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MONOTONICITY, CONTINUITY AND DIFFERENTIABILITY RESULTS FOR THE L^p HARDY CONSTANT

BY

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ABSTRACT

We consider the L^p Hardy inequality involving the distance to the boundary for a domain in the n-dimensional Euclidean space. We study the dependence on p of the corresponding best constant and we prove monotonicity, continuity and differentiability results. The focus is on non-convex domains in which case such a constant is in general not explicitly known.

1. Introduction

Given a bounded domain Ω in $\mathbb{R}^n$ and $p \in]1,\infty[$, we say that the L^p Hardy inequality holds in Ω if there exists $c > 0$ such that

$$\int_\Omega |\nabla u|^p dx \geq c \int_\Omega \frac{|u|^p}{d^p} dx\,, \qquad \text{for all } u \in C_c^\infty(\Omega), \tag{1.1}$$

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where $d(x) = \mathrm{dist}(x, \partial\Omega)$, $x \in \Omega$. The L^p Hardy constant of Ω is the best constant for inequality (1.1) and is denoted here by H_p.

It is well-known that the L^p Hardy inequality holds for all $p \in]1, \infty[$ under weak regularity assumptions on Ω, for example if Ω has a Lipschitz boundary. Moreover, if Ω is convex, and more generally if it is weakly mean convex, i.e. if $\Delta d \leq 0$ in the distributional sense in Ω, then $H_p = ((p-1)/p)^p$; see [20, 4]. If Ω is not weakly mean convex, little is known about the precise value of H_p and the available results only hold for $p = 2$ and for special domains, for example circular sectors and quadrilaterals in the plane. We refer to [2, 3, 4, 5, 6, 7, 9, 16, 20] for more information. We also refer to the monograph [14] for an introduction to the study of Hardy and Hardy-type inequalities with a historical perspective.

In this article we study the dependence of H_p upon variation of p and we prove four main results. First, we prove that $p(1 + H_p^{1/p})$ is a non-decreasing function of $p \in]1, \infty[$, and this is done without any smoothness assumption on Ω; see Theorem 2. In particular, it easily follows that H_p is right-continuous at any point $p \in]1, \infty[$. Second, we prove that if Ω is of class C^2 then H_p is also left-continuous, hence it is continuous on $]1, \infty[$; see Theorem 6. Third, we prove that if Ω is of class C^2 then H_p is differentiable at any point $p \in]1, \infty[$ such that $H_p < ((p-1)/p)^p$, and we compute a formula for the corresponding derivative; see Theorem 8.

We note that the proofs of our continuity and differentiability results exploit a result in [20], where it was shown in particular that if $H_p < ((p-1)/p)^p$ then equality is attained in (1.1) for some function $u_p \in W_0^{1,p}(\Omega)$ which behaves like d_Ω^α near $\partial\Omega$ for a suitable $\alpha \in]0, 1[$. Importantly, the results of [20] are proved under the assumption that Ω is of class C^2, and removing that assumption is not easy. The function u_p is uniquely identified by the extra normalizing conditions $u_p > 0$ and $\int_\Omega u_p^p / d^p dx = 1$. The fourth main result of the paper is a continuity result for the dependence of u_p and ∇u_p on p; see Theorem 7.

As is well-known, if equality is attained in (1.1) for some nontrivial function $u \in W_0^{1,p}(\Omega)$, then u is a minimizer for the Hardy quotient

$$R_p[u] := \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega \frac{|u|^p}{d^p} dx} \tag{1.2}$$

and solves the equation

$$-\Delta_p u = H_p \frac{|u|^{p-2}u}{d^p}, \tag{1.3}$$

where $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian.

