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Best constants for higher-order Rellich inequalities in $L^p(\Omega)$

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Abstract We obtain a series improvement to higher-order L^p-Rellich inequalities on a Riemannian manifold M. The improvement is shown to be sharp as each new term of the series is added.

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1 Introduction

In the article Davies and Hinz [7] proved higher-order L^p Rellich inequalities of the form

$$\int_{\mathbf{R}^N} \frac{|\Delta^m u|^p}{|x|^\gamma} \mathrm{d}x \geq A(2m,\gamma) \int_{\mathbf{R}^N} \frac{|u|^p}{|x|^{\gamma+2mp}} \mathrm{d}x, \tag{1}$$

and

$$\int_{\mathbf{R}^N} \frac{|\nabla\Delta^m u|^p}{|x|^\gamma} \mathrm{d}x \geq A(2m+1,\gamma) \int_{\mathbf{R}^N} \frac{|u|^p}{|x|^{\gamma+(2m+1)p}} \mathrm{d}x, \tag{2}$$

for all $u \in C_c^\infty(\mathbf{R}^N \setminus \{0\})$ with the sharp value for the constants $A(2m,\gamma)$ and $A(2m+1,\gamma)$. Their approach uses some integral inequalities involving $\Delta|x|^\sigma$ together with iteration, and is set initially in a Riemannian manifold context. One of the aims of the present paper is to improve inequalities (1) and (2) by adding a sharp non-negative term at the respective right-hand sides. In fact, this comes as a special – and most important – case of a more general theorem where instead of $|x|$ we have a distance function $\mathrm{d}(x) = \mathrm{dist}(x, K)$. Under a simple geometric assumption we establish an

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improved Rellich inequality of the form

$$\int_\Omega \frac{|\Delta^{m/2}u|^p}{d^\gamma}\,\mathrm{d}x \geq A(m,\gamma)\int_\Omega \frac{|u|^p}{d^{\gamma+2mp}}\,\mathrm{d}x + B(m,\gamma)\sum_{i=1}^{\infty}\int_\Omega V_i|u|^p\,\mathrm{d}x, \tag{3}$$

for all $u \in C_c^\infty(\Omega \setminus K)$, where at each step we have an optimal function $V_i(x)$ and a sharp constant $B(m,\gamma)$; see Theorem 2 for the precise statement. Here and below we interpret $|\Delta^{m/2}u|$ as $|\nabla\Delta^{(m-1)/2}u|$ when m is odd.

Improved versions of Hardy or Rellich inequlities have attracted considerable attention recently and especially for Hardy inequalities there is a substantial literature; see, e.g., [1,4,6,9,12] and references therein. The corresponding literature for Rellich inequalities is more restricted; see [3,5,8,9,11,12].

As was the case in [7], our results are formulated in a Riemannian manifold context, but we note that they are also new in the Euclidean case. We consider a Riemannian manifold M of dimension $N \geq 2$, a domain $\Omega \subset M$, a closed, piecewise smooth surface K of codimension k, $1 \leq k \leq N$, and the distance function $d(x) := \mathrm{dist}(x, K)$ which we assume to be bounded in Ω. We note that this last assumption is only needed for the improved inequality and not for the plain inequality where only the first term appears in the right-hand side of (3); to our knoweldge this is also new except in the case $M = \mathbf{R}^N$, $K = \{0\}$, studied in [7].

We define recursively

$$\begin{aligned} X_1(t) &= (1-\log t)^{-1}, \quad t \in (0,1], \\ X_i(t) &= X_1(X_{i-1}(t)), \quad i = 2,3,\ldots, \quad t \in (0,1]. \end{aligned} \tag{4}$$

These are iterated logarithmic functions that vanish at an increasingly slow rate at $t = 0$ and satisfy $X_i(1) = 1$.

