5/2} \left(1 - \frac{4}{|x + e_n|^2} \right) \left(n^2 + \sum_{i=1}^{\infty} X_1^2 \dots X_i^2 \right).$$

This is immediate if $|x + e_n| \leq 2$. Assuming $|x + e_n| \geq 2$ we will actually establish the stronger inequality (recalling that $\sum_k X_k^2 \leq 1/4$)

$$|x - e_n|^{1/2} \geq \frac{n^2 + \frac{1}{4}}{2} r^{1/2} |x + e_n|^{5/2} \left(1 - \frac{4}{|x + e_n|^2} \right). \quad (57)$$

But this is almost the same as inequality (31), the only difference being that in the place of n^2 we now have $(n^2 + 1/4)/2$; we omit further details.

The proof of (ii) is analogous, but we now use Lemma 3 instead of Lemma 1. Again, we may take $\rho = 1$. We then fix $r < 1$ and changing variables via T we obtain

$$\begin{aligned} & \int_{T(B_r^+)} |\nabla u|^2 dx \geq \frac{n^2}{4} \int_{T(B_r^+)} \frac{4u^2}{|x - e_n|^2 |x + e_n|^2} dx + \frac{1}{4} \sum_{i=1}^m \int_{T(B_r^+)} \frac{4u^2 X_1^2 \dots X_i^2}{|x - e_n|^2 |x + e_n|^2} dx \\ & + \frac{1}{16r^{1/2}} \int_{T(B_r^+)} \frac{4u^2}{|x - e_n|^{3/2} |x + e_n|^{5/2}} dx + C_n \left(\int_{T(B_r^+)} (X_1 \dots X_{m+1})^{\frac{2n-2}{n-2}} |u|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}} \end{aligned} \quad (58)$$

for all $u \in C_c^\infty(T(B_r^+))$; here $X_k = X_k(|x - e_n|/(\kappa r |x + e_n|))$. As in the proof of part (i), this is also true if the integrals are taken over $\mathcal{CB}_1 \cap B(e_n, r)$ and $u \in C_c^\infty(\mathcal{CB}_1 \cap B(e_n, r))$

Hence the result will follow once we establish for all $x \in \mathcal{CB}_1 \cap B(e_n, r)$ the inequality

$$\begin{aligned} & \frac{n^2}{4} \frac{4}{|x - e_n|^2 |x + e_n|^2} + \frac{1}{4} \sum_{i=1}^m \frac{4}{|x - e_n|^2 |x + e_n|^2} X_1^2 \dots X_i^2 \\ & + \frac{4}{16r^{1/2} |x - e_n|^{3/2} |x + e_n|^{5/2}} \geq \frac{n^2}{4|x - e_n|^2} + \frac{1}{4|x - e_n|^2} \sum_{i=1}^m X_1^2 \dots X_i^2. \end{aligned}$$

This is equivalent to

$$|x - e_n|^{1/2} \geq r^{1/2} |x + e_n|^{5/2} \left(1 - \frac{4}{|x + e_n|^2} \right) \left(n^2 + \sum_{i=1}^m X_1^2 \dots X_i^2 \right).$$

The argument now goes as in part (i); we omit further details. \square

Proof of Theorem 3. Without loss of generality we assume that $\rho = 1$. We use part (i) of Lemma 10 for $r = \sigma_n$ making a translation of (55) by $-e_n$. We obtain

$$\int_{\mathcal{CB}(-e_n, 1) \cap B(\sigma_n)} |\nabla u|^2 dx \geq \frac{n^2}{4} \int_{\mathcal{CB}(-e_n, 1) \cap B(\sigma_n)} \frac{u^2}{|x|^2} dx + \frac{1}{4} \sum_{i=1}^{\infty} \int_{\mathcal{CB}(-e_n, 1) \cap B(\sigma_n)} \frac{u^2}{|x|^2} X_1^2 \dots X_i^2 dx,$$

for all $u \in C_c^\infty(\mathcal{CB}(-e_n, 1) \cap B(\sigma_n))$, where $X_i = X_i(|x|/(3\kappa\sigma_n))$. Since $\Omega \subset \mathcal{CB}(-e_n, 1) \cap B(\sigma_n)$, the result follows. \square

Theorem 12. Let $n \geq 2$. There exists a positive constant λ_n that depends only on n such that, if the radius of the exterior ball satisfies $\rho < D/\sigma_n$, with σ_n as in Theorem 4, the following holds true:

$$\int_{\Omega} |\nabla u|^2 dx + \frac{\lambda_n}{\rho^2} \int_{\Omega} u^2 dx \geq \frac{n^2}{4} \int_{\Omega} \frac{u^2}{|x|^2} dx + \frac{1}{4} \sum_{i=1}^{\infty} \int_{\Omega} \frac{u^2}{|x|^2} X_1^2 \dots X_i^2 dx$$

for all $u \in C_c^\infty(\Omega)$; here $X_i = X_i(|x|/(3\kappa D))$. If in addition Ω satisfies an interior ball condition at 0 then the constants $1/4$ are sharp at each step.

