GEOMETRIC HARDY INEQUALITIES ON THE HEISENBERG GROUPS VIA CONVEXITY

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Abstract. We prove L^p-Hardy inequalities with distance to the boundary for domains in the Heisenberg group $\mathbb{H}^n$, $n \geq 1$. Our results are based on a geometric condition. This is first implemented for the Euclidean distance in certain non-convex domains. It is also implemented on half-spaces and convex polytopes for the distance defined by the gauge quasi-norm on $\mathbb{H}^n$ related to the fundamental solution of the horizontal Laplacian. In the more general context of a stratified Lie group of step two we study the superharmonicity and the weak H-concavity of the Euclidean distance to the boundary, thus obtaining an alternative proof for the L^2-Hardy inequality on convex domains. In all cases the constants are shown to be sharp.

1. Introduction

The classical L^p-Hardy inequality, $p > 1$, affirms that

$$\int_{\mathbb{R}^n} |\nabla u|^p \, \mathrm{d}x \geq \left| \frac{n-p}{p} \right|^p \int_{\mathbb{R}^n} \frac{|u|^p}{|x|^p} \, \mathrm{d}x \,,$$

for $u \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$, where the constant is sharp.

Another much studied type of Hardy inequality is where the Hardy potential is the distance to the boundary of a reference domain. A well known such result states that if $\Omega \subset \mathbb{R}^n$ is a convex domain and $d(x) = \mathrm{dist}(x, \partial\Omega)$, then for any $u \in C_c^\infty(\Omega)$ there holds

$$\int_\Omega |\nabla u|^p \, \mathrm{d}x \geq \left(\frac{p-1}{p} \right)^p \int_\Omega \frac{|u|^p}{d^p} \, \mathrm{d}x \,,$$

and the constant is sharp, cf. [MS97]. In [BFT04] the convexity condition was replaced by the more general notion of weak mean convexity, namely the requirement that $\Delta d \leq 0$ in the distributional sense in Ω. The above inequality is not valid without some geometric assumptions on Ω and for this reason inequalities of this type are often called geometric Hardy inequalities. The literature on geometric Hardy inequalities in Euclidean space is large and we refer the interested reader to the works [Ba24, BEL15, Dav98, RS19] which provide an overview of the topic.

On the other hand, subelliptic Hardy inequalities have been studied for quite a long time and the work [GL90] of Garofallo and Lanconelli in the 90's opened up the research in this direction. By subelliptic Hardy inequalities, we mean Hardy-type

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inequalities considered in the setting of homogeneous Lie groups, and in particular stratified groups. The systematic analysis of homogeneous Lie groups goes back to the seminal work [FS82] by Folland and Stein where the authors establish the corresponding "anisotropic" non-commutative harmonic analysis, see also [S93]. In view of their importance in the area of partial differential equations, stratified Lie groups, have been widely recognised as they play a key role in establishing subelliptic estimates for differential operators on general manifolds.

As in the Euclidean case, Hardy inequalities on stratified groups may involve either the distance to a point or the distance to the boundary. Moreover, one may use the Euclidean distance, the Carnot-Carathéodory distance or the distance related to the fundamental solution of the sub-Laplacian Δ_H, often called the gauge pseudodistance.

Concerning the distance to a point case we refer to [CCR15, D'A04, GL90, GKY17, FP21, RS17, Y13]. For an overview of the works in Hardy inequalities of all the above types we refer to the monograph [RS19]; see also the survey article [Su22].

For Hardy inequalities involving the distance to the boundary the literature in the stratified setting is limited and in most cases it involves the Euclidean distance. In [Lar16] $\mathbb{H}^n$, S. Larson shows that if $\Omega \subset \mathbb{H}^n$ is either a half space or a convex (in the Euclidean sense) domain, then for any $u \in C_c^\infty(\Omega)$ we have

$$\int_\Omega |\nabla_{\mathbb{H}^n} u|^2 \,\mathrm{d}x \geq \frac{1}{4} \int_\Omega \frac{|\nabla_{\mathbb{H}^n} d|^2}{d^2} u^2 \,\mathrm{d}x \,,$$

where $d(x)$, $x \in \Omega$, stands for the Euclidean distance to $\partial\Omega$ and the constant $1/4$ is the best possible. Later on in [RSS20] Ruzhansky et al. proved that if Ω is a half-space in any stratified group then for any $p > 1$ there holds

$$\int_\Omega |\nabla_H u|^p \mathrm{d}x \geq \Big(\frac{p-1}{p}\Big)^p \int_T \frac{|\nabla_H d|^p}{d^p} |u|^p \mathrm{d}x \,, \qquad u \in C_c^\infty(\Omega),$$

where ∇_H denotes the horizontal gradient on H. For other results in this direction see also [Rus18].

Our main interest in this work is to prove subelliptic geometric Hardy inequalities with best constant in the Heisenberg group $\mathbb{H}^n$ but also on any stratified group of step two. Our approach is based on the general method of [BFT04] and in particular in the L^p-superharmonicity of the distance function. The general result is then implemented in different contexts.

In the case of the Euclidean distance on the Heisenberg group $\mathbb{H}^n$ we prove the following result which goes beyond the convexity condition. The precise value of the constant $\beta(p,n)$ is given in Proposition 3.

Theorem A. *Let $R > \rho > 0$ and let T denote the torus*

$$T = \{\xi = (x,y,t) \in \mathbb{H}^n : \; (r-R)^2 + t^2 < \rho^2\}$$

where $r = \sqrt{|x|^2 + |y|^2}$. For any $p > 1$ there exists a positive constant $\beta(p,n)$ such that if

$$\text{(i)} \qquad R \geq \rho + \Big(\frac{(2n-1)\rho}{4}\Big)^{\frac{1}{3}}$$

$$\text{(ii)} \qquad R \geq \beta(p,n)\rho,$$

then

$$\int_T |\nabla_{\mathbb{H}^n} u|^p \mathrm{d}\xi \geq \Big(\frac{p-1}{p}\Big)^p \int_T \frac{|\nabla_{\mathbb{H}^n} d|^p}{d^p}|u|^p \mathrm{d}\xi\ , \qquad u \in C_c^\infty(T).$$

Another class of distances in a stratified group consists of those induced by any homogenous quasi-norm. In the case of the Heisenberg group $\mathbb{H}^n$ one important such quasi-norm is

$$N(\xi) = \Big((|x|^2 + |y|^2)^2 + t^2\Big)^{\frac{1}{4}}, \qquad \xi = (x, y, t) \in \mathbb{H}^n. \tag{1}$$

We recall if we set $Q = 2n + 2$,the homogeneous dimension of $\mathbb{H}^n$, then up to a multiplicative constant, N^{2-Q} is the fundamental solution of the sub-Laplacian on the Heisenberg group $\mathbb{H}^n$ [Fol73]. Denoting by d_N the distance to the boundary incuded by N, cf. (12) below, we have the following

Theorem B. *(i) Let $p > 1$ and let $D \subset \mathbb{H}^n$ be a half-space. There holds*

$$\int_D |\nabla_{\mathbb{H}^n} u|^p \mathrm{d}\xi \geq \left(\frac{p-1}{p}\right)^p \int_D \frac{|\nabla_{\mathbb{H}^n} d_N|^p}{d_N^p}|u|^p\, \mathrm{d}\xi\ , \qquad u \in C_c^\infty(D)\,.$$

(ii) In case p = 2 the above inequality is also valid for any bounded convex polytope. The constant is the best possible in both cases.

In Section 4 we extend our setting to that of an arbitrary stratified group of step two. In this setting a certain notion of convexity plays a central role. There are various notions of convexity in the sub-Riemannian setting and their properties can vary significantly [DGN03, DLZ24, LMS03]. Notably, in [MR03] R. Monty and M. Rickly prove that in the case of the Heisenberg group $\mathbb{H}$, if a set is geodesically convex and contains at least three points that do not lie on the same geodesic, then it necessarily coincides with $\mathbb{H}$.

In our context the relevant notions of convexity of sets and functions are the ones introduced at the same time by Lu, Manfredi and Stroffolini in [LMS03] on the Heisenberg group and by Danielli, Garofallo and Nhieu in [DGN03] on any stratified group. These notions are the analogues of the corresponding ones in the abelian case $\mathbb{R}^n$ but with a twist; the condition refers to a convex combination of two elements g, g' for which, additionally, $g' \in H_g$, i.e. g' lies in the horizontal plane passing through g; see Section 4.1 for the precise definitions. Exploring properties of the so-called weakly H-concave functions (see Definition 12) and using a result from [DGN03] we prove the following theorem, part (iii) of which is contained in [RSS20].

Theorem C. *Let G be a stratified group of step two and let $\Omega \subset G$ be a bounded domain which is convex in the Euclidean sense. Then*

(i) *The Euclidean distance to the boundary is weakly H-concave in Ω;*

(ii) $\Delta_{\mathbb{H}^n} d \leq 0$ *in the distributional sense in Ω;*

(iii) *The Hardy inequality*

$$\int_\Omega |\nabla_{\mathbb{H}^n} u|^2 \mathrm{d}g \geq \frac{1}{4}\int_\Omega \frac{|\nabla_{\mathbb{H}^n} d|^2}{d^2} u^2 \mathrm{d}g\ , \qquad u \in C_c^\infty(\Omega),$$

is valid.

2. Two general results on stratified groups

A stratified (or Carnot) group $G \equiv \mathbb{R}^n$ is naturally a homogeneous Lie group. Denoting by $\mathfrak{g}$ the corresponding Lie algebra we have $\dim(\mathfrak{g}) = n$ and $\mathfrak{g}$ admits a vector space decomposition of the form

$$\mathfrak{g} = \bigoplus_{j=1}^{r} V_j \,, \quad \text{such that} \quad \begin{cases} [V_1, V_{i-1}] = V_i \,, & 2 \leq i \leq r, \\ [V_1, V_r] = \{0\}, \end{cases} \tag{2}$$

where

$$[V_i, V_j] = \text{span}\{[X, Y] : \; X \in V_i, \; Y \in V_j\}.$$

Such a stratification naturally equips G with a non-anisotropic dilation structure $\delta_\lambda : G \to G$, $\lambda > 0$, and makes G a homogeneous Lie group. The vector spaces V_i are called the strata of the Lie algebra $\mathfrak{g}$. A symmetric homogeneous (quasi-)norm on G is a function $N : G \to [0, \infty)$ such that (i) $N(g) = 0$ if and only if $g = e$, where e is the identity element of G; (ii) $N(g) = N(g^{-1})$; and (iii) $N(\delta_\lambda(g)) = \lambda N(g)$. In this article we shall use the term quasi-norm to indicate a symmetric homogeneous quasi-norm.

