Spectral Stability under $L^p$-Perturbation of the Second-Order Coefficients*

G. Barbatis

Department of Mathematics, King's College London, London WC2R 2LS, United Kingdom

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INTRODUCTION

Let $H$ be a second order elliptic operator acting on a domain $\Omega \subseteq \mathbb{R}^N$. There has been a lot of work on the stability of the spectrum and the resolvent of $H$ under various sorts of perturbations. Perturbations of the 0th-order term are well studied. See for example [7]. There are also several results on perturbations of the higher-order terms. Classical perturbation theory can be applied in the case of uniform or asymptotic perturbations (see [5]). P. Deift [4] has obtained results for measurable perturbations in the context of scattering theory. More recent results ([3]) deal with boundary perturbations. In this paper we study $L^p$-perturbations of the second-order terms.

We work on a bounded Euclidean domain $\Omega \subseteq \mathbb{R}^N$ which we assume initially to have a $C^1$ boundary. The operators involved are uniformly elliptic with real measurable coefficients and satisfy Dirichlet boundary conditions. We first prove eigenvalue stability, but our main aim is the stability of the resolvent in trace classes.

If one is only interested in the fact of convergence rather than controlling the rate, then a simple approach based upon the monotone or dominated convergence theorem exists. However, quantitative control by such methods is not possible.

We emphasize the fact that we deal with operators with measurable coefficients. If one restricts attention to the smooth coefficient case, then standard methods of perturbation theory involve assuming uniform bounds on the first derivatives of the coefficients. Such bounds are not needed in our approach and are not always available in applications: apart from being more general, the measurable coefficients hypothesis is necessary for the study of the heat transport in a body with randomly distributed impurities. In particular we study the asymptotic form of the heat diffusion

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in a uniform medium containing a large number of impurities each of small volume; see Proposition 16.

We make use of a formula of Deift [4] that was used in the late seventies in scattering theory. Our main result is Theorem 9, where we establish Lipschitz continuity of the resolvent in trace ideals as the coefficients vary in $L^p$ spaces. It turns out that the proof depends heavily on $L^p$ bounds on the gradients of the eigenfunctions of the operators involved. Such bounds are available from [6] and that is where boundary regularity is needed. We show that the $C^1$ condition on the boundary can be weakened to a Lipschitz condition. In the last part of the paper we apply our main results to examine how the heat transport in a body is affected by small impurities. Finally, we identify the limit operators that describe the heat diffusion when the conductivity of the impurities becomes infinite or zero and we study the latter.

In a paper that will appear soon, we shall generalize these results in three directions: we shall be working on Riemannian manifolds, with weighted Laplace-Beltrami operators and Neumann and mixed boundary conditions.

**The Technical Setting**

Let $\Omega \subset \mathbb{R}^N$ be bounded with a $C^1$ boundary. For a positive definite matrix $a = \{a_{ij}(x)\}$ depending measurably upon $x \in \Omega$ we denote by $H_a$ the self-adjoint operator on $L^2(\Omega)$ given formally by

$$H_a = -\sum_i \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial}{\partial x_j} \right)$$

subject to Dirichlet boundary conditions.

We assume from now on that $H_a$ is uniformly elliptic, so that defining

$$d: W^{1,2}_0(\Omega) \to \bigoplus_{r=1}^N L^2(\Omega), \quad df = \nabla f,$$

we have

$$H_a f = d^* a d.$$ (1)

Equivalently, $f \in \text{Dom}(H_a)$ if and only if $f \in W^{1,2}_0(\Omega)$ and there exists $h \in L^2(\Omega)$ such that

$$\int_\Omega a \nabla f \cdot \nabla \phi \, dx = \int_\Omega h \phi \, dx \quad \text{all } \phi \in W^{1,2}_0(\Omega) \text{ (or } C_c^\infty(\Omega)\text{)},$$

in which case we define $H_a f = h.$
It is well known that such an operator has a discrete spectrum $0 < \lambda_{1,a} < \lambda_{2,a} \leq \lambda_{3,a} \leq \cdots$ and that there exists a complete orthonormal set of eigenfunctions $\{\phi_{n,a}\}$, $H_a \phi_{n,a} = \lambda_{n,a} \phi_{n,a}$.

We also set $R_a = (H_a + 1)^{-1}$. Finally, for any symmetric matrix $w$ defined on $\Omega$ and bounded away from $-\infty$ and $+\infty$ we denote by $Q_w$ the quadratic form

$$Q_w(f, g) = \int_{\Omega} w \nabla f \cdot \nabla g \, dx$$

with $\text{Dom}(Q_w) = W_0^{1,2}(\Omega)$ so that, in particular, $Q_a$ is the quadratic form associated to $H_a$.

We quote from [6] the following regularity result, which will turn out to be crucial for our results. See also [8, p 90] for a proof of the fact that the $C^1$ condition is sufficient.

**Theorem 1.** Let $\Omega \subset \mathbb{R}^N$ be bounded with a $C^1$ boundary and let $H_a$ be uniformly elliptic on $L^2(\Omega)$. There exists a $Q$, $2 < Q < \infty$, depending only on the ellipticity constants of $H_a$ such that for any $2 \leq q < Q$ the equation

$$H_a u = f$$

with $f \in L^N_{q(N+q)}$ has a unique solution $u \in W_0^{1,q}$ and

$$\|u\|_{W_0^{1,q}} \leq C_q \|f\|_{L^N_{q(N+q)}}.$$  \hspace{1cm} (2)

In fact, $Q$ depends only on the ratio of the ellipticity constants of $H_a$.

When the ratio is very large $Q$ is close to 2, while for a ratio very close to one $Q$ is close to $+\infty$.

**Eigenvalue Stability**

In this section we show how a simple application of the min-max principle yields stability estimates for the eigenvalues of a uniformly elliptic operator when the coefficients vary in $L^p$ spaces. We shall need the following

**Lemma 2.** Let $H$ be a positive self-adjoint operator with compact resolvent and let $Q$ be the corresponding quadratic form. If $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$ are the eigenvalues of $H$, then for any $k \in \mathbb{N}$

$$\sum_{n=1}^{k} \lambda_n = \inf_{n=1}^{k} \sum_{n=1}^{k} Q(f_n)$$  \hspace{1cm} (3)
where the infimum is taken over all orthonormal sets \( \{ f_1, \ldots, f_k \} \subseteq \operatorname{Dom}(H^{1/2}) \).