Problem (1.3) is a singular variant of the well-known eigenvalue problem for the Dirichlet p-Laplacian

$$-\Delta_p u = \lambda_p |u|^{p-2}u, \tag{1.4}$$

where H_p is replaced by the first eigenvalue λ_p of the p-Laplacian, which in turn is the minimum over $W_0^{1,p}(\Omega) \setminus \{0\}$ of the Rayleigh quotient

$$\frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega |u|^p dx}. \tag{1.5}$$

The study of the dependence of λ_p on p was initiated in the article [18] which has inspired many authors, ourselves included. We refer to [1, 10, 12] for recent, closely related results. In fact, the proofs of our monotonicity and continuity results exploit some ideas of [18]. However, we point out that although the two problems (1.3) and (1.4) look similar, they are radically different. For example, if Ω has finite Lebesgue measure, the Rayleigh quotient (1.5) has always a minimizer, and if Ω is also sufficiently smooth, the gradient of such minimizer does not blow up at the boundary. As is well-known, one of the main differences between the two problems is related to the lack of compactness for the embedding of the Sobolev space $W_0^{1,p}(\Omega)$ into the natural weighted space $L^p(\Omega, d^{-p}dx)$, which is also responsible for the appearance of a large essential spectrum for problem (1.3) in the case $p = 2$. Thus, the study of the dependence of H_p on p leads to a number of difficulties which require a detailed analysis.

We point out that our differentiability result can also be proved, with obvious simplifications, for the dependence of λ_p on p. Since we have not found such a result in the literature, we find it natural to state it in the Appendix.

2. Preliminaries

Unless otherwise indicated, by Ω we denote a bounded domain (i.e. a bounded open connected set) in $\mathbb{R}^n$. For $p \in]1, +\infty[$ we denote by $W^{1,p}(\Omega)$ the standard Sobolev space and by $W_0^{1,p}(\Omega)$ the closure in $W^{1,p}(\Omega)$ of the space $C_c^\infty(\Omega)$ of all C^∞-functions with compact support in Ω.

The L^p Hardy constant is defined by

$$H_p = \inf_{u \in W_0^{1,p}(\Omega), u \neq 0} R_p[u], \tag{2.1}$$

and if $H_p > 0$ we say that the L^p Hardy inequality is valid on Ω.

It is well-known that if Ω has a Lipschitz continuous boundary then $0 < H_p \leq ((p-1)/p)^p$. It is also known that if Ω is of class C^2 then there exists a minimizer u in (2.1) if and only if $H_p < ((p-1)/p)^p$; see [20, 21]. Moreover, the minimizer is unique up to a multiplicative constant, can be chosen to be positive and there exists $c > 0$ such that

$$c^{-1} d(x)^{\alpha_p} \leq u(x) \leq c d(x)^{\alpha_p}, \quad x \in \Omega, \tag{2.2}$$

where $\alpha_p \in](p-1)/p, 1[$ denotes the largest solution to the equation

$$(p-1)\alpha^{p-1}(1-\alpha) = H_p. \tag{2.3}$$

We set for simplicity

$$\mathcal{A} = \{p \in]1, \infty[: \ H_p < ((p-1)/p)^p\}.$$

In the sequel and provided Ω is C^2, for any $p \in \mathcal{A}$ we shall denote by u_p the positive minimizer normalized by the condition $\int_\Omega |u_p/d|^p dx = 1$. Inequalities (2.2) suggest that ∇u_p behaves like $d^{\alpha_p - 1}$ close to the boundary of Ω. In fact we can prove the following lemma which is a variant of [3, Thm. 4] providing further information on the dependence of the constants on p. We emphasize that in this lemma we do not assume that H_p depends continuously on p.

Lemma 1: *Assume that Ω is of class C^2 and $p_0 \in \mathcal{A}$. There exists $c > 0$ such that*

$$u_p(x) \leq c d^{\alpha_p}(x), \quad |\nabla u_p(x)| \leq c d^{\alpha_p - 1}(x), \tag{2.4}$$

for all $p \in \mathcal{A}$ sufficiently close to p_0 and for all $x \in \Omega$. In particular, $u_p \in W_0^{1,q}(\Omega)$ for all $q \in [1, 1/(1-\alpha_p)[$.

