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SPECTRAL STABILITY ESTIMATES FOR ELLIPTIC OPERATORS SUBJECT TO DOMAIN TRANSFORMATIONS WITH NON-UNIFORMLY BOUNDED GRADIENTS

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Abstract. We consider uniformly elliptic operators with Dirichlet or Neumann homogeneous boundary conditions on a domain Ω in $\mathbb{R}^N$. We consider deformations $\phi(\Omega)$ of Ω obtained by means of a locally Lipschitz homeomorphism ϕ and we estimate the variation of the eigenfunctions and eigenvalues upon variation of ϕ. We prove general stability estimates without assuming uniform upper bounds for the gradients of the maps ϕ. As an application, we obtain estimates on the rate of convergence for eigenvalues and eigenfunctions when a domain with an outward cusp is approximated by a sequence of Lipschitz domains.

§1. *Introduction.* Let Ω be a bounded domain (i.e., a bounded connected open set) in $\mathbf{R}^N$ and ϕ a locally Lipschitz homeomorphism between Ω and another bounded domain $\phi(\Omega)$ in $\mathbf{R}^N$. For fixed real coefficients A_{ij} defined in the whole of $\mathbf{R}^N$ with $A_{ij} = A_{ji}$ and satisfying the uniform ellipticity condition (2.2), we consider in $\phi(\Omega)$ the operator L defined formally by

$$Lu = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(A_{ij}(x)\frac{\partial u}{\partial x_j}\right), \tag{1.1}$$

and subject to the Dirichlet boundary condition $u = 0$ on $\partial\phi(\Omega)$ or the Neumann boundary condition

$$\sum_{i,j=1}^{N} A_{ij}\frac{\partial u}{\partial x_j}\nu_i = 0 \quad \text{on } \partial\phi(\Omega),$$

where $\nu = (\nu_1, \ldots, \nu_N)$ denotes the outer unit normal to $\partial\phi(\Omega)$. Following the approach developed in [1], we prove estimates for the deviation of the eigenvalues and eigenfunctions corresponding to the domain $\phi(\Omega)$ from those corresponding to a perturbation $\tilde{\phi}(\Omega)$ of $\phi(\Omega)$. In particular, we improve the results of [1] in two respects: we provide estimates which allow dealing with possibly singular maps ϕ and we improve the exponents appearing in the appropriate measures of vicinity of $\phi(\Omega)$ and $\tilde{\phi}(\Omega)$.

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The regular case of globally Lipschitz homeomorphisms ϕ was investigated in [1] where estimates for the variation of the resolvents, eigenfunctions and eigenvalues were proved under the assumption that the gradients of the maps ϕ and their inverses have a uniform upper bound. Those estimates can be applied for example to the case of uniform families of domains with Lipschitz continuous boundaries. However, if $\phi(\Omega)$ has boundary degenerations stronger than those of Ω (for example, Ω has a Lipschitz continuous boundary while $\phi(\Omega)$ has a cusp at the boundary) one cannot assume that ϕ has a bounded gradient. This problem might be overcome by approximating $\phi(\Omega)$ by means of suitable domains $\phi_\epsilon(\Omega)$, $\epsilon > 0$, where ϕ_ϵ are globally Lipschitz continuous maps: one would find estimates depending on ϵ and eventually would pass to the limit as $\epsilon \to 0$. However, the gradients of the maps ϕ_ϵ would not necessarily have a uniform upper bound, and hence the results of [1] could not be used in this limiting procedure. Thus, it is desirable to prove stability estimates independent of $\|\nabla\phi\|_{L^\infty(\Omega)}$. In this paper, we prove general stability estimates without using any uniform upper bound for $\|\nabla\phi\|_{L^\infty(\Omega)}$. These estimates are expressed in terms of a certain measure of vicinity $\delta_q(\phi, \tilde{\phi})$ of ϕ and $\tilde{\phi}$ which reduces to the Sobolev norm $\|\phi - \tilde{\phi}\|_{W^{1,q}(\Omega)}$ in regular cases; see (3.7) for the precise definition and Remark 4.

Similarly to [1] the estimates for the variation of eigenvalues and eigenfunctions are deduced from corresponding estimates for the variation of resolvent operators in the Hilbert–Schmidt class $\mathcal{C}^2$. Note that the resolvent $(L+1)^{-1}$ of the operator L belongs to the Schatten class $\mathcal{C}^r$, $1 \leqslant r < \infty$, if and only if the eigenvalues λ_n of L satisfy

$$\sum_{n=1}^{\infty} \frac{1}{(\lambda_n + 1)^r} < \infty,$$

and, in the case of smooth domains, this holds provided $r > N/2$. Condition $r > N/2$ is used in [1] and turns out to spoil the exponents in the stability estimates. If one is interested only in eigenvalues and eigenfunctions (and not in the solutions to the Poisson problem $Lu = f$), it is convenient to replace the resolvent $(L+1)^{-1}$ by suitable powers of it. Indeed, the operator $(L+1)^{-k}$ belongs to any fixed Schatten class $\mathcal{C}^r$ provided $k \in \mathbf{N}$ is large enough. The power k plays no essential role in the estimates for eigenvalues and eigenfunctions: this simple but crucial observation enables us to improve the estimates of [1].

In the case of transformations ϕ with uniformly bounded gradients considered in [1], the new estimate for eigenvalues reads

$$\left(\sum_{n=1}^{\infty}[(\tilde{\lambda}_n + 1)^{-k} - (\lambda_n + 1)^{-k}]^2\right)^{1/2} \leqslant c\delta_{2q_0/(q_0-2)}(\phi, \tilde{\phi}), \tag{1.2}$$

where λ_n, $\tilde{\lambda}_n$ denote the eigenvalues in $\phi(\Omega)$ and $\tilde{\phi}(\Omega)$ respectively; see Theorem 6. Here $q_0 \in]2, \infty]$ is a suitable parameter related to a summability assumption (property (P)) on the eigenfunctions and their gradients; see Definition 2. It turns out that in the case of sufficiently smooth domains (say, of class $C^{1,1}$) and sufficiently smooth coefficients A_{ij} (say, Lipschitz continuous), one can take $q_0 = \infty$, and hence the estimate (1.2) is expressed in terms of $\delta_2(\phi, \tilde{\phi})$. (Note that in [**1**] the best measure of vicinity appearing in the estimates is $\delta_{N+\epsilon}(\phi, \tilde{\phi})$, for any $\epsilon > 0$, and it is much worse than $\delta_2(\phi, \tilde{\phi})$ if $N > 2$.) In the general case of possibly singular transformations ϕ, the term $\delta_{2q_0/(q_0-2)}(\phi, \tilde{\phi})$ in (1.2) has to be replaced by $(1 + \delta_s(\phi, \tilde{\phi}))\delta_{2q_0/(q_0-2)}(\phi, \tilde{\phi})$ for a suitable $s \geqslant 1$, where the extra summand appears only for technical reasons and is not important for applications.

Estimate (1.2) is first applied to uniform families of domains with Lipschitz continuous boundaries as in [**1**]. In this case, the construction of appropriate transformations ϕ leads to the estimate

$$\left(\sum_{n=1}^{\infty}[(\lambda_n[\Omega_1]+1)^{-k} - (\lambda_n[\Omega_2]+1)^{-k}]^2\right)^{1/2} \leqslant c|\Omega_1 \vartriangle \Omega_2|^{1/2-1/q_0}, \quad (1.3)$$

provided Ω_1 and Ω_2 belong to the same Lipschitz class and the Lebesgue measure $|\Omega_1 \vartriangle \Omega_2|$ of the symmetric difference of Ω_1 and Ω_2 is small enough; see Theorem 11. Analogous estimates for the variation of the eigenfunctions are proved in Theorems 8 and 11.

We then apply our general stability estimates to the case of the Dirichlet Laplacian on a domain Ω with an exterior power-type cusp of exponent α sufficiently close to 1 (the case $\alpha = 1$ is clearly the regular Lipschitz case). We approximate Ω by a sequence Ω_δ, $\delta > 0$, of domains with Lipschitz continuous boundaries and estimate the rate of convergence of the eigenvalues and eigenfunctions in terms of $|\Omega \backslash \Omega_\delta|^{b(\alpha)}$, where $0 < b(\alpha) < 1$ is an explicit exponent depending only on N and α (with $b(1) = 1/2$ as expected from (1.3)). To do so, we establish the validity of property (P) in the domains Ω_δ by means of an *a priori* estimate of Maz'ya and Plamenevskii [**14**, p. 4] and a bootstrap argument; see Theorems 12 and 14. According to the strategy explained above, we then construct suitable maps $\phi_\epsilon : \Omega_\delta \to \Omega_\epsilon$ and obtain estimates in terms of $|\Omega_\epsilon \backslash \Omega_\delta|^{b(\alpha)}$. By letting $\epsilon \to 0$ we obtain the desired estimate.

We note that in the case of suitable uniform families of domains with Lipschitz continuous boundaries it was proved in [**5, 6**] that

$$|\lambda_n[\Omega_1] - \lambda_n[\Omega_2]| \leqslant c_n|\Omega_1 \vartriangle \Omega_2|^{1-2/q_0}, \quad (1.4)$$

where q_0 is as above. Moreover, in [**8**] it is proved that the exponent $1 - 2/q_0$ is sharp; see also [**12**]. Clearly, in estimate (1.3) we do not obtain the sharp exponent. The fact that our exponent is exactly half the sharp one seems to indicate that a variation of our method could lead to the optimal exponent. However, we note that our method has the advantage of providing stability estimates also for eigenfunctions and large enough powers of the resolvents. Such estimates cannot be obtained by the methods of [**4**] and [**5, 6**] which make use of the variational characterization of the eigenvalues. We note that while stability estimates for eigenvalues have been extensively studied in recent years, the corresponding problem for eigenfunctions is much less investigated. In this respect we mention the article of Pang [15] where probabilistic methods are used to obtain a stability estimate for the ground state of the Dirichlet Laplacian on a simply connected planar domain.

We note that estimates of the type (1.4) have been recently obtained by Lemenant and Milakis [13] for the first eigenvalue of the Dirichlet Laplacian in Reifenberg flat domains. We also note that stability estimates for the eigenvalues of uniformly elliptic operators with Dirichlet or Neumann boundary conditions on domains with continuous boundaries were proved in Burenkov and Davies [4] and in [7, 8] where the vicinity of the domains is expressed in terms of a variant of the Hausdorff distance. For more references on this subject we refer to [1] and to the survey paper [9].

This paper is organized as follows. In §2 we set the problem. In §3 we prove stability estimates for resolvents, eigenvalues and eigenfunctions in terms of $\delta_q(\phi, \tilde{\phi})$. In §4 we discuss some applications to domains with Lipschitz continuous boundaries as well as to domains with power-type cusps at the boundary, and we prove estimates in terms of the Lebesgue measure.

§2. *Elliptic operators and singular domain transformations.* Let Ω be an arbitrary bounded domain in $\mathbf{R}^N$. We consider a family of domains $\phi(\Omega)$ in $\mathbf{R}^N$ parametrized by locally Lipschitz homeomorphisms ϕ of Ω onto $\phi(\Omega)$. More precisely, we consider the family of transformations

$$\Phi(\Omega) := \{\phi \in (W^{1,\infty}_{\mathrm{loc}}(\Omega) \cap L^\infty(\Omega))^N : \text{the continuous representative of } \phi \text{ is injective and } \phi^{(-1)} \in (W^{1,\infty}_{\mathrm{loc}}(\phi(\Omega)))^N\}, \tag{2.1}$$

where $W^{1,\infty}_{\mathrm{loc}}(\Omega)$ denotes the Sobolev space of the functions in $L^\infty_{\mathrm{loc}}(\Omega)$ which have weak derivatives of first order in $L^\infty_{\mathrm{loc}}(\Omega)$. Observe that if $\phi \in \Phi(\Omega)$ then ϕ is locally Lipschitz continuous. Note also that if $\phi \in \Phi(\Omega)$ then $\phi(\Omega)$ is also a bounded domain. Moreover, any transformation $\phi \in \Phi(\Omega)$ allows changing variables in integrals in the standard way.