**Proof.** It is clear that the left-hand side in (3) is larger than the right-hand side. Let \( \{ \phi_n \}_{n=1}^\infty \) be a complete orthonormal system of eigenfunctions of \( H \) with \( H\phi_n = \lambda_n \phi_n \). Let \( \{ f_n \}_{n=1}^k \) be an orthonormal subset of \( \operatorname{Dom}(H^{1/2}) \) and write

\[
 f_n = \sum_{m=1}^\infty a_{nm} \phi_m, \quad n = 1, \ldots, k.
\]

Then

\[
 \sum_{n=1}^k Q(f_n) = \sum_{n=1}^k \sum_{m=1}^k |a_{nm}|^2 \lambda_m
\]

and the result follows since the orthonormality of \( \{ f_n \} \) implies

\[
 \sum_{n=1}^k |a_{nm}|^2 \leq 1. \quad \square
\]

Let \( H_a, H_b \) be uniformly elliptic. By \( c_a \) we shall denote any constant that depends only on the ellipticity constants of \( H_a \). More generally, if \( w \) is any scalar- or matrix-valued function on \( \Omega \), a subindex \( w \) shall indicate dependence only on \( \|w\|_\infty \) and \( \|w^{-1}\|_\infty \).

We have the following

**Theorem 3.** There exists a number \( P = P_{a,b}, \; 2 < P < + \infty \), such that for all \( p > P \) and all \( k \in \mathbb{N} \) we have

\[
 \left| \sum_{n=1}^k \lambda_{n,b} - \lambda_{n,a} \right| \leq c_{p,a,b} \|b - a\|_p k^{1+2/N+1/p}.
\] (4)

**Proof.** From the lemma we have

\[
 \sum_{n=1}^k Q_a(\phi_{n,a}) = \sum_{n=1}^k \lambda_{n,a} \leq \sum_{n=1}^k Q_a(\phi_{n,b}),
\]

\[
 \sum_{n=1}^k Q_b(\phi_{n,b}) = \sum_{n=1}^k \lambda_{n,b} \leq \sum_{n=1}^k Q_b(\phi_{n,a}).
\]

Subtracting we get

\[
 \sum_{n=1}^k Q_{b-a}(\phi_{n,b}) \leq \sum_{n=1}^k \lambda_{n,b} - \lambda_{n,a} \leq \sum_{n=1}^k Q_{b-a}(\phi_{n,a})
\] (5)
and hence
\[
\left| \sum_{n=1}^{k} \hat{\lambda}_{n,b} - \hat{\lambda}_{n,a} \right| \leq \max \left\{ \left\| \sum_{n=1}^{k} Q_{b-a}(\phi_{n,a}) \right\|, \left\| \sum_{n=1}^{k} Q_{b-a}(\phi_{n,b}) \right\| \right\} \\
\leq \sum_{n=1}^{k} \{ |Q_{b-a}(\phi_{n,a})| + |Q_{b-a}(\phi_{n,b})| \}. \tag{6}
\]

Denoting by $\phi_n$ either $\phi_{n,a}$ or $\phi_{n,b}$ we have for any $1 < p < \infty$
\[
|Q_{b-a}(\phi_n)| = \left| \int_{\Omega} (b-a) \nabla \phi_n \cdot \nabla \phi_n \, dx \right| \\
\leq \|b-a\|_p \|\nabla \phi_n\|_2^{2p'}.
\tag{7}
\]

where $p' = p/(p-1)$.

If the constants $Q_a$, $Q_b$ are as in Theorem 1 and we set $Q_0 = \min\{Q_a, Q_b\}$, then (2) implies
\[
\|d\phi_n\|_q \leq c_{q,a} \|\phi_n\|_{Nq/(N+q)}, \quad \text{all} \ q < Q_0. \tag{8}
\]

Now, it is a standard ultracontractivity result (see [2, p. 63]) that for any $2 \leq s \leq \infty$ the semigroup $e^{-H_s t}$ maps $L^2$ into $L^s$ and in fact
\[
\|e^{-H_s t} f\|_s \leq c t^{-N(s-2)/4s}, \quad \text{all} \ f \in L^2. \tag{9}
\]
In particular
\[
\|\phi_n\|_s \leq c e^{2s/n(N-2)/4s}
\]
so that optimising over $t > 0$ we conclude that
\[
\|\phi_n\|_s \leq c \hat{\lambda}_n^{N(s-2)/4s}, \quad 2 \leq s \leq \infty.
\]

It follows from (8) that
\[
\|d\phi_n\|_q \leq c_{q,a} \hat{\lambda}_n^{(N+2-2N/q)/4}, \quad \text{all} \ q < Q_0 \tag{10}
\]
and so
\[
\|d\phi_n\|_2^{2p'} \leq c_{p,a} \hat{\lambda}_n^{(1/2) + (N/4P)}, \quad \text{all} \ p > P \tag{11}
\]
where $P$ is such that $2P' = Q_0$.

Since $\hat{\lambda}_n \leq c_{n,b} n^{2/N}$, (11) implies
\[
\|d\phi_n\|_2^{2p} \leq c_{p,a,b} n^{1/N + (1/2P)}, \quad \text{all} \ p > P.
\]
It follows that
\[ \left| \sum_{n=1}^{k} \lambda_{n,b} - \lambda_{n,a} \right| \leq c_{p,a,b} \| b - a \|_p \sum_{n=1}^{k} n^{(2/N) + (1/p)}, \quad \text{all } n \in \mathbb{N} \]
which proves the theorem since
\[ \sum_{n=1}^{k} n^{(2/N) + (1/p)} \sim k^{1 + (2/N) + (1/p)} \] as \( k \to \infty \).

Here and below, the symbol \( \sim \) indicates that either the ratio of two quantities converges to one, or that it is bounded away from zero and infinity. The meaning intended will be clear from the context.

Standard perturbation theory arguments can also be used to estimate the differences \( \lambda_{n,b} - \lambda_{n,a} \), and one can prove that there exists \( \hat{P} = \hat{P}_{a,b} \) such that
\[ |\lambda_{n,b} - \lambda_{n,a}| \leq c_{p,a,b} \| b - a \|_p n^{(2/N) + (1/p)}, \quad \text{all } p > P. \]
This, of course, implies (4) with \( P \) replaced by \( \hat{P} \).

This theorem says a lot about the stability of eigenvalues, but there are further questions that one can pose. In our main theorem we establish stability of the resolvents in trace classes. Such a result not only implies eigenvalue stability, it also yields stability of the spectral projections and, hence, of eigenspaces and eigenfunctions.