If G is a stratified group, the system $\{X_1, \ldots, X_m\}$, $m \leq n$, of vector fields in the first stratum V_1 of $\mathfrak{g}$ generates, after iterated commutators, the whole of $\mathfrak{g}$, and so it is a system of Hörmander vector fields on $\mathbb{R}^n$. The vector space spanned by $\{X_1, \ldots, X_m\}$ is referred to as the horizontal hyperplane.

The first-order vector-valued differential operator

$$\nabla_H = (X_1, \ldots, X_m)$$

is then called the horizontal gradient on G (or the subgradient on G). Similarly div_H will denote the horizontal divergence given by

$$\text{div}_H(f_1, \ldots, f_m) = X_1 f_1 + \ldots X_m f_m.$$

The second-order differential operator

$$\Delta_H = X_1^2 + \ldots + X_m^2$$

is called the horizontal Laplacian (or sublaplacian) on G and is the sub- Riemannian analogue of the Laplacian on $\mathbb{R}^n$. By Hörmander's Theorem, see [Hör67], the operator Δ_H is hypoelliptic. For $p > 1$ we also have the associated horizontal p-Laplacian given by

$$\Delta_{p,H}\, u = \text{div}_H\big(|\nabla_H u|^{p-2} \nabla_H u\big).$$

Finally, let us also recall that the (bi-invariant) Haar measure in the case of a stratified group is just, up to multiplication by a constant, the Lebesgue measure on the underlying manifold $\mathbb{R}^n$.

We first prove a general theorem which will be later applied in the case of the Euclidean distance and of the pseudodistance induced by the quasi-norm (1).

In what follows, we will say that a function is CC-Lipschitz if it is Lipschitz with respect to the Carnot-Carathéodory distance (equivalently, with respect to the distance induced by any homogeneous quasi-norm).

Part (a) of the next theorem is essentially contained in [RSS20] but we include the short proof of it because of its central role in the present article.

Theorem 1. *Let G be a stratified group and let $\Omega \subset G$ be open and connected. Let $p > 1$ and let $d : \Omega \to (0, \infty)$ be a positive, locally CC-Lipschitz function.*
(a) Assume that

$$\Delta_{p,H}\, d \le 0 \quad \text{in } \Omega\,,$$

where the inequality is understood in the distributional sense. Then

$$\int_\Omega |\nabla_H u|^p \mathrm{d}x \ge \Big(\frac{p-1}{p}\Big)^p \int_\Omega \frac{|\nabla_H d|^p}{d^p} |u|^p \,\mathrm{d}x\,, \qquad u \in C_c^\infty(\Omega).$$

(b) Assume that there exist $x_0 \in \partial\Omega$ and two neighbourhoods A, A' of x_0 in G with $A' \subset\subset A$ and such that

(i) *There exists $c > 0$ such that $|\nabla_H d| \ge c$ in $A \cap \Omega$.*

(ii) *The integral $\int_{A'\cap\Omega} d^{-1+\epsilon}\mathrm{d}x$ is finite for $\epsilon > 0$ and diverges to $+\infty$ as $\epsilon \to 0$.*

Then

$$\inf_{u \in C_c^\infty(\Omega)} \frac{\int_\Omega |\nabla_H u|^p \mathrm{d}x}{\int_\Omega \frac{|\nabla_H d|^p}{d^p}|u|^p \mathrm{d}x} \le \Big(\frac{p-1}{p}\Big)^p.$$

Proof. First we note that in view of [MSC01, Theorem 2.5] $\nabla_H d$ exists a.e. in Ω. Let T be a vector field in $L^1_{\text{loc}}(\Omega)$ and $u \in C_c^\infty(\Omega)$. Using an argument from [BFT04], with the only difference that differential operators are replaced by the corresponding horizontal ones, we obtain that

$$\int_\Omega |\nabla_H u|^p \mathrm{d}x \ge \int_\Omega \Big(\mathrm{div}_H T - (p-1)|T|^{\frac{p}{p-1}}\Big)|u|^p \mathrm{d}x,$$

where $\mathrm{div}_H T$ is understood in the distributional sense. We now make the particular choice

$$T = -\Big(\frac{p-1}{p}\Big)^{p-1} \frac{1}{d^{p-1}} |\nabla_H d|^{p-2} \nabla_H d.$$

For this choice we have

$$\mathrm{div}_H T - (p-1)|T|^{\frac{p}{p-1}} = \Big(\frac{p-1}{p}\Big)^p \frac{|\nabla_H d|^p}{d^p} - \Big(\frac{p-1}{p}\Big)^{p-1} \frac{1}{d^p}\,\Delta_{p,H} d\,,$$

hence (a) follows.

To prove (b), let ψ be a smooth cut-off function supported in A and satisfying $0 \le \psi \le 1$ and $\psi(x) = 1$ in A'. We fix $\epsilon > 0$, which will eventually tend to zero, and we define

$$u_\epsilon(x) = d(x)^{\frac{p-1}{p}+\epsilon}\psi(x), \qquad x \in \Omega.$$

A standard argument shows that u_ϵ can be used as a test function. Applying the elementary inequality $|a+b|^p \le |a|^p + c_p(|a|^{p-1}|b| + |b|^p)$, $a, b \in \mathbb{R}^n$, we obtain

$$\begin{aligned} |\nabla u_\epsilon|^p &= \Big|\big(\frac{p-1}{p} + \epsilon\big) d^{-\frac{1}{p}+\epsilon}\psi \nabla_H d + d^{\frac{p-1}{p}+\epsilon}\nabla_H \psi\Big|^p \\ &\le \Big(\frac{p-1}{p} + \epsilon\Big)^p d^{-1+\epsilon p}\psi^p |\nabla_H d|^p + c_p' d^{\epsilon p}|\nabla_H d|^{p-1}|\nabla_H \psi| + c_p d^{p-1+\epsilon p}|\nabla_H \psi|^p. \end{aligned}$$

It follows that

$$\begin{aligned} \int_\Omega |\nabla_H u_\epsilon|^p \mathrm{d}x &\le \Big(\frac{p-1}{p} + \epsilon\Big)^p \int_\Omega d^{-1+\epsilon p}\psi^p |\nabla_H d|^p \mathrm{d}x \\ &\quad + c_p' \int_\Omega d^{\epsilon p}|\nabla_H d|^{p-1}|\nabla_H \psi|\,\mathrm{d}x + c_p \int_\Omega d^{p-1+\epsilon p}|\nabla_H \psi|^p \mathrm{d}x. \end{aligned}$$

The last two integrals stay bounded as $\epsilon \to 0$, so

$$\int_\Omega |\nabla_H u_\epsilon|^p \mathrm{d}x \leq \Big(\frac{p-1}{p} + \epsilon\Big)^p \int_\Omega d^{-1+\epsilon p} \psi^p |\nabla_H d|^p \mathrm{d}x + O(1).$$

We also have

$$\int_\Omega \frac{|\nabla_H d|^p}{d^p} u_\epsilon^p \mathrm{d}x = \int_\Omega |\nabla_H d|^p d^{-1+\epsilon p} \psi^p \mathrm{d}x\,.$$

Hence, since the last integral diverges to infinity as $\epsilon \to 0$, we arrive at

$$\frac{\int_\Omega |\nabla_H u_\epsilon|^p \mathrm{d}x}{\int_\Omega \frac{|\nabla_H d|^p}{d^p} u_\epsilon^p \mathrm{d}x} = \Big(\frac{p-1}{p} + \epsilon\Big)^p + o(1), \qquad \text{as } \epsilon \to 0.$$

Letting $\epsilon \to 0$ concludes the proof. □

Given a quasi-norm N we may define a pseudodistance by

$$d_N(x,y) = N(y^{-1}x)\,, \qquad x, y \in G.$$

The following proposition allows us to apply Theorem 1 in the case of the gauge quasi-norm (1).

Proposition 2. *Let S be a closed set in a stratified group G and let N be any quasi-norm on G that is smooth out of the origin. Then the pseudodistance*

$$d_{N,S}(g) = \inf_{b \in S} d_N(b,g) = \inf_{b \in S} N(g^{-1}b)$$

is CC-Lipschitz.

Proof. By the equivalence of all quasi-norms on a stratified group it is enough to show that for $g, g' \in S$ we have

$$|d(g) - d(g')| \leq K d_N(g^{-1}g')\,, \tag{3}$$

for some $K > 0$. By [BLU07, Proposition 5.14.1] there exists $\beta \geq 1$ such that

$$N(xy) \leq \beta N(x) + N(y)\,, \qquad \text{for all } x, y \in G\,.$$

Now, let $(b_n), (b_n)' \subset S$ be such that

$$d(g) = \lim N(g^{-1}b_n)\,, \qquad d(g') = \lim N(g'^{-1}b'_n)\,,$$

We then have

$$\begin{aligned} d(g) - d(g') &\leq \liminf \big[N(g^{-1}b'_n) - N(g'^{-1}b'_n)\big] \\ &= \liminf \big[N(g^{-1}g'g'^{-1}b'_n) - N(g'^{-1}b'_n)\big] \\ &\leq \liminf \big[\beta N(g^{-1}g') + N(g'^{-1}b'_n) - N(g'^{-1}b'_n)\big] \\ &= \beta N(g^{-1}g')\,. \end{aligned}$$

Similarly we can show that

$$d(g') - d(g) \leq C\beta\, d_N g', g)\,,$$

and (3) follows. □

3. Geometric Hardy inequalities on the Heisenberg group

In the section we consider geometric Hardy inequalities on the Heisenberg group $\mathbb{H}^n$. In the first part we consider the Euclidean distance and prove the validity of the Hardy inequality with best constant on certain torii under suitable assumptions on the radii. In the second part we study the geometric Hardy inequality for the pseudodistance induced by the quasi-norm (1).