Let $A = (A_{ij})_{i,j=1,\ldots,N}$ be a real symmetric matrix-valued function defined on $\mathbf{R}^N$ such that $A_{ij} \in L^\infty(\mathbf{R}^N)$ for all $i, j = 1, \ldots, N$ and

$$\theta^{-1}|\xi|^2 \leqslant \sum_{i,j=1}^N A_{ij}(x)\xi_i\xi_j \leqslant \theta|\xi|^2, \tag{2.2}$$

for all $x, \xi \in \mathbf{R}^N$ and some $\theta \geqslant 1$. This matrix will be fixed throughout the paper.

Let $\phi \in \Phi(\Omega)$ and let $\mathcal{W}$ denote either $W^{1,2}_0(\phi(\Omega))$ or $W^{1,2}(\phi(\Omega))$. Here $W^{1,2}(\phi(\Omega))$ denotes the standard Sobolev space of functions in $L^2(\phi(\Omega))$ with first order weak derivatives in $L^2(\phi(\Omega))$ endowed with its usual norm, and $W^{1,2}_0(\phi(\Omega))$ denotes the closure in $W^{1,2}(\phi(\Omega))$ of the C^∞-functions with compact support in Ω. We consider a non-negative self-adjoint operator L on $L^2(\phi(\Omega))$ given formally by (1.1) and satisfying Dirichlet or Neumann boundary conditions on $\partial\phi(\Omega)$. More precisely, L is defined as the self-adjoint operator on $L^2(\phi(\Omega))$ canonically associated with the quadratic form Q_L given by

$$\mathrm{Dom}(Q_L) = \mathcal{W}, \qquad Q_L(v) = \int_{\phi(\Omega)} \sum_{i,j=1}^{N} A_{ij}(y) \frac{\partial v}{\partial y_i} \frac{\partial \bar{v}}{\partial y_j}\, dy, \tag{2.3}$$

for all $v \in \mathcal{W}$. We now consider the operator H on $L^2(\Omega)$ obtained by pulling back L to Ω as follows. Let C_ϕ be the operator from $L^2(\phi(\Omega))$ to $L^2(\Omega)$ defined by $C_\phi v = v \circ \phi$ for all $v \in L^2(\phi(\Omega))$. Let $v \in W^{1,2}(\phi(\Omega))$ be given and let $u = C_\phi v$. Observe that

$$\int_{\phi(\Omega)} |v|^2\, dy = \int_\Omega |u|^2 |\det \nabla\phi(x)|\, dx.$$

Moreover a simple computation shows that

$$\int_{\phi(\Omega)} \sum_{i,j=1}^{N} A_{ij}(y) \frac{\partial v}{\partial y_i} \frac{\partial \bar{v}}{\partial y_j}\, dy = \int_\Omega \sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial \bar{u}}{\partial x_j} |\det \nabla\phi(x)|\, dx,$$

where $a = (a_{ij})_{i,j=1,\dots,N}$ is the matrix-valued function defined on Ω by

$$\begin{aligned} a_{ij} &= \sum_{r,s=1}^{N} \left(A_{rs} \frac{\partial \phi_i^{(-1)}}{\partial y_r} \frac{\partial \phi_j^{(-1)}}{\partial y_s} \right) \circ \phi \\ &= ((\nabla\phi)^{-1} A(\phi) (\nabla\phi)^{-t})_{ij}. \end{aligned}$$

Here $(\nabla\phi)^{-t}$ denotes the transpose of the inverse of the matrix $\nabla\phi$. The operator H is defined as the non-negative self-adjoint operator on the Hilbert space $L^2(\Omega, |\det \nabla\phi(x)|\, dx)$ associated with the closure of the quadratic form Q_H with $\mathrm{Dom}(Q_H) = C_\phi[\mathcal{W}]$ and

$$Q_H(u) = \int_\Omega \sum_{i,j=1}^{N} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial \bar{u}}{\partial x_j} |\det \nabla\phi(x)|\, dx, \quad u \in \mathrm{Dom}(Q_H).$$

We note that H is not necessarily uniformly elliptic. We also note that, equivalently, H can be defined as

$$H = C_\phi L C_{\phi^{(-1)}}.$$

In particular H and L are unitarily equivalent and the operator H has compact resolvent if and only if L has compact resolvent. We set

$$g(x) := |\det \nabla\phi(x)|,$$

for all $x \in \Omega$, and we denote by $\langle \cdot, \cdot \rangle_g$ the inner product in $L^2(\Omega, g\, dx)$ and also in $(L^2(\Omega, g\, dx))^N$.

§3. *Stability estimates.* In this section we shall consider maps ϕ with the properties described in Section 2, and we make the additional assumption that ϕ and its inverse $\phi^{(-1)}$ are Lipschitz continuous. We note that in this case

$$C_\phi[W^{1,2}(\phi(\Omega))] = W^{1,2}(\Omega) \quad \text{and} \quad C_\phi[W_0^{1,2}(\phi(\Omega))] = W_0^{1,2}(\Omega).$$

In this context we give an additional definition. We define $T : L^2(\Omega, g\,dx) \to (L^2(\Omega, g\,dx))^N$ to be the operator with domain $\mathrm{Dom}(T) = C_\phi[\mathcal{W}]$ and $Tu = a^{1/2}\nabla u$. We then have

$$H = T^{(*)_g} T. \tag{3.1}$$

Here the adjoint $T^{(*)_g}$ of T is understood with respect to the inner product of $L^2(\Omega, g\,dx)$ and this has been emphasized in the notation. Subsequently, however, we shall simply write T^* instead of $T^{(*)_g}$, unless it is necessary to distinguish two different scalar products.

Let ϕ and $\tilde{\phi}$ be two such maps on Ω and let L and $\tilde{L}$ be the corresponding operators on $\phi(\Omega)$ and $\tilde{\phi}(\Omega)$ defined as in §2. We assume that either L and $\tilde{L}$ both satisfy Dirichlet boundary conditions or L and $\tilde{L}$ both satisfy Neumann boundary conditions. We shall use a tilde to distinguish the objects corresponding to L from those corresponding to $\tilde{L}$. Our aim is to compare L and $\tilde{L}$ and to do this we shall compare the respective pull-backs H and $\tilde{H}$. Since H and $\tilde{H}$ act on different Hilbert spaces—$L^2(\Omega, g\,dx)$ and $L^2(\Omega, \tilde{g}\,dx)$—we shall use the canonical unitary operator,

$$w : L^2(\Omega, g\,dx) \longrightarrow L^2(\Omega, \tilde{g}\,dx), \qquad u \mapsto wu,$$

defined as the multiplication by the function $w := g^{1/2}\tilde{g}^{-1/2}$. We shall retain the same symbol w to stand for multiplication operator by the function w from $L^2(\Omega, \rho_1\,dx)$ to $L^2(\Omega, \rho_2\,dx)$ for other weights ρ_1 and ρ_2. These weights will always be bounded away from zero and infinity, so the actual action of the operator will always be the same. We also define the matrix-valued function

$$S := w^{-2} a^{-1/2} \tilde{a} a^{-1/2} \tag{3.2}$$

and use the symbol S to stand for the associated multiplication operator between various spaces $(L^2(\Omega, \rho\,dx))^N$ with weights ρ that will be again bounded away from zero and infinity.

As will subsequently be clear, in order to compare H and $\tilde{H}$ we shall also need the auxiliary operator T^*ST. Since

$$\|S^{1/2}Tu\|^2_{L^2(\Omega, g\,dx)} = \int_\Omega (\tilde{a}\nabla u \cdot \nabla \bar{u})\tilde{g}\,dx, \quad u \in C_\phi[\mathcal{W}],$$

T^*ST is the non-negative self-adjoint operator in $L^2(\Omega, g\,dx)$ canonically associated with the closure of the quadratic form

$$\int_\Omega (\tilde{a}\nabla u \cdot \nabla \bar{u})\tilde{g}\,dx, \quad u \in C_\phi[\mathcal{W}].$$

Hence $\tilde{H}$ and T^*ST have the same quadratic form, but they act on different Hilbert spaces: $L^2(\Omega, \tilde{g}\, dx)$ and $L^2(\Omega, g\, dx)$ respectively. It is easily seen that the operator T^*ST is the pull-back to Ω via $\tilde{\phi}$ of the operator

$$\hat{L} := \frac{\tilde{g} \circ \tilde{\phi}^{(-1)}}{g \circ \tilde{\phi}^{(-1)}} \tilde{L}. \tag{3.3}$$

Thus we shall deal with the operators L, $\tilde{L}$ and $\hat{L}$ and the respective pull-backs H, $\tilde{H}$ and T^*ST. We shall repeatedly use the fact that these operators are pairwise unitarily equivalent.

Throughout this section we assume that these operators have compact resolvent and that their non-zero eigenvalues λ_n satisfy the estimate

$$\lambda_n \geqslant C_1 n^{1/\alpha}, \tag{3.4}$$

for some positive constants α and C_1.

Remark 1. We recall that if Ω is a bounded domain with Lipschitz continuous boundary then (3.4) is satisfied with $\alpha = N/2$ (no restrictions on the boundary are required in the case of Dirichlet boundary conditions); see [**1**] for references.

Subsequently we shall denote by $\lambda_n[E]$, $n \in \mathbf{N}$, the eigenvalues of a non-negative self-adjoint operator E with compact resolvent, arranged in non-decreasing order and repeated according to multiplicity, and by $\psi_n[E]$, $n \in \mathbf{N}$, a corresponding orthonormal sequence of eigenfunctions.

We introduce the following property which will be important in what follows.

Definition 2. Let U be an open set in $\mathbf{R}^N$ and $\rho > 0$ be a measurable function on U, and let E be a non-negative self-adjoint operator on $L^2(U, \rho\, dx)$ with compact resolvent and $\mathrm{Dom}(E) \subset W^{1,1}_{\mathrm{loc}}(U)$. Let $q_0 \in\,]2, \infty]$, $\gamma, C_2 \in\,]0, \infty[$. We say that E satisfies property (P1) with the parameters q_0, γ and C_2 if

$$\|\psi_n[E]\|_{L^{q_0}(U,\rho\, dx)} \leqslant C_2 \lambda_n[E]^{\gamma}, \tag{P1}$$

for all $n \in \mathbf{N}$ such that $\lambda_n[E] \neq 0$. We say that E satisfies property (P2) with the parameters q_0, γ and C_2 if

$$\|\nabla\psi_n[E]\|_{L^{q_0}(U,\rho\, dx)} \leqslant C_2 \lambda_n[E]^{1/2+\gamma}, \quad n \in \mathbf{N}. \tag{P2}$$

Finally, we say that E satisfies property (P) with the parameters q_0, γ and C_2 if it satisfies both (P1) and (P2) with these parameters.

The next lemma involves the Schatten norms $\|\cdot\|_{\mathcal{C}^r}$, $1 \leqslant r \leqslant \infty$. For a compact operator E on a Hilbert space they are defined by $\|E\|_{\mathcal{C}^r} = (\sum_n \mu_n(E)^r)^{1/r}$, if $r < \infty$, and $\|E\|_{\mathcal{C}^\infty} = \|E\|$, where $\mu_n(E)$ are the singular values of E, i.e., the non-zero eigenvalues of $(E^*E)^{1/2}$; the Schatten space $\mathcal{C}^r$, defined as the space of those compact operators for which the Schatten norm $\|\cdot\|_{\mathcal{C}^r}$ is finite, is a Banach space; see Reed and Simon [**16**] or Simon [**17**] for details.

Let $F := TT^*$, $F_S := S^{1/2}TT^*S^{1/2}$. It is well known that $\sigma(F)\setminus\{0\} = \sigma(H)\setminus\{0\}$ and similarly $\sigma(F_S)\setminus\{0\} = \sigma(T^*ST)\setminus\{0\}$; see [**10**, Theorem 2]. Moreover, we note that

$$\tilde{H} = (\tilde{a}^{1/2}\nabla)^{(*)_{\tilde{g}}}\tilde{a}^{1/2}\nabla = w^2(\tilde{a}^{1/2}\nabla)^* w^{-2}\tilde{a}^{1/2}\nabla = w(wT^*STw)w^{-1}, \quad (3.5)$$

and therefore the eigenvalues of the operator wT^*STw coincide with the eigenvalues of $\tilde{H}$.