Proof. The existence of a constant $c = c(p) > 0$ for each $p \in \mathcal{A}$ such that the first inequality in (2.4) holds has been proved in [20, Lemma 9] and [21, Lemma 5.2]. The existence of a constant $c = c(p) > 0$ for each $p \in \mathcal{A}$ such that the second inequality in (2.4) holds has been proved in [3, Theorem 4]. We shall now show that $c(p)$ can be chosen so that it is locally bounded with respect to $p \in \mathcal{A}$.

Let $p \in \mathcal{A}$ and let $u \in W_0^{1,p}(\Omega)$ be a positive minimizer of the L^p Hardy constant normalized by $\int_\Omega u^p/d^p dx = 1$. Let α be as in (2.3). For any $\beta > 0$ we set $\Omega_\beta = \{x \in \Omega : \ d(x) < \beta\}$. Let $\beta_0 > 0$ be small enough so that $d(x)$ is twice continuously differentiable in $\Omega_{2\beta_0}$. Following [20, 21], we define

$$v = d^\alpha(1-d).$$

A direct computation gives that in $\Omega_{2\beta_0}$,

$$\begin{aligned} &-\Delta_p v - H_p\frac{v^{p-1}}{d^p} \\ &\quad = (p-1)\alpha^{p-1}d^{\alpha p-\alpha-p}\Big\{(1-\alpha)\Big[\Big(1-\Big(1+\frac{1}{\alpha}\Big)d\Big)^{p-1} - (1-d)^{p-1}\Big] \\ &\qquad\qquad + \Big(1+\frac{1}{\alpha}\Big)\Big(1-\Big(1+\frac{1}{\alpha}\Big)d\Big)^{p-2}d\Big\} \\ &\qquad - \alpha^{p-1}d^{\alpha p-\alpha-p+1}\Big(1-\Big(1+\frac{1}{\alpha}\Big)d\Big)^{p-1}\Delta d \\ &\quad = d^{\alpha p-\alpha-p}(A + B\, d\Delta d)\,, \end{aligned} \tag{2.5}$$

where terms in A do not involve Δd. We expand A in powers of d and obtain

$$\begin{aligned} A &= (p-1)\alpha^{p-2}(\alpha p - p + 2)d + O(d^2) \\ &\geq (p-1)\alpha^{p-2}d + O(d^2). \end{aligned}$$

It can easily be verified that the coefficient of d^2 is locally bounded with respect to $p \in]1,+\infty[$. Hence there exists $\beta_1 \in]0,\beta_0[$ which is locally bounded away from zero with respect to p such that

$$A \geq \frac{(p-1)\alpha^{p-2}}{2}d, \quad \text{in } \Omega_{\beta_1}. \tag{2.6}$$

Since Δd is bounded in Ω_{β_0}, it follows from (2.5) and (2.6) that there exists $\beta_2 \in]0,\beta_1[$ bounded away from zero locally in $p \in \mathcal{A}$ such that

$$-\Delta_p v - H_p\frac{v^{p-1}}{d^p} \geq 0, \quad \text{in } \Omega_{\beta_2}.$$

Now let

$$C_1(p) = \sup\left\{u(x) \ : \ x \in \{d(x) = \beta_2\}\right\}.$$

The constant $C_1(p)$ is finite by standard regularity results for quasilinear elliptic equations. Looking e.g. at the proofs of Theorems 1 and 2 of the classical paper of Serrin [22] we can trace the dependence of $C_1(p)$ in p for $p \leq n$ and see that it is locally bounded for $p \leq n$. As mentioned in [22], the case $p > n$ is simpler since the result follows by the Sobolev embedding. We note that the fact that the Sobolev constant blows-up as $p \to n^+$ is not a problem, since the argument used in [22, Theorem 2] for $p = n$ can be extended without changes to include all p in a neighborhood of n. We omit the details.

Defining next $C^* = C_1/(\beta_2^\alpha(1-\beta_2))$, we then have

$$C^* = \sup\left\{\frac{u(x)}{v(x)},\ x \in \{d(x) = \beta_2\}\right\}.$$

Applying [21, Proposition 3.1] we conclude that

$$u(x) \le C^* v(x) \le C^* d^\alpha, \quad \text{in } \Omega_{\beta_2}.$$

This estimate clearly holds true also in $\Omega \setminus \Omega_{\beta_2}$, with a constant C^* still remaining locally bounded with respect to $p \in \mathcal{A}$, completing the proof of the first estimate of (2.4).