**Preliminary Results**

In order to compare the resolvents of two operators we shall need the following well known result, a proof of which can be found in [4].

**Proposition 4.** Let \( \mathcal{H}_1 \to \mathcal{H}_2 \) be closed and densely defined. Let \( H = T^*T, \ F = TT^* \). Then \( \text{Sp}(H) \cup \{ 0 \} = \text{Sp}(F) \cup \{ 0 \} \) and if \( \mu \notin \text{Sp}(H), \mu \neq 0 \), then
(i) \( \mu(H + \mu)^{-1} + T^*(F + \mu)^{-1}T = 1 \)
(ii) \( (F + \mu)^{-1}T = T(H + \mu)^{-1} \)
and dually
(i)' \( \mu(F + \mu)^{-1} + T(H + \mu)^{-1}T^* = 1 \)
(ii)' \( (H + \mu)^{-1}T^* = T^*(F + \mu)^{-1} \).
Moreover, $\mu \neq 0$ is an eigenvalue of $H$ if and only if it is an eigenvalue of $F$ and so the two multiplicities coincide.

Note that not only we may have $0 \in \text{Sp}(F)$ while $0 \notin \text{Sp}(H)$, it may also be the case that $\text{Ker}(F)$ is infinite dimensional. This will turn out to be an important feature of this problem.

Equation (12) below is an analogue of the resolvent formula

$$( -A + V_1 )^{-1} - ( -A + V_2 )^{-1} = ( -A + V_1 )^{-1} ( V_1 - V_2 ) ( -A + V_2 )^{-1}$$

which is useful for the study of 0th-order perturbations. We set

$$r_a = a^{1/2} d : L^2(\Omega) \to \mathbb{C} L^2(\Omega),$$

so that $H_a = r_a^* r_a$ by (1), and define

$$F_a = r_a r_a^* : \mathbb{C} L^2(\Omega) \to \mathbb{C} L^2(\Omega).$$

**Lemma 5.** There exist partial isometries $U_a, U_b : \mathbb{C} L^2(\Omega) \to \mathbb{C} L^2(\Omega)$ such that

$$R_b - R_a = U_b^* G(F_b) b^{1/2} ( b^{-1} - a^{-1} ) a^{1/2} G(F_a) U_a$$

(12)

where $G(t) = t^{1/2} / (t + 1)$.

**Proof.** From Proposition 4 and we have

$$R_b - R_a = -r_b^* (F_b + 1)^{-1} r_b + r_a^* (F_a + 1)^{-1} r_a$$

$$= -d^* [ (dd^* + b^{-1} )^{-1} - (dd^* + a^{-1} )^{-1} ] d$$

$$= d^* (dd^* + b^{-1} )^{-1} ( b^{-1} - a^{-1} ) (dd^* + a^{-1} )^{-1} d$$

$$= r_b^* (F_b + 1)^{-1} b^{1/2} ( b^{-1} - a^{-1} ) a^{1/2} (F_a + 1)^{-1} r_a.$$

Using polar decomposition we can write $r_a = |r_a^*| U_a$ and observe that $|r_a^*| = F_a^{1/2}$.

As we shall see later, the fact that $G(0) = 0$ is crucial, in view of the fact that the kernels of $F_a$ and $F_b$ are infinite dimensional.

For $1 \leq r < \infty$ let $\mathcal{E}^r$ denote the trace ideal

$$\mathcal{E}^r = \{ A \in \mathcal{B}(L^2(\Omega)) | \text{tr} |A|^r < \infty \}$$

normed by

$$\|A\|_{\mathcal{E}^r} = (\text{tr} |A|^r)^{1/r}$$
or, equivalently,
\[ \| A \|_{W^r} = \left\{ \sum_n \mu_n(A)^r \right\}^{1/r} \] (13)
where \( \{ \mu_n(A) \} \) are the singular values of \( A \).

Lemma 5 yields at once the following

**Corollary 6.** For any \( r > N/2 \) there exists a constant \( c_{r,a,b} \) such that
\[ \| R_b - R_a \|_{W^r} \leq c_{r,a,b} \| b - a \|_{\infty}. \] (14)

**Proof.** Using Hölder’s inequality for trace ideals, we have for any \( 1 \leq r \leq \infty \),
\[ \| R_b - R_a \|_r \leq \| G(F_b) \|_{2r} \| b^{1/2}(b^{-1} - a^{-1})a^{1/2} \|_{\infty} \| G(F_a) \|_{2r} \]
Since \( G(0) = 0 \), Proposition 4 and (13) imply that
\[ \| G(F_a) \|_{2r} = \left( \sum_n G(\lambda_n a)^{2r} \right)^{1/2r} \]
where \( \{ \lambda_n \} \) are the eigenvalues of \( H_a \). Since \( \lambda_n \sim n^{2/N} \) as \( n \to \infty \), we conclude that \( \| G(F_a) \|_{2r} < \infty \) if and only if \( r > N \), and the same holds for \( \| G(F_b) \|_{2r} \). (14) follows if we note that
\[ \| b^{1/2}(b^{-1} - a^{-1})a^{1/2} \|_{\infty} < c_{a,b} \| b - a \|_{\infty}. \]

What we are interested in is to obtain estimates of this type but with the \( L^p \) norm being replaced by some other \( L^p \) norm, with \( p \) being as small as possible. See Proposition 16 for a physical interpretation of this requirement.

**A Negative Result**

Before continuing, let us see why a certain approach which seems, probably, more natural and efficient does not actually work in the problem we are interested in.