LEMMA 3. (i) *Let E be a non-negative self-adjoint operator on $L^2(\Omega, \rho\, dx)$ whose eigenvalues satisfy inequality (3.4) for some $\alpha, C_1 > 0$. Assume that E satisfies property (P1) for some q_0, γ and C_2. Then for large enough $k \in \mathbf{N}$, depending only on α and γ, there exists $c > 0$ such that for all measurable functions R on Ω,*

$$\|R(E+1)^{-k}\|_{\mathcal{C}^2} \leqslant c\|R\|_{L^{2q_0/(q_0-2)}(\Omega,\rho\, dx)}.$$

The constant c depends only on k, α, γ, C_1, C_2 and, if $\lambda_1[E] = 0$ and has multiplicity m, also on $\|\psi_i[E]\|_{L^{q_0}(\Omega,\rho\, dx)}$, $i = 1, \ldots, m$.

(ii) *Assume that H (respectively T^*ST) satisfies property (P2) for some q_0, γ and C_2. Then for large enough $k \in \mathbf{N}$, depending only on α and γ, there exists $c > 0$ such that for all measurable matrix-valued functions R on Ω,*

$$\|R(F+1)^{-k}F^{1/2}\|_{\mathcal{C}^2} \leqslant c\|Ra^{1/2}\|_{L^{2q_0/(q_0-2)}(\Omega,g\, dx)}.$$

$$\text{(respectively} \quad \|R(F_S+1)^{-k}F_S^{1/2}\|_{\mathcal{C}^2} \leqslant c\|RS^{1/2}a^{1/2}\|_{L^{2q_0/(q_0-2)}(\Omega,g\, dx)}).$$

The constant c depends only on k, α, γ, C_1 and C_2.

Proof. We first prove statement (i). Assume for simplicity that $\lambda_1[E] \neq 0$. We have

$$\begin{aligned}
\|R(E+1)^{-k}\|^2_{\mathcal{C}^2} &= \sum_{n=1}^{\infty} \|R(E+1)^{-k}\psi_n[E]\|^2_{L^2(\Omega,\rho\, dx)} \\
&= \sum_{n=1}^{\infty} (\lambda_n[E]+1)^{-2k}\|R\psi_n[E]\|^2_{L^2(\Omega,\rho\, dx)} \\
&\leqslant \|R\|^2_{L^{2q_0/(q_0-2)}(\Omega,\rho\, dx)} \sum_{n=1}^{\infty} (\lambda_n[E]+1)^{-2k}\|\psi_n[E]\|^2_{L^{q_0}(\Omega,\rho\, dx)} \\
&\leqslant c\|R\|^2_{L^{2q_0/(q_0-2)}(\Omega,\rho\, dx)} \sum_{n=1}^{\infty} (\lambda_n[E]+1)^{-2k}\lambda_n[E]^{2\gamma} \\
&\leqslant c\|R\|^2_{L^{2q_0/(q_0-2)}(\Omega,\rho\, dx)}, \qquad (3.6)
\end{aligned}$$

provided k is large enough. In the case $\lambda_1[E] = 0$ and has multiplicity m, one has simply to take into account the first m summands in (3.6).

We now prove statement (ii). We only consider F, the operator F_S is treated similarly. We note that $(\lambda_n[H]^{-1/2}T\psi_n[H])$ is an orthonormal basis of $\mathrm{Ker}(F)^{\perp}$. Hence

$$\begin{aligned}
\|R(F+1)^{-k}F^{1/2}\|_{\mathcal{C}^2} &= \sum_{n=1}^{\infty} \lambda_n[E]^{-1}\|R(F+1)^{-k}F^{1/2}T\psi_n[E]\|^2_{L^2(\Omega,g\,dx)} \\
&= \sum_{n=1}^{\infty}(\lambda_n[E]+1)^{-2k}\|RT\psi_n[E]\|^2_{L^2(\Omega,g\,dx)} \\
&\leqslant \|Ra^{1/2}\|^2_{L^{2q_0/(q_0-2)}(\Omega,g\,dx)} \\
&\quad \times \sum_{n=1}^{\infty}(\lambda_n[E]+1)^{-2k}\|\nabla\psi_n[E]\|^2_{L^{q_0}(\Omega,g\,dx)} \\
&\leqslant c\|Ra^{1/2}\|^2_{L^{2q_0/(q_0-2)}(\Omega,g\,dx)} \\
&\quad \times \sum_{n=1}^{\infty}(\lambda_n[E]+1)^{-2k}\lambda_n[E]^{2\gamma+1} \\
&\leqslant c\|Ra^{1/2}\|^2_{L^{2q_0/(q_0-2)}(\Omega,g\,dx)},
\end{aligned}$$

provided k is large enough. This completes the proof. □

Our stability estimates are expressed in terms of the following measure of vicinity of ϕ and $\tilde{\phi}$ (we recall that $w := g^{1/2}\tilde{g}^{-1/2}$):

$$\delta_q(\phi,\tilde{\phi}) = \delta_q^{(1)}(\phi,\tilde{\phi}) + \delta_q^{(2)}(\phi,\tilde{\phi}), \tag{3.7}$$

where

$$\begin{aligned}
\delta_q^{(1)}(\phi,\tilde{\phi}) &= \|w-1\|_{L^q(\Omega,g\,dx)} + \|w^{-1}-1\|_{L^q(\Omega,g\,dx)}, \\
\delta_q^{(2)}(\phi,\tilde{\phi}) &= \|(S^{1/2}-S^{-1/2})a^{1/2}\|_{L^q(\Omega,g\,dx)} + \|(S-I)a^{1/2}\|_{L^q(\Omega,g\,dx)}.
\end{aligned}$$

Remark 4. Note that if we consider maps ϕ $\tilde{\phi}$ belonging to a family of transformations φ satisfying the uniform estimate

$$c^{-1} \leqslant \text{ess inf}\,|\det\nabla\varphi|, \qquad \|\nabla\varphi\|_{L^\infty} \leqslant c$$

for a fixed $c>0$, and the coefficients A_{ij} are Lipschitz continuous, then

$$\delta_q(\phi,\tilde{\phi}) \leqslant C\|\phi-\tilde{\phi}\|_{W^{1,q}(\Omega)}.$$

THEOREM 5 (Stability of resolvents). *Assume that the operators H and T^*ST satisfy properties (P1) and (P2) and that $w^{-1}\tilde{H}w$ satisfies property (P1), for the same parameters q_0, γ and C_2. Then for all large enough $k\in\mathbf{N}$ depending only on α and γ and for any $s>q_0(\alpha+2\gamma)/(q_0-2)$, there exists $c>0$ such that*

$$\|(w^{-1}\tilde{H}w+1)^{-k}-(H+1)^{-k}\|_{\mathcal{C}^2} \leqslant c[1+\delta_s(\phi,\tilde{\phi})]\delta_{2q_0/(q_0-2)}(\phi,\tilde{\phi}). \tag{3.8}$$

The constant c depends only on α, k, γ, s, C_1, C_2 and, in the case of Neumann boundary conditions, also on $\|g\|_{L^{q_0}(\Omega)}$.

Note. The factor $1+\delta_s(\phi,\tilde{\phi})$ appears for technical reasons and is not of importance for applications.

Proof. We fix $k\in\mathbf{N}$ large enough so that part (i) of Lemma 3 can be applied to the operators H, T^*ST and $w^{-1}\tilde{H}w$ and part (ii) of the same lemma can be applied to the operators H and T^*ST. Since $w^{-1}\tilde{H}w=wT^*STw$, we can write

$$(w^{-1}\tilde{H}w+1)^{-k}-(H+1)^{-k}=A+B,$$

where

$$A=(wT^*STw+1)^{-k}-(T^*ST+1)^{-k},$$
$$B=(T^*ST+1)^{-k}-(T^*T+1)^{-k}.$$

We first estimate A in terms of $\delta_q^{(1)}(\phi,\tilde{\phi})$. We have

$$A=-\sum_{i=0}^{k-1}(T^*ST+1)^{-i}[(T^*ST+1)^{-1} - (wT^*STw+1)^{-1}](wT^*STw+1)^{-(k-1-i)}. \tag{3.9}$$

First we estimate the terms in the sum (3.9) corresponding to $i\leqslant[k/2]$. A direct computation shows that

$$(T^*ST+1)^{-1}-(wT^*STw+1)^{-1}=D_1+D_2+D_3+D_4+D_5,$$

where

$$D_1=(w-1)(wT^*STw+1)^{-1},$$
$$D_2=(w-1)(wT^*STw+1)^{-1}(w-1),$$
$$D_3=(wT^*STw+1)^{-1}(w-1),$$
$$D_4=(T^*ST+1)^{-1}(w-w^{-1})(wT^*STw+1)^{-1}(1-w),$$
$$D_5=(T^*ST+1)^{-1}(w^{-1}-w)(wT^*STw+1)^{-1}.$$

Hence we need to estimate the terms $A_1,\ldots,A_5$ defined by

$$A_j=\sum_{i=0}^{[k/2]}(T^*ST+1)^{-i}D_j(wT^*STw+1)^{-(k-1-i)}.$$

Applying Lemma 3(i) for $E=wT^*STw$ we obtain that if k is large enough, then

$$\|A_1\|_{\mathcal{C}^2}\leqslant c\|w-1\|_{L^{2q_0/(q_0-2)}(\Omega,g\,dx)}. \tag{3.10}$$

Now, applying [1, Lemma 4.5] with $p=q_0/(q_0-2)$ we obtain

$$\|(w-1)(wT^*STw+1)^{-1}\|\leqslant c\|w-1\|_{L^s(\Omega,g\,dx)},$$

for any $s>q_0(\alpha+2\gamma)/(q_0-2)$, and hence

$$\|A_2\|_{\mathcal{C}^2}\leqslant c\|w-1\|_{L^s(\Omega,g\,dx)}\|w-1\|_{L^{2q_0/(q_0-2)}(\Omega,g\,dx)}.$$

The remaining terms A_3, A_4 and A_5 are estimated similarly.

In order to estimate the terms in the sum (3.9) corresponding to $i > [k/2]$ it is possible to proceed as above by swapping (T^*ST+1) and (wT^*STw+1) and using the following decomposition:

$$(T^*ST+1)^{-1} - (wT^*STw+1)^{-1} = D_1' + D_2' + D_3' + D_4' + D_5',$$

where

$$\begin{aligned}
D_1' &= (1-w^{-1})(T^*ST+1)^{-1},\\
D_2' &= (w^{-1}-1)(T^*ST+1)^{-1}(1-w^{-1}),\\
D_3' &= (T^*ST+1)^{-1}(1-w^{-1}),\\
D_4' &= (w^{-1}-1)(T^*ST+1)^{-1}(w^{-1}-w)(wT^*STw+1)^{-1},\\
D_5' &= (T^*ST+1)^{-1}(w^{-1}-w)(wT^*STw+1)^{-1}.
\end{aligned}$$

We now consider the term B. We write

$$B = \sum_{i=0}^{k-1}(T^*ST+1)^{-i}[(T^*ST+1)^{-1} - (T^*T+1)^{-1}](T^*T+1)^{-(k-1-i)}.$$

Let B_i denote the ith summand. We have (see [1])

$$\begin{aligned}
&(T^*ST+1)^{-1} - (T^*T+1)^{-1}\\
&\quad = T^*S^{1/2}(F_S+1)^{-1}(S^{-1/2}-S^{1/2})(F+1)^{-1}T.
\end{aligned}$$

It is also known [10] that $T(T^*T+1)^{-m} = (TT^*+1)^{-m}T$, $m \in \mathbf{N}$, with a similar relation, of course, for $S^{1/2}T$. Hence

$$\begin{aligned}
B_i &= (T^*ST+1)^{-i}T^*S^{1/2}(F_S+1)^{-1}\\
&\quad \times (S^{-1/2}-S^{1/2})(F+1)^{-1}T(T^*T+1)^{-(k-1-i)}\\
&= T^*S^{1/2}(F_S+1)^{-i-1}(S^{-1/2}-S^{1/2})(F+1)^{-k-i}T.
\end{aligned}$$

Using polar decomposition for $S^{1/2}T$ we note that $\|T^*S^{1/2}(F_S+1)^{-i-1}\| \leqslant 1$. Using also polar decomposition for T and applying Lemma 3 (ii) for F, we therefore obtain that, for $i \leqslant [k/2]$,

$$\begin{aligned}
\|B_i\|_{\mathcal{C}^2} &\leqslant \|T^*S^{1/2}(F_S+1)^{-i-1}\|\|(S^{-1/2}-S^{1/2})(F+1)^{-(k-i)}F^{1/2}\|_{\mathcal{C}^2}\\
&\leqslant \|(S^{-1/2}-S^{1/2})(F+1)^{-(k-i)}F^{1/2}\|_{\mathcal{C}^2}\\
&\leqslant c\|(S^{-1/2}-S^{1/2})a^{1/2}\|_{L^{2q_0/(q_0-2)}(\Omega,g\,dx)},
\end{aligned}$$

provided k is large enough. For the terms with $i > [k/2]$, we argue similarly (but now use F_S instead of F) and obtain

$$\|B_i\|_{\mathcal{C}^2} \leqslant c\|(S-I)a^{1/2}\|_{L^{2q_0/(q_0-2)}(\Omega,g\,dx)}.$$

This concludes the proof of the theorem. □

As in [1], from Theorem 5 we immediately deduce the following theorem.