We recall that the Heisenberg group $\mathbb{H}^n$ is the manifold

$$\mathbb{H}^n = \{\xi = (x, y, t) : x, y \in \mathbb{R}^n,\ t \in \mathbb{R}\}$$

equipped with the group operation

$$\xi\xi' = \big(x + x', y + y', t + t' + 2(x \cdot y' - y \cdot x')\big).$$

The left-invariant vector fields

$$X_i = \partial_{x_i} + 2y_i\partial_t\ , \qquad Y_i = \partial_{y_i} - 2x_i\partial_t\ , \quad i = 1, \dots, n,$$

form the canonical basis basis of the first stratum and the associated horizontal gradient and horizontal Laplacian on $\mathbb{H}^n$ are given respectively by

$$\nabla_{\mathbb{H}^n} = (X_1, \dots, X_n, Y_1, \dots, Y_n),$$

and

$$\Delta_{\mathbb{H}^n} = \sum_{i=1}^{n}(X_i^2 + Y_i^2)\,.$$

So in the current section we denote the horizontal gradient and Laplacian by $\nabla_{\mathbb{H}^n}$ and $\Delta_{\mathbb{H}^n}$, respectively, to emphasize that the obtained results refer to the particular case of $\mathbb{H}^n$.

3.1. **Hardy inequalities with respect to the Euclidean distance on a torus.**

We will see here that Theorem 1 can be applied in the case of the Euclidean distance on a torus and thus goes beyond the convexity assumption of [Lar16].

We first note that for any $u \in C^2(\mathbb{H}^n)$ we have

$$\Delta_{\mathbb{H}^n} u = \sum_{i=1}^{n}(u_{x_ix_i} + u_{y_iy_i}) + 4(|x|^2 + |y|^2)u_{tt} - 2\sum_{i=1}^{n}(y_iu_{x_it} - x_iu_{y_it}) \tag{4}$$

We now use cylindrical coordinates (r, ω, t) in $\mathbb{H}^n$, that is spherical coordinates (r, ω) in $\mathbb{R}^{2n}$,

$$(x, y) = r\omega\ , \quad r > 0,\ \omega \in \mathrm{S}^{2n-1},$$

where S^{2n-1} denotes the unit sphere in $\mathbb{R}^{2n}$. The Euclidean gradient in $\mathbb{R}^{2n}$ is then given by

$$\nabla u = u_r\,\omega + \frac{1}{r}\nabla_\omega u.$$

Suppose now that a function $u \in C^2(\mathbb{H}^n)$ is independent of ω, that is $u = u(r, t)$. In this case $\nabla u = u_r\,\omega$, so

$$\sum_{i=1}^{n}(y_iu_{x_i} - x_iu_{y_i}) = \sum_{i=1}^{n}(y_i\frac{u_r}{r}x_i - x_i\frac{u_r}{r}y_i) = 0.$$

Hence, for such functions, (4) gives

$$\Delta_{\mathbb{H}^n} u = u_{rr} + \frac{2n-1}{r}u_r + 4r^2u_{tt}.$$

Consider now a torus $T \subset \mathbb{H}^n$ which is symmetric with respect to the t-axis and is centered at the origin. Letting R, ρ $(R > \rho)$ denote the two radii, T is described in cylindrical coordinates as

$$T = \{\xi = (r, \omega, t) : \ (r-R)^2 + t^2 < \rho^2\}. \tag{5}$$

The Euclidean distance to the boundary is given by

$$d(\xi) = \rho - \sqrt{(r-R)^2 + t^2}, \qquad \xi \in T,$$

and is smooth in T except on the $(2n-1)$-dimensional 'circle'

$$S = \{\xi = (r, \omega, t) : \ t = 0, \ r = R\}.$$

The horizontal Laplacian of d is then given by

$$\Delta_{\mathbb{H}^n} d = d_{rr} + \frac{2n-1}{r} d_r + 4r^2 d_{tt}, \qquad \text{in } T \setminus S. \tag{6}$$

In $T \setminus S$ we have

$$\begin{aligned} d_r &= -\frac{r-R}{\sqrt{(r-R)^2 + t^2}} \\ d_{rr} &= -\big((r-R)^2 + t^2\big)^{-\frac{3}{2}} t^2 \\ d_{tt} &= -\big((r-R)^2 + t^2\big)^{-\frac{3}{2}} (r-R)^2. \end{aligned}$$

Substituting in (6) we conclude that in $T \setminus S$ there holds

$$\begin{aligned} \Delta_{\mathbb{H}^n} d = &- \big((r-R)^2 + t^2\big)^{-\frac{3}{2}} \frac{1}{r} \\ &\times \Big\{ \Big[(2n-1)(r-R) + 4r^3\Big](r-R)^2 + \Big[r + (2n-1)(r-R)\Big] t^2 \Big\}. \end{aligned} \tag{7}$$

Proposition 3. *Let $p > 1$. Let $R > \rho > 0$ and let T be the torus* (5)*. Then there exists a positive constant $\beta(p,n)$ such that if*

$$\begin{aligned} &\text{(i)} \qquad R \geq \rho + \Big(\frac{(2n-1)\rho}{4}\Big)^{\frac{1}{3}} \\ &\text{(ii)} \qquad R \geq \beta(p,n)\rho, \end{aligned}$$

then $\Delta_{p,\mathbb{H}^n} d \leq 0$ in $T \setminus S$. Moreover we can take

$$\beta(p,n) = \begin{cases} \max\Big\{\frac{2n+p-2}{p-1}, \ \frac{2n-p+1}{2(2-p)}\Big\}, & \text{if } 1 < p < 2, \\ 2n & \text{if } p = 2, \\ 2n + p - 2 & \text{if } p \geq \frac{13+\sqrt{32n-7}}{8}. \end{cases}$$

whereas for $2 < p < \frac{13+\sqrt{32n-7}}{8}$ we have $\beta(p,n) = 1 + \frac{1}{a(p,n)}$, where $a(p,n)$ is the positive solution of

$$(2n+p-3)^2 a^2 + 4\Big((p-2)(2n+p-3) + (p-1)(2n-1)\Big)a - 4(2p-3) = 0$$

Proof. Let $A = |\nabla_{\mathbb{H}^n} d|^2$. We then have

$$\Delta_{p,\mathbb{H}^n} d = A^{\frac{p-4}{2}} \Big(A\, \Delta_{\mathbb{H}^n} d + \frac{p-2}{2}(d_r A_r + 4r^2 d_t A_t)\Big). \tag{8}$$

Now, simple computations give

$$A = d_r^2 + 4r^2 d_t^2 = \frac{(r-R)^2 + 4r^2 t^2}{(r-R)^2 + t^2}. \tag{9}$$

and

$$A_r = \frac{2(r-R)(1-4rR)t^2 + 8rt^4}{\big((r-R)^2+t^2\big)^2}, \qquad A_t = \frac{2t(r-R)^2(4r^2-1)}{\big((r-R)^2+t^2\big)^2}.$$

Hence

$$d_r A_r + 4r^2 d_t A_t = -\Big((r-R)^2+t^2\Big)^{-\frac{5}{2}}\Big\{(r-R)^2t^2(2-8r^2-8rR+32r^4)+8r(r-R)t^4\Big\}.$$

Substituting in (8) and recalling (7) we obtain after some more computations that

$$\Delta_{p,\mathbb{H}^n} d = -\,A^{\frac{p-4}{2}}\big((r-R)^2+t^2\big)^{-\frac{5}{2}}W\,,$$

where

$$\begin{aligned} W = \frac{1}{r}\Big((r-R)^2+4r^2t^2\Big)\Big((r-R)^2\Big[(2n-1)(r-R)+4r^3\Big] \\ + \Big[(2n-1)(r-R)+r\Big]t^2\Big) \\ + (p-2)\Big((r-R)^2\Big[1-4rR-4r^2+16r^4\Big]t^2+4r(r-R)t^4\Big). \end{aligned}$$

In case $p=2$ we note that our assumption implies that

$$(2n-1)(r-R)+4r^3 \geq 0\,, \quad (2n-1)(r-R)+r \geq 0$$

in T, hence $W \geq 0$, as required.

For $p \neq 2$ we collect similar powers of t to obtain

$$\begin{aligned} rW &= (r-R)^4\Big((2n-1)(r-R)+4r^3\Big) \\ &\quad + (r-R)^2\Big\{\Big(2n-1+4(2n+p-3)r^2\Big)(r-R) \\ &\quad + 16(p-1)r^5 - 8(p-2)r^3 + (p-1)r\Big\}t^2 \\ &\quad + 4r^2\Big((2n+p-3)(r-R)+r\Big)t^4 \\ &=: C_0 + C_1t^2 + C_2t^4. \end{aligned}$$

Assumption (i) implies that $C_0 \geq 0$. Similarly, assumption (ii) implies that $R \geq (2n+p-2)\rho$ in all cases and therefore $C_2 \geq 0$.

We shall prove that $C_1 \geq 0$. Equivalently, that

$$r - R + r\,\frac{16(p-1)r^4 - 8(p-2)r^2 + p - 1}{2n-1+4(2n+p-3)r^2} \geq 0, \qquad \text{for all } R-\rho \leq r \leq R+\rho.$$

For this we shall find a positive constant $a = a(p,n)$ such that

$$\frac{16(p-1)r^4 - 8(p-2)r^2 + p - 1}{2n-1+4(2n+p-3)r^2} \geq a\,, \quad r > 0,$$

or equivalently

$$16(p-1)r^4 - \Big(8(p-2)+4a(2n+p-3)\Big)r^2 + p - 1 - a(2n-1) \geq 0\,, \qquad r > 0. \tag{10}$$

If such an a has been found then we shall have $C_1 \geq 0$ provided the radii of T satisfy

$$r - R + ar \geq 0, \qquad \text{for all } r \in [R-\rho, R+\rho],$$

which is equivalent to

$$R \geq (1+\frac{1}{a})\rho.$$

At this point we need to distinguish different cases.

Case $1<p<2$. In this case we choose

$$a = a_1(p,n) := \min\Big\{\frac{p-1}{2n-1},\ \frac{2(2-p)}{2n+p-3}\Big\}$$

which makes all coefficients in (10) non-negative. The requirement on the radii then is

$$R \geq \Big(1+\frac{1}{a_1(p,n)}\Big)\rho = \max\Big\{\frac{2n+p-2}{p-1}\,,\ \frac{2n-p+1}{2(2-p)}\Big\}\,\rho,$$

and it is satisfied by our assumptions.

