THE HARDY CONSTANT: A REVIEW

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ABSTRACT. We present a review of results that have been obtained in the past twenty-five years concerning the L^p-Hardy inequality with distance to the boundary. We concentrate on results where the best Hardy constant is either computed exactly or estimated from below.

1. INTRODUCTION

In this artricle we present a review of some of the results that have been obtained in the past twenty- five years concerning the Hardy inequality

$$\int_\Omega |\nabla u|^p dx \geq c \int_\Omega \frac{|u|^p}{d^p}\, dx \ , \qquad u \in C_c^\infty(\Omega). \tag{1.1}$$

Here Ω is an open and connected subset of $\mathbb{R}^n$ with non-empty boundary and $d(x) = \mathrm{dist}(x, \partial\Omega)$, $x \in \Omega$, denotes the distance to the boundary of Ω. Hardy inequalities involving the function $d(x)$ are sometimes called geometric Hardy inequalities in order to distinguish them from Hardy inequalities involving the distance to an interior point.

Since the publication of the review article [20] the literature related to inequality (1.1) has grown significantly. Several aspects of this inequality as well as other related inequalities have been extensively studied: weighted inequalities, Rellich inequalities, improved inequalities, inequalities on non-Euclidean settings, fractional inequalities and more. The publication of three books [7, 26, 40] is indicative of the recent interest on this area.

In the present article we shall be primarily concerned with the best constant for inequality (1.1). Hence we shall present results where the best constant is precisely computed as well as results where lower estimates are obtained. Some mention of improved Hardy inequalities will also be made. At the end of the article we present some open problems.

2. THE HARDY CONSTANT

The L^p-Hardy inequality involving the distance to the boundary reads

$$\int_\Omega |\nabla u|^p dx \geq c \int_\Omega \frac{|u|^p}{d^p}\, dx \ , \qquad u \in C_c^\infty(\Omega). \tag{2.1}$$

Here $p > 1$, $\Omega \subset \mathbb{R}^n$ is a domain and $d(x) = \mathrm{dist}(x, \partial\Omega)$, $x \in \Omega$.

We say that the L^p-Hardy inequality is valid for the domain Ω if there exists $c > 0$ such that (2.1) holds true. We denote by $H_p(\Omega)$ the best constant for (2.1), the L^p-Hardy constant of the domain Ω. In case $p = 2$ we shall simply write $H(\Omega)$.

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There are various sufficient conditions as well as necessary conditions for the validity of the Hardy inequality. These are typically related to some regularity of the domain. As already mentioned, we shall be primarily concerned with the precise value of the Hardy constant as well as with explicit lower estimates.

2.1. **Domains with critical Hardy constant.** For $p>1$ we set

$$\alpha_p = \Big(\frac{p-1}{p}\Big)^p .$$

This constant plays a special role for the L^p-Hardy inequality. Besides being the Hardy constant in dimension one, it is also the case that $H_p(\Omega) \leq \alpha_p$ when some part of $\partial\Omega$ is C^2; indeed a lower regularity is enough, see [34, Theorem 5].

The importance of the value α_p is also indicated by the following dichotomy which has been obtained in [34, 35].

Theorem 2.1. *Let Ω be a bounded domain with C^2 boundary. There holds $H_p(\Omega) < \alpha_p$ if and only if the L^p Hardy quotient admits a minimizer in $W_0^{1,p}(\Omega)$.*

This has been generalized in [30] to bounded domains with $C^{1,\gamma}$ boundary. We note that for the 'only if' part lower boundary regularity is enough, see [41].

There are several conditions under which the L^p Hardy constant is equal to α_p. The one that is probably best known is the convexity of the domain Ω. This has been long known in case $p=2$ while the case of general $p>1$ was obtained in [36].

A more general condition was established in [11] where it was proved that if the domain Ω is such that

$$\Delta d \leq 0, \qquad \text{in } \Omega \tag{2.2}$$

(in the distributional sense) then

$$\int_\Omega |\nabla u|^p dx \geq \Big(\frac{p-1}{p}\Big)^p \int_\Omega \frac{|u|^p}{d^p} dx\ , \quad u \in C_c^\infty(\Omega).$$

Indeed the following series improvement was established in [12]. Define $X_1(t) = (1-\log t)^{-1}$, $X_k(t) = X_1(X_{k-1}(t)$, $k \geq 2$, $t \in (0,1)$. (These are iterated logarithmic functions that vanish at $t=0$ at a rate that becomes slower as k increases.)