A theorem of Birman and Solomjak [1] asserts that the eigenvalues \( \{ \lambda_n \} \) of \( H_a \) satisfy
\[ \lambda_n \sim c_N \left\{ \left( \frac{\det a^{-1}}{\lambda_2} \right)^{1/2} \right\}^{-2/N} n^{2/N} \] as \( n \to \infty \) (15)

...
so that, in particular,

\[ \lambda_n^{-1} \leq c_n ||a^{-1}||_{N/2}^2. \]

Hence, for any \( q > N/2 \),

\[
\| H_a^{-1} \|_q = \left( \sum_n \lambda_n^{-q} \right)^{1/q} \\
\leq c_N \| a^{-1} \|_{N/2} \left( \sum_n n^{-2q/N} \right)^{1/q} \\
= c_{N,q} \| a^{-1} \|_{N/2}.
\] (16)

If the map \( a^{-1} \mapsto H_a^{-1} = (d^*ad)^{-1} \) were linear, which seems as though it could be in view of what the formula

\[
(H_b + \mu)^{-1} - (H_a + \mu)^{-1} = d^*(dd^* + \mu b^{-1})^{-1}(b^{-1} - a^{-1})(dd^* + \mu a^{-1})^{-1} d
\]

looks like as \( \mu \to 0 \), then we would conclude that

\[
\| H_b^{-1} - H_a^{-1} \|_q \leq c_q \| b^{-1} - a^{-1} \|_{N/2}, \quad \text{all } q > N/2,
\]
a much better result than Theorem 9 in that the range of the parameters is better and no regularity of the boundary is needed. However, we have the following

**Lemma 7.** The map \( a^{-1} \mapsto H_a^{-1} \) is not linear.

**Proof.** For \( \mu > 0 \) we have

\[
(H_b + \mu)^{-1} - (H_a + \mu)^{-1} = U_b^* G_p(F_b) b^{1/2}(b^{-1} - a^{-1}) a^{1/2} G_p(F_a) U_a
\]

where \( G_p(t) = t^{1/2}/(t + \mu) \).

Defining \( P_a : \bigoplus L^2 \to \bigoplus L^2 \) by

\[
P_a f = \begin{cases} 0 & \text{if } f \in \text{Ker } F_a \\ \lambda_n^{-1/2} f & \text{if } f \in \text{Ker}(F_a - \lambda_n) \end{cases}
\]

we can easily check that

\[
\| G_p(F_a) - P_a \| \to 0 \quad \text{as } \mu \to 0
\]

and conclude that

\[
H_b^{-1} - H_a^{-1} = T_b(b^{-1} - a^{-1}) T_a
\] (17)
where $T_a = a^{1/2} P_a U_a$. Replacing by $kb$, $k \in \mathbb{N}$, and observing that $T_k = T_{kb}$ we have

$$H_{kb}^{-1} - H_a^{-1} = T_{kb}(k^{-1}b^{-1} - a^{-1}) T_a$$

and letting $k \to \infty$

$$H_a^{-1} = T_b a^{-1} T_a = T_{a^{-1}} T_a$$

where $T =: T_j$. Now, suppose that $T_a$ does not depend on $a$. Then

$$T_a T_a^*$$

is independent of $a$

$$a^{1/2} P_a^2 a^{1/2}$$

is independent of $a$

$$P_a^2 = a^{-1/2} Q a^{-1/2}, \quad \text{some } Q$$

$$a^{-1/2} Q a^{-1/2} f = 0, \quad \forall f \in \text{Ker } F_a$$

$$Q g = 0, \quad \forall g \text{ such that } d^* a g = 0$$

which is a contradiction.

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**Resolvent Stability**

To prove our main result, Theorem 9, we need some trace estimates for operators of the form $\tilde{V} G(F_a)$ acting on $L^2(\Omega)$. Let $2 < p < \infty$ be a parameter. We think of $\text{Sp}(F_a)$ as a measure space with each eigenvalue $\lambda_{n,a}$ carrying a weight $\lambda_{n,a}^N p \times m(\lambda_{n,a})$, where $m$ stands for the multiplicity of the eigenvalue, while to 0 we assign weight $+\infty$ reflecting the fact that $\text{Ker}(F_a)$ is infinite dimensional. Associated to this discrete measure space are the corresponding $l^q$ spaces, $1 \leq q \leq \infty$, defined by

$$l^q = \left\{ \tilde{G} : \text{Sp}(F_a) \to \mathbb{R} \ | \ \tilde{G}(0) = 0, \sum_{n} |\tilde{G}(\lambda_{n,a})|^q \lambda_{n,a}^N p < \infty \right\}, \quad 1 \leq q \leq \infty$$

and

$$l^\infty = \left\{ \tilde{G} : \text{Sp}(F_a) \to \mathbb{R} \ | \ \sup_{n} |\tilde{G}(\lambda_{n,a})| < \infty \right\}.$$
Lemma 8. Let $\mathcal{V}$ be a measurable matrix-valued map and $\mathcal{G} : \text{Sp}(F_a) \to \mathbb{R}$. There exists $P_a < \infty$ such that for $p > P_a$ and $1 \leq r \leq \infty$, $\mathcal{V} \in L^p(\Omega)$ and $\mathcal{G} \in l^2(\text{Sp}(F_a), \lambda_n^{-1/2})$ imply $\| \mathcal{V} \mathcal{G}(F_a) \|_2 \leq c_{p,a} \| \mathcal{V} \|_p \| \mathcal{G} \|_2$.

Proof. It is enough to prove the result for $r = 1$, $\infty$ since we can then use interpolation. The case $r = \infty$ is trivial.

Let $\{ \psi_{n,a} \}$ be a complete orthonormal system of eigenfunctions of $H_a$, say $H_a \psi_{n,a} = \lambda_{n,a} \psi_{n,a}$. Then

$$\psi_{n,a} = \lambda_{n,a}^{-1/2} \mathcal{G}_a \phi_n,$$

is an orthonormal system in $\oplus L^2(\Omega)$ that satisfies $F_a \psi_{n,a} = \lambda_{n,a} \psi_{n,a}$ and spans $(\text{Ker} F_a)^\perp$. Since $\mathcal{G}(0) = 0$, we have

$$\| \mathcal{V} \mathcal{G}(F_a) \|_2^2 = \sum_n \| \mathcal{V} \mathcal{G}(F_a) \psi_{n,a} \|_2^2$$

$$= \sum_n | \mathcal{G}(\lambda_{n,a}) |^2 \| \mathcal{G} \psi_{n,a} \|_2^2$$

$$\leq \| \mathcal{V} \|_p^2 \sum_n | \mathcal{G}(\lambda_{n,a}) |^2 \| \psi_{n,a} \|_{2p/(p-2)}^2$$

$$\leq c_a \| \mathcal{V} \|_p^2 \sum_n | \mathcal{G}(\lambda_{n,a}) |^2 \lambda_{n,a}^{-1/2} \| \phi_{n,a} \|_{2p/(p-2)}^2.$$ Setting $P_a = 2Q_a/(Q_a - 2)$ it follows from (10) that

$$\| \phi_{n,a} \|_{2p/(p-2)} \leq c_{p,a} \lambda_{n,a}^{(p+N)/2p}, \quad \text{all } p > P_a$$

and we conclude that

$$\| \mathcal{V} \mathcal{G}(F_a) \|_2^2 \leq c_{p,a} \| \mathcal{V} \|_p^2 \sum_n | \mathcal{G}(\lambda_{n,a}) |^2 \lambda_{n,a}^{-1/2}$$

as required. \[ \Box \]