Proof. We argue as in the proof of Theorem 2. This time however we also use the global estimate $\sum_i X_1^2 \dots X_i^2 \leq 1/4$ in order to estimate uniformly the constant in front of the L^2 term. We omit further details. The sharpness of the constants $1/4$ has already been proved. \square

Proof of Theorem 4. We argue as in the proof of Theorem 3, using now part (ii) of Lemma 10. To prove the sharpness of the constants $1/4$ and the exponent $(2n-2)/(n-2)$ we argue as in the proof of Theorem 2; we omit the details. \square

In case the exterior ball is small, working as in Theorem 9 we have the following

Theorem 13. Let $n \geq 3$. There exist positive constants λ_n and C_n that depend only on n such that, if the radius of the exterior ball satisfies $\rho < D/\sigma_n$, with σ_n as in Theorem 4, then for any $m \in \mathbb{N}$ the following holds true:

$$\begin{aligned} \int_{\Omega} |\nabla u|^2 dx + \frac{\lambda_n}{\rho^2} \int_{\Omega} u^2 dx &\geq \frac{n^2}{4} \int_{\Omega} \frac{u^2}{|x|^2} dx + \frac{1}{4} \sum_{i=1}^m \int_{\Omega} \frac{u^2}{|x|^2} X_1^2 \dots X_i^2 dx \\ &\quad + C_n \left(\int_{\Omega} (X_1 \dots X_{m+1})^{\frac{2n-2}{n-2}} |u|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}}, \end{aligned}$$

for all $u \in C_c^\infty(\Omega)$; here $X_i = X_i(|x|/(3\kappa D))$. If in addition Ω satisfies an interior ball condition at 0 then the exponents $(2n-2)/(n-2)$ of X_i are also sharp.

6.2. Maximal logarithmic potentials

Here we characterize maximal potentials in the context of logarithmically improved Hardy inequalities. Our starting point in this subsection is the improved Hardy inequality contained in Theorem 4,

$$\lambda \int_{\Omega} u^2 dx + \int_{\Omega} |\nabla u|^2 dx \geq \frac{n^2}{4} \int_{\Omega} \frac{u^2}{|x|^2} dx + \frac{1}{4} \sum_{i=1}^m \int_{\Omega} \frac{u^2}{|x|^2} X_1^2 \dots X_i^2 dx \quad (59)$$

for all $u \in C_c^\infty(\Omega)$; here $X_i = X_i(|x|/3\kappa\tilde{D})$ where $\tilde{D} \geq D$. We shall be interested in the problem of improvements of (59) and whether the corresponding best constants are attained.

The analysis that will follow is analogous to that of Section 5; for this reason we shall avoid the details in cases where the arguments are quite similar.

Definition 3. A non-negative potential $V \in L_{\text{loc}}^{n/2}(\bar{\Omega} \setminus \{0\})$ is called *m-admissible* if there exist $\lambda \geq 0$, $\tilde{D} \geq D$ and $C > 0$ such that

$$\lambda \int_{\Omega} u^2 dx + \int_{\Omega} |\nabla u|^2 dx \geq \frac{n^2}{4} \int_{\Omega} \frac{u^2}{|x|^2} dx + \frac{1}{4} \sum_{i=1}^m \int_{\Omega} \frac{u^2}{|x|^2} X_1^2 \dots X_i^2 dx + C \int_{\Omega} V u^2 dx, \quad u \in C_c^\infty(\Omega), \quad (60)$$

where $X_i = X_i(|x|/3\kappa\tilde{D})$. The class of all *m-admissible* potentials for the domain Ω is denoted by $\mathcal{A}_m(\Omega)$.

We note that there is a big variety of *m-admissible* potentials. For example if V satisfies

$$\int_{\Omega} V^{\frac{n}{2}} (X_1 \dots X_{m+1})^{1-n} dx < +\infty, \quad (61)$$

where $X_i = X_i(|x|/3\kappa D)$, $i = 1, \dots, m+1$, then V is *m-admissible* by Theorem 4.

For a given $V \in \mathcal{A}_m(\Omega)$ we denote by $b_m(\lambda) > 0$ the best constant C of inequality (60). We next address the question whether inequality (60) with best constant $b_m(\lambda)$ can be further improved. That is, whether there exists potential $W \in \mathcal{A}_m(\Omega)$ and a positive constant C such that the following inequality holds true as well

$$\lambda \int_{\Omega} u^2 dx + \int_{\Omega} |\nabla u|^2 dx \geq \frac{n^2}{4} \int_{\Omega} \frac{u^2}{|x|^2} dx + \sum_{i=1}^m \int_{\Omega} \frac{u^2}{|x|^2} X_1^2 \dots X_i^2 dx + b_m(\lambda) \int_{\Omega} V u^2 dx + C \int_{\Omega} W u^2 dx,$$

for $u \in C_c^\infty(\Omega)$. In case there does not exist such a potential W , we say that the potential

$$\frac{n^2}{4} \frac{1}{|x|^2} + \sum_{i=1}^m \frac{X_1^2 \dots X_i^2}{|x|^2} + b_m(\lambda) V(x),$$

is an *m-maximal potential*. Our next goal is to characterize *m-maximal* potentials. In this direction for $V \in \mathcal{A}_m(\Omega)$ and small $r > 0$ we define

$$C_{m,r}(V) = \inf_{u \in C_c^\infty(\Omega \cap B_r)} \frac{\lambda \int_{\Omega \cap B_r} u^2 dx + \int_{\Omega \cap B_r} |\nabla u|^2 dx - \frac{n^2}{4} \int_{\Omega \cap B_r} \frac{u^2}{|x|^2} dx + \frac{1}{4} \sum_{i=1}^m \int_{\Omega \cap B_r} \frac{u^2}{|x|^2} X_1^2 \dots X_i^2 dx}{\int_{\Omega \cap B_r} V u^2 dx},$$

where $X_i = X_i(|x|/3\kappa\tilde{D})$ and $\tilde{D} \geq D$. We also define

$$\mathcal{C}_m^0(V) = \lim_{r \rightarrow 0+} C_{m,r}(V).$$

Arguing as in Section 5 we can see that $\mathcal{C}_m^0(V)$ is independent of the specific choice of $\lambda \geq 0$ and $\tilde{D} \geq D$.