Case $p>2$. In this case the coefficient of r^2 in (10) is negative, so we consider the discriminant. We have

$$\begin{aligned}&\Big(8(p-2)+4a(2n+p-3)\Big)^2-64(p-1)\Big(p-1-a(2n-1)\Big)\\ =&16\Big\{-4(2p-3)+4\Big((p-2)(2n+p-3)+(p-1)(2n-1)\Big)a+(2n+p-3)^2a^2\Big\}.\end{aligned}$$

We now choose $a=a(p,n)$ to be the positive root of the quadratic polynomial above. So the requirement on the radii for (10) is

$$R \geq \Big(1+\frac{1}{a(p,n)}\Big)\rho$$

and $\beta(p,n)$ is given by

$$\beta(p,n) = \max\Big\{2n+p-2\,,\ 1+\frac{1}{a(p,n)}\Big\}.$$

Finally we to note that for $p>2$,

$$\begin{aligned}2n+p-2 \geq 1+\frac{1}{a(p,n)} &\iff a(p,n) \geq \frac{1}{2n+p-3}\\ &\iff 4p^2-13p+11-2n \geq 0\\ &\iff p \geq \frac{13+\sqrt{32n-7}}{8}.\end{aligned}$$

This completes the proof. □

Remark 1. For $1<p<2$ we have that

$$\frac{2n-p+1}{2(2-p)} \geq 2n+p-2 \quad \text{iff} \quad \frac{3}{2} \leq p < 2.$$

Moreover if we define $p_0=\sqrt{9n^2-8n+2}-3(n-1)$ then $3/2<p_0<2$ and for $1<p<2$ we have

$$\max\Big\{\frac{2n+p-2}{p-1}\,,\ \frac{2n-p+1}{2(2-p)}\Big\} = \begin{cases}\frac{2n+p-2}{p-1}, & \text{if } 1<p\leq p_0,\\ \frac{2n-p+1}{2(2-p)}, & \text{if } p_0\leq p<2.\end{cases}$$

Theorem 4. *Let $R > \rho > 0$ and let T denote the torus* (5). *Let $p > 1$ and assume that conditions (i) and (ii) of Proposition 3 are satisfied. Then*

(i) *There holds $\Delta_{p,\mathbb{H}^n} d \leq 0$ in the distributional sense in T.*

(ii) *For any $u \in C_c^\infty(T)$ there holds*

$$\int_T |\nabla_{\mathbb{H}^n} u|^p \mathrm{d}\xi \geq \Big(\frac{p-1}{p}\Big)^p \int_T \frac{|\nabla_{\mathbb{H}^n} d|^p}{d^p} |u|^p \mathrm{d}\xi.$$

Moreover the constant in (ii) is the best possible.

Proof. Let $p > 1$. By Proposition 3 we have $\Delta_{p,\mathbb{H}^n} d \leq 0$ in $T \setminus S$. Hence the inequality in (ii) for any $u \in C_c^\infty(T \setminus S)$ follows from Theorem 1.

In order to extend this to any $u \in C_c^\infty(T)$ it is enough to establish that $\Delta_{p,\mathbb{H}^n} d \leq 0$ in the distributional sense in T. That is, we must prove that given a non-negative function $\phi \in C_c^\infty(T)$ there holds

$$\int_T |\nabla_{\mathbb{H}^n} d|^{p-2} \nabla_{\mathbb{H}^n} d \cdot \nabla_{\mathbb{H}^n} \phi \, \mathrm{d}\xi \geq 0. \tag{11}$$

For this we shall use a standard approximation argument. Let

$$q(\xi) = \sqrt{(r-R)^2 + t^2}\,, \qquad \xi = (r, \omega, t) \in T,$$

be the (Euclidean) distance of $\xi \in T$ to the 'circle' S. For $\epsilon > 0$ small we consider a smooth function ψ_ϵ on T such that

$$\psi_\epsilon(\xi) = \begin{cases} 0, & \text{if } q(\xi) < \epsilon, \\ 1, & \text{if } q(\xi) > 2\epsilon \end{cases}$$

and $|\nabla \psi_\epsilon| \leq c/\epsilon$. Then $\phi_\epsilon := \psi_\epsilon \phi$ is a non-negative smooth function in $C_c^\infty(T \setminus S)$ and hence, by Proposition 3,

$$\int_T |\nabla_{\mathbb{H}^n} d|^{p-2} \nabla_{\mathbb{H}^n} d \cdot \nabla_{\mathbb{H}^n} \phi_\epsilon \, \mathrm{d}\xi \geq 0.$$

Since $|\nabla_{\mathbb{H}^n} d|$ is bounded, in order to complete the proof it is enough to show that

$$\int_T |\nabla_{\mathbb{H}^n} \phi_\epsilon - \nabla_{\mathbb{H}^n} \phi| \, \mathrm{d}\xi \longrightarrow 0\,, \qquad \text{as } \epsilon \to 0,$$

since (11) will then follow by letting $\epsilon \to 0$.

In fact, since $|\nabla_{\mathbb{H}^n} u| \leq c|\nabla u|$ in T, it is enough to consider the Euclidean gradient. We have

$$\|\nabla \phi_\epsilon - \nabla \phi\|_{L^1(T)} \leq \|(1-\psi_\epsilon)\phi\|_{L^1(T)} + \|\phi \nabla \psi_\epsilon\|_{L^1(T)}.$$

The first norm in the RHS tends to zero as $\epsilon \to 0$ by the Dominated Convergence Theorem. For the second one we have

$$\begin{aligned} \int_T |\phi \nabla \psi_\epsilon| \mathrm{d}\xi &\leq \frac{c}{\epsilon} \int_{\{\epsilon < q(\xi) < 2\epsilon\}} \mathrm{d}\xi \\ &\leq \frac{c_1}{\epsilon} \epsilon^2 \\ &\to 0. \end{aligned}$$

Hence the Hardy inequality (ii) has been proved.

To establish the optimality of the constant we apply part (b) of Theorem 1. Assumption (i) is satisfied by (9). The fact that (ii) is satisfied is well known, see [BFT04, Lemma 5.2]. This completes the proof of the theorem. □

Remark 2. It is evident from the argument in the above proof that if $\Omega \subset \mathbb{H}^n$ is a domain with C^2 boundary then the corresponding Hardy constant connot be larger than $((p-1)/p)^p$, provided there exists a point $\xi_0 \in \partial\Omega$ such that $\nabla_{\mathbb{H}^n} d(\xi_0) \neq 0$. This, of course, is very generic. We do not pursue this any further here, but we make two comments.

1. Assume that Ω possesses cylindrical symmetry so that $d = d(r,t)$. Suppose that at a point $\xi_0 \in \partial\Omega$ we have $\nabla_{\mathbb{H}^n} d(\xi_0) = 0$, that is

$$d_{x_i} + 2y_i d_t = 0 \ , \qquad d_{y_i} - 2x_i d_t = 0 \ , \qquad i = 1, \dots, n.$$

Multiplying by x_i and y_i respectively and adding we obtain

$$\sum_{i=1}^{n} (x_i d_{x_i} + y_i d_{y_i}) = 0, \qquad \text{at the point } \xi_0.$$

Using cylindrical coordinates (r, ω, t) we then obtain that $d_r(\xi_0) = 0$. By cylindrical symmetry we have

$$d_r^2 + d_t^2 = |\nabla d|^2 = 1 \ , \qquad \text{in } T,$$

hence $d_t(\xi_0) = 1$ and therefore the tangent hyperplane at ξ_0 must be parallel to the hyperplane $\{t = 0\}$. Hence generically we have $\nabla_{\mathbb{H}^n} d \neq 0$ on $\partial\Omega$

2. Let Ω be a domain which does not necessarily possess some kind of symmetry. If $\xi_0 \in \partial\Omega$ is a boundary point with $\nabla_{\mathbb{H}^n} d(\xi_0) = 0$, then we have

$$d_{x_i}^2 = 4y_i^2 d_t^2 \ , \qquad d_{y_i}^2 = 4x_i^2 d_t^2 \ , \quad i = 1, \dots, n, \qquad \text{at the point } \xi_0.$$

Since $|\nabla d| = 1$, adding implies

$$(1 + 4r^2) d_t^2 = 1, \qquad \text{at the point } \xi_0.$$

It follows that if in addition the tangent hyperplane at ξ_0 is parallel to the hyperplane $\{t = 0\}$ (and so $d_t(\xi_0) = 1$), then we must necessarily have $r = 0$, that is the point ξ_0 must lie on the t-axis.

3.2. **Hardy inequalities with respect to the gauge pseudodistance.** In this section we consider geometric Hardy inequalities in the Heiseberg group with respect to the gauge quasi-norm

$$N(\xi) = \Big((|x|^2 + |y|^2)^2 + t^2 \Big)^{\frac{1}{4}}, \qquad \xi = (x, y, t) \in \mathbb{H}^n \ .$$

For a given domain $\Omega \subset \mathbb{H}^n$ the induced distance to the boundary is given by

$$d_N(\xi) = \mathrm{dist}_N(\xi, \partial\Omega) = \inf\{N((\xi')^{-1}\xi), \ \xi' \in \partial\Omega\}, \qquad \xi \in \Omega. \tag{12}$$

We first consider the case where our domain is the half-space

$$\Pi_0 = \{(x, y, t) \in \mathbb{H}^n : t > 0\}.$$

It is then easy to see that for $\xi = (x, y, t) \in \Pi_0$ and $\xi' = (x', y', 0) \in \partial\Pi_0$ we have

$$d_N(\xi, \xi') = \left(\Big(|x' - x|^2 + |y' - y|^2 \Big)^2 + \Big(t + 2(x \cdot y' - y \cdot x') \Big)^2 \right)^{\frac{1}{4}}. \tag{13}$$