For the second inequality we apply the regularity estimates of [11, Theorems 1.1 and 1.2], as was done in [3]. The constants involved are locally bounded in p (see in particular [11, Remark 5.1]). This completes the proof. ∎

3. Monotonicity and continuity of the Hardy constant

The following theorem holds without any smoothness assumption of Ω (not even the boundedness of Ω is actually required) and is inspired by the monotonicity result proved in Lindqvist [18] for the first eigenvalue of the p-Laplacian.

Theorem 2: *The function*

$$p \mapsto p(1 + H_p^{1/p})$$

is non-decreasing in $]1, +\infty[$.

Proof. Let $1 < p < s$ and let $\psi \in C_c^\infty(\Omega)$. Then the function

$$u = |\psi|^{\frac{s}{p}} d^{1-\frac{s}{p}}$$

belongs to $W_0^{1,p}(\Omega)$ and

$$\begin{aligned}\Big(\int_\Omega |\nabla u|^p dx\Big)^{1/p} &= \Big(\int_\Omega \Big|\frac{s}{p}\Big(\frac{|\psi|}{d}\Big)^{\frac{s}{p}-1}\nabla\psi + (1-\frac{s}{p})\Big(\frac{|\psi|}{d}\Big)^{\frac{s}{p}}\nabla d\Big|^p dx\Big)^{1/p}\\ &\le \frac{s}{p}\Big(\int_\Omega \Big(\frac{|\psi|}{d}\Big)^{s-p}|\nabla\psi|^p dx\Big)^{1/p} + \frac{s-p}{p}\Big(\int_\Omega \Big(\frac{|\psi|}{d}\Big)^{s} dx\Big)^{1/p}\\ &\le \frac{s}{p}\Big(\int_\Omega |\nabla\psi|^s dx\Big)^{1/s}\Big(\int_\Omega \Big(\frac{|\psi|}{d}\Big)^{s} dx\Big)^{\frac{1}{p}-\frac{1}{s}} + \frac{s-p}{p}\Big(\int_\Omega \Big(\frac{|\psi|}{d}\Big)^{s} dx\Big)^{1/p}.\end{aligned}$$

This implies

$$H_p^{1/p} \le R_p[u]^{1/p} \le \frac{s}{p}R_s[\psi]^{1/s} + \frac{s-p}{p}.$$

Taking the infimum over all $\psi \in C_c^\infty(\Omega)$ we conclude that

$$H_p^{1/p} \le \frac{s}{p}H_s^{1/s} + \frac{s-p}{p},$$

and the result follows. ∎

Remarks: (1) For $\alpha \in [0,1]$ let

$$\lambda_{\alpha,p} = \inf_{u\in W_0^{1,p}(\Omega),u\neq 0} \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega \frac{|u|^p}{d^{\alpha p}}dx};$$

so $\lambda_{1,p} = H_p$ and $\lambda_{0,p} = \lambda_p$ is the first eigenvalue of the Dirichlet p-Laplacian in Ω (see the Introduction). It has been shown in [18, Theorem 3.2] that the function $p \mapsto p\lambda_{0,p}^{1/p}$ is non-decreasing in $]1,\infty[$. In view of this and Theorem 2 it is tempting to believe that for any fixed $\alpha \in [0,1]$ the map $p \mapsto p(\alpha + \lambda_{\alpha,p}^{1/p})$ is non-decreasing in $]1,\infty[$. However, it can be seen that the method of proof fails for $\alpha \in]0,1[$.