Definition 4. The potential $V \in \mathcal{A}_m(\Omega)$ is *m-subcritical* if $\mathcal{C}_m^0 = +\infty$.

Lemma 11. *Let V be a non-negative potential satisfying*

$$\int_{\Omega} V^{n/2} (X_1 \dots X_{m+1})^{1-n} dx < +\infty,$$

where $X_i = X_i(|x|/3\kappa D)$, $i = 1, \dots, m+1$. Then V is an m -subcritical potential.

Proof. The proof is quite similar to the proof of Lemma 6 and makes use of Theorem 4 to establish the inequality

$$\begin{aligned} & \frac{\int_{\Omega \cap B_r} |\nabla u|^2 dx - \frac{n^2}{4} \int_{\Omega \cap B_r} \frac{u^2}{|x|^2} dx - \frac{1}{4} \sum_{k=1}^m \int_{\Omega \cap B_r} \frac{u^2}{|x|^2} X_1^2 \dots X_k^2 dx}{\int_{\Omega \cap B_r} V u^2 dx} \\ & \geq \frac{1}{C_n} \left(\int_{\Omega \cap B_r} V^{n/2} (X_1 \dots X_{m+1})^{1-n} dx \right)^{-\frac{2}{n}}. \end{aligned}$$

The result then follows. \square

As in Section 5 we shall also consider the following more general situation. We consider non-negative potentials $V, W_1, W_2 \in \mathcal{A}_m(\Omega)$ and assume that there exist $c > 0$ and a radius $R > 0$ so that

$$\begin{aligned} \int_{\Omega \cap B_R} W_1 u^2 dx + \int_{\Omega \cap B_R} |\nabla u|^2 dx & \geq \frac{n^2}{4} \int_{\Omega \cap B_R} \frac{u^2}{|x|^2} dx + \frac{1}{4} \sum_{i=1}^m \int_{\Omega \cap B_R} \frac{u^2}{|x|^2} X_1^2 \dots X_i^2 dx \\ & + \int_{\Omega \cap B_R} W_2 u^2 dx + c \int_{\Omega \cap B_R} V u^2 dx \end{aligned} \quad (62)$$

for all $u \in C_c^\infty(\Omega \cap B_R)$. For $0 < r \leq R$ we define

$$C_{m,r}(W_1, W_2; V) = \inf_{C_c^\infty(\Omega \cap B_r)} \frac{\int_{\Omega \cap B_r} W_1 u^2 dx + \int_{\Omega \cap B_r} |\nabla u|^2 dx - \frac{n^2}{4} \int_{\Omega \cap B_r} \frac{u^2}{|x|^2} dx - \frac{1}{4} \sum_{i=1}^m \int_{\Omega \cap B_r} \frac{u^2}{|x|^2} X_1^2 \dots X_i^2 dx - \int_{\Omega \cap B_r} W_2 u^2 dx}{\int_{\Omega \cap B_r} V u^2 dx}$$

and we denote

$$\mathcal{C}_m^0(W_1, W_2; V) = \lim_{r \rightarrow 0^+} C_{m,r}(W_1, W_2; V).$$

The proof of the following lemma is similar to the proof of Lemma 7 and is omitted.

Lemma 12. *Let V, W_1, W_2 be non-negative potentials in $\mathcal{A}_m(\Omega)$ and assume that there exist $R > 0$ and $c > 0$ such that (62) holds true. If in addition W_1, W_2 are subcritical then $\mathcal{C}_m^0(W_1, W_2; V) = \mathcal{C}_m^0(V)$.*

Let $\tilde{D} \geq D$ be fixed. Given $u \in C_c^\infty(\Omega)$ we define the function w by

$$\begin{aligned} u(x) &= \frac{|x + \rho e_n|^2 - \rho^2}{|x|^{\frac{n}{2}} |x + 2\rho e_n|^{\frac{n}{2}}} X_1^{-\frac{1}{2}} \dots X_m^{-\frac{1}{2}} w(x) \\ &=: \phi_m(x) w(x), \end{aligned}$$

where $X_i = X_i(|x|/3\kappa\tilde{D})$, $i = 1, \dots, m$. Then $w \in C_c^\infty(\Omega)$ by our assumption that $B(-2\rho e_n, 2\rho)$ is an exterior ball. After some computations we arrive at

$$\begin{aligned} \int_{\Omega} |\nabla u|^2 dx &= \int_{\Omega} \phi_m^2 |\nabla w|^2 dx + n^2 \rho^2 \int_{\Omega} \frac{\phi_m^2 w^2}{|x|^2 |x + 2\rho e_n|^2} dx \\ &\quad + \int_{\Omega} \frac{\phi_m^2 w^2}{|x|^2} \left\{ \frac{1}{4} \sum_{k=1}^m X_1^2 \dots X_k^2 - \frac{n}{2} \frac{|x|^2 + 2\rho x_n}{|x + 2\rho e_n|^2} \sum_{k=1}^m X_1 \dots X_k + \frac{|x|^2}{|x|^2 + 2\rho x_n} \sum_{k=1}^m X_1 \dots X_k \right\} dx, \end{aligned}$$