Lemma 5. *Let* $\Pi_0 = \{(x, y, t) \in \mathbb{H}^n : t > 0\}$ *and let*

$$d_N(\xi) = \inf\{N((\xi')^{-1}\xi), \ \xi' \in \partial\Pi_0\} \ , \qquad \xi \in \Pi_0, \tag{14}$$

denote the corresponding pseudodistance to the boundary. Then, for any $\xi = (x, y, t) \in \Pi_0$, $d_N(\xi)$ *depends only on* $r = \sqrt{|x|^2 + |y|^2}$ *and* $t > 0$. *More precisely, we have*

$$d_N(\xi) = d_N(r,t) = \begin{cases} \left(2r^4s^2 - 3tr^2s + t^2\right)^{\frac{1}{4}}, & r > 0,\ t > 0, \\ t^{\frac{1}{2}}, & r = 0,\ t > 0, \end{cases}$$

where for fixed $r, t > 0$ *the real number* $s \in \mathbb{R}$ *is the unique solution of the equation*

$$s^3 + 2s - \frac{t}{r^2} = 0\,. \tag{15}$$

Proof. The case $r = 0$ is immediate, so we assume that $r > 0$. By (13) the infimum in (14) is attained at a point (x', y') which is a critical point of the function

$$F(x', y') = \Big(|x' - x|^2 + |y' - y|^2\Big)^2 + \Big(t + 2(x \cdot y' - y \cdot x')\Big)^2, \quad (x', y') \in \mathbb{R}^{2n}.$$

For $i = 1, \ldots, n$ we have

$$F_{x'_i} = 4\Big(|x' - x|^2 + |y' - y|^2\Big)(x'_i - x_i) - 4\Big(t + 2(x \cdot y' - y \cdot x')\Big)y_i\,,$$
$$F_{y'_i} = 4\Big(|x' - x|^2 + |y' - y|^2\Big)(y'_i - y_i) + 4\Big(t + 2(x \cdot y' - y \cdot x')\Big)x_i\,.$$

Assume now that (x', y') is a critical point of F. Then necessarily $(x', y') \neq (x, y)$. From the last two relations we then obtain

$$x_i(x'_i - x_i) + y_i(y'_i - y_i) = 0, \qquad i = 1, \ldots, n.$$

We set

$$s = \frac{t + 2(x \cdot y' - y \cdot x')}{|x' - x|^2 + |y' - y|^2}.$$

and note that

$$x'_i - x_i = sy_i\,, \qquad y'_i - y_i = -sx_i\,, \qquad i = 1, \ldots, n. \tag{16}$$

We then have

$$x \cdot y' - y \cdot x' = \sum_{j=1}^{n} \left[x_j(y'_j - y_j) - y_j(x'_j - x_j)\right] = -r^2 s \tag{17}$$

and

$$|x' - x|^2 + |y' - y|^2 = s^2 \sum_{j=1}^{n} (y_j^2 + x_j^2) = r^2 s^2.$$

Hence for $i = 1, \ldots, n$, we have

$$F_{x'_i} = 4y_i(r^2s^3 + 2r^2s - t), \qquad F_{y'_i} = -4x_i(r^2s^3 + 2r^2s - t)$$

and we thus conclude that s must solve (15).

Since the cubic equation has a unique solution, there exists a unique critical point (x', y') of F given by (16).

Finally, by (17) we have

$$\begin{aligned} d_N^4(\xi) &= \Big(|x'-x|^2+|y'-y|^2\Big)^2+\Big(t+2(x\cdot y'-x'\cdot y)\Big)^2 \\ &= \Big(|x'-x|^2+|y'-y|^2\Big)^2+\Big(t-2r^2s\Big)^2 \\ &= 2r^4s^2-3tr^2s+t^2, \qquad (18) \end{aligned}$$

where we have also used (15). This completes the proof.

□

Proposition 6. *Let $\Pi\subset\mathbb{H}^n$ be an arbitrary half-space and let $d_N(\xi)$, $\xi\in\Pi$, denote the pseudodistance to the boundary with respect to the quasi-norm N. Then*

(i) *The function d_N is C^1 in Π.*

(ii) *For any $p>1$ there holds $\Delta_{p,\mathbb{H}^n}d_N\le 0$ in the distributional sense in Π.*

Proof. For simplicity we will write d instead of d_N. By group action (see also [Lar16, p340]) it is enough to consider the case $\Pi=\Pi_0$. Also, it is preferable to work with the function

$$G(r,t)=d(r,t)^4$$

instead of $d(r,t)$. To compute the various derivatives of $G(r,t)$ we recall from Lemma 5 that

$$G(r,t)=2r^4s^2-3tr^2s+t^2=(t-r^2s)(t-2r^2s) \quad \text{in } \Pi_0\setminus\{r=0\}, \qquad (19)$$

where $s=s(r,t)$ is defined by (15). Since $t=r^2(s^3+2s)$ we may eliminate t from (19) and we obtain

$$G(r,t)=r^4s^4(s^2+1). \qquad (20)$$

By (15) we have

$$s_t=\frac{1}{r^2(3s^2+2)}\,, \qquad s_r=-\frac{2s(s^2+2)}{r(3s^2+2)}. \qquad (21)$$

Hence

$$\begin{aligned} G_r &= 4r^3s^4(s^2+1)+r^4(6s^5+4s^3)s_r \\ &= 4r^3s^4(s^2+1)-r^4(6s^5+4s^3)\frac{2s(s^2+2)}{r(3s^2+2)} \\ &= -4r^3s^4. \qquad (22) \end{aligned}$$

Similarly we obtain

$$\begin{aligned} G_t &= 2r^2s^3, & G_{rr} &= -\frac{4r^2s^4(s^2-10)}{3s^2+2} \qquad (23)\\ G_{tt} &= \frac{6s^2}{3s^2+2}, & G_{rt} &= -\frac{16rs^3}{3s^2+2}. \end{aligned}$$

It is clear from (19) that to prove (i) we only need to restrict out attention near the half-line $\{r=0,\ t>0\}$.

From (15) we easily find that for fixed $t>0$ we have

$$s=\frac{t^{\frac13}}{r^{\frac23}}-\frac23\frac{r^{\frac23}}{t^{\frac13}}+o(r^{\frac23}), \qquad \text{as } r\to 0.$$

It then follows from (20) that

$$G(r,t) = t^2 - 3t^{\frac{4}{3}} r^{\frac{4}{3}} + o(r^{\frac{4}{3}}), \qquad \text{as } r \to 0.$$

For the partial derivatives of G we only need to consider the first asymptotic term. From (22) and (23) we arrive at

$$G_r(r,t) = -4t^{\frac{4}{3}} r^{\frac{1}{3}} + o(r^{\frac{1}{3}}) \ , \qquad G_t(r,t) = 2t + o(1).$$

It follows that

$$G_r(0,t) = 0 = \lim_{r\to 0} G_r(r,t) \ , \qquad G_t(0,t) = 2t = \lim_{r\to 0} G_t(r,t).$$

Since the various $o(1)$ above are locally uniform with respect to $t > 0$, we conclude that G is C^1 in Π_0.

To prove (ii) we first note that setting $A = |\nabla_{\mathbb{H}^n} d|^2$ we have, cf. (8),

$$\Delta_{p,\mathbb{H}^n} d = A^{\frac{p-4}{2}} \Big(A\, \Delta_{\mathbb{H}^n} d + \frac{p-2}{2} \big(d_r A_r + 4r^2 d_t A_t \big) \Big). \tag{24}$$

We have

$$d_r = \frac{1}{4} G^{-\frac{3}{4}} G_r, \qquad d_t = \frac{1}{4} G^{-\frac{3}{4}} G_t, \qquad d_{rr} = -\frac{3}{16} G^{-\frac{7}{4}} G_r^2 + \frac{1}{4} G^{-\frac{3}{4}} G_{rr}$$

and

$$d_{tt} = -\frac{3}{16} G^{-\frac{7}{4}} G_t^2 + \frac{1}{4} G^{-\frac{3}{4}} G_{tt}, \qquad d_{rt} = -\frac{3}{16} G^{-\frac{7}{4}} G_r G_t + \frac{1}{4} G^{-\frac{3}{4}} G_{rt}.$$

Moreover

$$A = d_r^2 + 4r^2 d_t^2 = \frac{1}{16} G^{-\frac{3}{2}} (G_r^2 + 4r^2 G_t^2) = G^{-\frac{3}{2}} r^6 s^6 (s^2+1),$$

$$\begin{aligned} A_r &= 2 d_r d_{rr} + 8r d_t^2 + 8r^2 d_t d_{rt} \\ &= G^{-\frac{5}{2}} \Big(-\frac{3}{32} G_r^3 + \frac{1}{8} G\, G_r G_{rr} + \frac{r}{2} G\, G_t^2 - \frac{3r^2}{8} G_r G_t^2 + \frac{r^2}{2} G\, G_t G_{rt} \Big) \\ &= G^{-\frac{5}{2}} \, \frac{2r^9 s^{12} (s^2+2)(s^2+1)}{3s^2+2} \end{aligned}$$

and

$$\begin{aligned} A_t &= 2 d_r d_{rt} + 8r^2 d_t d_{tt} \\ &= G^{-\frac{5}{2}} \Big(-\frac{3}{32} G_r^2 G_t + \frac{1}{8} G\, G_r G_{rt} - \frac{3r^2}{8} G_t^3 + \frac{r^2}{2} G\, G_t G_{tt} \Big) \\ &= -G^{-\frac{5}{2}} \, \frac{r^8 s^{11} (s^2+1)}{3s^2+2}. \end{aligned}$$

Combining the above we arrive at

$$d_r A_r + 4r^2 d_t A_t = -2\, G^{-\frac{13}{4}} \, \frac{r^{12} s^{14} (s^2+1)^3}{3s^2+2}. \tag{25}$$

On the other hand in $\Pi_0 \setminus \{r = 0\}$ we have, cf. (6),

$$\begin{aligned} \Delta_{\mathbb{H}^n} d &= d_{rr} + \frac{2n-1}{r} d_r + 4r^2 d_{tt} \\ &= \frac{1}{4} G^{-\frac{7}{4}} \Big\{ G G_{rr} - \frac{3}{4} G_r^2 + \frac{2n-1}{r} G G_r + 4r^2 G G_{tt} - 3r^2 G_t^2 \Big\} \\ &= -G^{-\frac{7}{4}} \, \frac{r^6 s^8 (s^2+1)\big((6n-2)s^2 + 4n - 3 \big)}{3s^2+2}. \end{aligned} \tag{26}$$

From (24), (25) and (26) we obtain

$$\Delta_{p,\mathbb{H}^n} d = -G^{-\frac{3p+1}{4}} \frac{r^{3p} s^{3p+2}(s^2+1)^{\frac{p}{2}}\big((6n+p-4)s^2+4n+p-5\big)}{3s^2+2} \leq 0,$$

and the desired inequality has been proved pointwise in $\Pi_0 \setminus \{r=0\}$ (where d_N is smooth). To complete the proof we argue as in the proof of Theorem 4, using in particular functions ψ_ϵ, $\epsilon > 0$, as in that proof. Part (i) is also used at this point since the local boundedness of $|\nabla_H d|$ is required when letting $\epsilon \to 0$. □

Proposition 7. *Let $\Pi_0 = \{(x,y,t) \in \mathbb{H}^n : t > 0\}$ and let $d_N = d_N(r,t)$ denote the corresponding gauge pseudodistance to the boundary $\partial\Pi_0$ of the point $\xi = (r,\omega,t) \in \Pi_0$ expressed in cylindrical coordinates. Then for any fixed $r \neq 0$ we have*

$$\text{(i)} \quad d_N(r,t) = \frac{t}{2r} + O(t^3)$$

$$\text{(ii)} \quad |\nabla_{\mathbb{H}^n} d_N(r,t)| = 1 + O(t^2)$$

as $t \to 0+$.