(2) It follows from Theorem 2 that the function $p \mapsto H_p$ has one-sided limits at every $p > 1$ and

$$\lim_{s\to p-} H_s \le H_p \le \lim_{s\to p+} H_s. \tag{3.1}$$

Lemma 3: *We have*

$$\limsup_{s\to p} H_s = \lim_{s\to p+} H_s = H_p.$$

Proof. Given any $u \in C_c^\infty(\Omega)$ we have $H_s \leq R_s[u]$ and therefore

$$\limsup_{s\to p} H_s \leq R_p[u].$$

Taking the infimum over all $u \in C_c^\infty(\Omega)$ we obtain $\limsup_{s\to p} H_s \leq H_p$ which, combined with (3.1), yields the result. ∎

In order to prove Theorem 6 we need the following lemmas. The first can be proved simply by differentiating under the integral sign.

Lemma 4: *Let $u \in W_0^{1,p}(\Omega)$ be fixed. The functions defined by*

$$N(s) = \int_\Omega |\nabla u|^s dx, \qquad D(s) = \int_\Omega \frac{|u|^s}{d^s} dx$$

are differentiable in $]1, p[$ and

$$N'(s) = s \int_\Omega |\nabla u|^s \ln |\nabla u| dx, \qquad D'(s) = s \int_\Omega \frac{|u|^s}{d^s} \ln\Big(\frac{|u|}{d}\Big) dx$$

for all $1 < s < p$.

Lemma 5: *Assume that Ω is of class C^2. We have*

$$\liminf_{s\to p} H_s \geq H_p.$$

Proof. It follows from (3.1) that

$$\liminf_{s\to p} H_s = \liminf_{s\to p-} H_s.$$

Suppose by contradiction that this liminf is a number $L < H_p$. Let s_n, $n \in \mathbb{N}$, be an increasing sequence of exponents with $s_n \to p$ and $H_{s_n} \to L$ as $n \to \infty$. Then, since $L < H_p \leq (\frac{p-1}{p})^p$, we have that $H_{s_n} < (\frac{s_n-1}{s_n})^{s_n}$ for all $n \in \mathbb{N}$ sufficently large and therefore the L^{s_n}-Hardy quotient has a positive minimizer u_{s_n}. Let α_{s_n} be the corresponding exponents defined as in (2.3). It then follows that $\lim_{n\to\infty} \alpha_{s_n} > (p-1)/p$. Applying Lemma 1 we thus obtain that

$$\|u_{s_n}\|_{W_0^{1,p+\epsilon}(\Omega)} \leq M \tag{3.2}$$

for some fixed $\epsilon, M > 0$ and all $n \in \mathbb{N}$ sufficiently large. Hence

$$\begin{aligned} H_p &\leq \liminf_{n\to\infty} R_p[u_{s_n}] \\ &= \liminf_{n\to\infty} \big(R_{s_n}[u_{s_n}] + \big\{R_p[u_{s_n}] - R_{s_n}[u_{s_n}]\big\}\big) \\ &= L + \liminf_{n\to\infty} \big(R_p[u_{s_n}] - R_{s_n}[u_{s_n}]\big). \end{aligned}$$

To reach a contradiction it is enough to prove that the last liminf is zero. Now, by Lemma 4 and (3.2) the function $s \mapsto R_s[u_{s_n}]$ is differentiable in (s_n, p) for each fixed $n \in \mathbb{N}$. Hence by the Mean Value Theorem, for each $n \in \mathbb{N}$ there exists $\xi_n \in (s_n, p)$ such that

$$R_p[u_{s_n}] - R_{s_n}[u_{s_n}] = (p - s_n) \frac{dR_p[u_{s_n}]}{dp}\bigg|_{p=\xi_n}.$$

From Lemma 4 and (3.2) it easily follows that $\frac{dR_p[u_{s_n}]}{dp}\Big|_{p=\xi_n}$ remains bounded as $n \to \infty$. This concludes the proof. ∎

Theorem 6: *Let Ω be bounded with C^2 boundary. Then the function $p \mapsto H_p$ is continuous on $]1, \infty[$.*

Proof. It follows from Lemmas 3 and 5. ∎

4. Differentiability of the Hardy constant

We recall that $\mathcal{A} = \{p \in]1, \infty[\, : \, H_p < ((p-1)/p)^p\}$. The proof of the following theorem is based on adapting the arguments of Lindqvist [18, Thm. 3.6].