so inequality (60) is written

$$\begin{aligned} &\lambda \int_{\Omega} \phi_m^2 w^2 dx + \int_{\Omega} \phi_m^2 |\nabla w|^2 dx + \int_{\Omega} \frac{\phi_m^2 w^2}{|x|^2} \left\{ -\frac{n}{2} \frac{|x|^2 + 2\rho x_n}{|x + 2\rho e_n|^2} + \frac{|x|^2}{|x|^2 + 2\rho x_n} \right\} \left(\sum_{k=1}^m X_1 \dots X_k \right) dx \\ &\geq \frac{n^2}{4} \int_{\Omega} \frac{|x|^2 + 4\rho x_n}{|x|^2 |x + 2\rho e_n|^2} \phi_m^2 w^2 dx + c \int_{\Omega} V \phi_m^2 w^2 dx. \end{aligned} \quad (63)$$

It is clear from the above that (63) is valid for all functions $w \in C_c^\infty(\Omega)$ and moreover, for a fixed $\lambda \geq 0$ the best constants c for inequalities (60) and (63) coincide. That common best constant shall be denoted by $b_m = b_m(\lambda)$.

Defining

$$Q_m(x) = \frac{n^2}{4} \frac{|x|^2 + 4\rho x_n}{|x|^2 |x + 2\rho e_n|^2} + \frac{1}{|x|^2} \left\{ \frac{n}{2} \frac{|x|^2 + 2\rho x_n}{|x + 2\rho e_n|^2} - \frac{|x|^2}{|x|^2 + 2\rho x_n} \right\} \left(\sum_{k=1}^m X_1 \dots X_k \right)$$

it then follows that

$$C_{m,r}(V) = \inf_{W_0^{1,2}(\Omega \cap B_r; \phi_m^2)} \frac{\lambda \int_{\Omega \cap B_r} \phi_m^2 w^2 dx + \int_{\Omega \cap B_r} \phi_m^2 |\nabla w|^2 dx - \int_{\Omega \cap B_r} Q_m \phi_m^2 w^2 dx}{\int_{\Omega \cap B_r} \phi_m^2 V w^2 dx}$$

where $W_0^{1,2}(\Omega \cap B_r; \phi_m^2)$ denotes the closure of $C_c^\infty(\Omega \cap B_r)$ under the norm

$$\int_{\Omega \cap B_r} \phi_m^2 |\nabla w|^2 dx + \int_{\Omega \cap B_r} \phi_m^2 w^2 dx.$$

Similarly to Section 5 we shall use a simpler way for expressing $\mathcal{C}_m^0(V) = \lim_{r \rightarrow 0+} C_{m,r}(V)$. For this we define

$$C_{m,r}(V; \phi_m^2) = \inf_{W_0^{1,2}(\Omega \cap B_r; \phi_m^2)} \frac{\int_{\Omega \cap B_r} \phi_m^2 |\nabla w|^2 dx}{\int_{\Omega \cap B_r} V \phi_m^2 w^2 dx},$$

and

$$\mathcal{C}_m^0(V; \phi_m^2) = \lim_{r \rightarrow 0+} C_{m,r}(V; \phi_m^2).$$

Lemma 13. *Let V be a non-negative potential in $\mathcal{A}_m(\Omega)$. Then $\mathcal{C}_m^0(V) = \mathcal{C}_m^0(V; \phi_m^2)$.*

Proof. The proof is quite similar to the proof of Lemma 8. In particular we make use of the fact that

$$\int_{\Omega} \left(\frac{X_1}{|x|^2 + 2x_n} \right)^{n/2} (X_1 \dots X_{m+1})^{1-n} < +\infty,$$

from which it easily follows that $|Q_m|$ is an m -subcritical potential. We omit further details. \square

One important consequence of m -subcriticality is the following compactness property whose proof is similar to that of Lemma 9 and is therefore omitted.

Lemma 14. *Assume that the positive potential $V \in \mathcal{A}_m(\Omega)$ is m -subcritical. Then for any sequence (w_k) which is bounded in $W_0^{1,2}(\Omega; \phi_m^2)$ there exists a subsequence, also denoted by (w_k) , and a function $w_0 \in W_0^{1,2}(\Omega; \phi_m^2)$ so that*

$$\begin{aligned} \text{(i)} \quad & w_k \rightharpoonup w_0 \text{ in } W_0^{1,2}(\Omega; \phi_m^2) \\ \text{(ii)} \quad & \int_{\Omega} V \phi_m^2 (w_k - w_0)^2 dx \longrightarrow 0. \end{aligned}$$

We can now state and prove the main theorems of this section.

Theorem 14. *Let $V \in \mathcal{A}_m(\Omega)$ and $\lambda \geq 0$ be given and let $b_m(\lambda)$ be the best constant for the inequality*

$$\lambda \int_{\Omega} \phi_m^2 w^2 dx + \int_{\Omega} \phi_m^2 |\nabla w|^2 dx \geq \int_{\Omega} Q_m \phi_m^2 w^2 dx + c \int_{\Omega} V \phi_m^2 w^2 dx, \quad w \in W_0^{1,2}(\Omega; \phi_m^2), \quad (64)$$

where

$$\phi_m(x) = \frac{|x + \rho e_n|^2 - \rho^2}{|x|^{\frac{n}{2}} |x + 2\rho e_n|^{\frac{n}{2}}} X_1^{-\frac{1}{2}} \dots X_m^{-\frac{1}{2}}$$

and

$$Q_m(x) = \frac{n^2}{4} \frac{|x|^2 + 4\rho x_n}{|x|^2 |x + 2\rho e_n|^2} + \frac{1}{|x|^2} \left\{ \frac{n}{2} \frac{|x|^2 + 2\rho x_n}{|x + 2\rho e_n|^2} - \frac{|x|^2}{|x|^2 + 2\rho x_n} \right\} \left(\sum_{k=1}^m X_1 \dots X_k \right),$$

with $X_i = X_i(|x|/3\kappa\tilde{D})$, $\tilde{D} \geq D$. If in addition

$$b_m(\lambda) < C_m^0$$

then the best constant $b_m(\lambda)$ in (64) is realized by a function $w_0 \in W_0^{1,2}(\Omega; \phi_m^2)$. In particular the best constant $b_m(\lambda)$ is realized if V is an m -subcritical potential.