Given $m \in \mathbf{N}$ and $\gamma \geq 0$ we also define

$$\begin{aligned} A'(m,\gamma) &= \prod_{i=0}^{[(m-1)/2]}\Big(\frac{k-\gamma-(m-2i)p}{p}\Big)^p, \\ A''(m,\gamma) &= \prod_{j=1}^{[m/2]}\Big(\frac{pk-k+\gamma+(m-2j)p}{p}\Big)^p, \\ A(m,\gamma) &= A'(m,\gamma)A''(m,\gamma), \\ B(m,\gamma) &= \frac{p-1}{2p}A(m,\gamma)\Big\{\sum_{i=0}^{[(m-1)/2]}\Big(\frac{k-\gamma-(m-2i)p}{p}\Big)^{-2} \\ &\qquad + \sum_{j=1}^{[m/2]}\Big(\frac{pk-k+\gamma+(m-2j)p}{p}\Big)^{-2}\Big\}. \end{aligned} \tag{5}$$

Concerning the above definitions, we adopt the convention that empty sums are equal to zero and empty products are equal to one; this of course refers to the sum or product over j when $m = 1$. To state our first theorem we introduce the following technical hypothesis:

$$\left(\begin{array}{ll} \gamma \neq \frac{3pk-8p^2-2k+6p}{4p-2}, & \text{if } m \text{ is even} \\ \gamma + p \neq \frac{3pk-8p^2-2k+6p}{4p-2}, & \text{if } m \text{ is odd, } m \geq 3 \end{array}\right) \quad \text{or} \quad p > \frac{13+\sqrt{105}}{4} \qquad (*).$$

We then have

Theorem 1 (Improved Rellich inequality) *Let $m \in N$ and assume that $d(\cdot)$ is bounded in Ω. Let $\gamma \geq 0$ be such that $k - \gamma - mp > 0$ and suppose that $(*)$ is satisfied. Assume moreover that*

$$d\Delta d - k + 1 \geq 0, \quad in\ \Omega \setminus K \tag{6}$$

in the distributional sense. Then there exists a $D_0 \geq \sup_{x\in\Omega} d(x)$ such that for any $D \geq D_0$ there holds

$$\int_\Omega d^{-\gamma}|\Delta^{m/2}u|^p \mathrm{d}x \geq A(m,\gamma)\int_\Omega d^{-\gamma-mp}|u|^p \mathrm{d}x \\ +B(m,\gamma)\sum_{i=1}^{\infty}\int_\Omega \mathrm{d}^{-\gamma-mp}X_1^2X_2^2\cdots X_i^2|u|^p\mathrm{d}x, \tag{7}$$

for all $u \in C_c^\infty(\Omega \setminus K)$, where $X_j = X_j(d(x)/D)$.

We present some examples where the geometric condition (6) is satisfied:

Example 1 Suppose that $M = \mathbf{R}^N$ and that K is affine. Then (6) is satisfied as an equality (this includes the case where K consists of a single point).

Example 2 Suppose that M is a Cartan–Hadamard manifold, that is, a simply connected geodesically complete non-compact manifold with non-positive sectional curvature. If $K = \{x_0\}$ (some point in M) then (6) is satisfied; see [10].

Example 3 Suppose $M = M_1 \times M_2$ where M_1 is a Cartan–Hadamard manifold of dimension k. If $K = \{x_0\} \times M_2$ for some $x_0 \in M_1$, then, in an obvious notation, $d(x,y)(\Delta d)(x,y) = d(x,y)(\Delta d_1)(x) \geq d_1(x)(\Delta d_1)(x) \geq k-1$, so (6) is satisfied for any $\Omega \subset M$.

Concerning the important case $M = \mathbf{R}^N$, $K = \partial\Omega$, condition (6) is satisfied if Ω is the complement of a convex domain. This however is excluded from our theorem due to the assumptions $k - \gamma - mp > 0$, $\gamma \geq 0$;[1] on the other hand, these conditions are not needed for Theorem 2 below. It should be noted that Rellich inequalities involving $\mathrm{dist}(x, \partial\Omega)$ present surprising difficulties when $p \neq 2$. In particular, it is not known whether the inequality $\int_\Omega |\Delta u|^p \mathrm{d}x \geq \{(p-1)(2p-1)/p^2\}^p \int_\Omega |u|^p \mathrm{d}^{-2p}\mathrm{d}x$ is valid when Ω is bounded and convex with a smooth boundary; see also [12, Chapter 2] for results in this direction.

In our second theorem we prove the optimality of the constants and exponents of Theorem 1. This is quite technical and shall be established only in the case where $M = \mathbf{R}^N$ and K is affine (or, indeed, has an affine part, since the argument is local); we believe that extra effort should yield the result in the general case, but we have not pursued this. This would require in particular estimates on the behavior of higher-order derivatives of $d(x)$ near K; see [2, Theorem 3.2].