THEOREM 6 (Stability of eigenvalues). *Assume that the operators H and T^*ST satisfy properties (P1) and (P2) and that $w^{-1}\tilde{H}w$ satisfies property (P1), for the same parameters q_0, γ and C_2. Then for all large enough $k \in \mathbf{N}$ depending only on α, γ and for any $s > q_0(\alpha + 2\gamma)/(q_0 - 2)$ there exists $c > 0$ such that*

$$\left(\sum_{n=1}^{\infty}[(\lambda_n[\tilde{L}] + 1)^{-k} - (\lambda_n[L] + 1)^{-k}]^2\right)^{1/2} \leqslant c[1 + \delta_s(\phi, \tilde{\phi})]\delta_{2q_0/(q_0-2)}(\phi, \tilde{\phi}). \quad (3.11)$$

The constant c depends only on α, k, γ, s, C_1, C_2 and, in the case of Neumann boundary conditions, also on $\|g\|_{L^{q_0}(\Omega)}$.

In order to estimate the variation of the eigenfunctions, we need the following lemma.

LEMMA 7. *Let A, B be compact self-adjoint and positive operators in a Hilbert space $\mathcal{H}$. Let λ_n, μ_n, $n \in \mathbf{N}$, be the eigenvalues of A, B respectively. Let ϕ_n, ψ_n, $n \in \mathbf{N}$, be orthonormal sequences of eigenfunctions corresponding to λ_n, μ_n respectively. Let ν be an eigenvalue of A, $\Lambda = \{n \in \mathbf{N} : \lambda_n = \nu\}$ and $d = \min\{|\lambda_i - \nu| : i \in \mathbf{N}\backslash\Lambda\}$. Let P, Q be the orthogonal projectors of $\mathcal{H}$ onto* $\text{span}\{\phi_n : n \in \Lambda\}$ *and* $\text{span}\{\psi_n : n \in \Lambda\}$, *respectively.*

If $\|A - B\| < d/2$ then

$$\|P - Q\| < \frac{2(1 + |\Lambda|)}{d}\|A - B\|.$$

Proof. Note that by the min–max principle it follows that $|\lambda_i - \mu_i| \leqslant \|A - B\|$ for all $i \in \mathbf{N}$; thus, if $\|A - B\| < d/2$ then $|\mu_n - \nu| < d/2$ for all $n \in \Lambda$ and $|\mu_i - \nu| > d/2$ for all $i \in \mathbf{N}\backslash\Lambda$.

Let $u \in \text{Ran}(P)$, $\|u\| \leqslant 1$. Then

$$\begin{aligned}\|A - B\|^2 &\geqslant \|Au - Bu\|^2 = \|\nu u - Bu\|^2 \\ &= \left\|\nu\sum_{i=1}^{\infty}\langle u, \psi_i\rangle\psi_i - \sum_{i=1}^{\infty}\mu_i\langle u, \psi_i\rangle\psi_i\right\|^2 \\ &\geqslant \sum_{i\notin\Lambda}(\nu - \mu_i)^2|\langle u, \psi_i\rangle|^2 > \frac{d^2}{4}\sum_{i\notin\Lambda}|\langle u, \psi_i\rangle|^2,\end{aligned}$$

that is, $\|(I - Q)u\| < (2/d)\|A - B\|$. Thus

$$\|(I - Q)P\| < \frac{2}{d}\|A - B\|. \quad (3.12)$$

Now, let $n \in \Lambda$. Then

$$\|A - B\|^2 \geqslant \|A\psi_n - B\psi_n\|^2 = \left\| \sum_{i=1}^{\infty} \lambda_i \langle \psi_n, \phi_i \rangle \phi_i - \mu_n \psi_n \right\|^2$$
$$= \sum_{i=1}^{\infty} (\lambda_i - \mu_n)^2 |\langle \psi_n, \phi_i \rangle|^2$$
$$\geqslant \sum_{i \notin \Lambda} (\lambda_i - \mu_n)^2 |\langle \psi_n, \phi_i \rangle|^2$$
$$> \frac{d^2}{4} \sum_{i \notin \Lambda} |\langle \psi_n, \phi_i \rangle|^2 = \frac{d^2}{4} \|(I - P)\psi_n\|^2.$$

Hence

$$\|Q(I - P)\| = \|(I - P)Q\| \leqslant \frac{2|\Lambda| \|A - B\|}{d}. \tag{3.13}$$

The proof follows by combining (3.12) and (3.13). □

In the following theorem it is understood that $\psi_k[L]$ and $\psi_k[\tilde{L}]$ are extended by zero outside $\phi(\Omega)$ and $\tilde{\phi}(\Omega)$ respectively.

THEOREM 8 (Stability of eigenfunctions). *Assume that the operators H and T^*ST satisfy properties (P1) and (P2) and that $w^{-1}\tilde{H}w$ satisfies property (P1), for the same parameters q_0, γ and C_2. Let λ be an eigenvalue of L (respectively $\tilde{L}$) of multiplicity m and let $n \in \mathbf{N}$ be such that $\lambda = \lambda_n = \cdots = \lambda_{n+m-1}$. Then for any $s > q_0(\alpha + 2\gamma)/(q_0 - 2)$ there exists $c > 0$ depending only on α, q_0, γ, C_1, C_2, λ_{n-1}, λ, λ_{n+m} and, in case of Neumann boundary conditions, $\|g\|_{L^{q_0}(\Omega)}$ such that the following is true: if $[1 + \delta_s(\phi, \tilde{\phi})]\delta_{2q_0/(q_0-2)}(\phi, \tilde{\phi}) \leqslant c^{-1}$ and $\psi_n[\tilde{L}], \ldots, \psi_{n+m-1}[\tilde{L}]$ (respectively $\psi_n[L], \ldots, \psi_{n+m-1}[L]$) are orthonormal eigenfunctions of $\tilde{L}$ in $L^2(\tilde{\phi}(\Omega))$ (respectively L in $L^2(\phi(\Omega))$), then there exist orthonormal eigenfunctions $\psi_n[L], \ldots, \psi_{n+m-1}[L]$ of L in $L^2(\phi(\Omega))$ (respectively $\psi_n[\tilde{L}], \ldots, \psi_{n+m-1}[\tilde{L}]$ of $\tilde{L}$ in $L^2(\tilde{\phi}(\Omega))$) such that*

$$\begin{aligned}
&\|\psi_l[L] - \psi_l[\tilde{L}]\|_{L^2(\phi(\Omega) \cup \tilde{\phi}(\Omega))} \\
&\quad \leqslant c([1 + \delta_s(\phi, \tilde{\phi})]\delta_{2q_0/(q_0-2)}(\phi, \tilde{\phi}) \\
&\qquad + \|g^{1/2}\psi_l[L] \circ \phi - \tilde{g}^{1/2}\psi_l[L] \circ \tilde{\phi}\|_{L^2(\Omega)} \\
&\qquad + \|g^{1/2}\psi_l[\tilde{L}] \circ \phi - \tilde{g}^{1/2}\psi_l[\tilde{L}] \circ \tilde{\phi}\|_{L^2(\Omega)}),
\end{aligned} \tag{3.14}$$

for all $l = n, \ldots, n + m - 1$.

Proof. Let $k \in \mathbf{N}$ be large enough so that estimate (3.8) holds. Let $\lambda = \lambda_n = \cdots = \lambda_{n+m-1}$ be an eigenvalue of L of multiplicity m. By applying Lemma 7 with $\mathcal{H} = L^2(\Omega, g\,dx)$, $A = (H + 1)^{-k}$, $B = (w^{-1}\tilde{H}w + 1)^{-k}$ and $\nu = (\lambda + 1)^{-k}$, it follows that there exists $c > 0$ as in the statement such that if $[1 + \delta_s(\phi, \tilde{\phi})]\delta_{2q_0/(q_0-2)}(\phi, \tilde{\phi}) \leqslant c^{-1}$ then

$$\|P - Q\| \leqslant c[1 + \delta_s(\phi, \tilde{\phi})]\delta_{2q_0/(q_0-2)}(\phi, \tilde{\phi}), \tag{3.15}$$

where P, Q are the orthonormal projectors in $L^2(\Omega, g\,dx)$ as in Lemma 7.

Now, given eigenfunctions $\psi_l[\tilde{L}]$ as in the statement, we set $\psi_l[\tilde{H}] = \psi_l[\tilde{L}] \circ \tilde{\phi}$ and we note that $w^{-1}\psi_l[\tilde{H}]$ are eigenfunctions of $w^{-1}\tilde{H}w$. Proceeding as in [**1**, the proof of Theorem 5.6], using the selection Lemma [**1**, Lemma 5.4] and estimate (3.15), we have that, possibly by enlarging c, if $[1 + \delta_s(\phi, \tilde{\phi})]\delta_{2q_0/(q_0-2)}(\phi, \tilde{\phi}) < c^{-1}$ there exist eigenfunctions $\psi_n[H], \ldots, \psi_{n+m-1}[H]$ such that

$$\|\psi_l[H] - w^{-1}\psi_l[\tilde{H}]\|_{L^2(\Omega, g\,dx)} \leqslant c[1 + \delta_s(\phi, \tilde{\phi})]\delta_{2q_0/(q_0-2)}(\phi, \tilde{\phi}), \quad (3.16)$$

for all $l = n, \ldots, n + m - 1$. We set $\psi_l[L] = \psi_l[H] \circ \phi^{(-1)}$ for all $l = n, \ldots, n + m - 1$. By changing variables in the left-hand side of (3.16) we obtain

$$\begin{aligned} &\|\psi_l[L] - w^{-1} \circ \phi^{(-1)}\psi_l[\tilde{H}] \circ \phi^{(-1)}\|_{L^2(\phi(\Omega))} \\ &\quad \leqslant c[1 + \delta_s(\phi, \tilde{\phi})]\delta_{2q_0/(q_0-2)}(\phi, \tilde{\phi}). \end{aligned}$$

Hence

$$\begin{aligned} \|\psi_l[L] - \psi_l[\tilde{L}]\|_{L^2(\phi(\Omega))} &\leqslant c[1 + \delta_s(\phi, \tilde{\phi})]\delta_{2q_0/(q_0-2)}(\phi, \tilde{\phi}) \\ &\quad + \|\psi_l[\tilde{L}] - w^{-1} \circ \phi^{(-1)}\psi_l[\tilde{H}] \circ \phi^{(-1)}\|_{L^2(\phi(\Omega))} \end{aligned}$$

and similarly

$$\begin{aligned} \|\psi_l[\tilde{L}] - \psi_l[L]\|_{L^2(\tilde{\phi}(\Omega))} &\leqslant c[1 + \delta_s(\phi, \tilde{\phi})]\delta_{2q_0/(q_0-2)}(\phi, \tilde{\phi}) \\ &\quad + \|\psi_l[L] - w \circ \tilde{\phi}^{(-1)}\psi_l[H] \circ \tilde{\phi}^{(-1)}\|_{L^2(\tilde{\phi}(\Omega))}, \end{aligned}$$

for all $l = n, \ldots, n + m - 1$. Estimate (3.14) follows by combining the previous two inequalities and changing variables in integrals again. If λ is an eigenvalue of $\tilde{L}$ we work similarly. □

§4. *Applications.* In this section we apply Theorems 6 and 8 in order to obtain explicit stability estimates in terms of Lebesgue measure. This will be carried out by showing that condition (P) is satisfied in suitable classes of domains and by constructing appropriate transformations ϕ.