Proof. The proof is similar to the proof of Theorem 10 so we shall only give a sketch of the proof. What is important for our argument is that the potential Q_m is a subcritical potential. We consider a minimizing sequence (w_k) for (64) and without loss of generality we assume that

$$\int_{\Omega} Q_m \phi_m^2 w_k^2 dx + b_m(\lambda) \int_{\Omega} V \phi_m^2 w_k^2 dx = 1, \quad \lambda \int_{\Omega} \phi_m^2 w_k^2 dx + \int_{\Omega} \phi_m^2 |\nabla w_k|^2 dx \longrightarrow 1, \quad \text{as } k \rightarrow \infty.$$

Since (w_k) is bounded in $W_0^{1,2}(\Omega; \phi_m^2)$, it has a subsequence, which we assume is (w_k) itself, which converges weakly to some $w_0 \in W_0^{1,2}(\Omega; \phi_m^2)$. We define $v_k = w_k - w_0$.

We consider a small enough $r > 0$ so that $C_{m,r}(V; \phi_m^2) > b_m(\lambda)$ and a smooth cut-off function ψ such that $\psi = 1$ in $B_{r/2}$ and $\psi = 0$ outside B_r . Arguing as in the proof of Theorem 10 we obtain

$$\int_{\Omega} \phi_m^2 |\nabla v_k|^2 dx \geq C_{m,r}(V; \phi_m^2) \int_{\Omega} V \phi_m^2 v_k^2 dx + o(1) \quad (65)$$

and also

$$\int_{\Omega} Q_m \phi_m^2 w_0^2 dx + b_m(\lambda) \int_{\Omega} V \phi_m^2 w_0^2 dx + b_m(\lambda) \int_{\Omega} V \phi_m^2 v_k^2 dx = 1 + o(1) \quad (66)$$

and

$$\lambda \int_{\Omega} \phi_m^2 w_0^2 dx + \int_{\Omega} \phi_m^2 |\nabla w_0|^2 dx + \int_{\Omega} \phi_m^2 |\nabla v_k|^2 dx = 1 + o(1). \quad (67)$$

From (65) and (66) we obtain

$$\int_{\Omega} \phi_m^2 |\nabla v_k|^2 dx \geq \frac{C_{m,r}(V; \phi_m^2)}{b_m(\lambda)} \left(1 - \int_{\Omega} Q_m \phi_m^2 w_0^2 dx + b_m(\lambda) \int_{\Omega} V \phi_m^2 w_0^2 dx \right) + o(1).$$

Writing (64) for $w = w_0$ we obtain from (67) that

$$\int_{\Omega} \phi_m^2 |\nabla v_k|^2 dx \leq 1 - \int_{\Omega} Q_m \phi_m^2 w_0^2 dx - b_m(\lambda) \int_{\Omega} V \phi_m^2 w_0^2 dx + o(1)$$

and arguing as before we conclude that

$$\int_{\Omega} Q_m \phi_m^2 w_0^2 dx + b_m(\lambda) \int_{\Omega} V \phi_m^2 w_0^2 dx = 1,$$

that is w_0 is a minimizer for (64). \square

Finally, the next is a direct consequence of Theorem 14.

Theorem 15. *Let $n \geq 3$. V is a non-negative potential in $\mathcal{A}_m(\Omega)$ and $\tilde{D} \geq D$.*

(a) *Let $\lambda \geq 0$ and $b_m(\lambda) > 0$ be the best constant for the following inequality*

$$\lambda \int_{\Omega} u^2 dx + \int_{\Omega} |\nabla u|^2 dx \geq \frac{n^2}{4} \int_{\Omega} \frac{u^2}{|x|^2} dx + \frac{1}{4} \sum_{k=1}^m \int_{\Omega} \frac{u^2}{|x|^2} X_1^2 \dots X_k^2 dx + b_m(\lambda) \int_{\Omega} V u^2 dx, \quad u \in C_c^\infty(\Omega), \quad (68)$$

where $X_i = X_i(|x|/3\kappa\tilde{D})$. If in addition

$$b_m(\lambda) < C_m^0,$$

then the potential $n^2/4|x|^2 + 1/4|x|^2 \sum_{k=1}^m X_1^2 \dots X_k^2 + b_m(\lambda)V(x)$ is a maximal potential, that is, inequality (68) cannot be improved by adding a non-negative potential W in the RHS.

(b) If V is an m -subcritical potential then there exist $\lambda \geq 0$ and a best constant $b_m(\lambda) > 0$ such that (68) is true. Moreover the potential $n^2/4|x|^2 + 1/4|x|^2 \sum_{k=1}^m X_1^2 \dots X_k^2 + b_m(\lambda)V(x)$ is a maximal potential.