[1] We note here that the condition $\gamma \geq 0$ of Theorem 1 can be weakened to $pk - k + \gamma > 0$. We have not persued this since our main goal is to obtain inequalities whose left-hand sides do not involve any weight (and this only requires $\gamma \geq 0$; see the proof of Theorem 1 below). We note however that if negative γ were allowed, then condition $(*)$ would need to be strengthened; see the comments in the beginning of the proof of Theorem 1.

Theorem 2 (Optimality) *Let $\Omega \subset \mathbf{R}^N$ and let K be an affine hypersurface of codimension $k \in \{1,\ldots,N\}$ such that $K \cap \Omega \neq \emptyset$. Assume that for some $\gamma \in \mathbf{R}$, $D \geq \sup_\Omega d(x)$, $r \geq 1$ and some $\theta \in \mathbf{R}$, $C > 0$ there holds*

$$\int_\Omega d^{-\gamma}|\Delta^{m/2}u|^p \mathrm{d}x \geq |A(m,\gamma)| \int_\Omega d^{-\gamma-mp}|u|^p \mathrm{d}x$$

$$+|B(m,\gamma)|\sum_{i=1}^{r-1}\int_\Omega d^{-\gamma-mp}X_1^2X_2^2\cdots X_i^2|u|^p\mathrm{d}x + C\int_\Omega d^{-\gamma-mp}X_1^2X_2^2\cdots X_{r-1}^2X_r^\theta|u|^p\mathrm{d}x,$$

for all $u \in C_c^\infty(\Omega)$, where $X_j = X_j(d(x)/D)$. Then

(i) $\theta \geq 2$,
(ii) *if* $\theta = 2$ *then* $C \leq |B(m,\gamma)|$.

We note that $X_j(d(x)/D_1)/X_j(d(x)/D_2) \to 1$ as $d(x) \to 0$, for any $D_1, D_2 \geq \sup_\Omega d$ and in this sense the precise value of D_0 in Theorem 1 does not affect the optimality of the theorem. We also note that by a standard argument, if $k-\gamma-mp > 0$ then the validity of (7) for all $u \in C_c^\infty(\Omega \setminus K)$ implies its validity for all $u \in C_c^\infty(\Omega)$.

The proof of Theorem 1 is given in Sect. 2 and uses some of the ideas of [3] and, in particular, induction on m. However, it is more technical due to $p \neq 2$ and the extra parameter k; moreover, the proof in [3] uses one-dimensional arguments and depends on the Euclidean structure. The proof of Theorem 2 is given in Sect. 3 and uses an appropriately chosen minimising sequence.

2 The Rellich inequality

Throughout the paper we shall repeatedly use the differentiation rule

$$\frac{\mathrm{d}}{\mathrm{d}t}X_i^\beta(t) = \frac{\beta}{t}X_1(t)X_2(t)\cdots X_{i-1}(t)X_i^{\beta+1}(t), \quad i = 1,2,\ldots, \quad \beta \in \mathbf{R}, \tag{8}$$

which is easily proved by induction on $i \in \mathbf{N}$. Let us define the functions

$$\eta(t) = \sum_{i=1}^\infty X_1X_2\cdots X_i, \quad \zeta(t) = \sum_{i=1}^\infty X_1^2X_2^2\cdots X_i^2,$$

$$\theta(t) = \sum_{i=1}^\infty\sum_{j=1}^i X_1^3\cdots X_j^3X_{j+1}^2\cdots X_i^2,$$

(see [3] for a detailed discussion of the convergence of these series). It follows from (8) that

$$\eta'(t) = \frac{\eta^2(t)+\zeta(t)}{2t}, \quad \zeta'(t) = \frac{2\theta(t)}{t}, \ t \in (0,1). \tag{9}$$

Note. In the sequel we shall use the symbol X_i for $X_i(t)$, $t \in (0,1]$, for $X_i(d(x)/D)$, $x \in \Omega$, $D \geq \sup_\Omega d$, and also for $X_i(t/D)$, $t \in (0, \sup_\Omega d)$, $D \geq \sup_\Omega d$. It will always be made explicit which meaning is intended. The same also holds for the functions η, ζ and θ.