Now we can prove the following

Theorem 9. There exists $P_0$ satisfying $2 < P_0 < \infty$ and depending only on $\Omega$ and the ellipticity constants of $H_a$ and $H_b$ such that if

\begin{enumerate}
  \item [i)] \hspace{1cm} $p > P_0$ \hspace{3cm} (18)
\end{enumerate}

and

\begin{enumerate}
  \item [ii)] \hspace{1cm} $r > \frac{N}{2} + \frac{N}{p}$ \hspace{3cm} (19)
\end{enumerate}
then
\[ \|R_b - R_a\|_r \leq c_{p,r,a,b} \|b - a\|_{p^{r/2}}. \] (20)

Proof. We may assume that \( W = b - a \geq 0 \) and hence write \( W = V^2 \), since otherwise we can write \( W = W_+ - W_- \) and use the fact that \( \|W_\pm\|_q \leq \|W\|_q \).

For any \( 1 \leq r \leq \infty \) we then have from Lemma 5
\[ \|R_b - R_a\|_r \leq \|G(F_b)\|_{b^{1/2}V} \|V^{-1/2}G(F_a)\|_{2r} \]
so that if \( p > P_0 = \max\{P_a, P_b\} \), then
\[ \|R_b - R_a\|_r \leq c_{p,r,a,b} \|V\|_{pr}^{2} \|G(F_b)\|_{2r} \|G(F_a)\|_{2r} \]
by Lemma 8. We have \( G(0) = 0 \) and, in fact,
\[ \|G\|_{2r} < \infty \iff \sum_n |G(\lambda_{n,a})|^{2r} \lambda_n^{N/p} < \infty \]
\[ \iff \sum_n \lambda_{n,a}^{1/r + (N/p)} < \infty \]
\[ \iff \sum_n n^{2/(N(-r+(N/p)))} < \infty \]
\[ \iff \frac{1}{2} + \frac{1}{p} \leq \frac{r}{N} \]
which proves the theorem.

Remark. If the boundary \( \partial \Omega \) and the matrices \( a \) and \( b \) are sufficiently smooth, then the constant \( Q \) in Theorem 1 can be taken to be equal to \( +\infty \). This implies that the index \( P_0 \) in the Theorem can be taken to be equal to 2 in that case.

Remark. If we do not make any regularity assumptions on \( \partial \Omega \), then a variation of Theorem 1 exists (Theorem 2 of [6]) that involves local rather than global Sobolev estimates. Hence the conclusion of Lemma 8, and hence of Theorem 9, is still valid under the additional assumption that the difference \( b - a \) has compact support in \( \Omega \).

Our theorem establishes the Lipschitz continuity of the map
\[ L^{p^{r/2}} \ni a \mapsto R_a \in \mathcal{C}^r \]
when \( p \) and \( r \) satisfy (i) and (ii) and the \( L^\infty \) norms of the matrices \( a \) and \( a^{-1} \) are bounded away from infinity. It is possible however to improve the
range of both parameters $p$ and $r$ at the cost of replacing Lipschitz continuity by Hölder continuity.

**Corollary 10.** If $p$, $r$, $s$ and $\gamma$ satisfy

$$\begin{align*}
p > P_0, \quad r > \frac{N}{2} + \frac{N}{p}, \quad \frac{N}{2} < s < r, \quad \gamma < \frac{r(2s-N)}{s(2r-N)}
\end{align*}$$

(21)

then there exists a constant $c_{p,r,s,\gamma,a,b} < \infty$ such that

$$\| R_b - R_a \| \leq c_{p,r,s,\gamma,a,b} \| b - a \|^{2\gamma/p}_t$$

(22)

for all $1 \leq t \leq pr/2$.

**Proof.** From (17) we have

$$\| R_b - R_a \|_q \leq c_{q,a,b}, \quad \text{all } q > N/2.$$  

(23)

Interpolation between (20) and (23) yields

$$\| R_b - R_a \| \leq c_{p,r,s,\gamma,a,b} \| b - a \|^{2\gamma/p}_t$$

(24)

and the result follows if we apply the formula

$$\| u \|_p \leq \| u \|_{(p-t)^p}^{(p-t)}, \quad \text{for all } 1 \leq t \leq p.$$  

which is valid for all $1 \leq t \leq p$. 

From the above estimate we can immediately obtain stability estimates for the eigenvalues: let $\{ \tilde{\lambda}_{n,a} \}, \{ \tilde{\lambda}_{n,b} \}$ be the eigenvalues of $H_a$ and $H_b$ respectively.

**Corollary 11.** If $p$, $r$, $s$, $\gamma$ and $t$ are as in Corollary 10 then

$$\left( \sum_n | \tilde{\lambda}_{n,b} - \tilde{\lambda}_{n,a} |^{s} n^{-4s/n} \right)^{1/s} \leq c_{p,r,s,\gamma,a,b} \| b - a \|^{2\gamma/p}_t.$$  

(25)

**Proof.** This follows from the fact that if $A$, $B$ are any two compact operators acting on a Hilbert space $\mathcal{H}$, then

$$\left( \sum_n | \mu_n(A) - \mu_n(B) |^{s} n^{-4s/n} \right)^{1/s} \leq \| A - B \|_{\mathcal{L}(\mathcal{H})},$$

where $\{ \mu_n(A) \}, \{ \mu_n(B) \}$ are the singular values of the operators. See [9, p. 20] for a proof.
A comparison of Theorem 3 with Theorem 9 is natural. First, we note that the constant $P$ of Theorem 3 is equal to $P_0/2$, where $P_0$ is as in Theorem 9.