7. Maximal potentials in finite cones

In the previous two sections we characterized maximal potentials in bounded domains satisfying the exterior and interior ball condition. Analogous results also hold true in the case of finite cones.

Let \mathcal{C}_1 be the cone determined by the domain $\Sigma \subset S^{n-1}$ as defined in Section 2; more generally we set $\mathcal{C}_r := \mathcal{C} \cap B_r$. In this subsection we shall initially be interested in the question of characterizing maximal potentials for improved versions of inequality (2).

Definition 5. A non-negative potential $V \in L_{\text{loc}}^{n/2}(\overline{\mathcal{C}_1} \setminus \{0\})$ is called *admissible* if there exists $c > 0$ such that

$$\int_{\mathcal{C}_1} |\nabla u|^2 dx \geq \left(\left(\frac{n-2}{2} \right)^2 + \mu_1(\Sigma) \right) \int_{\mathcal{C}_1} \frac{u^2}{|x|^2} dx + c \int_{\mathcal{C}_1} V u^2 dx, \quad u \in C_c^\infty(\mathcal{C}_1). \quad (69)$$

We denote by $\mathcal{A}(\mathcal{C}_1)$ the class of all admissible potentials.

Once again there is a big variety of admissible potentials. For example if V satisfies $\int_{\mathcal{C}_1} V^{n/2} X_1^{1-n} dx < +\infty$ where $X_1 = X_1(|x|)$, then V is admissible by Theorem 5.

Given $V \in \mathcal{A}(\mathcal{C}_1)$ and $r \in (0, 1)$ we define

$$C_r(V) = \inf_{u \in C_c^\infty(\mathcal{C}_r)} \frac{\int_{\mathcal{C}_r} |\nabla u|^2 dx - \left(\left(\frac{n-2}{2} \right)^2 + \mu_1(\Sigma) \right) \int_{\mathcal{C}_r} \frac{u^2}{|x|^2} dx}{\int_{\mathcal{C}_r} V u^2 dx}$$

and

$$\mathcal{C}^0(V) = \lim_{r \rightarrow 0^+} C_r(V).$$

Definition 6. We say that the potential $V \in \mathcal{A}(\mathcal{C}_1)$ is subcritical if $\mathcal{C}^0(V) = +\infty$.

The analogue of Lemma 6 is the following

Lemma 15. Let V be a non-negative potential satisfying

$$\int_{\mathcal{C}_1} V^{n/2} X_1^{1-n} dx < +\infty,$$

where $X_1 = X_1(|x|)$. Then V is a subcritical potential.

Given $u \in C_c^\infty(\mathcal{C}_1)$ we define the function w by

$$u(x) = |x|^{-\frac{n-2}{2}} \phi_1(\omega) w(x) =: \psi(x) w(x), \quad (70)$$

where $\omega := \frac{x}{|x|} \in \Sigma$ and $\phi_1(\omega)$, is the first eigenfunction of the Dirichlet Laplacian in Σ . After some computations we obtain

$$\int_{\mathcal{C}_1} |\nabla u|^2 dx = \int_{\mathcal{C}_1} \psi^2 |\nabla w|^2 dx + \left(\left(\frac{n-2}{2} \right)^2 + \mu_1(\Sigma) \right) \int_{\mathcal{C}_1} \psi^2 w^2 dx.$$

As usual we denote by $W_0^{1,2}(\mathcal{C}_1; \psi^2)$ the closure of $C_c^\infty(\mathcal{C}_1)$ under the norm

$$\int_{\mathcal{C}_1} \psi^2 |\nabla w|^2 dx + \int_{\mathcal{C}_1} \psi^2 w^2 dx.$$

It is easily seen that inequality (69) under the change of variables (70) is equivalent to

$$\int_{\mathcal{C}_1} \psi^2 |\nabla w|^2 dx \geq c \int_{\mathcal{C}_1} V \psi^2 w^2 dx, \quad w \in W_0^{1,2}(\mathcal{C}_1; \psi^2).$$

The analogues of Theorems 10 and 11 read as follows:

Theorem 16. *Let $V \in \mathcal{A}(\mathcal{C}_1)$ and suppose that b is the best constant for the inequality*

$$\int_{\mathcal{C}_1} \psi^2 |\nabla w|^2 dx \geq b \int_{\mathcal{C}_1} V \psi^2 w^2 dx, \quad w \in W_0^{1,2}(\mathcal{C}_1; \psi^2). \quad (71)$$

If in addition $b < C^0(V)$ then the best constant b in (71) is realized by a function $w_0 \in W_0^{1,2}(\mathcal{C}_1; \psi^2)$. In particular the best constant b is realized if the potential V is subcritical.

Theorem 17. *Let V be a non-negative potential in $\mathcal{A}(\mathcal{C}_1)$.*

(a) Let $b > 0$ be the best constant in the following inequality

$$\int_{\mathcal{C}_1} |\nabla u|^2 dx \geq \left(\left(\frac{n-2}{2} \right)^2 + \mu_1(\Sigma) \right) \int_{\mathcal{C}_1} \frac{u^2}{|x|^2} dx + b \int_{\mathcal{C}_1} V u^2 dx, \quad u \in C_c^\infty(\mathcal{C}_1). \quad (72)$$

If in addition $b < C^0(V)$ then the potential $\left[\left(\frac{n-2}{2} \right)^2 + \mu_1(\Sigma) \right] |x|^{-2} + bV(x)$ is a maximal potential, that is inequality (72) cannot be improved by adding a non-negative potential W in the RHS.