For the sake of simplicity we work with real-valued functions, noting that with minor modifications the proofs also work in the complex case. As in [7], the proof of Theorem 1 uses iteration and for this we shall need to consider first the case $m = 2$. The following proposition has been obtained in [5] for $\gamma = \mu = 0$. We set

$$Q = \frac{(k-\gamma-2p)(pk-k+\gamma)}{p^2},$$

Proposition 3 *Let $\gamma, \mu \geq 0$ be given and assume that $k-\gamma-2p > 0$. Suppose that*

$$d\Delta d - k + 1 \geq 0, \quad \text{in } \Omega \setminus K \tag{10}$$

in the distributional sense and assume also that $(4p-2)\gamma \neq 3pk - 8p^2 - 2k + 6p$ or $p > (13+\sqrt{105})/4$. Then there exists $D_0 \geq \sup_\Omega d(x)$ such that for all $D \geq D_0$ and all $\mu \geq 0$ there holds

$$\int_\Omega d^{-\gamma}(1+\mu\zeta)|\Delta u|^p \mathrm{d}x \geq Q^p \int_\Omega d^{-\gamma-2p}|u|^p \mathrm{d}x \\ + \left(\frac{p-1}{2p} Q^p \left\{ \left(\frac{k-\gamma-2p}{p}\right)^{-2} + \left(\frac{pk-k+\gamma}{p}\right)^{-2} \right\} + Q^p \mu \right) \int_\Omega d^{-\gamma-2p} \zeta |u|^p \mathrm{d}x, \tag{11}$$

for all $u \in C_c^\infty(\Omega \setminus K)$; here $\zeta = \zeta(d(x)/D)$.

Proof Let $u \in C_c^\infty(\Omega \setminus K)$ be fixed. For a positive, locally bounded function ϕ with $|\nabla \phi| \in L^1_{\mathrm{loc}}(\Omega \setminus K)$ we have

$$\begin{aligned} -\int_\Omega \Delta\phi |u|^p \mathrm{d}x &= p\int_\Omega \nabla\phi \cdot (|u|^{p-2}u\nabla u)\mathrm{d}x \\ &= -p\int_\Omega \phi|u|^{p-2}u\Delta u\, \mathrm{d}x - p(p-1)\int_\Omega \phi|u|^{p-2}|\nabla u|^2 \mathrm{d}x \\ &\leq p\left(\frac{p-1}{p}\int_\Omega d^{\frac{\gamma}{p-1}}(1+\mu\zeta)^{-\frac{1}{p-1}}\phi^{\frac{p}{p-1}}|u|^p \mathrm{d}x \right. \\ &\quad \left. + \frac{1}{p}\int_\Omega d^{-\gamma}(1+\mu\zeta)|\Delta u|^p \mathrm{d}x \right) - p(p-1)\int_\Omega \phi|u|^{p-2}|\nabla u|^2 \mathrm{d}x, \end{aligned}$$

from which follows that

$$\int_\Omega d^{-\gamma}(1+\mu\zeta)|\Delta u|^p \mathrm{d}x \geq T_1 + T_2 + T_3, \tag{12}$$

where

$$T_1 = p(p-1)\int_\Omega \phi |u|^{p-2}|\nabla u|^2 \mathrm{d}x\,,$$

$$T_2 = -\int_\Omega \Delta\phi |u|^p \mathrm{d}x\,,$$

$$T_3 = -(p-1)\int_\Omega d^{\frac{\gamma}{p-1}}(1+\mu\zeta)^{-\frac{1}{p-1}}\phi^{\frac{p}{p-1}}|u|^p \mathrm{d}x\,.$$

We next choose $\phi = \lambda d^{-\gamma-2p+2}(1+\alpha\eta+\beta\eta^2)$ where $\lambda > 0$ and $\alpha,\beta \in \mathbf{R}$ are to be determined and $\eta = \eta(d(x)/D)$ with $D \geq \sup_\Omega d$ also yet to be determined. To estimate T_1 we set $v = |u|^{p/2}$ and apply [5, Theorem 1] obtaining

$$\begin{aligned} T_1 &= \frac{4(p-1)\lambda}{p}\int_\Omega d^{-\gamma-2p+2}(1+\alpha\eta+\beta\eta^2)|\nabla v|^2 \mathrm{d}x \\ &\geq \frac{4(p-1)\lambda}{p}\int_\Omega d^{-\gamma-2p}\Big\{\frac{(k-\gamma-2p)^2}{4}+\frac{(k-\gamma-2p)^2\alpha}{4}\eta \\ &\quad +\Big(\frac{(k-\gamma-2p)\alpha}{4}+\frac{(k-\gamma-2p)^2\beta}{4}\Big)\eta^2 \\ &\quad +\Big(\frac{1}{4}+\frac{(k-\gamma-2p)\alpha}{4}\Big)\zeta\Big\}|u|^p \mathrm{d}x. \end{aligned} \tag{13}$$