4.1. *Spectral stability for smooth and Lipschitz domains.* In this subsection we consider bounded domains Ω in $\mathbf{R}^N$ of class $C^{m,1}$ for $m = 0, 1$, i.e., bounded domains which are locally subgraphs of $C^{m,1}$ functions. In this context, domains of class $C^{1,1}$ represent the smooth case.

THEOREM 9. *The following statements hold.*

(i) *Let Ω be a bounded domain in $\mathbf{R}^N$ of class $C^{1,1}$ and let A_{ij} be Lipschitz functions defined on Ω satisfying (2.2). Then the operator (1.1) subject either to Dirichlet or Neumann boundary conditions on Ω satisfies property (P) with $q_0 = \infty$ and $\gamma = N/4$.*

(ii) *Let* Ω *be a bounded domain in* $\mathbf{R}^N$ *of class* $C^{0,1}$ *and let* A_{ij} *be measurable functions defined on* Ω *satisfying (2.2). Then the operator (1.1) subject either to Dirichlet or Neumann boundary conditions on* Ω *satisfies property (P) with some* $q_0 > 2$ *and* $\gamma = N(q_0 - 2)/(4q_0)$.

(iii) *The Laplace operator subject to Dirichlet boundary conditions on a bounded domain in* $\mathbb{R}^N$ *of class* $C^{0,1}$ *satisfies property (P) with some* $q_0 > 4$ *if* $N = 2$ *and some* $q_0 > 3$ *if* $N \geqslant 3$.

Statement (i) is well known; see [**1**] for references. For a proof of statement (ii) we refer, e.g., to [**1**, Remark 6.5]. Statement (iii) is a consequence of the work of Jerison and Kenig [**11**].

Definition 10. Let V be a bounded open cylinder; i.e., there exists a rotation R such that $R(V) = W \times]a, b[$, where W is a bounded convex open set in $\mathbf{R}^{N-1}$. Let $M, \rho > 0$. We say that a bounded domain $\Omega \subset \mathbf{R}^N$ belongs to $\mathcal{C}_M^{m,1}(V, R, \rho)$ if Ω is of class $C^{m,1}$ and there exists a function $g \in C^{m,1}(\overline{W})$ such that $a + \rho \leqslant g \leqslant b$, $|g|_{m,1} := \sum_{0<|\alpha|\leqslant m+1} \|D^\alpha g\|_{L^\infty(W)} \leqslant M$, and

$$R(\Omega \cap V) = \{(\bar{x}, x_N) : \bar{x} \in W, a < x_N < g(\bar{x})\}. \tag{4.1}$$

In the following theorem we denote by $\lambda_n[L]$, $\lambda_n[\tilde{L}]$ the eigenvalues of the operator (1.1) subject to Dirichlet or Neumann boundary conditions on Ω and $\tilde{\Omega}$ respectively. Similarly, the eigenfunctions are denoted by $\psi_n[L]$ and $\psi_n[\tilde{L}]$. Moreover, by V_ρ we denote the set $\{x \in V : d(x, \partial V) > \rho\}$.

THEOREM 11. *Let* A_{ij}, $i, j = 1, \ldots, N$ *be measurable functions defined on* $\mathbf{R}^N$ *satisfying* $A_{ij} = A_{ji}$ *and the ellipticity condition (2.2). Let* $\Omega \in \mathcal{C}_M^{0,1}(V, R, \rho)$. *Then there exists* $2 < q_0 \leqslant \infty$ *such that the following statements hold.*

(i) *For all large enough* $k \in \mathbf{N}$ *there exists* $c_1 > 0$ *such that*

$$\left(\sum_{n=1}^{\infty} |(\lambda_n[L] + 1)^{-k} - (\lambda_n[\tilde{L}] + 1)^{-k}|^2\right)^{1/2} \leqslant c_1 |\Omega \vartriangle \tilde{\Omega}|^{1/2 - 1/q_0}, \tag{4.2}$$

for all $\tilde{\Omega} \in \mathcal{C}_M^{0,1}(V, R, \rho)$ *such that* $\tilde{\Omega} \cap (V_\rho)^c = \Omega \cap (V_\rho)^c$.

(ii) *Let* $\lambda[L]$ *be an eigenvalue of multiplicity* m *and let* $n \in \mathbf{N}$ *be such that* $\lambda[L] = \lambda_n[L] = \cdots = \lambda_{n+m-1}[L]$. *There exists* $c_2 > 0$ *such that the following is true: if* $\tilde{\Omega} \in \mathcal{C}_M^{0,1}(V, R, \rho)$, $\Omega \cap (V_\rho)^c = \tilde{\Omega} \cap (V_\rho)^c$, $|\Omega \vartriangle \tilde{\Omega}| \leqslant c_2^{-1}$, *then, given orthonormal eigenfunctions* $\psi_n[\tilde{L}], \ldots, \psi_{n+m-1}[\tilde{L}]$ *in* $L^2(\tilde{\Omega})$, *there exist corresponding orthonormal eigenfunctions* $\psi_n[L], \ldots, \psi_{n+m-1}[L]$ *in* $L^2(\Omega)$ *such that*

$$\|\psi_n[L] - \psi_n[\tilde{L}]\|_{L^2(\Omega \cup \tilde{\Omega})} \leqslant c_2 |\Omega \vartriangle \tilde{\Omega}|^{1/2 - 1/q_0}.$$

If in addition $A_{ij} \in C^{0,1}(\mathbf{R}^N)$ *and* $\Omega, \tilde{\Omega} \in \mathcal{C}_M^{1,1}(V, R, \rho)$ *then statements* (i) *and* (ii) *hold with* $q_0 = \infty$.

Proof. Let $\tilde{\Omega} \in \mathcal{C}_M^{0,1}(V, R, \rho)$. By [**1**, Lemma 7.4] there exists a bi-Lipschitz map Φ from Ω onto $\tilde{\Omega}$ such that

$$\tau^{-1} \leqslant \operatorname*{ess\,inf}_{\Omega} |\det \nabla \Phi| \quad \text{and} \quad \|\nabla \Phi\|_{L^\infty(\Omega)} \leqslant \tau, \tag{4.3}$$

where $\tau > 0$ depends only on N, V, M, ρ and such that there exists $\hat{\Omega} \subset \Omega$ satisfying the following properties:

$$\Phi(x) = x \quad \text{for all } x \in \hat{\Omega} \qquad \text{and} \qquad |\Omega \backslash \hat{\Omega}| \leqslant 2|\Omega \vartriangle \tilde{\Omega}|. \tag{4.4}$$

As in [**1**, Theorem 7.3] we apply Theorems 6 and 8 with $\phi = \mathrm{Id}$ and $\tilde{\phi} = \Phi$. By Remark 1, condition (3.4) is satisfied for $\alpha = N/2$. Moreover, by Theorem 9 and [**1**] the operators L, $\tilde{L}$ and T^*ST satisfy property (P) for some $q_0 > 2$; hence also H, $\tilde{H}$ satisfy property (P) and $w^{-1}\tilde{H}w$ satisfies property (P1). Thus Theorems 6 and 8 apply, and estimates (3.11), (3.14) hold. By (4.3) and (4.4) it follows that

$$\delta_{2q_0/(q_0-2)}(\phi, \tilde{\phi}) \leqslant c|\Omega \vartriangle \tilde{\Omega}|^{1/2-1/q_0},$$

which combined with (3.11), (3.14) easily implies the validity of statements (i) and (ii) (the last two terms in the right-hand side of (3.14) are estimated by means of the Hölder inequality and property (P1)). In the case of open sets of class $C^{1,1}$ it is enough to observe that by Theorem 9 and [**1**] it is possible to choose $q_0 = \infty$ and proceed as above. □

4.2. *An abstract regularity theorem.* We prove a theorem on the regularity of eigenfunctions of a general operator H which will be used in the next subsection. This theorem is a generalization of [**6**, Theorem 5.1] which was concerned with domains satisfying a uniform cone condition. The theorem has two main assumptions: a general multiplicative Sobolev inequality (which is an assumption on the underlying domain Ω and replaces the standard multiplicative Sobolev inequality used in [**6**]) and an *a priori* estimate on the operator H. More precisely, we need to consider the following properties.

(A) Sobolev inequality. Let $m \in \mathbf{N}$, $M > 0$. If $1 \leqslant p, q \leqslant \infty$ and β is a multi-index of length $|\beta| < m$ such that

$$\frac{1}{q} \geqslant \frac{1}{p} - \frac{m - |\beta|}{M},$$

where, in the case of equality, $1 < p < q < \infty$, then there exists $\tau = \tau(m, \beta, M, p, q) \in]0, 1]$ and $C_4 = C_4(m, \beta, M, p, q, \Omega)$ such that for all $u \in W^{m,p}(\Omega)$,

$$\|D^\beta u\|_{L^q(\Omega)} \leqslant C_4 \|u\|^{\tau}_{W^{m,p}(\Omega)} \|u\|^{1-\tau}_{L^p(\Omega)}. \tag{4.5}$$

(B) A priori estimate. Let $m \in \mathbf{N}$, $1 < p_0 < \infty$ and $H : \mathrm{Dom}(H) \to L^1_{\mathrm{loc}}(\Omega)$ where $\mathrm{Dom}(H) \subset L^{p_0}(\Omega)$. For all $p_0 \leqslant p < \infty$ there exists $A_p < \infty$ such that if $u \in \mathrm{Dom}(H)$ and $Hu \in L^p(\Omega)$ then $u \in W^{m,p}(\Omega)$ and

$$\|u\|_{W^{m,p}(\Omega)} \leqslant A_p \|Hu\|_{L^p(\Omega)}. \tag{4.6}$$

Then, following the bootstrap argument used in [6], we prove the following theorem.

THEOREM 12. *Let $1 < p_0 < \infty$ and let H be an operator with* $\mathrm{Dom}(H) \subset L^{p_0}(\Omega)$. *Assume that the Sobolev inequality (A) and the a priori estimate (B) are satisfied for some $m \in \mathbf{N}$, $M > 0$. Assume further that $\tau(m, 0, M, p, q) = (M/m)(1/p - 1/q)$. Then for any eigenfunction ϕ of H, $H\phi = \lambda\phi$, the following statements hold.*

(i) *For any $p_0 \leqslant p < \infty$, $\phi \in W^{m,p}(\Omega)$ and there exists $B_p < \infty$ such that*

$$\|\phi\|_{W^{m,p}(\Omega)} \leqslant B_p |\lambda|^{1+(M/m)(1/p_0 - 1/p)} \|\phi\|_{L^{p_0}(\Omega)}. \tag{4.7}$$

(ii) *Let β be a multi-index with $|\beta| < m$, and define*

$$\rho = \inf\left\{ \tau(m, \beta, M, p, \infty) + \frac{M}{m}\left(\frac{1}{p_0} - \frac{1}{p}\right) : p > \max\left\{ \frac{M}{m - |\beta|}, p_0 \right\} \right\}.$$

Then for any $\eta > 0$ there exists $B_{m,\eta} < \infty$ such that

$$\|D^\beta \phi\|_{L^\infty(\Omega)} \leqslant B_{m,\eta} (1 + |\lambda|)^{\rho + \eta} \|\phi\|_{L^{p_0}(\Omega)}. \tag{4.8}$$

Remark 13. It follows immediately that if

$$\tau(m, \beta, M, p, \infty) = \frac{A + \alpha M/p}{A + \alpha(m - |\beta|)},$$

for some $A, \alpha > 0$, and $|\beta| < m$ and $p_0 \leqslant M/(m - |\beta|)$, then by (4.8) it follows that

$$\|D^\beta \phi\|_{L^\infty(\Omega)} \leqslant B_{m,\eta} (1 + |\lambda|)^{(1/m)(M/p_0 + |\beta|) + \eta} \|\phi\|_{L^{p_0}(\Omega)}, \tag{4.9}$$

for any $\eta > 0$.