Theorem 2.2. *Assume that for the domain $\Omega \subset \mathbb{R}^n$ condition (2.2) is satisfied and also assume that $\sup_\Omega d < +\infty$. Then for any $p>1$ there exists a constant $D \geq \sup_\Omega d$ such that*

$$\int_\Omega |\nabla u|^p dx \geq \Big(\frac{p-1}{p}\Big)^p \int_\Omega \frac{|u|^p}{d^p} dx + \frac{p-1}{2p}\Big(\frac{p-1}{p}\Big)^{p-2} \sum_{k=1}^\infty \int_\Omega \frac{|u|^p}{d^p} X_1^2 X_2^2 \dots X_k^2 dx,$$

for all $u \in C_c^\infty(\Omega)$; here $X_j = X_j(d(x)/D)$. Moreover the inequality is sharp as each new term of the series is added.

Theorem 2.2 is one amongst several results where an improvement of a Hardy inequality with sharp constant is obtained. To our knowledge the first such result is contained in [37]. Improved Hardy inequalities can be of two types: the added term may contain a weighted L^p norm or some weighted L^q norm with $p<q\leq p^*$ where $p^* = np/(n-p)$ is the Sobolev exponent. Improvements of the first type, also called homogeneous improvements, have been obtained, among others, in [6, 11, 12, 13, 18, 24, 29, 42]. Concerning Sobolev improvements see [11, 17, 18, 23, 24, 25, 27]; we shall not go into any further details regarding improved Hardy inequalities.

Going back to condition (2.2), we note that it is also mentioned in [20]. Domains for which (2.2) is valid are called weakly mean convex domains. Any convex domain is weakly mean convex. It has been proved independently in [33, 39] that if Ω is bounded with C^2 boundary then weak mean convexity is equivalent to mean convexity, that is to the mean curvature of $\partial\Omega$ being non-negative. In two dimensions and for bounded domains with C^2 boundary mean convexity is equivalent to convexity [2]. This is not true in higher dimensions.

In case $p = 2$ there are other domains that are known to have Hardy constant equal to 1/4. One such domain is the annulus $\{x \in \mathbb{R}^n : r < |x| < R\}$ in dimension $n \geq 3$ [34]. This covers also the case $R = \infty$, i.e. the complement of a ball. The latter case can also be obtained as a special case of a more general result in [27] where it is shown that if the domain $\Omega \subset \mathbb{R}^n$ satisfies

$$-\Delta d + (n-1)\frac{\nabla d \cdot x}{|x|^2} \geq 0 , \qquad \text{in } \Omega$$

then the Hardy inequality with constant 1/4 is valid for Ω.

Using different methods the following theorem has been obtained in [3, 4]:

Theorem 2.3. *Let $n \geq 2$. There exists a number $\Lambda_n > 0$ (expressed by means of a certain hypergeometric equation) such that: if $\Omega \subset \mathbb{R}^n$ is a bounded domain and if for each point $y \in \partial\Omega$ there exists a ball B_r of radius $r = (\sup_\Omega d)/\Lambda_n$ with $y \in \partial B_r$ and $B_r \subset \Omega^c$, then $H(\Omega) = 1/4$.*

An analogous result for $p \in (1, 2)$ is contained in [5].

The following theorem about cones has been proved in [19]; see also [22].

Theorem 2.4. *Let U be an open connected subset of the unit sphere and let Ω be the corresponding infinite cone, given in spherical coordinates by $\Omega = \{(r, \omega) : r > 0 , \omega \in U\}$. Let $\delta : U \to \mathbb{R}$ be such that $d = r\delta(\omega)$ on Ω. Then the Hardy constant of Ω coincides with the best constant k_U for the inequality*

$$\int_U |\nabla_\omega g|^2 dS + \Big(\frac{n-2}{2}\Big)^2 \int_U g^2 dS \geq k_U \int_U \frac{g^2}{\delta^2} dS, \qquad g \in C_c^\infty(U).$$

3. The Hardy constant in two dimensions

There is more that can be said about the L^2-Hardy constant when we consider domains in $\mathbb{R}^2$. This is due to the availability of tools from complex analysis but also to the fact that explicit computations are simpler than in higher dimensions.

The following well known result of Ancona [1] is proved by a suitable application of Koebe's 1/4 theorem.