(b) If V is a subcritical potential then there exists a best constant $b > 0$ such that (72) is true. Moreover the potential $\left[\left(\frac{n-2}{2} \right)^2 + \mu_1(\Sigma) \right] |x|^{-2} + bV(x)$ is a maximal potential.

The proofs of these theorems are quite similar and slightly simpler to the proofs of Theorems 10 and 11.

In analogy to the results of Section 6 we have similar theorems for the improved Hardy inequality involving logarithmic corrections. In particular we have

Definition 7. A non-negative potential $V \in L_{\text{loc}}^{n/2}(\overline{\mathcal{C}_1} \setminus \{0\})$ is called *m-admissible* if there exists $c > 0$ such that for $u \in C_c^\infty(\mathcal{C}_1)$, there holds

$$\int_{\mathcal{C}_1} |\nabla u|^2 dx \geq \left(\left(\frac{n-2}{2} \right)^2 + \mu_1(\Sigma) \right) \int_{\mathcal{C}_1} \frac{u^2}{|x|^2} dx + \frac{1}{4} \sum_{i=1}^m \int_{\mathcal{C}_1} \frac{u^2}{|x|^2} X_1^2 \dots X_i^2 dx + c \int_{\mathcal{C}_1} V u^2 dx. \quad (73)$$

We denote by $\mathcal{A}_m(\mathcal{C}_1)$ the class of all *m-admissible* potentials.

Given $V \in \mathcal{A}_m(\mathcal{C}_1)$ and $r \in (0, 1)$ we define

$$C_{m,r}(V) = \inf_{u \in C_c^\infty(\mathcal{C}_r)} \frac{\int_{\mathcal{C}_r} |\nabla u|^2 dx - \left(\left(\frac{n-2}{2} \right)^2 + \mu_1(\Sigma) \right) \int_{\mathcal{C}_r} \frac{u^2}{|x|^2} dx - \frac{1}{4} \sum_{i=1}^m \int_{\mathcal{C}_r} \frac{u^2}{|x|^2} X_1^2 \dots X_i^2 dx}{\int_{\mathcal{C}_r} V u^2 dx}$$

and

$$\mathcal{C}_m^0(V) = \lim_{r \rightarrow 0^+} C_{m,r}(V).$$

Definition 8. We say that the potential $V \in \mathcal{A}_m(\mathcal{C}_1)$ is m -subcritical if $\mathcal{C}_m^0(V) = +\infty$.

Changing variables by

$$u(x) = |x|^{-\frac{n-2}{2}} \phi_1(\omega) X_1^{-\frac{1}{2}} \dots X_m^{-\frac{1}{2}} w(x) =: \psi_m(x) w(x),$$

inequality (73) is equivalent to

$$\int_{\mathcal{C}_1} \psi_m^2 |\nabla w|^2 dx \geq c \int_{\mathcal{C}_1} V \psi_m^2 w^2 dx, \quad w \in W_0^{1,2}(\mathcal{C}_1; \psi_m^2).$$

Now the analogues of Theorems 14 and 15 are as follows

Theorem 18. Let $V \in \mathcal{A}_m(\mathcal{C}_1)$ and let b_m be the best constant for the inequality

$$\int_{\mathcal{C}_1} \psi_m^2 |\nabla w|^2 dx \geq b_m \int_{\mathcal{C}_1} V \psi_m^2 w^2 dx, \quad w \in W_0^{1,2}(\Omega; \psi_m^2). \quad (74)$$

If in addition $b_m < \mathcal{C}_m^0(V)$ then the best constant b_m in (74) is realized in $W_0^{1,2}(\mathcal{C}_1; \psi_m^2)$. In particular the best constant b_m is realized if the potential V is m -subcritical.

Theorem 19. Let V be a non-negative potential in $\mathcal{A}_m(\mathcal{C}_1)$.

(a) Let $b_m > 0$ be the best constant in the following inequality

$$\int_{\mathcal{C}_1} |\nabla u|^2 dx \geq \left(\left(\frac{n-2}{2} \right)^2 + \mu_1(\Sigma) \right) \int_{\mathcal{C}_1} \frac{u^2}{|x|^2} dx + \frac{1}{4} \sum_{i=1}^m \int_{\mathcal{C}_1} \frac{u^2}{|x|^2} X_1^2 \dots X_i^2 dx + b_m \int_{\mathcal{C}_1} V u^2 dx, \quad (75)$$

where $u \in C_c^\infty(\mathcal{C}_1)$.

If in addition $b_m < \mathcal{C}_m^0(V)$ then the potential $\left[\left(\frac{n-2}{2} \right)^2 + \mu_1(\Sigma) + \frac{1}{4} \sum_{i=1}^m X_1^2 \dots X_i^2 \right] |x|^{-2} + b_m V(x)$ is a maximal potential, that is inequality (75) cannot be improved by adding a non-negative potential W in the RHS.

(b) If V is a subcritical potential then there exists a best constant $b_m > 0$ such that (75) is true. Moreover the potential $\left[\left(\frac{n-2}{2} \right)^2 + \mu_1(\Sigma) + \frac{1}{4} \sum_{i=1}^m X_1^2 \dots X_i^2 \right] |x|^{-2} + b_m V(x)$ is a maximal potential.