Proof. We set $s(q) = Mq/(M - mq)$ if $0 < q < M/m$, and $s(q) = \infty$ if $M/m \leqslant q \leqslant \infty$. Since

$$\lim_{k \to \infty} \underbrace{s(\cdots(s(q))\cdots)}_{k} = \infty,$$

one can apply the bootstrap argument used in [6, Theorem 5.1] and prove that $\phi \in L^p(\Omega)$ for any $p_0 \leqslant p \leqslant \infty$; see [6, Remark 5.9]. Applying (4.6) we obtain

$$\|\phi\|_{W^{m,p}(\Omega)} \leqslant A_p |\lambda| \|\phi\|_{L^p(\Omega)}, \qquad p_0 \leqslant p < \infty. \tag{4.10}$$

If $p = p_0$, then (4.7) is an immediate consequence of (4.10), so we assume that $p_0 < p < \infty$. Let us define $\sigma(t) = Mt/(M + mt)$, for all $t \geqslant 0$. Note that $\sigma(t) = s^{(-1)}(t)$ for all $t \geqslant 0$. We define a sequence $(p_k)_{k \geqslant 1}$ by

$$p_1 = p, \qquad p_{k+1} = \max\{p_0, \tfrac{1}{2}(\sigma(p_k) + p_k)\}.$$

We note that

$$\frac{1}{2}(\sigma(p_k) + p_k) = \left(M + \frac{mp_k}{2}\right)(M + mp_k)^{-1} p_k \leqslant \nu p_k,$$

where $\nu = (M + mp_0/2)(M + mp_0)^{-1} < 1$; hence $p_k = p_0$ from some k onwards. Let κ be the first such k. We then have $p = p_1 > p_2 > \cdots > p_\kappa = p_0$. Moreover, $p_{k+1} > \sigma(p_k)$, $k = 1, \ldots, \kappa - 1$. Inverting we then obtain $p_k < s(p_{k+1})$, $k = 1, \ldots, \kappa - 1$. Applying (4.5) for $q = p_k$, $p = p_{k+1}$, $\beta = 0$ and (4.10) we obtain, with $\tau_k = \tau(m, 0, M, p_{k+1}, p_k)$ and $c_1(k) = C_4(m, 0, M, p_{k+1}, p_k)A_{p_{k+1}}^{\tau_k}$,

$$\begin{aligned}\|\phi\|_{L^{p_k}(\Omega)} &\leqslant C_4(m, 0, M, p_{k+1}, p_k)\|\phi\|_{W^{m,p_{k+1}}(\Omega)}^{\tau_k}\|\phi\|_{L^{p_{k+1}}(\Omega)}^{1-\tau_k}\\ &\leqslant C_4(m, 0, M, p_{k+1}, p_k)A_{p_{k+1}}^{\tau_k}|\lambda|^{\tau_k}\|\phi\|_{L^{p_{k+1}}(\Omega)}^{\tau_k}\|\phi\|_{L^{p_{k+1}}(\Omega)}^{1-\tau_k}\\ &= c_1(k)|\lambda|^{\tau_k}\|\phi\|_{L^{p_{k+1}}(\Omega)}.\end{aligned}$$

Hence

$$\begin{aligned}\|\phi\|_{L^{p_1}(\Omega)} &\leqslant c_2(1)|\lambda|^{\tau_1}\|\phi\|_{L^{p_2}(\Omega)}\\ &\leqslant c_1(1)c_1(2)|\lambda|^{\tau_1+\tau_2}\|\phi\|_{L^{p_3}(\Omega)},\end{aligned}$$

and, by iteration,

$$\|\phi\|_{L^{p_1}(\Omega)} \leqslant c_2(p)|\lambda|^{\tau_1+\tau_2+\cdots+\tau_{\kappa-1}}\|\phi\|_{L^{p_\kappa}(\Omega)},$$

where $c_2(p) = \prod_{i=1}^{\kappa-1} c_1(i)$. Recalling that $p_1 = p$, $p_\kappa = p_0$ and $\tau_1 + \cdots + \tau_{\kappa-1} = (M/m)(1/p_0 - 1/p)$, this takes the form

$$\|\phi\|_{L^p(\Omega)} \leqslant c_2(p)|\lambda|^{(M/m)(1/p_0-1/p)}\|\phi\|_{L^{p_0}(\Omega)}. \tag{4.11}$$

Plugging this back into (4.10) we obtain (4.7), with $B_p = c_2(p)A_p$.

We now prove (ii). Let $|\beta| < m$, $p > \max\{M/(m - |\beta|), p_0\}$ and $\tau = \tau(m, \beta, M, p, \infty)$, $C_4 = C_4(m, \beta, M, p, \infty, \Omega)$. By (4.5), (4.7) and (4.11) we then have

$$\begin{aligned}\|D^\beta\phi\|_{L^\infty(\Omega)} &\leqslant C_4\|\phi\|_{W^{m,p}(\Omega)}^{\tau}\|\phi\|_{L^p(\Omega)}^{1-\tau}\\ &\leqslant C_4(B_p|\lambda|^{1+(M/m)(1/p_0-1/p)}\|\phi\|_{L^{p_0}(\Omega)})^{\tau}\\ &\qquad\times (c_2(p)|\lambda|^{(M/m)(1/p_0-1/p)}\|\phi\|_{L^{p_0}(\Omega)})^{1-\tau}\\ &= c_3(p)|\lambda|^{\rho}\|\phi\|_{L^{p_0}(\Omega)},\end{aligned} \tag{4.12}$$

where $c_3(p) = C_4B_p^{\tau}c_2(p)^{1-\tau}$ and

$$\rho = \tau\left(1 + \frac{M}{m}\left(\frac{1}{p_0} - \frac{1}{p}\right)\right) + (1-\tau)\frac{M}{m}\left(\frac{1}{p_0} - \frac{1}{p}\right) = \tau + \frac{M}{m}\left(\frac{1}{p_0} - \frac{1}{p}\right).$$

Optimizing over p we obtain (4.8). □

4.3. *Spectral stability for domains with outward cusps.* Let $0 < \alpha < 1$. Let $\Omega \subset \mathbf{R}^N$, $N \geqslant 2$, be a domain the boundary of which is C^2 apart from a single outward cusp. More precisely, we assume that

$$\Omega \cap]-1, 1[^N = \{(\bar{x}, x_N) \in]-1, 1[^N : x_N < 1 - |\bar{x}|^\alpha\}$$

and that $\partial\Omega$ is C^2 outside $]-1, 1[^N$; here $\bar{x} = (x_1, \ldots, x_{N-1})$.

Our aim is to obtain stability estimates for the deviation of the eigenvalues and eigenfunctions of the Dirichlet Laplacian L on Ω from the eigenvalues and eigenfunctions of the Dirichlet Laplacian L_ϵ on the domain Ω_ϵ defined for $\epsilon \in]0, 1/2[$ by

$$\Omega_\epsilon \cap]-1,1[^N = \{(\bar{x}, x_N) \in]-1,1[^N : x_N < \min\{1-\epsilon, 1-|\bar{x}|^\alpha\}\},$$
$$\Omega_\epsilon \setminus]-1,1[^N = \Omega \setminus]-1,1[^N.$$

First we apply Theorem 12 in the case of the Dirichlet Laplacian on the domain Ω. Let

$$N_\alpha = N + (N-1)\left(\frac{1}{\alpha} - 1\right),$$

and for a multi-index $\beta = (\beta_1, \dots, \beta_N)$, let

$$|\beta|_\alpha = \beta_1 + \cdots + \beta_{N-1} + \alpha\beta_N.$$

THEOREM 14. *The Dirichlet Laplacian on Ω satisfies property (P) for $q_0 = \infty$ and any $\gamma > N_\alpha/4$.*

Proof. We shall apply Theorem 12. By [**2**, p. 239] the Sobolev inequality (A) is satisfied with $M = N_\alpha$ and

$$\tau(m, \beta, M, p, q) = \frac{|\beta|_\alpha + \alpha N_\alpha(1/p - 1/q)}{|\beta|_\alpha + \alpha(m - |\beta|)}. \tag{4.13}$$

Moreover, by [**14**, Theorem 9.1], the Dirichlet Laplacian satisfies the *a priori* estimate (B) for the parameters $m = 2$ and arbitrary $1 < p_0 < \infty$. Hence the result follows by applying (4.9) (see Remark 13). □

Now let $\epsilon_0 \in]0, 1/2[$ be fixed and let $\epsilon \in [0, \epsilon_0]$. We define

$$\hat{\Omega}_\epsilon = (\Omega \setminus]-1,1[^N) \cup \{(\bar{x}, x_N) \in]-1,1[^N : \ x_N < h_\epsilon(\bar{x})\}$$

where $h_\epsilon :]-1,1[^{N-1} \to]-1, 1-\epsilon_0[$ is the function implicitly defined by

$$h_\epsilon(\bar{x}) = 1 - 2\epsilon_0 + [(1-\epsilon_0 - h_\epsilon(\bar{x}))^4 + \max\{|\bar{x}|^2, \epsilon^{2/\alpha}\}]^{\alpha/2} \tag{4.14}$$

for all $|\bar{x}| < \epsilon_0^{1/\alpha}$ and by $h_\epsilon(\bar{x}) = 1 - |\bar{x}|^\alpha$ for all $|\bar{x}| \geqslant \epsilon_0^{1/\alpha}$. It is easily seen that the function h_ϵ is Lipschitz continuous (see also (4.16) below). Let ϕ_ϵ be the map from Ω_{ϵ_0} to Ω_ϵ defined by

$$\phi_\epsilon(\bar{x}, x_N) \equiv \begin{cases} (\bar{x}, x_N) & \text{if } (\bar{x}, x_N) \in \hat{\Omega}_\epsilon, \\ (\bar{x}, -1 + 2\epsilon_0 + 2x_N & \\ \quad -[(1-\epsilon_0 - h_\epsilon(\bar{x}))^2(1-\epsilon_0 - x_N)^2 & \\ \quad + \max\{|\bar{x}|^2, \epsilon^{2/\alpha}\}]^{\alpha/2}) & \text{if } (\bar{x}, x_N) \in \Omega_{\epsilon_0} \setminus \hat{\Omega}_\epsilon. \end{cases} \tag{4.15}$$

We note that $\phi_\epsilon(\Omega_{\epsilon_0}) = \Omega_\epsilon$ and $\phi_\epsilon(\bar{x}, h_\epsilon(\bar{x})) = (\bar{x}, h_\epsilon(\bar{x}))$, and hence $\phi_\epsilon \in \Phi(\Omega_{\epsilon_0})$. Moreover, we note that $\hat{\Omega}_{\epsilon_0} = \Omega_{\epsilon_0}$ and $\phi_{\epsilon_0} = \mathrm{Id}$.

LEMMA 15. *Assume that* $\frac{1}{2} < \alpha \leqslant 1$ *and* $0 < \epsilon_0 \leqslant \frac{1}{4}$. *There exists a constant* $c > 0$ *depending only on* α *such that*

$$\frac{\det \nabla\phi_\epsilon}{\det \nabla\phi_{\epsilon'}} \leqslant c, \tag{4.16}$$

for all $0 \leqslant \epsilon' < \epsilon \leqslant \epsilon_0$.

Proof. We first prove that

$$C_\alpha(\epsilon_0 - \max\{|\bar{x}|^\alpha, \epsilon\}) \leqslant 1 - \epsilon_0 - h_\epsilon(\bar{x}) \leqslant \epsilon_0 - \max\{|\bar{x}|^\alpha, \epsilon\}, \tag{4.17}$$

for all $0 \leqslant \epsilon \leqslant \epsilon_0$, where $C_\alpha = 1 - 1/2^{2\alpha-1}$. Indeed, from (4.14) we have

$$\begin{aligned} 1 - \epsilon_0 - h_\epsilon &= \epsilon_0 - [(1-\epsilon_0-h_\epsilon)^4 + \max\{|\bar{x}|^2, \epsilon^{2/\alpha}\}]^{\alpha/2} \\ &\leqslant \epsilon_0 - \max\{|\bar{x}|^\alpha, \epsilon\}, \end{aligned}$$

which is the second inequality in (4.17). It then follows that

$$1 - \epsilon_0 - h_\epsilon \geqslant \epsilon_0 - [(\epsilon_0 - \max\{|\bar{x}|^\alpha, \epsilon\})^4 + \max\{|\bar{x}|^2, \epsilon^{2/\alpha}\}]^{\alpha/2},$$

and hence (4.17) will be proved if we show that

$$\epsilon_0 - [(\epsilon_0 - y)^4 + y^{2/\alpha}]^{\alpha/2} \geqslant C_\alpha(\epsilon_0 - y), \tag{4.18}$$

for all $0 < y < \epsilon_0$. To prove (4.18) it suffices to note that since $\epsilon_0 \leqslant 1/4$

$$\epsilon_0 - [(\epsilon_0 - y)^4 + y^{2/\alpha}]^{\alpha/2} \geqslant \epsilon_0 - y - (\epsilon_0 - y)^{2\alpha} \geqslant C_\alpha(\epsilon_0 - y).$$

Hence (4.17) is proved.