Theorem 7: *Let Ω be of class C^2 and $p_0 \in \mathcal{A}$. Then for all p sufficiently close to p_0 we have $p \in \mathcal{A}$ and $u_p, u_{p_0} \in W^{1,\max\{p_0,p\}}(\Omega)$. Moreover*

$$\lim_{p \to p_0} \|u_p - u_{p_0}\|_{W^{1,\max\{p_0,p\}}(\Omega)} = 0. \tag{4.1}$$

Proof. Theorem 6 and Lemma 1 easily imply that for p close enough to p_0 we have $p \in \mathcal{A}$ and, moreover, $u_p \in W^{1,p_0}(\Omega)$ and $u_{p_0} \in W^{1,p}(\Omega)$.

We now prove (4.1). Let $\delta > 0$ be fixed in such a way that $p_0 + 2\delta < 1/(1 - \alpha_{p_0})$. By Theorem 6 and Lemma 1 it follows that there exists a constant $c > 0$ independent of p such that

$$\|u_p\|_{W^{1,p_0+\delta}(\Omega)} \le c,$$

for all $p \in \mathcal{A}$ sufficiently close to p_0. Moreover, since Ω has C^2 boundary we have $u_p \in W_0^{1,p_0+\delta}(\Omega)$ for any such p.

By the reflexivity of the space $W_0^{1,p_0+\delta}(\Omega)$ and the Rellich–Kondrachov Theorem it follows that there exists $\tilde{u} \in W_0^{1,p_0+\delta}(\Omega)$ such that, up to taking a subsequence, $\nabla u_p \rightharpoonup \nabla \tilde{u}$ weakly in $L^{p_0+\delta}(\Omega)$ and $u_p \to \tilde{u}$ in $L^{p_0+\delta}(\Omega)$ as $p \to p_0$. Note that $\int_\Omega |\tilde{u}|^{p_0}/d^{p_0} dx = 1$, which can be deduced by passing to the limit as $p \to p_0$ in the equality $\int_\Omega |u_p|^p/d^p dx = 1$ and using the Dominated Convergence Theorem combined with estimates (2.4). In particular $\tilde{u} \neq 0$. Clearly, $\nabla u_p \rightharpoonup \nabla \tilde{u}$ weakly in $L^{p_0}(\Omega)$, hence

$$\int_\Omega |\nabla \tilde{u}|^{p_0} dx \le \liminf_{p \to p_0} \int_\Omega |\nabla u_p|^{p_0} dx \tag{4.2}$$

as $p \to p_0$. By the Mean Value Theorem and Lemma 4 we have that

$$\begin{aligned} \int_\Omega |\nabla u_p|^{p_0} dx &= \int_\Omega |\nabla u_p|^p dx + (p_0 - p) \int_\Omega s_p |\nabla u_p|^{s_p} \ln |\nabla u_p| dx \\ &= H_p + (p_0 - p) \int_\Omega s_p |\nabla u_p|^{s_p} \ln |\nabla u_p| dx, \end{aligned} \tag{4.3}$$

for some real number s_p between p_0 and p. It is clear that by the uniform boundedness of the norms of u_p in $W_0^{1,p_0+\delta}(\Omega)$, the integrals $\int_\Omega s_p |\nabla u_p|^{s_p} \ln |\nabla u_p| dx$ are uniformly bounded for p close enough to p_0. Thus, by passing to the limit as $p \to p_0$ in (4.3) and using the continuity of the map $p \mapsto H_p$ it follows that

$$\lim_{p \to p_0} \int_\Omega |\nabla u_p|^{p_0} dx = \lim_{p \to p_0} H_p = H_{p_0}. \tag{4.4}$$

This, combined with (4.2) and the condition $\int_\Omega |\tilde{u}|^{p_0}/d^{p_0} dx = 1$, implies that $\int_\Omega |\nabla \tilde{u}|^{p_0} = H_{p_0}$. Thus, $\tilde{u} = u_{p_0}$.