Theorem 3.1. *If $\Omega \subset \mathbb{R}^2$ is a simply connected domain then*

$$\int_\Omega |\nabla u|^2 dx \geq \frac{1}{16} \int_\Omega \frac{u^2}{d^2}\, dx , \qquad u \in C_c^\infty(\Omega).$$

In [32] a modified version of Koebe's 1/4 theorem was used to prove the next theorem which involves a quantified measure of non-convexity.

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^2$ be simply connected and satisfy an external cone condition: each $y \in \partial\Omega$ is the vertex of an infinite cone C of angle θ with $\Omega \subset C$. Then*

$$\int_\Omega |\nabla u|^2 dx \geq \frac{\pi^2}{4\theta^2} \int_\Omega \frac{u^2}{d^2}\, dx , \qquad \textit{for all } u \in C_c^\infty(\Omega).$$

We note that if Ω is convex then one recaptures the constant 1/4.

3.1. **Sectors in $\mathbb{R}^2$.** Let $\beta \in [\pi, 2\pi]$ and let Λ_β denote the infinite sector of angle β,

$$\Lambda_\beta = \{(r,\theta) : r > 0\,, \ \ 0 < \theta < \beta\}.$$

Then $d^{-2} = r^{-2} V_\beta(\theta)$ where

$$V_\beta(\theta) = \begin{cases} \frac{1}{\sin^2\theta}, & 0 < \theta < \frac{\pi}{2}, \\ 1, & \frac{\pi}{2} < \theta < \beta - \frac{\pi}{2}, \\ \frac{1}{\sin^2(\beta-\theta)}, & \beta - \frac{\pi}{2} < \theta < \beta. \end{cases}$$

It follows from Theorem 2.4 that the Hardy constant $H(\Lambda_\beta)$ coincides with the best constant c_β for the Hardy-type inequality

$$\int_0^\beta g'(\theta)^2 d\theta \geq c_\beta \int_0^\beta g(\theta)^2 V_\beta(\theta) d\theta\ , \quad g \in C_c^\infty(0,\beta).$$

Equivalently, c_β is the largest constant c for which the boundary value problem

$$\begin{cases} \psi''(\theta) + cV_\beta(\theta)\psi(\theta) = 0, & 0 \leq \theta \leq \beta, \\ \psi(0) = \psi(\beta) = 0, \end{cases} \tag{3.1}$$

has a positive solution in $(0,\beta)$.

The qualitative behavior of c_β as a function of β was studied in [19] where it was proved that there exists a critical angle $\beta_{cr} \in (4, 2\pi)$ such that $c_\beta = 1/4$ if $\beta \in [\pi, \beta_{cr}]$ while c_β is strictly decreasing in the interval $[\beta_{cr}, 2\pi]$. Numerical computations give $\beta_{cr} \simeq 1.546\pi$ and $c_{2\pi} \simeq 0.205$. In [15] the boundary value problem (3.1) was further analyzed to obtain the following explicit description of c_β:

Theorem 3.3. *The critical angle β_{cr} is the unique solution in the interval $(\pi, 2\pi)$ of the equation*

$$\tan\Big(\frac{\beta_{cr} - \pi}{4}\Big) = 4\bigg(\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}\bigg)^2.$$

Moreover for $\beta \in [\beta_{cr}, 2\pi]$ the constant c_β is the unique solution to the equation

$$\sqrt{c_\beta}\tan\Big(\sqrt{c_\beta}\big(\frac{\beta-\pi}{2}\big)\Big) = 2\left(\frac{\Gamma(\frac{3+\sqrt{1-4c_\beta}}{4})}{\Gamma(\frac{1+\sqrt{1-4c_\beta}}{4})}\right)^2.$$

The next theorem was obtained in [15] and provides an explicit description of the Hardy constant of an arbitrary non-convex quadrilateral.

Theorem 3.4. *The Hardy constant of a non-convex quadrilateral is equal to c_β where $\beta \in (\pi, 2\pi)$ is the size of the non-convex angle.*

The proof of this theorem makes a combined use of the distance function and the solution $\psi(\theta)$ of (3.1). Following this method the Hardy constants of other domains were computed in [16].

Some open problems.

We have made a brief exposition of some of the results that have been obtained in the past twenty-five years concerning geometric Hardy inequalities. Naturally, several interesting problems remain open. We close this short review article by presenting three such problems.