We now prove (4.16). We restrict our attention to $\bar{x} \in]-1, 1[^{N-1}$ with $|\bar{x}| \leqslant \epsilon_0^{1/\alpha}$, since $\phi_\epsilon = \phi_{\epsilon'}$ when $|\bar{x}| > \epsilon_0^{1/\alpha}$. We set for simplicity $J = 1 - \epsilon_0 - x_N$. A direct computation together with (4.17) gives

$$\begin{aligned} &\frac{\det \nabla\phi_\epsilon}{\det \nabla\phi_{\epsilon'}} \\ &= \frac{2 + \alpha J[(1-\epsilon_0-h_\epsilon)^2 J^2 + \max\{|\bar{x}|^2, \epsilon^{2/\alpha}\}]^{(\alpha-2)/2}(1-\epsilon_0-h_\epsilon)^2}{2 + \alpha J[(1-\epsilon_0-h_{\epsilon'})^2 J^2 + \max\{|\bar{x}|^2, \epsilon'^{2/\alpha}\}]^{(\alpha-2)/2}(1-\epsilon_0-h_{\epsilon'})^2} \\ &\leqslant C_\alpha^{-2} \frac{2 + \alpha J[(\epsilon_0 - \max\{|\bar{x}|^2, \epsilon^{2/\alpha}\})^2 J^2 + \max\{|\bar{x}|^2, \epsilon^{2/\alpha}\}]^{(\alpha-2)/2}(\epsilon_0 - \max\{|\bar{x}|^2, \epsilon^{2/\alpha}\})^2}{2 + \alpha J[(\epsilon_0 - \max\{|\bar{x}|^2, \epsilon'^{2/\alpha}\})^2 J^2 + \max\{|\bar{x}|^2, \epsilon'^{2/\alpha}\}]^{(\alpha-2)/2}(\epsilon_0 - \max\{|\bar{x}|^2, \epsilon'^{2/\alpha}\})^2} \\ &\leqslant C_\alpha^{-2}\left\{1 + \frac{[(\epsilon_0 - \max\{|\bar{x}|^2, \epsilon^{2/\alpha}\})^2 J^2 + \max\{|\bar{x}|^2, \epsilon'^{2/\alpha}\}]^{(\alpha-2)/2}(\epsilon_0 - \max\{|\bar{x}|^2, \epsilon^{2/\alpha}\})^2}{[(\epsilon_0 - \max\{|\bar{x}|^2, \epsilon'^{2/\alpha}\})^2 J^2 + \max\{|\bar{x}|^2, \epsilon'^{2/\alpha}\}]^{(\alpha-2)/2}(\epsilon_0 - \max\{|\bar{x}|^2, \epsilon'^{2/\alpha}\})^2}\right\} \\ &\leqslant 2C_\alpha^{-2}, \end{aligned}$$

where we have used the simple fact that the function $f(t) = (at+b)^{(\alpha-2)/2}t$, $t \geqslant 0$ with $a, b \geqslant 0$, is monotone increasing. □

Recall that L, L_ϵ denote the Dirichlet Laplacians on Ω, Ω_ϵ respectively, as defined in the beginning of this section.

THEOREM 16. *Let* $\alpha \in]1 - N/15, 1[$. *Then the following statements hold.*

(i) *For all sufficiently large* $k \in \mathbf{N}$ *and any* $\eta > 0$ *there exists* $c > 0$ *depending only on* Ω, k *and* η *such that if* $|\Omega \backslash \Omega_\epsilon| < c^{-1}$ *then the eigenvalues* $\lambda_n[L_\epsilon]$, $\lambda_n[L]$ *satisfy the estimate*

$$\left(\sum_{n=1}^{\infty}\left|(\lambda_n[L_\epsilon]+1)^{-k} - (\lambda_n[L]+1)^{-k}\right|^2\right)^{1/2} \leqslant c|\Omega\backslash\Omega_\epsilon|^{1/2-5(1-\alpha)/(N-1+\alpha)-\eta}. \quad (4.19)$$

(ii) *Let* λ *be an eigenvalue of multiplicity* m *of* L *and let* $n \in \mathbf{N}$ *be such that* $\lambda = \lambda_n[L] = \cdots = \lambda_{n+m-1}[L]$. *For any* $\eta > 0$ *there exists* $c > 0$ *depending only on* Ω, λ, $\lambda_{n-1}[L]$, $\lambda_{n+m}[L]$ *and* η *such that the following is true: if* $|\Omega\backslash\Omega_\epsilon| \leqslant c^{-1}$, *then, given orthonormal eigenfunctions* $\psi_n[L_\epsilon], \ldots, \psi_{n+m-1}[L_\epsilon]$ *of* L_ϵ, *there exist corresponding orthonormal eigenfunctions* $\psi_n[L], \ldots, \psi_{n+m-1}[L]$ *of* L *such that*

$$\|\psi_n[L_\epsilon] - \psi_n[L]\|_{L^2(\Omega)} \leqslant c|\Omega\backslash\Omega_\epsilon|^{1/2-5(1-\alpha)/(N-1+\alpha)-\eta}. \quad (4.20)$$

Proof. Step 1. Let $0 < \epsilon' < \epsilon \leqslant \epsilon_0 < 1/4$. We apply Theorem 5 with the maps $\phi = \phi_\epsilon : \Omega_{\epsilon_0} \to \Omega_\epsilon$ and $\tilde{\phi} = \phi_{\epsilon'} : \Omega_{\epsilon_0} \to \Omega_{\epsilon'}$. The pull-back to Ω_{ϵ_0} of L_ϵ via ϕ_ϵ is denoted by H_ϵ; similarly, the corresponding matrix S and the function w defined in §3 are denoted by $S_{\epsilon,\epsilon'}$ and $w_{\epsilon,\epsilon'}$ respectively, and the operator $(w_{\epsilon,\epsilon'}^{-2} \circ \tilde{\phi}^{(-1)})L_{\epsilon'}$ defined in (3.3) is denoted by $\hat{L}_{\epsilon,\epsilon'}$; the matrix $(\nabla\phi_\epsilon)^{-1}(\nabla\phi_\epsilon)^{-t}$ is denoted by a_ϵ and the operator $a_\epsilon^{1/2}\nabla$ is denoted by T_ϵ. This notation will be used later in Step 3 also for the limiting case $\epsilon' = 0$.

Note that $\det\nabla\phi_\epsilon \geqslant 1$ and for each $q \in [1, N/(1-\alpha)[$ there exists $M > 0$ independent of ϵ such that

$$\|\nabla\phi_\epsilon\|_{L^q(\Omega_{\epsilon_0})}, \qquad \|\mathrm{Adj}(\nabla\phi_\epsilon)\|_{L^q(\Omega_{\epsilon_0})}, \qquad \|\det\nabla\phi_\epsilon\|_{L^q(\Omega_{\epsilon_0})} \leqslant M, \quad (4.21)$$

where $\mathrm{Adj}(\nabla\phi_\epsilon)$ denotes the adjugate matrix of $\nabla\phi_\epsilon$. Similar computations show that if $q+1 < N/(1-\alpha)$ then

$$\|\nabla\phi_\epsilon\|_{L^q(\Omega_{\epsilon_0}, g_\epsilon)}, \; \|\det\nabla\phi_\epsilon\|_{L^q(\Omega_{\epsilon_0}, g_\epsilon)} \leqslant M. \quad (4.22)$$

We now verify that the assumptions of Theorem 5 are satisfied. It is well known that L_ϵ and $L_{\epsilon'}$ satisfy inequality (3.4) with $\alpha = N/2$ and C_1 independent of ϵ_0, ϵ and ϵ'. Moreover, it follows from [**3**, Theorem 3.1] that there exists C_1 independent of ϵ_0, ϵ and ϵ' such that (cf. (3.3))

$$\lambda_n[T_\epsilon^* S_{\epsilon,\epsilon'} T_\epsilon] = \lambda_n[\hat{L}_{\epsilon,\epsilon'}] \geqslant C_1 n^{2/N},$$

i.e. $T_\epsilon^* S_{\epsilon,\epsilon'} T_\epsilon$ satisfies (3.4) with the same parameters.

Now, it is standard that L_ϵ and $L_{\epsilon'}$ satisfy property (P1) with $q_0 = \infty$ and $\gamma = N/4$ (see e.g., [**1**]). Since $\psi_n[H_\epsilon] = \psi_n[L_\epsilon] \circ \phi_\epsilon$, H_ϵ also satisfies property

(P1) with $q_0 = \infty$ and $\gamma = N/4$. Moreover, using also (4.21), we have, for q_0 with $(q_0+2)/2 < N/(1-\alpha)$,

$$\begin{aligned}\|\psi_n[w_{\epsilon,\epsilon'}^{-1} H_{\epsilon'} w_{\epsilon,\epsilon'}]\|_{L^{q_0}(\Omega_{\epsilon_0}, g_\epsilon)} &= \|w_{\epsilon,\epsilon'}^{-1}\psi_n[H_{\epsilon'}]\|_{L^{q_0}(\Omega_{\epsilon_0}, g_\epsilon)} \\ &\leqslant c\lambda_n[H_{\epsilon'}]^{N/4}\|g_{\epsilon'}^{1/2}\|_{L^{q_0}(\Omega_{\epsilon_0}, g_\epsilon)} \\ &\leqslant c\lambda_n[w_{\epsilon,\epsilon'}^{-1} H_{\epsilon'} w_{\epsilon,\epsilon'}]^{N/4}.\end{aligned}$$

Hence the operator $w_{\epsilon,\epsilon'}^{-1} H_{\epsilon'} w_{\epsilon,\epsilon'}$ satisfies property (P1) for any $q_0 < 2(N-1+\alpha)/(1-\alpha)$ and $\gamma = N/4$, uniformly in ϵ_0, ϵ, ϵ'.

By the argument in [**14**, Theorem 9.1] (which deals with the case of a cusp) it follows that the operator L_ϵ satisfies the *a priori* estimate (B) with any $p_0 > 1$, $m = 2$ and A_p independent of ϵ. Since the Sobolev inequality (A) is also valid with $M = N_\alpha$ and τ defined by (4.13), and C_4 independent of ϵ, by Theorem 12 it follows that the operator L_ϵ satisfies property (P2) for $q_0 = \infty$ and any $\gamma > N_\alpha/4$, uniformly in ϵ (see also Theorem 14). Since $\nabla\psi_n[H_\epsilon] = (\nabla\psi_n[L_\epsilon]\circ\phi_\epsilon)\nabla\phi_\epsilon$, we have, for any q_0 with $q_0 + 1 < N/(1-\alpha)$ (cf. (4.22)) and any $\eta > 0$,

$$\begin{aligned}\|\nabla\psi_n[H_\epsilon]\|_{L^{q_0}(\Omega_{\epsilon_0}, g_\epsilon)} &\leqslant c\lambda_n[H_\epsilon]^{1/2+N_\alpha/4+\eta}\|\nabla\phi_\epsilon\|_{L^{q_0}(\Omega_{\epsilon_0}, g_\epsilon)} \\ &\leqslant c\lambda_n[H_\epsilon]^{1/2+N_\alpha/4+\eta},\end{aligned}$$

uniformly in ϵ, ϵ_0. Hence H_ϵ satisfies property (P2) for any $q_0 < (N-1+\alpha)/(1-\alpha)$ and any $\gamma > N_\alpha/4$.