Hence, while (4) says that
\[
\sum_{n=1}^{k} \lambda_{n,b} - \lambda_{n,a} \leq c \| b - a \|_p k^{1 + (1/N) + (1/p)}, \quad \text{all } p > P_0/2. \tag{26}
\]
from (25) we have
\[
\sum_{n=1}^{k} |\lambda_{n,b} - \lambda_{n,a}| \leq c \| b - a \|_{pr/2} k^{1 + (1/N) - (1/r)} \tag{27}
\]
whenever $p > P_0$ and $r > N/2 + N/p$. Choosing any $p > P_0$ and
\[
r = \frac{N}{2} + \frac{N}{p} + \varepsilon
\]
(27) gives
\[
\sum_{n=1}^{k} |\lambda_{n,b} - \lambda_{n,a}| \leq c \| b - a \|_{(pN/4) + (N/2) + \varepsilon} k^{1 + (1/N) - (2pN(2 + p)) + \varepsilon} \]
while (26) for $q = pN/4 + N/2$ gives
\[
\sum_{n=1}^{k} \lambda_{n,b} - \lambda_{n,a} \leq c \| b - a \|_{(pN/4) + (N/2)} k^{1 + (2/N) + (4pN + 2N)}.
\]
Interestingly enough, one observes that
\[
1 + \frac{4}{N} - \frac{2p}{N(2 + p)} = 1 + \frac{2}{N} + \frac{4}{pN + 2N},
\]
so that, in some sense, the two methods yield equally good results. However, while (26) has the disadvantage that it involves first summing and then taking absolute values, it is better in that $q$ can be smaller than $pr/2$ in (27) and in that it does not produce the factor $ke$ that (27) gives.

**Lipschitz Domains**

One of the assumptions of Theorem 9 was that the boundary $\partial \Omega$ is $C^1$. At the cost of a larger $P_0$ it is possible to replace that hypothesis by a weaker one.
Definition. Let $D \subset \mathbb{R}^N$ be bounded. We say that $\Omega$ has the global Lipschitz property if there exists $\pi: D \to \mathbb{R}^N$ bi-Lipschitz such that $\pi(D) =: \tilde{D}$ has a $C^1$ boundary.

Suppose that $\Omega$ is globally Lipschitz. There are two ways to proceed. One is to establish $W^{1,p}$ bounds on the eigenfunctions of the operators using Lemma 12 below and then proceed as before to prove Theorem 10. The second, which we follow, is to make use of Theorem 9. In both cases the proof is based on the fact that uniform ellipticity is preserved under globally Lipschitz transformations.

Let $M_1, M_2$ be the two Lipschitz constants of $\pi$:

$$\frac{1}{M_2} |x - y| \leq |\pi(x) - \pi(y)| \leq M_1 |x - y|, \quad \text{all } x, y \in \Omega.$$

For $x \in \Omega$ set $\tilde{x} = \pi(x)$ and for a function (or a matrix) $f \in L^1_{\text{loc}}(\Omega)$ define $\tilde{f}$ on $\Omega$ by

$$\tilde{f}(\tilde{x}) = f(x), \quad \text{all } x \in \Omega.$$

Let $J\pi, J\pi^{-1}$ be the Jacobian matrices of $\pi$ and $\pi^{-1}$ respectively, so that

$$\frac{1}{cM_1} \leq |\det(J\pi)| \leq cM_1, \quad \frac{1}{cM_2} \leq |\det(J\pi^{-1})| \leq cM_2. \quad (28)$$

Using (28), (29) and the chain rule one can easily check that for $1 \leq p \leq \infty$

$$f \in W^{1,p}(\Omega) \Rightarrow \tilde{f} \in W^{1,p}(\Omega)$$

and

$$\frac{1}{cM_1^{N/p} M_2} \|f\|_{1,p} \leq \|\tilde{f}\|_{1,p} \leq cM_1 M_2^{N/p} \|f\|_{1,p}. \quad (30)$$

The following lemma describes the operator on $L^2(\Omega)$ induced by $H_\alpha$.

Set $\delta_\alpha = |\det(J\pi)|^{-1}$.

Lemma 12. Define the matrix $a_1$ on $\Omega$ by

$$a_1 = \delta_\alpha (J\pi)^* a(J\pi).$$

and set $\tilde{a} = \tilde{a}_1$. Then

$$f \in \text{Dom}(H_\alpha) \Rightarrow \tilde{f} \in \text{Dom}(H_\tilde{a})$$
and if \( f \in \text{Dom}(H_a) \) then

\[
H_a \tilde{f} = (\delta_a H_a f).
\] (31)

**Proof.** First we note that \( H_a \) is uniformly elliptic and in fact

\[
\|a\|_\infty \leq \|\delta_a\|_\infty \|J_a\|_\infty^{\frac{2}{m}} \|a\|_\infty, \tag{32}
\]

\[
\|a^{-1}\|_\infty \leq \|\delta_a^{-1}\|_\infty \|J_a^{-1}\|_\infty^{\frac{2}{m}} \|a^{-1}\|_\infty. \tag{33}
\]

It is enough to prove the one implication since the statement is symmetric. So, let \( f \in \text{Dom}(H_a) \), \( H_a f = h \). Then

\[
\int_\Omega a \nabla f \cdot \nabla \phi \, dx = \int_\Omega h \phi \, dx, \quad \text{all } \phi \in C^\infty_c(\Omega)
\]

\[
\Leftrightarrow \int_\Omega a_i (J_\pi)^{-1} \nabla f \cdot (J_\pi)^{-1} \nabla \phi \, dx = \int_\Omega h \phi \, dx, \quad \text{all } \phi \in C^\infty_c(\Omega)
\]

\[
\Leftrightarrow \int_\Omega \tilde{a} \tilde{f} \tilde{\nabla} \phi \, dx = \int_\Omega \delta_a \tilde{h} \phi \, dx, \quad \text{all } \phi \in C^\infty_c(\Omega)
\]

and the result follows since \( \phi \in C^\infty_c(\Omega) \) implies \( \tilde{\phi} \in W^{1,\infty}(\Omega) \subset W^{1,2}(\tilde{\Omega}) \).

If for \( g \in L^2(\Omega) \) we define \( \tilde{g} = \delta_a g \in L^2(\tilde{\Omega}) \), it follows from Lemma 12 that

\[
(R_a g) \sim = R_a \tilde{g}, \quad \text{all } g \in L^2(\Omega). \tag{34}
\]

**Lemma 13.** For any \( 1 \leq r \leq \infty \) we have

\[
\|R_b - R_a\|_r \leq cM_1^{N/2}M_2^{3N/2} \|R_b - R_a\|_r.
\]

**Proof.** For any \( g \in L^2(\Omega) \) we have

\[
\|(R_b - R_a) g\| \leq cM_2^{N/2} \|\{(R_b - R_a) g\}\| \sim = cM_2^{N/2} \|\{(R_b - R_a) \tilde{g}\}\|
\]

and so

\[
\frac{\|(R_b - R_a) g\|}{\|g\|} \leq cM_1^{N/2}M_2^{3N/2} \|\{(R_b - R_a) \tilde{g}\}\| / \|\tilde{g}\|. \tag{35}
\]

Since \( g \mapsto \tilde{g} \) is invertible, for any subspace \( L \) of \( L^2(\Omega) \) we have

\[
\dim L = \dim \{ \tilde{g} | g \in L \}
\]

and so the lemma follows from (35) using the min-max theorem.
Theorem 14. There exists $P_0 < \infty$, $P = P_{a,b,Jn}$, such that if

(i)  
\[ p > P_0 \]

and

(ii)  
\[ r > \frac{N}{2} + \frac{N}{p}. \]

then

\[ \| R_b - R_a \|_r \leq c_{r,p,a,b,Jn} \| b - a \|_{p/2}. \]

Proof. This follows directly from Lemmas 12 and 13 and Theorem 9.