As in [18, Thm. 3.6] we now use Clarkson's inequalities. If $\max\{p_0, p\} \ge 2$ we have

$$\begin{aligned} &\int_\Omega \left| \frac{\nabla u_p - \nabla u_{p_0}}{2} \right|^{\max\{p_0,p\}} dx \\ &\le \frac{1}{2} \int_\Omega |\nabla u_p|^{\max\{p_0,p\}} dx + \frac{1}{2} \int_\Omega |\nabla u_{p_0}|^{\max\{p_0,p\}} dx \\ &\quad - \int_\Omega \left| \frac{\nabla u_p + \nabla u_{p_0}}{2} \right|^{\max\{p_0,p\}} dx \le \frac{1}{2} \int_\Omega |\nabla u_p|^{\max\{p_0,p\}} dx \\ &\quad + \frac{1}{2} \int_\Omega |\nabla u_{p_0}|^{\max\{p_0,p\}} dx - H_{\max\{p,p_0\}} \int_\Omega \left| \frac{u_p + u_{p_0}}{2d} \right|^{\max\{p_0,p\}} dx. \end{aligned} \tag{4.5}$$

By the continuity of the L^p-norm, it follows that

$$\lim_{p \to p_0} \int_\Omega |\nabla u_{p_0}|^{\max\{p_0,p\}} dx = \int_\Omega |\nabla u_{p_0}|^{p_0} dx = H_{p_0}. \tag{4.6}$$

Moreover, using the Dominated Convergence Theorem combined with estimates (2.4) yields

$$\lim_{p \to p_0} \int_\Omega \left| \frac{u_p + u_{p_0}}{2d} \right|^{\max\{p_0,p\}} dx = \int_\Omega \left| \frac{u_{p_0}}{d} \right|^{p_0} dx = 1.$$

We then deduce from (4.4)–(4.6) and Theorem 6 that

$$\int_\Omega \left|\frac{\nabla u_p - \nabla u_{p_0}}{2}\right|^{\max\{p_0,p\}} dx \to 0$$

as required. The case $p_0 < 2$ can be treated in a similar way using the appropriate Clarkson inequality for $p < 2$. ∎

THEOREM 8: *Let Ω be of class C^2. Then the map $p \mapsto H_p$ is of class C^1 on $\mathcal{A}$ and*

$$H_p' = p\int_\Omega |\nabla u_p|^p \ln|\nabla u_p| dx - pH_p \int_\Omega \frac{u_p^p}{d^p} \ln\frac{u_p}{d} dx\,, \quad p \in \mathcal{A}. \tag{4.7}$$

Proof. Let $p_0 \in \mathcal{A}$ be fixed. Since $\mathcal{A}$ is an open set, if $p > 1$ is sufficiently close to p_0 we have that $p \in \mathcal{A}$, hence the minimizer u_p exists. Moreover, by Lemma 1 and Theorem 6, there exist $\epsilon, \delta > 0$ such that $p < 1/(1-\alpha_{p_0}) + \epsilon$ and

$$u_p \in W^{1,1/(1-\alpha_{p_0})+\epsilon}(\Omega), \tag{4.8}$$

for all $p \in]p_0 - \delta, p_0 + \delta[$. Since u_{p_0} and u_p minimize the corresponding Rayleigh quotients, we have

$$R_p[u_p] - R_{p_0}[u_p] \le H_p - H_{p_0} \le R_p[u_{p_0}] - R_{p_0}[u_{p_0}]. \tag{4.9}$$

By (4.8) and Lemma 4 we have that for any fixed $p \in]p_0 - \delta, p_0 + \delta[$, the map $q \mapsto R_q[u_p]$ is differentiable on $]p_0 - \delta, p_0 + \delta[$, hence (4.9) implies that

$$R'_{p_\xi}[u_p](p - p_0) \le H_p - H_{p_0} \le R'_{p_\eta}[u_{p_0}](p - p_0) \tag{4.10}$$

for some p_ξ, p_η between p_0 and p. By Theorem 7 and estimates (2.4) one can prove that

$$R'_{p_\xi}[u_p],\ R'_{p_\eta}[u_{p_0}] \to R'_{p_0}[u_{p_0}], \quad \text{as } p \to p_0. \tag{4.11}$$

Indeed, by (4.1) it follows that possibly passing to subsequences $\lim_{p\to p_0} u_p(x) = u_{p_0}(x)$ a.e. in Ω which, combined with estimates (2.4), allows passing to the limit under the integral signs in order to get (4.11). Thus, (4.10) and (4.11) imply that H_p is differentiable at $p = p_0$. Formula (4.7) for $p = p_0$ is then easily proved by using the formulas provided by Lemma 4.