We finally consider $T_\epsilon^* S_{\epsilon,\epsilon'} T_\epsilon$. By Lemma 15, $w_{\epsilon,\epsilon'}^2 \leqslant c$ and hence the operator $\hat{L}_{\epsilon,\epsilon'} = (w_{\epsilon,\epsilon'}^{-2}\circ\phi_{\epsilon'}^{(-1)})L_{\epsilon'}$, which is self-adjoint on $L^2(\Omega_{\epsilon'}, w_{\epsilon,\epsilon'}^2\circ\phi_{\epsilon'}^{(-1)})$, also satisfies the *a priori* estimate (B), for the same parameters as $L_{\epsilon'}$. Since the Sobolev inequality (A) is also valid (cf. Theorem 14), we can apply Theorem 12 and (4.9) and conclude that any eigenfunction $\psi_n[\hat{L}_{\epsilon,\epsilon'}]$ of $\hat{L}_{\epsilon,\epsilon'}$ satisfies

$$\|D^\beta\psi_n[\hat{L}_{\epsilon,\epsilon'}]\|_{L^\infty(\Omega_{\epsilon'})} \leqslant c\lambda_n[\hat{L}_{\epsilon,\epsilon'}]^{|\beta|/2+N_\alpha/2p_0+\eta}\|\psi_n[\hat{L}_{\epsilon,\epsilon'}]\|_{L^{p_0}(\Omega_{\epsilon'})}, \quad (4.23)$$

for all multi-indices β with $|\beta| \leqslant 1$, all $p_0 > 1$ and any $\eta > 0$, uniformly in ϵ_0, ϵ, ϵ'. Now, for any p_0 with $1 < p_0 < 2(1-(1-\alpha)/N)$ we have, by the Hölder inequality,

$$\begin{aligned}&\|\psi_n[\hat{L}_{\epsilon,\epsilon'}]\|_{L^{p_0}(\Omega_{\epsilon'})} \\ &\quad\leqslant \|\psi_n[\hat{L}_{\epsilon,\epsilon'}](w_{\epsilon,\epsilon'}\circ\phi_{\epsilon'}^{(-1)})\|_{L^2(\Omega_{\epsilon'})}\|w_{\epsilon,\epsilon'}^{-1}\circ\phi_{\epsilon'}^{(-1)}\|_{L^{2p_0/(2-p_0)}(\Omega_{\epsilon'})} \\ &\quad= \|w_{\epsilon,\epsilon'}^{-1}\circ\phi_{\epsilon'}^{(-1)}\|_{L^{2p_0/(2-p_0)}(\Omega_{\epsilon'})} \\ &\quad\leqslant \|g_{\epsilon'}^{1/2}\circ\phi_{\epsilon'}^{(-1)}\|_{L^{2p_0/(2-p_0)}(\Omega_{\epsilon'})} \\ &\quad= \left(\int_{\Omega_{\epsilon_0}} g_{\epsilon'}^{2/(2-p_0)}\,dx\right)^{(2-p_0)/2p_0} \\ &\quad\leqslant c, \qquad (4.24)\end{aligned}$$

uniformly in $\epsilon_0, \epsilon, \epsilon'$. Now, we have $\psi_n[T_\epsilon^* S_{\epsilon,\epsilon'} T_\epsilon] = \psi_n[\hat{L}_{\epsilon,\epsilon'}] \circ \phi_{\epsilon'}$. Hence (4.23) implies that $T_\epsilon^* S_{\epsilon,\epsilon'} T_\epsilon$ satisfies (P1) for $q_0 = \infty$ and any $\gamma > N_\alpha N/(4(N-1+\alpha))$. Moreover, for any p_0 as in (4.24), any $\eta > 0$ and any q_0 with $q_0 + 1 < N/(1-\alpha)$ we have, also using (4.22),

$$\begin{aligned}\|\nabla\psi_n[T_\epsilon^* S_{\epsilon,\epsilon'} T_\epsilon]\|_{L^{q_0}(\Omega_{\epsilon_0}, g_\epsilon)} &\leqslant c\lambda_n[T_\epsilon^* S_{\epsilon,\epsilon'} T_\epsilon]^{1/2+N_\alpha/2p_0+\eta} \|\nabla\phi_{\epsilon'}\|_{L^{q_0}(\Omega_{\epsilon_0}, g_\epsilon)} \\ &\leqslant c\lambda_n[T_\epsilon^* S_{\epsilon,\epsilon'} T_\epsilon]^{1/2+N_\alpha/2p_0+\eta}.\end{aligned}$$

Hence $T_\epsilon^* S_{\epsilon,\epsilon'} T_\epsilon$ satisfies property (P2) for any $q_0 < (N-1+\alpha)/(1-\alpha)$ and any $\gamma > N_\alpha N/(4(N-1+\alpha))$.

Summing up, Theorem 5 can be applied for any $q_0 < (N-1+\alpha)/(1-\alpha)$ and any $\gamma > N_\alpha N/(4(N-1+\alpha))$.

Applying the theorem we obtain that for any $q_0 < (N-1+\alpha)/(1-\alpha)$ and $k \in \mathbf{N}$ sufficiently large there holds

$$\|(w_{\epsilon,\epsilon'}^{-1} H_{\epsilon'} w_{\epsilon,\epsilon'} + 1)^{-k} - (H_\epsilon + 1)^{-k}\|_{\mathcal{C}^2(L^2(\Omega_{\epsilon_0}), g_\epsilon)} \leqslant c\delta_{2q_0/(q_0-2)}(\phi_\epsilon, \phi_{\epsilon'}), \tag{4.25}$$

uniformly in ϵ_0, ϵ, ϵ', provided $\delta_{2q_0/(q_0-2)}(\phi_\epsilon, \phi_{\epsilon'}) < c^{-1}$. Since $w_{\epsilon_0,\epsilon'} = w_{\epsilon,\epsilon'} w_{\epsilon_0,\epsilon}$, by unitary equivalence and (4.25) we obtain

$$\begin{aligned}&\|(w_{\epsilon_0,\epsilon'}^{-1} H_{\epsilon'} w_{\epsilon_0,\epsilon'} + 1)^{-k} - (w_{\epsilon_0,\epsilon}^{-1} H_\epsilon w_{\epsilon_0,\epsilon} + 1)^{-k}\|_{\mathcal{C}^2(L^2(\Omega_{\epsilon_0}))} \\ &\quad \leqslant c\delta_{2q_0/(q_0-2)}(\phi_\epsilon, \phi_{\epsilon'}),\end{aligned} \tag{4.26}$$

uniformly in $\epsilon_0, \epsilon, \epsilon'$. In particular, for $\epsilon = \epsilon_0$

$$\|(w_{\epsilon_0,\epsilon'}^{-1} H_{\epsilon'} w_{\epsilon_0,\epsilon'} + 1)^{-k} - (H_{\epsilon_0} + 1)^{-k}\|_{\mathcal{C}^2(L^2(\Omega_{\epsilon_0}))} \leqslant c\delta_{2q_0/(q_0-2)}(\phi_{\epsilon_0}, \phi_{\epsilon'}). \tag{4.27}$$

Step 2. We now estimate the right-hand side of (4.26). We note that

$$\begin{aligned}|S_{\epsilon,\epsilon'}^{1/2}| &\leqslant c|\nabla\phi_\epsilon|\,|\mathrm{Adj}(\nabla\phi_{\epsilon'})|, \\ |S_{\epsilon,\epsilon'}^{-1/2}| &\leqslant c|\nabla\phi_{\epsilon'}|\,|\mathrm{Adj}(\nabla\phi_\epsilon)|, \\ |a_\epsilon^{1/2}| &\leqslant c|\mathrm{Adj}(\nabla\phi_\epsilon)|\end{aligned} \tag{4.28}$$

for some constant $c > 0$. Note also that $\phi_\epsilon = \phi_{\epsilon'}$ on $U_\epsilon = \{(\bar{x}, x_N) \in \Omega_{\epsilon_0} : |\bar{x}| > \epsilon^{1/\alpha}\}$. Thus, by (4.21), (4.28) and the Hölder inequality it follows that if $1 < s < q_0/6 < q_0 < (N-1+\alpha)/(1-\alpha)$ then

$$\begin{aligned}&\|(S_{\epsilon,\epsilon'}^{1/2} - S_{\epsilon,\epsilon'}^{-1/2}) a_\epsilon^{1/2}\|_{L^s(\Omega_{\epsilon_0}, g_\epsilon)} \\ &\quad \leqslant |\Omega_{\epsilon_0} \backslash U_\epsilon|^{1/s-4/q_0} \|(S_{\epsilon,\epsilon'}^{1/2} - S_{\epsilon,\epsilon'}^{-1/2}) a_\epsilon^{1/2} g_\epsilon^{1/s}\|_{L^{q_0/4}(\Omega_{\epsilon_0} \backslash U_\epsilon)} \\ &\quad \leqslant c|\Omega_{\epsilon_0} \backslash U_\epsilon|^{1/s-4/q_0},\end{aligned}$$

and

$$\begin{aligned}&\|(S_{\epsilon,\epsilon'} - I) a_\epsilon^{1/2}\|_{L^s(\Omega_{\epsilon_0}, g_\epsilon)} \\ &\quad \leqslant |\Omega_{\epsilon_0} \backslash U_\epsilon|^{1/s-6/q_0} \|(S_{\epsilon,\epsilon'} - I) a_\epsilon^{1/2} g_\epsilon^{1/s}\|_{L^{q_0/6}(\Omega_{\epsilon_0} \backslash U_\epsilon)} \\ &\quad \leqslant c|\Omega_{\epsilon_0} \backslash U_\epsilon|^{1/s-6/q_0},\end{aligned}$$

for some constant $c > 0$. One can similarly estimate the other summands in (3.7) and obtain

$$\delta_s(\phi_\epsilon, \phi_{\epsilon'}) \leqslant c|\Omega_{\epsilon_0} \backslash U_\epsilon|^{1/s-6/q_0}. \tag{4.29}$$

uniformly in ϵ_0, ϵ and ϵ', for s and q_0 as above.

Step 3. Since $\alpha > 1 - N/15$, it is possible to choose $14 < q_0 < (N - 1 + \alpha)/(1-\alpha)$ which guarantees that $2q_0/(q_0 - 2) < q_0/6$; thus choosing $s = 2q_0/(q_0 - 2)$ in (4.29) it follows in particular that $\delta_s(\phi_\epsilon, \phi_{\epsilon'}) \to 0$ as $\epsilon \to 0$, uniformly in $\epsilon' \in (0, \epsilon)$. This combined with (4.26) implies that the sequence $(w_{\epsilon_0,\epsilon}^{-1} H_\epsilon w_{\epsilon_0,\epsilon} + 1)^{-k}$ is Cauchy in ϵ for $\epsilon \to 0$. Thus, by passing to the limit in (4.27) as $\epsilon' \to 0$ we obtain

$$\|(w_{\epsilon_0,0}^{-1} H_0 w_{\epsilon_0,0} + 1)^{-k} - (H_{\epsilon_0} + 1)^{-k}\|_{\mathcal{C}^2(L^2(\Omega_{\epsilon_0}))} \leqslant c\delta_{2q_0/(q_0-2)}(\phi_{\epsilon_0}, \phi_0). \tag{4.30}$$

Taking into account that $\phi_{\epsilon_0} = \mathrm{Id}$, using (4.28) and proceeding as in Step 2, we obtain

$$\delta_s(\phi_{\epsilon_0}, \phi_0) \leqslant c|\Omega_{\epsilon_0} \backslash \hat{\Omega}_0|^{1/s-4/q_0} \tag{4.31}$$

for all $1 < s < q_0/4 < q_0 < (N - 1 + \alpha)/(1 - \alpha)$. Since $1 - \epsilon_0 - h_0 \leqslant \epsilon_0 - |\bar{x}|^\alpha$ (cf. (4.17) with $\epsilon = 0$), we obtain $|\Omega_{\epsilon_0} \backslash \hat{\Omega}_0| \leqslant |\Omega \backslash \Omega_{\epsilon_0}|$. By (4.30), (4.31) and choosing $s = 2q_0/(q_0 - 2)$ it follows that

$$\|(w_{\epsilon_0,0}^{-1} H_0 w_{\epsilon_0,0} + 1)^{-k} - (H_{\epsilon_0} + 1)^{-k}\|_{\mathcal{C}^2(L^2(\Omega_{\epsilon_0}))} \leqslant c|\Omega \backslash \Omega_{\epsilon_0}|^{(q_0-10)/2q_0}. \tag{4.32}$$

In order to finish, it suffices to observe that $(q_0 - 10)/2q_0 \to 1/2 - 5(1 - \alpha)/(N - 1 + \alpha)$ as $q_0 \to (N - 1 + \alpha)/(1 - \alpha)$ and proceed as in the proof of Theorems 6 and 8. □

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