Corollary 15. Corollaries 10 and 11 are also valid if $\Omega$ is globally Lipschitz.

Applications and a Limit Case

A typical application of the above results examines how the heat diffusion on a body is affected by the existence of impurities of different conductivity. Using Theorem 14 we establish spectral stability as the impurities become small in volume.

Let $\Omega \subset \mathbb{R}^N$ be bounded with a globally Lipschitz boundary. Let $H$ denote the Dirichlet Laplacian on $\Omega$ so that if $\Omega$ is homogeneous of conductivity one, then the Dirichlet heat diffusion is described by the equation $\partial u / \partial t = -Hu$.

Now suppose instead that there are disjoint connected sets $S_k \subset \Omega$, $k = 1, 2, \ldots, r$, of conductivity $\alpha_k \neq 1$. Then the heat equation becomes

\[ \partial u / \partial t = -Ha u \]  \hspace{1cm} (36)

where $a(x)$ is the scalar matrix given by

\[ a(x) = \begin{cases} 
1 & \text{if } x \in \Omega \setminus \bigcup S_k \\
\alpha_k & \text{if } x \in S_k.
\end{cases} \]

Set $S = \bigcup S_k$ and let $|S|$ denote the Lebesgue measure of $S$. From Theorem 14 we immediately deduce the following

Proposition 16. For $P_0, p$ and $r$ as in Theorem 10, we have

\[ \| R - R_a \|_r \leq c \max_k |1 - \alpha_k| \ |S|^{2/np}. \]  \hspace{1cm} (37)
Note that the fact that $p$ in (16) can be strictly smaller than $+\infty$ is crucial in establishing stability of the resolvent as $S$ shrinks to a set of measure zero.

For the sake of simplicity we assume from now on that $\alpha_1 = \alpha_2 = \cdots = \alpha_r =: \alpha > 1$.

**Example 1.** Suppose that there are $M$ disjoint balls $B(u_k, \rho)$, $k = 1, \ldots, M$, with centre $u_k$ and the same small radius $\rho$. The following expresses the intuition that the effect of many small randomly distributed spherical impurities depends upon balancing the number $M$ against the radius $\rho$: if $P_0$ is as above and we fix $r > N/2$, then

$$\|R - R_n\|_r \leq c |1 - \alpha| o(1) \quad \text{as} \quad \rho \to 0$$

provided $M = o(\rho^{-N})$.

**Example 2.** More detailed information can be obtained if we are only interested in eigenvalue stability. If $\{\lambda_n\}$, $\{\lambda_{n,a}\}$ are the corresponding eigenvalues, equation (5) and a simple argument yield

$$0 \leq \lambda_{n,a} - \lambda_n \leq 2(\alpha - 1) \int_S (|\nabla \phi_1|^2 + \cdots + |\nabla \phi_n|^2) \, dx \quad (38)$$

where $\{\phi_n\}$ are the eigenfunctions of $H$.

This estimate can be improved if additional information is available. If $\Omega$ is regular enough so that $\|\nabla \phi_k\|_\infty < \infty$, $1 \leq k \leq n$, (38) gives

$$0 \leq \lambda_{n,a} - \lambda_n \leq 2(\alpha - 1)(\|\nabla \phi_1\|_\infty^2 + \cdots + \|\nabla \phi_n\|_\infty^2)|S| \quad (39)$$

If we assume that $S$ shrinks uniformly in the sense that it is contained in the ball of fixed centre $x_0$ and radius $\epsilon$, then taking the Taylor expansion of $\phi_n$ around $x_0$ yields the asymptotic inequality

$$0 \leq \lambda_{n,a} - \lambda_n \leq 2(\alpha - 1)\omega_n (|\nabla \phi_1(x_0)|^2 + \cdots + |\nabla \phi_n(x_0)|^2) \epsilon^N + O(\epsilon^{N+2})$$

as $\epsilon \to 0$, where $\omega_n$ denotes the volume of the unit ball.

**Example 3.** Let $\Omega$ be the unit ball in $\mathbb{R}^N$ and let

$$S = \{x \in \Omega \mid |x| < \epsilon\}.$$

Let $H_\epsilon$ denote the corresponding operators so that $H_0 = -\Delta$. The bottom eigenvalue $\lambda_\epsilon$ of $H_\epsilon$ as well as the ground state can be calculated explicitly using separation of variables and one can see that

$$\lambda_\epsilon = \lambda_0 + k_N (1 - \alpha^{-1}) \epsilon^{N+2} + O(\epsilon^{N+4}), \quad \text{as} \quad \epsilon \to 0. \quad (41)$$
The constant \( k_N \) can be explicitly computed: if 
\[
J_n(y) = y^n \sum_{i=0}^{\infty} c_{i,n} y^{2i}, \quad n \in \mathbb{R}
\]
denotes the Bessel functions of the first kind of order \( n \), then for \( N \) odd we get 
\[
k_N = \frac{\gamma_{(N-2)/2}}{g_N(\lambda_0)}
\]
where 
\[
\gamma_n = c_{n+1,1} - 2 c_{n+1,2} c_{n,2}
\]
and 
\[
g_N(y) = \frac{J_{(N-2)/2}(y^{1/2})}{J_{-(N-2)/2}(y^{1/2})} y^{-(N+2)/2}.
\]
\( \lambda_0 \) being the first eigenvalue of \( H_0 \) or, equivalently, the least positive solution of \( J_{(N-2)/2}(y^{1/2}) = 0 \). The result is also valid for \( N \) even with the only difference that the functions \( J_n \) must be replaced by Weber’s Bessel functions of the second kind \( N_n \). The reason why we have an \( \lambda = N+2 \) in (41) and not an \( \lambda = N \) as (40) suggests is that \( \{\phi_n\} \) is not orthonormal.