Finally, in order to prove that the map $p \mapsto H_p'$ is continuous on $\mathcal{A}$, one has simply to apply again Theorem 7 combined with estimates (2.4) as above. ∎

Remarks: (1) We note explicitly that since $H_p = \int_\Omega |\nabla u_p|^p dx$ we have that

$$\begin{aligned}(4.12)\quad &\int_\Omega |\nabla k u_p|^p \ln |\nabla k u_p| dx - H_p \int_\Omega \frac{|k u_p|^p}{d^p} \ln \frac{|k u_p|}{d} dx \\ &= |k|^p \Big(\int_\Omega |\nabla u_p|^p \ln |\nabla u_p| dx - H_p \int_\Omega \frac{|u_p|^p}{d^p} \ln \frac{|u_p|}{d} dx \Big)\end{aligned}$$

for any $k \in \mathbb{R}$, with $k \neq 0$. In particular, it follows that if we consider a minimizer u for H_p which is not necessarily normalized as u_p, then

$$(4.13)\qquad H_p' = \frac{p \int_\Omega |\nabla u|^p \ln |\nabla u| dx}{\int_\Omega \frac{|u|^p}{d^p} dx} - \frac{p H_p \int_\Omega \frac{|u|^p}{d^p} \ln \frac{|u|}{d} dx}{\int_\Omega \frac{|u|^p}{d^p} dx}.$$

(2) For all $p \in \mathcal{A}$ any minimizer u for H_p satisfies the following inequality:

$$(4.14)\qquad H_p \int_\Omega \frac{|u|^p}{d^p} \ln \frac{|u|}{d} dx \leq \frac{H_p + H_p^{\frac{p-1}{p}}}{p} \int_\Omega \frac{|u|^p}{d^p} dx + \int_\Omega |\nabla u|^p \ln |\nabla u| dx.$$

Indeed, by Theorems 2 and 8 the derivative of the function $p \mapsto p(1 + H_p^{1/p})$ is non-negative, hence inequality (4.14) follows by formula (4.13).

5. Appendix

The proof of Theorem 8 can be carried out also in the case of the first eigenvalue λ_p of the p-Laplacian defined by

$$(5.1)\qquad \lambda_p = \inf_{v \in W_0^{1,p}(\Omega),\ v \neq 0} \frac{\int_\Omega |\nabla v|^p dx}{\int_\Omega |v|^p dx};$$

see the Introduction. Recall that if Ω is a domain with finite measure, then there exists a unique minimizer v_p in (5.1) satisfying the normalizing conditions $v_p > 0$ and $\int_\Omega v_p^p dx = 1$. See the classical paper [19] and also [13] for further discussions.

By using the same argument of the proof of Theorem 8 combined with the results in [18] concerning the continuous depencence of v_p on p (we refer in particular to the local convergence result [18, Thm. 6.3] which by [17] admits a natural global version in the case of domains of class $C^{1,\beta}$) one can prove the following theorem.

THEOREM 9: *Let Ω be a bounded domain in $\mathbb{R}^n$ of class $C^{1,\beta}$ with $\beta \in]0,1]$. Then the function $p \mapsto \lambda_p$ is of class C^1 on $]1,\infty[$ and*

$$\lambda_p' = p \int_\Omega |\nabla v_p|^p \ln |\nabla v_p| dx - p\lambda_p \int_\Omega v_p^p \ln v_p dx, \quad p \in]1,\infty[\,. \tag{5.2}$$

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