Problem 1. Let H^* be the largest constant such that

$$\int_\Omega |\nabla u|^2 dx \geq H^* \int_\Omega \frac{u^2}{d^2}\, dx$$

for all simply connected domains $\Omega \subset \mathbb{R}^2$ and for all $u \in C_c^\infty(\Omega)$. Determining the exact value of H^* is an open problem first posed, to our knowledge, in [31]; see also [8]. What is currently known is that $H^* \in [\frac{1}{16}, c_{2\pi}]$ and the exact computation of H^* seems very difficult.

A more realistic immediate target would be to determine the Hardy constant of more planar domains and/or to narrow the above interval for H^*. The fact that for the proof of Theorem 3.4 five different types of quadrilaterals were distinguished is indicative of how challenging even this simpler problem is.

Problem 2. Consider the inequality

$$\int_\Omega |\nabla u|^2 dx \geq \alpha \int_\Omega \frac{u^2}{d^2} - \beta \int_\Omega u^2 dx \tag{3.2}$$

where α, β are real numbers and $u \in C_c^\infty(\Omega)$. The number

$$H_w(\Omega) = \sup\big\{\alpha \in \mathbb{R} : \text{there exists } \beta \in \mathbb{R} \text{ such that (3.2) holds for all } u \in C_c^\infty(\Omega)\big\}$$

is called the weak Hardy constant of the domain $\Omega \subset \mathbb{R}^n$.

The weak Hardy constant has been studied in detail in [19]. It was shown in particular that for any bounded domain Ω there holds $H_w(\Omega) = \min\{h(y) : y \in \partial\Omega\}$ where $h(y)$ is the local Hardy constant at the point $y \in \partial\Omega$ defined as

$$\begin{aligned} h(y) \quad = \quad \lim_{r\to 0} \sup\big\{\alpha \in \mathbb{R} &: \text{there exists } \beta \in \mathbb{R} \text{ such that} \\ &\text{(3.2) holds for all } u \in C_c^\infty(\Omega) \cap B_r(y)\big\}. \end{aligned}$$

It was conjectured in [19] that $h(y) \leq 1/4$ for any bounded domain Ω and any $y \in \partial\Omega$. This remains open to this day. Indeed we are not aware of any result stating that $H(\Omega) \leq 1/4$ for any bounded domain $\Omega \subset \mathbb{R}^n$, a weaker version of the above conjecture.

Problem 3. From [14, Theorem 1] and [28, Exercise 4.2.10] easily follows the following

Theorem 3.5. *Let $p > 1$ and $n, m \in \mathbb{N}$, $n, m \geq 2$. There exists a constant $c(n,m,p)$ such that for any weakly mean convex domain $\Omega \subset \mathbb{R}^n$ and any $u \in C_c^\infty(\Omega)$ there holds* [1]

$$\int_\Omega |\Delta^{m/2} u|^p dx \geq c(n,m,p) \int_\Omega \frac{|u|^p}{d^{mp}} dx \tag{3.3}$$

[1] Here $|\Delta^{m/2} u|$ stands for $|\nabla \Delta^{(m-1)/2} u|$ when m is odd.

The best value of the constant $c(m,n,p)$ is not known. Even for the seemingly simple inequality

$$\int_{\mathbb{R}^n_+} |\Delta u|^p dx \geq c \int_{\mathbb{R}^n_+} \frac{|u|^p}{x_n^{2p}} dx \ , \qquad u \in C_c^\infty(\mathbb{R}^n_+)$$

the best constant is not known except when $p=2$. This is in sharp contrast to the case of Rellich inequalities with distance to an interior point where for $m \in \mathbb{N}$ and $p>1$ satisfying $mp<n$ the best constant $A(m,p,n)$ for the inequality

$$\int_\Omega |\Delta^{m/2} u|^p dx \geq A(m,p,n) \int_\Omega \frac{|u|^p}{|x|^{mp}} dx \ , \quad C_c^\infty(\Omega),$$

is known [21] and an infinite series improvement with iterated logarithms similar to that of Theorem 2.2 has been obtained in the framework of a general Cartan-Hadamard manifold [10]. We note that in case $p=2$ and for convex Ω the best constant in (3.3) and a sharp series improvement have been obtained in [9, 14, 38]. Whether the latter estimates are valid for mean convex domains is also an open problem.

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