A similar formula can be obtained for higher-order eigenvalues in the same way. The calculations as well as the results will involve additional parameters related to the angular momentum of the corresponding eigenfunctions.

**Degenerate Operators.** There are further questions that one can pose concerning the operators \( H_\alpha \) above. It is natural to ask what happens when the constant \( \alpha \) converges either to zero or to infinity.

We assume from now on that \( S \) is an open subset of \( \Omega \) with locally Lipschitz boundary and that \( S \subset \Omega \). We also set \( U = \Omega \setminus S \).

For \( 0 < \alpha < \infty \), let \( Q_{\alpha} \) be the quadratic form on \( L^2(\Omega) \) associated with the operator \( H_\alpha \), so that 
\[
\text{Dom}(Q_{\alpha}) = W^{1,2}(\Omega).
\]
\[
Q_{\alpha}(f) = \int_{\Omega} a(x) |\nabla f|^2 \, dx, \quad \text{all } x \in W^{1,2}_0.
\]
Let $Q_\infty$ be the limit of the $Q_a$'s as $a \to \infty$. Clearly

$$\text{Dom}(Q_\infty) = \{ f \in W_0^{1,2}(\Omega) \mid f \text{ is constant on each } S_k \}$$

and

$$Q_\infty(f) = \int_U |\nabla f|^2 \, dx, \quad \text{all } f \in \text{Dom}(Q_\infty).$$

This form is closed but not densely defined. It is however densely defined if considered as acting on the Hilbert space $L^2(A, \mu) \subset L^2(\Omega)$ where $A =: \{ \sigma_1 \cup \cdots \cup \sigma_r \} \cup U$ is the measure space defined by

$$\begin{align*}
\mu(\sigma_k) &= |S_k|, & k = 1, \ldots, r \\
\mu(E) &= |E|, & \text{all } E \subset U
\end{align*}$$

and one easily checks that the Lipschitz condition implies that the space

$$\mathcal{D} =: \text{Dom}(Q_\infty) \cap C_c^\infty(\Omega)$$

is a core of $Q_\infty$.

The study of the operator $H_\infty$ on $L^2(A)$ associated to the form $Q_\infty$ and its dependence on the $S_k$'s and their relative position is a very interesting problem which is under investigation.

The case $a \to 0$ is much simpler and in fact the Lipschitz condition on $S$ can be weakened to the condition $|\partial S| = 0$. Let $Q'_a = \lim_{a \to 0} Q_a$ so that

$$\text{Dom}(Q'_a) = W_0^{1,2}(\Omega)$$

and

$$Q'_a(f) = \int_U |\nabla f|^2 \, dx, \quad \text{all } f \in \text{Dom}(Q'_a).$$

Let $V$ be the closure in $W^{1,2}(U)$ of the set

$$V_0 = \{ f \mid_U \in C_c^\infty(\Omega) \}$$

and define the quadratic form $Q_a$ on $L^2(\Omega)$ by

$$\text{Dom}(Q_a) = \{ f \in L^2(\Omega) \mid f \mid_U \in V \}$$

and

$$Q_a(f) = \int_U |\nabla f|^2 \, dx, \quad \text{all } f \in \text{Dom}(Q_a).$$

The proof of the following proposition is simple and is omitted:
Proposition 17. (i) $Q_n$ is densely defined and closed and (ii) $C^\infty_c(\Omega)$ is a core of $Q_n$ and (hence) $Q_n$ is the form closure of $Q_n$.

Let $H_n$ be the self-adjoint operator on $L^2(\Omega)$ associated with the form $Q_n$. Formally, one can write

$$H_n = -\sum_i \frac{\partial}{\partial x_i} \left( a_0(x) \frac{\partial}{\partial x_i} \right)$$

where

$$a_0(x) = \begin{cases} 1 & \text{if } x \in U \\ 0 & \text{if } x \in S. \end{cases}$$

Let $H_V$ be the self-adjoint operator on $L^2(U)$ associated with the quadratic form $Q_V$, where

$$\text{Dom}(Q_V) = V$$

and

$$Q_V(f) = \int_U |\nabla f|^2 \, dx, \quad \text{all } f \in V.$$ 

$H_V$ is the Laplacian on $U$ satisfying Dirichlet boundary conditions on $\partial \Omega$ and Neumann boundary conditions on $\partial S$.

In accordance with our intuition we have the following

Theorem 18. The spaces $L^2(S)$ and $L^2(U)$ are invariant under the action of $H_n$ and writing

$$L^2(\Omega) = L^2(S) \oplus L^2(U) \quad \text{(42)}$$

we have

(i) $\text{Dom}(H_n) = L^2(S) \oplus \text{Dom}(H_V)$ \quad \text{(43)}

(ii) $H_n = 0 \oplus H_V$. \quad \text{(44)}

Proof. First, we note that whether or not a function in $L^2(\Omega)$ lies in $\text{Dom}(Q_n)$ or $\text{Dom}(H_n)$ depends only on its restriction on $U$. Now, let $f \in \text{Dom}(H_n)$, $H_n f = g$. Then $f \in \text{Dom}(Q_n)$ and

$$\int_U \nabla f \cdot \nabla \phi \, dx = \int_{\Omega} g \phi \, dx, \quad \text{all } \phi \in C^\infty_c(\Omega).$$
It follows that $g|_{S} = 0$ and
\[ \int_{U} \nabla f \cdot \nabla \phi \, dx = \int_{U} g \phi \, dx, \quad \text{all } \phi \in V'. \quad (45) \]

Since $f \in \operatorname{Dom}(Q_{a})$ implies $f|_{U} \in V$, it follows that $f|_{U} \in \operatorname{Dom}(H_{V})$ and
\[ H_{V}(f|_{U}) = g|_{U}. \quad (46) \]

Conversely, let $f \in L^{2}(\Omega)$ be such that $f|_{U} = f_{1} \in \operatorname{Dom}(H_{V})$, $H_{V}f_{1} = g$, say. Then $f_{1} \in V$ and
\[ \int_{U} \nabla f_{1} \cdot \nabla \phi \, dx = \int_{U} g \phi \, dx, \quad \text{all } \phi \in V \]
which clearly implies that $f \in \operatorname{Dom}(H_{a})$ and $H_{a}f = 0 \oplus g$.

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