Proof. For simplicity we write d instead of d_N. Differentiating (18) we get

$$4d^3 d_t = 2t - 3r^2 s + (4r^4 s - 3tr^2)s_t.$$

Now using the first part of (21) and the fact that s solves (15) we obtain

$$d_t = \frac{6ts^2 - 2r^2 s + t - 9r^2 s^3}{4d^3(3s^2+2)} = \frac{6ts^2 + 16r^2 s - 8t}{4d^3(3s^2+2)}.$$

Similarly we find that

$$d_r = \frac{24r^4 s^4 + 16r^4 s^2 - 18tr^2 s^3 - 20tr^2 s + 6t^2}{4d^3 r(3s^2+2)} = \frac{40r^2 ts - 32r^4 s^2 - 12t^2}{4d^3 r(3s^2+2)}.$$

We now let $t \to 0+$. From (15) we find

$$s = \frac{t}{2r^2} - \frac{t^3}{16r^6} + O(t^5).$$

Plugging this in (18) we have

$$d^4(r,t) = \frac{t^4}{16r^4} - \frac{t^6}{64r^8} + O(t^8)$$

and (i) follows. We then also have

$$\frac{1}{d^6(r,t)} = \frac{64r^6}{t^6} + O(\frac{1}{t^4})$$

and combining the above we obtain

$$d_t^2 = \frac{1}{4r^2} + O(t^2)\,, \qquad d_r^2 = \frac{t^2}{4r^4} + O(t^4).$$

We thus conclude that

$$|\nabla_{\mathbb{H}^n} d|^2 = d_r^2 + 4r^2 d_t^2 = 1 + O(t^2),$$

as required. □

Theorem 8. *Let $p > 1$ and Π be an arbitrary half-space in $\mathbb{H}^n$. Let $d_N(\xi) = \mathrm{dist}_N(\xi, \partial\Pi)$ denote the corresponding pseudodistance of $\xi \in \Pi$ to the boundary $\partial\Pi$. Then there holds*

$$\int_{\Pi} |\nabla_{\mathbb{H}^n} u|^p \mathrm{d}\xi \geq \Big(\frac{p-1}{p}\Big)^p \int_{\Pi} \frac{|\nabla_{\mathbb{H}^n} d_N|^p}{d_N^p} |u|^p \mathrm{d}\xi \ , \quad u \in C_c^\infty(\Pi).$$

Moreover the constant is the best possible.

Proof. Action by an appropriate group element reduces the proof to the case $\Pi = \Pi_0 = \{(x, y, t) : t > 0\}$. The validity of the Hardy inequality is a consequence of Theorem 1 (a) and Proposition 6. The sharpness of the constant follows from the second part of Theorem 1 (b) and Proposition 7. □

In case $p = 2$ we will extend the above to the case of a bounded convex polytope. For this we will need the following lemma where, as above, $\Pi_0 = \{(x, y, t) \in \mathbb{H}^n : t > 0\}$.

Lemma 9. *Any point $\xi \in \Pi_0$ has a unique nearest boundary point $(x', y', 0) \in \partial\Pi$. Moreover, given a point $\xi' = (x', y', 0) \in \partial\Pi_0$ and $\rho > 0$, there exists a unique point $\xi = (x, y, t) \in \Pi_0$ whose nearest boundary point is ξ' and for which $d_N(\xi) = \rho$.*

Proof. We have already seen in the proof of Lemma 5 that given $\xi \in \Pi_0$ the nearest boundary point $\xi' \in \partial\Pi$ is uniquely defined.

Suppose now that a point $\xi' = (x', y', 0) \in \partial\Pi_0$ and $\rho > 0$ are given. Assume that $\xi \in \Pi_0$ has ξ' as its nearest boundary point and that $d_N(\xi) = \rho$. Denoting $r^2 = |x|^2 + |y|^2$ and $r'^2 = |x'|^2 + |y'|^2$ we have from (16) that

$$r'^2 = (1 + s^2) r^2 \ ,$$

where $s > 0$ is defined in terms of r, t by (15). We also have (cf. (20)) $\rho^4 = r^4 s^4 (1 + s^2)$. We thus conclude that

$$\frac{r'^4}{\rho^4} = \frac{1 + s^2}{s^4},$$

and this relation uniquely determines $s > 0$. Now going back to (16) we obtain

$$x_i = \frac{x_i' - s y_i'}{1 + s^2}, \qquad y_i = \frac{s x_i' + y_i'}{1 + s^2}.$$

We also have $t = r^2(s^3 + 2s)$, hence the point $\xi = (x, y, t) \in \Pi_0$ has been uniquely determined. It is not difficult now to see that this point has indeed $(x', y', 0)$ as its nearest boundary point and $d_N(\xi) = \rho$. This completes the proof. □

Remark 3. Let us point out that the convexity of a set $\Omega \subset G$ in a stratified group G is a genuine geometric notion in the sense that it is invariant under left translations; i.e., if $\Omega \subset G$ is convex and $g \in G$, then $g\Omega$ is also convex.

Theorem 10. *Let $\Omega \subset \mathbb{H}^n$ be a bounded convex polytope and let $d_N(\xi)$, $\xi \in \Omega$, denote the corresponding gauge pseudodistance to the boundary. Then*

(i) $\Delta_{\mathbb{H}^n} d_N \leq 0$ *in the distributional sense in* Ω.

(ii) *The Hardy inequality*

$$\int_\Omega |\nabla_{\mathbb{H}^n} u|^2 \mathrm{d}\xi \geq \frac{1}{4} \int_\Omega \frac{|\nabla_{\mathbb{H}^n} d_N|^2}{d_N^2} u^2 \mathrm{d}\xi \ , \quad u \in C_c^\infty(\Omega),$$

is valid.

Proof. Let $E_1, \ldots, E_m$ denote the sides of Ω. We define

$$A_k = \{\xi \in \Omega : d(\xi) = \mathrm{dist}_N(\xi, E_k), \}, \qquad k = 1, \ldots, m.$$

Hence the sets A_k have pairwise disjoint interiors and $\cup_{k=1}^m A_k = \Omega$. Let Π_k, $k = 1, \ldots, m$, denote the half-spaces determined by Ω so that

$$E_k \subset \partial\Pi_k, \ \ k = 1, \ldots, m, \quad \text{and} \quad \Omega = \bigcap_{k=1}^m \Pi_k.$$

We claim that

$$d_N(\xi) = \mathrm{dist}_N(\xi, \partial\Pi_k) =: d_k(\xi) \ , \qquad \text{for all } \xi \in A_k. \tag{27}$$

Indeed, suppose to the contary that (27) is not true for some k and some $\xi_0 \in A_k$. Then there exists $\xi_0' \in \partial\Pi_k \setminus E_k$ such that

$$\mathrm{dist}_N(\xi_0, \partial\Pi_k) = d_N(\xi_0, \xi_0').$$

By Lemma 9 there exists a continuous curve joining ξ_0 and ξ_0' with the property that each point on the curve has ξ_0' as its nearest point on $\partial\Pi_k$. Since $\xi_0' \notin E_k$, that curve necessarily intersects some A_j, $j \neq k$, which is a contradiction.

It immediately follows from (27) that

$$d_N(\xi) = \min_{1 \leq k \leq m} d_k(\xi), \qquad \xi \in \Omega.$$

To prove (i) we fix a non-negative function $\phi \in C_c^\infty(\Omega)$ and we aim to show that

$$\int_\Omega \nabla_{\mathbb{H}^n} d \cdot \nabla_{\mathbb{H}^n} \phi \, \mathrm{d}\xi \geq 0.$$

We recall the divergence theorem in the stratified setting: if the vector field F takes values in the first stratum and sufficient regularity is assumed then

$$\int_A \mathrm{div}_{\mathbb{H}^n} F \, \mathrm{d}\xi = \int_{\partial A} F \cdot \nu_H dS \, ,$$

where $\nu_H = (\nu_x + 2\nu_t y, \nu_y - 2\nu_t x)$ and $\nu = (\nu_x, \nu_y, \nu_t)$ denotes the usual outer normal.

Let $k \in \{1, \ldots, m\}$ be fixed. Integrating by parts and using Proposition 6 we obtain

$$\begin{aligned} \int_{A_k} \nabla_{\mathbb{H}^n} d \cdot \phi \, \mathrm{d}\xi &= -\int_{A_k} \phi \, \Delta_{\mathbb{H}^n} d_k \, \mathrm{d}\xi + \int_{\partial A_k} \phi \, \nabla_{\mathbb{H}^n} d \cdot \nu_H dS \\ &\geq \int_{\partial A_k} \phi \, \nabla_{\mathbb{H}^n} d_k \cdot \nu_H dS. \end{aligned}$$

We then add over all $k = 1, \ldots, m$.