Remark. All the above results involve a single point singularity on the boundary. Similar results however can be obtained when there are multiple singularities. For instance we have the following result

Theorem 20. Assume that $\Omega \subset \mathbb{R}^n$, $n \geq 3$, is a bounded domain that satisfies an exterior ball condition at each of the points $a_1, \dots, a_m \in \partial\Omega$. Then there exist a positive constant $c = c(n, m)$ depending only on n and m and a positive constant λ such that

$$\lambda \int_{\Omega} u^2 dx + \int_{\Omega} |\nabla u|^2 dx \geq \frac{n^2}{4} \sum_{k=1}^m \int_{\Omega} \frac{u^2}{|x - a_k|^2} dx + c \left(\int_{\Omega} |u|^{\frac{2n}{n-2}} W dx \right)^{\frac{n-2}{n}}, \quad u \in C_c^\infty(\Omega),$$

where

$$W = W(x) := \prod_{k=1}^m X_1^{\frac{2n-2}{n-2}} \left(\frac{|x - a_k|}{D} \right), \quad \text{and} \quad D := \max_{k=1, \dots, m} \sup_{x \in \Omega} |x - a_k|.$$

The proof uses ideas that we have used so far in connection with standard partition of unity arguments; we omit further details.

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References

- [1] Adimurthi, N. Chaudhuri, N. Ramaswamy, An improved Hardy–Sobolev inequality and its applications, *Proc. Am. Math. Soc.* 130 (2002) 489–505.
- [2] Adimurthi, S. Filippas, A. Tertikas, On the best constant of Hardy–Sobolev inequalities, *Nonlinear Anal.* 70 (2009) 2826–2833.
- [3] G. Barbatis, S. Filippas, A. Tertikas, A unified approach to improved L^p Hardy inequalities with best constants, *Trans. Am. Math. Soc.* 356 (2004) 2169–2196.
- [4] G. Barbatis, G.S. Filippas, A. Tertikas, Series expansion for L^p Hardy inequalities, *Indiana Univ. Math. J.* 52 (2003) 171–190.
- [5] H. Brezis, E.H. Lieb, Sobolev inequalities with remainder terms, *J. Funct. Anal.* 62 (1985) 73–86.
- [6] H. Brezis, L. Nirenberg, Positive solutions of nonlinear elliptic problems involving critical exponents, *Commun. Pure Appl. Math.* 36 (1983) 437–477.
- [7] H. Brezis, J.L. Vázquez, Blow-up solutions of some nonlinear elliptic equations, *Rev. Mat. Complut.* 10 (1997) 443–469.
- [8] C. Cazacu, On Hardy inequalities with singularities on the boundary, *C. R. Acad. Sci. Paris Sér. I* 349 (2011) 273–277.
- [9] C. Cazacu, Hardy inequalities and Pohozaev identity for operators with boundary singularities: some applications, *C. R. Acad. Sci. Paris Sér. I* 349 (2011) 1167–1172.
- [10] C. Cazacu, Schrödinger operators with boundary singularities: Hardy inequality, Pohozaev identity and controllability results, *J. Funct. Anal.* 263 (2012) 3741–3783.
- [11] C. Cazacu, Controllability of the heat equation with an inverse square potential localized on the boundary, *SIAM J. Control Optim.* 52 (2014) 2055–2089.
- [12] J.L. Chern, C.S. Lin, Minimizers of Caffarelli–Kohn–Nirenberg inequalities with the singularity on the boundary, *Arch. Ration. Mech. Anal.* 197 (2010) 401–432.
- [13] B. Devyver, A spectral result for Hardy inequalities, *J. Math. Pures Appl.* 102 (2014) 813–853.
- [14] M.M. Fall, On the Hardy–Poincaré inequality with boundary singularities, *Commun. Contemp. Math.* 14 (2012) 1250019.
- [15] M.M. Fall, R. Musina, Hardy Poincaré inequality with boundary singularities, *Proc. R. Soc. Edinb. A* 142 (2012) 769–786.
- [16] S. Filippas, L. Moschini, A. Tertikas, Sharp two-sided heat kernel estimates for critical Schrödinger operators on bounded domains, *Commun. Math. Phys.* 273 (2007) 237–281.
- [17] S. Filippas, A. Tertikas, Optimizing improved Hardy inequalities, *J. Funct. Anal.* 192 (2002) 186–233, *J. Funct. Anal.* 255 (2008) 2095 (Corrigendum).
- [18] S. Filippas, A. Tertikas, J. Tidblom, On the structure of Hardy–Sobolev–Maz’ya inequalities, *J. Eur. Math. Soc.* 11 (2009) 1165–1185.
- [19] N. Ghoussoub, X.S. Kang, Hardy–Sobolev critical elliptic equations with boundary singularities, *Ann. Inst. Henri Poincaré, Anal. Non Linéaire* 21 (2004) 767–793.
- [20] N. Ghoussoub, F. Robert, The effect of curvature on the best constant in the Hardy–Sobolev inequalities, *Geom. Funct. Anal.* 16 (2006) 1201–1245.
- [21] N. Ghoussoub, F. Robert, Sobolev inequalities for the Hardy–Schrödinger operator: extremals and critical dimensions, *Bull. Math. Sci.* 6 (2016) 89–144.
- [22] A.I. Nazarov, Hardy Sobolev Inequalities in a cone, *J. Math. Sci.* 132 (2006) 419–426.
- [23] A. Tertikas, Critical phenomena in linear elliptic problems, *J. Funct. Anal.* 154 (1998) 42–66.
- [24] J.L. Vázquez, E. Zuazua, The Hardy inequality and the asymptotic behaviour of the heat equation with an inverse-square potential, *J. Funct. Anal.* 173 (2000) 103–153.