Now, each boundary ∂A_k consists of outer parts where ϕ vanishes as well as of common boundaries with other sets A_j, $j \neq k$. Let us fix such a set A_j, $j \neq k$. Denoting by ν_H the horizontal normal vector which is outer with respect to A_k, we conclude that the two contributions on the surface S from A_k and A_j add up to

$$\int_S \phi\,(\nabla_{\mathbb{H}^n} d_k - \nabla_{\mathbb{H}^n} d_j)\cdot \nu_H dS.$$

The surface S is a level set for the function $d_k - d_j$ and at each point $\xi \in S$ there holds $\nabla d_k - \nabla d_j = \lambda\nu$ where $\lambda = \lambda(\xi) \geq 0$. We therefore have

$$\begin{aligned}
&(\nabla_{\mathbb{H}^n} d_k - \nabla_{\mathbb{H}^n} d_j)\cdot \nu_H \\
&= \big(\nabla_x d_k - \nabla_x d_j + 2d_{k,t}y - 2d_{j,t}y, \nabla_y d_k - \nabla_y d_j - 2d_{k,t}x + 2d_{j,t}x\big) \\
&\quad \cdot \big(\nu_x + 2y\nu_t, \nu_y - 2x\nu_t\big) \\
&= \big(\lambda\nu_x + 2\lambda\nu_t y, \lambda\nu_y - 2\lambda\nu_t x\big)\cdot\big(\nu_x + 2\nu_t y, \nu_y - 2\nu_t x\big) \\
&= \lambda\Big(|\nu_x + 2\nu_t y|^2 + |\nu_y - 2\nu_t x|^2\Big) \\
&\geq 0.
\end{aligned}$$

Combining the above completes the proof of (i). Part (ii) is an immediate consequence of part (i) and Theorem 1. □

As an immediate consequence of Theorem 10 we obtain the geometric uncertainty principle on the convex set $\Omega \subset \mathbb{H}^n$ with respect to the gauge pseudodistance on $\mathbb{H}^n$.

Corollary 11. *Let $D \subset \mathbb{H}^n$ be either a bounded convex polytope or an arbitary half-space in $\mathbb{H}^n$ and let*

$$d_N(\xi) = \mathrm{dist}_N(\xi, \partial D)$$

denote the corresponding pseudodistance of $\xi \in D$ to the boundary ∂D. Then for any $u \in C_c^\infty(D)$ we have

$$\left(\int_D |\nabla_{\mathbb{H}^n} u|^2 \mathrm{d}\xi\right)^{\frac{1}{2}} \left(\int_D d_N^2 u^2 \mathrm{d}\xi\right)^{\frac{1}{2}} \geq \frac{1}{2}\int_D u^2 \mathrm{d}\xi\,.$$

Proof. A combination of Theorem 10, Part (ii) and of the Cauchy-Schwarz inequality yields

$$\begin{aligned}
\left(\int_D |\nabla_{\mathbb{H}^n} u|^2 \mathrm{d}\xi\right)\left(\int_D d_N^2 u^2 \mathrm{d}\xi\right) &\geq \frac{1}{4}\left(\int_D \frac{u^2}{d_N^2}\mathrm{d}\xi\right)\left(\int_D d_N^2 u^2 \mathrm{d}\xi\right) \\
&\geq \frac{1}{4}\left(\int_D u^2 \mathrm{d}\xi\right)^2.
\end{aligned}$$

□

4. Hardy inequalities on stratified groups of step two

In this section we consider stratified groups of step two. If $G \equiv \mathbb{R}^n$ is such a group with (cf. (2)) $\dim(V_1) = m < n$, then each element $g \in G$ can be written as

$$g = (g^{(1)}, g^{(2)}) = (g_1, \cdots, g_m, g_{m+1}, \cdots, g_n),$$

where $g^{(1)} \in \mathbb{R}^m$ and $g^{(2)} \in \mathbb{R}^{n-m}$ belong in the first and the second stratum of G, respectively. It is known that the group law has the form

$$(g'g)_i = \begin{cases} g_i + g'_i\,, & i = 1, \cdots, m, \\ g_i + g'_i + \frac{1}{2}\langle B^{(i)} g'^{(1)}, g^{(1)}\rangle\,, & i = m+1, \cdots, n, \end{cases} \tag{28}$$

where the $B^{(i)}$'s are $m \times m$ matrices, and $\langle \cdot, \cdot \rangle$ stands for the standard inner product in $\mathbb{R}^m$, see e.g. [BLU07, Remark 17.3.1]. The group law (28) can also be written as

$$g'g = (g^{(1)} + g'^{(1)}, g^{(2)} + g'^{(2)} + \frac{1}{2}\langle B g'^{(1)}, g^{(1)}\rangle)\,, \tag{29}$$

where $\langle B g^{(1)}, g'^{(1)}\rangle$ denotes the $(n-m)$-tuple

$$(\langle B^{(m+1)} g^{(1)}, g'^{(1)}\rangle, \cdots, \langle B^{(n)} g^{(1)}, g'^{(1)}\rangle)\,.$$

The inverse element is then given by

$$(g^{(1)}, g^{(2)})^{-1} = (-g^{(1)}, -g^{(2)} + \frac{1}{2}\langle B g^{(1)}, g^{(1)}\rangle)\,.$$

We note that the (anisotropic) dilations on a stratified group G of step two are given by the maps δ_λ, $\lambda > 0$, defined by

$$\delta_\lambda((g^{(1)}, g^{(2)})) = (\lambda g^{(1)}, \lambda^2 g^{(2)}).$$

In the first part of this section we prove some results on the concavity, in the sense of [DGN03, LMS03], of the Euclidean distance to the boundary on a convex set $\Omega \subset G$. This, combined with results for [DGN03], yields Theorem C of the introduction and in particular the L^2 Hardy inequality on convex domains $\Omega \subset G$.

4.1. **On the distance function from the boundary of bounded convex domains in stratified groups.** To develop the subsequent analysis we first need to clarify the notions of convexity of sets and functions in the stratified setting. Even though, as mentioned above, these notions were introduced at the same time in [DGN03] and in [LMS03], here we adopt the notation of [DGN03] since in [LMS03] these notions are developed in the viscosity sense, while for us the weak sense is more suitable. To this end let us first introduce the following auxiliary notion.

Let G be a stratified group with $\dim(G) = n$ and $\dim(V_1) = m$. Given a point $g \in G$ the horizontal plane H_g passing through g is defined by

$$H_g = L_g(\exp(V_1 \oplus \{0\}))\,,$$

where L_g denotes the left translation by $g \in G$ and $\exp : \mathfrak{g} \to G$ is the exponential map for the group G. In particular, we have that

$$H_e = \exp(V_1 \oplus \{0\})\,,$$

where $e \in G$ is the identity element of G.

Following [DGN03], for given $g, g' \in G$ and $\lambda \in [0, 1]$ we denote by g_λ the anisotropic analogue of the standard Euclidean convex combination, that is

$$g_\lambda = g_\lambda(g; g') := g\delta_\lambda(g^{-1}g').$$

The following definition was given in [DGN03, Definition 5.5].

Definition 12. A function $u : G \to (-\infty, \infty]$ is called weakly H-convex if $\{g \in G : u(g) = \infty\} \neq G$, and if for every $g \in G$ and $g' \in H_g$ one has

$$u(g_\lambda) \leq u(g) + \lambda(u(g') - u(g))\,, \qquad \lambda \in [0, 1].$$

The notion of a weakly H-concave function can be defined accordingly.

In [DGN03, Definition 7.1] the authors introduced the following definition of convexity of sets in the stratified setting.

Definition 13. A subset $\Omega \subset G$ of a stratified group G is called weakly H-convex if for any $g \in \Omega$ and for any $g' \in \Omega \cap H_g$ one has $g_\lambda \in \Omega$ for every $\lambda \in [0,1]$.

Remark 4. It is easy to prove that if $\Omega \subset \mathbb{R}^n$ is convex in the Euclidean sense then Ω is a weakly H-convex set in a stratified group $G \equiv \mathbb{R}^n$ of step two. To see this we first observe that by the identification

$$\exp(g_1 X_1 + \cdots + g_n X_n) = (g_1, \cdots, g_n)\,,$$

between G and the corresponding Lie algebra $\mathfrak{g}$ via the exponential map, we have $g \in H_e$ if and only if g is of the form $g = (g_1, \cdots, g_m, 0, \cdots, 0)$. Since $H_g = L_g H_e$, we obtain from (29) that $g' \in H_g$ if and only if g' is of the form

$$g' = (g'^{(1)}, g'^{(2)}) = (g^{(1)} + v^{(1)}, g^{(2)} + \frac{1}{2}\langle Bg^{(1)}, v^{(1)}\rangle) \tag{30}$$

for some $v^{(1)} \in \mathbb{R}^m$. Suppose now that $g \in \Omega$ and let $g' \in \Omega \cap H_g$. Then $g^{-1}g' \in H_e$, which in turn implies that

$$\delta_\lambda(g^{-1}g') = (\lambda(g'^{(1)} - g^{(1)}), 0)\,.$$

So

$$\begin{aligned} g_\lambda &= (g^{(1)} + \lambda(g'^{(1)} - g^{(1)}), g^{(2)} + \frac{1}{2}\langle Bg^{(1)}, \lambda(g'^{(1)} - g^{(1)})\rangle) \\ &= (g^{(1)} + \lambda(g'^{(1)} - g^{(1)}), g^{(2)} + \frac{1}{2}\lambda\langle Bg^{(1)}, v^{(1)}\rangle\,) \end{aligned}$$

since by (30) we have $g'^{(1)} = g^{(1)} + v^{(1)}$, for some $v \in H_e$. Using (30) we conclude that

$$g_\lambda = (1-\lambda)g + \lambda g'\,. \tag{31}$$

Hence if Ω is convex (in the Euclidean sense) it is also weakly H-convex.

In the following example we show that the distance to a hyperplane with respect to the quasi-norm (1) is not weakly H-concave.

Example 1. Let $\mathbb{H}^n$ be the Heisenberg group and $\Pi_0 = \{(x,y,t) \in \mathbb{H}^n : t > 0\}$. We shall prove that the the distance d_N (cf. (12)) is not necessarily weakly H-concave.

Actually, we shall show that the weak H-concavity fails in a neighbourhood of any boundary point. Indeed, let $\xi = (x,y,t) \in \Pi_0$, and for fixed $\alpha > 0$ let $\xi' = (x', y', t) = (ax, ay, t)$. Then $\xi^{-1}\xi' = (x'-x, y'-y, 0) \in H_e$, hence $\xi' \in H_\xi$. Given $\lambda \in (0,1)$ we have by (31)

$$\xi_\lambda = \big(\lambda x' + (1-\lambda)x,\ \lambda y' + (1-\lambda)y,\ t\big).$$

We use cylindrical coordinates (cf. Section 3) and write

$$\xi = (r, \omega, t), \quad \xi' = (r', \omega, t), \quad \xi_\lambda = (r_\lambda, \omega, t)$$

Assume now for contradiction that d_N is weakly H-convex. Then

$$d_N(r_\lambda, t) \geq (1-\lambda)d_N(r,t) + \lambda d_N(r', t).$$

Using the asymptotics of Proposition 7 we then have

$$\frac{t}{2r_\lambda} \geq (1-\lambda)\frac{t}{2r} + \lambda\frac{t}{2r'} + O(t^3), \qquad \text{as } t \to 0+\,.$$

Hence
$$\frac{1}{r_\lambda} \geq (1-\lambda)\frac{1}{r} + \lambda\frac{1}{r'} \ ,$$
which contradicts the strict convexity of the function $1/r$. We note that the above argument can be implemented in a small neighbourood of any boundary point $\xi_0 \in \partial\Pi_0$.

4.2. **Hardy inequalities with respect to the Euclidean distance.** In this section we prove that the Euclidean distance to the boundary on a convex, bounded domain Ω is weakly H-concave and superharmonic. This provides an alternative proof of the L^2-Hardy inequality for such domains.

Theorem 14. *Let G be a stratified group of step two and let $\Omega \subset G$ be a convex, in the Euclidean sense, bounded domain in G. Then the Euclidean distance to the boundary is a weakly H-concave function on Ω.*

Proof. Let Ω be as in the hypothesis and let $g, g' \in \Omega$, with $g' \in H_g$. We want to show that for any $\lambda \in [0,1]$ we have

$$d(g\delta_\lambda(g^{-1}g')) \geq (1-\lambda)d(g) + \lambda d(g') \,. \tag{32}$$

Notice that showing $B_{r_\lambda}(g_\lambda) \subset \Omega$, where $B_{r_\lambda}(g_\lambda)$ is the Euclidean ball of radius $r_\lambda = (1-\lambda)d(g) + \lambda d(g')$ centered at g_λ, we would have the desired inequality (32). Let $h \in B_{r_\lambda}(g_\lambda)$. Then $|h - g_\lambda| = \rho \leq r_\lambda$. We define
$$v = \frac{h - g_\lambda}{\rho} \,, \qquad g_1 = g + \rho_1 v \,, \qquad g_1' = g' + \rho_2 v,$$
where
$$\rho_1 := \frac{d(g)}{(1-\lambda)d(g) + \lambda d(g')}\rho \,, \quad \text{and} \quad \rho_2 := \frac{d(g')}{(1-\lambda)d(g) + \lambda d(g')}\rho \,.$$
Then $g_1 \in B_{d(g)}(g) \subset \Omega$ and $g_1' \in B_{d(g')}(g') \subset \Omega$, since $|v| = 1$, $\rho_1 \leq d(g)$, and $\rho_2 \leq d(g')$. Recalling also (31) we then have
$$\begin{aligned}
(1-\lambda)g_1 + \lambda g_1' &= (1-\lambda)(g + \rho_1 v) + \lambda(g' + \rho_2 v) \\
&= (1-\lambda)g + \lambda g' + (1-\lambda)\rho_1 v + \lambda \rho_2 v \\
&= g_\lambda + \rho v \\
&= h \,,
\end{aligned}$$
where the last inequality follows by the choice of v. Hence, by the Euclidean convexity of Ω, we have $h \in \Omega$, and the proof is complete. □

From Theorem 14 we immediately obtain the following result; the sharpness of the constant $1/4$ follows under the hypotheses of part (b) of Theorem 1.

Theorem 15. *Let G be a stratified group of step two and let $\Omega \subset G$ be a bounded domain which is convex in the Euclidean sense. Then*

(i) $\Delta_H d \leq 0$ *in the distributional sense in* Ω;

(ii) *The Hardy inequality*
$$\int_\Omega |\nabla_H u|^2 \mathrm{d}g \geq \frac{1}{4}\int_\Omega \frac{|\nabla_H d|^2}{d^2} u^2 \mathrm{d}g \ , \quad u \in C_c^\infty(\Omega),$$
is valid.

Proof. By Theorem 14 the distance function is weakly H-concave. Let $X_1, \ldots,$ X_m be the vector fields that generate the first stratum V_1 of the corresponding Lie algebra. By Theorem [DGN03, Theorem 8.1] each $X_k^2 d$, $k = 1, \ldots, m$, is a non-positive Radon measure on Ω; this proves (i). Part (ii) now follows from Theorem 1. □

References

[BEL15] A.A. Balinsky, W.D. Evans, R.T. Lewis. The analysis and geometry of Hardy's inequality, Universitext Springer, Cham, 2015, xv+263 pp.

[Ba24] G. Barbatis. The Hardy Constant: A Review. In: Chatzakou, M., Restrepo, J., Ruzhansky, M., Torebek, B., Van Bockstal, K. (eds) Modern Problems in PDEs and Applications. Trends in Mathematics, vol 4. Birkhäuser, 2024.

[BFT04] G. Barbatis, S. Filippas, and A. Tertikas. A unified approach to improved L^p Hardy inequalities with best constants. *Trans. Amer. Math. Soc.*, 356:2169–2196, 2004.

[BLU07] A. Bonfiglioli, E. Lanconelli and F. Uguzzoni. *Startified Lie groups and potential theory for their sub-Laplacians.* Springer Monographs in Mathematics, Springer, 2007.

[CCR15] P. Ciatti, M. G. Cowling and F. Ricci. Hardy and uncertainty inequalities on stratified Lie groups. *Adv. Math.*, 277:365–387, 2015.

[D'A04] L. D'Ambrosio. Some Hardy inequalities on the Heisenberg group.*Differential Equations*, 40(4):552–564, 2004.

[DGN03] D. Danielli, N. Garofalo and D.M. Nhieu. Notions of convexity in Carnot groups. *Commun. Anal. Geom.*, 11(2):263–342, 2003

[Dav98] E. B. Davies. A review of Hardy inequalities. *Oper. Theory Adv. Appl.*, 110:55–67, 1998.

[DLZ24] F. Dragoni, G. Liu and Y. Zhang. Horizontal semiconcavity for the square of Carnot Cathéodory distance on 2 step Carnot groups and applications to Hamilton Jacobi equations. *arXiv: 402.19164v2*, 2024.

[Fol73] G. Folland. A fundamental solution for a subelliptic operator. *Bull. Amer. Math. Soc.*, 79(2): 373-376, 1973.

[FS82] G. Folland and E. M. Stein. *Hardy spaces on homogeneous groups.* Vol. 28 of Math. Notes, Princeton University Press, Princeton, N.J., 1982.

[FP21] V. Franceschi and D. Prandi. Hardy-Type inequalities for the Carnot–Carathéodory distance in the Heisenberg group. *J Geom. Anal.*, 31, 2455–2480 (2021).

[GL90] N. Garofalo and E. Lanconelli. Frequency functions on the Heisenberg group, the uncertainty principle and unique continuation. *Ann. Inst. Fourier* (Grenoble), 40:313–356, 1990.

[GKY17] J.A. Goldstein, I. Kombe, and A. Yener. A unified approach to weighted Hardy type inequalities on Carnot groups. *Discrete Contin. Dyn. Syst.*, 37(4):2009–2021, 2017.

[Hör67] L. Hörmander. Hypoelliptic second order differential equations, *Acta Mathematica*, 119 (1967), Issue 1, 147-171.

[Lar16] S. Larson. Geometric Hardy inequalities for the sub-elliptic Laplacian on convex domains in the Heisenberg group. *Bull. Math. Sci.*, 6:335–352, 2016.

[LMS03] G. Lu, J. J. Manfredi, and B. Stroffolini. Convex functions on the Heisenberg group. *Calc. Var. Partial Differ. Equ.*, 19:1–22, 2003.

[MS97] T. Matskewich T. and P.E. Sobolevskii. The best possible constant in generalized Hardy's inequality for convex domain in $\mathbb{R}^n$. *Nonlinear Anal.*, Theory, Methods & Appl., 28: 1601-1610, 1997.

[MSC01] R. Monti and F. Serra Cassano. Surface measures in Carnot-Carathéodory spaces. *Calc. Var.*, 13, 339-376, 2001.

[MR03] R. Monti and M. Rickly. Geodetically convex sets in the Heisenberg group. *J. Convex Anal.*, 12 (2005), no. 1, 187–196.

[S93] E. M. Stein. *Harmonic Analysis: Real-variable Methods, Orthogonality, and Oscillatory Integrals.* Princeton Mathematical Series, 43, Princeton University Press, Princeton, NJ, 1993

[Rus18] B. Ruszkowski. Hardy inequalities for the Heisenberg Laplacian on convex bounded polytopes. *Math. Scand.*, 123:101–120, 2018.

[RS17] M. Ruzhansky and D. Suragan. On horizontal Hardy, Rellich, Caffarelli-Kohn-Nirenberg and p-sub-Laplacian inequalities on stratified groups. *Differ. Equ.*, 262(3):799-1821, 2017.

[RS19] M. Ruzhansky and D. Suragan. *Hardy inequalities on homogeneous groups: 100 years of Hardy inequalities*. Progress in Math., Vol. 327, Birkhäuser/Springer, Cham, 2019. xvi+571pp. (N)

[RSS20] M. Ruzhansky, B. Sabitbek, and D. Suragan. Geometric Hardy and Hardy-Sobolev inequalities on Heisenberg groups. *Bull. Math. Sci.*, 10 (2020), no. 3, 17 pp.

[Su22] D. Suragan. A survey of Hardy type inequalities on homogeneous groups Springer Proc. Math. Stat., 385, Springer, Cham, 2022, 99–122.

[Y13] Q. H. Yang. Hardy type inequalities related to Carnot-Carathéodory distance on the Heisenberg group.(English summary). *Proc. Amer. Math. Soc.*, 141 (2013), no. 1, 351–362.

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