To estimate T_2 we define $f(t) = \lambda t^{-\gamma-2p+2}(1+\alpha\eta(t/D)+\beta\eta^2(t/D))$, $t \in (0, \sup_\Omega d)$, so that $\phi(x) = f(d(x))$. We then have

$$\begin{aligned} f'(t) = \lambda t^{-\gamma-2p+1}\Big\{&(-\gamma-2p+2)+(-\gamma-2p+2)\alpha\eta+\Big[\frac{\alpha}{2}+(-\gamma-2p+2)\beta\Big]\eta^2 \\ &+\frac{\alpha}{2}\zeta+\beta\eta^3+\beta\eta\zeta\Big\} \end{aligned} \tag{14}$$

and

$$\begin{aligned} f''(t) = \lambda t^{-\gamma-2p}\Big\{&(\gamma+2p-1)(\gamma+2p-2)+(\gamma+2p-1)(\gamma+2p-2)\alpha\eta \\ &+\Big[-\frac{(2\gamma+4p-3)\alpha}{2}+(\gamma+2p-1)(\gamma+2p-2)\beta\Big]\eta^2 \\ &+\Big[-\frac{(2\gamma+4p-3)\alpha}{2}\Big]\zeta+\Big[\frac{\alpha}{2}-(2\gamma+4p-3)\beta\Big](\eta^3+\eta\zeta)+\alpha\theta\Big\} \end{aligned} \tag{15}$$

(where the argument of η, ζ and θ in (14) and (15) is t/D). Since $f'(t) \leq 0$ for large D, we have from (10)

$$-\Delta\phi = -f''(d) - f'(d)\Delta d \geq -f''(d) - \frac{k-1}{d}f'(d), \tag{16}$$

in the distributional sense in $\Omega \setminus K$. Combining (14), (15) and (16) we conclude that

$$
\begin{aligned}
T_2 \geq \lambda \int_\Omega d^{-\gamma-2p} \Big\{ & (k-\gamma-2p)(\gamma+2p-2) + (k-\gamma-2p)(\gamma+2p-2)\alpha\eta \\
& + \left(\frac{2\gamma+4p-k-2}{2}\alpha + (k-\gamma-2p)(\gamma+2p-2)\beta\right)\eta^2 \\
& + \left(\frac{2\gamma+4p-k-2}{2}\alpha\right)\zeta + \left((2\gamma+4p-k-2)\beta - \frac{\alpha}{2}\right) \\
& \times(\eta^3+\eta\zeta) - \alpha\theta + O(\eta^4)\Big\} \mathrm{d}x.
\end{aligned} \tag{17}
$$

As for T_3, we use Taylor's theorem to obtain after some simple calculations

$$
\begin{aligned}
&(1+\mu\zeta)^{-\frac{1}{p-1}}\phi^{\frac{p}{p-1}} \\
&\quad = \lambda^{\frac{p}{p-1}} d^{-\frac{(\gamma+2p-2)p}{p-1}} \left\{ 1 + \frac{p\alpha}{p-1}\eta + \left(\frac{p\beta}{p-1} + \frac{p\alpha^2}{2(p-1)^2}\right)\eta^2 \right. \\
&\quad \left. -\frac{\mu}{p-1}\zeta + \left(\frac{p\alpha\beta}{(p-1)^2} - \frac{p(p-2)\alpha^3}{6(p-1)^3}\right)\eta^3 - \frac{p\alpha\mu}{(p-1)^2}\eta\zeta + O(\eta^4) \right\},
\end{aligned}
$$

and thus conclude that

$$
\begin{aligned}
T_3 = -(p-1)\lambda^{\frac{p}{p-1}} \int_\Omega d^{-\gamma-2p} \Big\{ & 1 + \frac{p\alpha}{p-1}\eta + \left(\frac{p\beta}{p-1} + \frac{p\alpha^2}{2(p-1)^2}\right)\eta^2 - \frac{\mu}{p-1}\zeta \\
& + \left(\frac{p\alpha\beta}{(p-1)^2} - \frac{p(p-2)\alpha^3}{6(p-1)^3}\right)\eta^3 - \frac{p\alpha\mu}{(p-1)^2}\eta\zeta + O(\eta^4)\Big\} \mathrm{d}x.
\end{aligned} \tag{18}
$$

Using (13), (17) and (18) we arrive at

$$
\int_\Omega d^{-\gamma}(1+\mu\zeta)|\Delta u|^p \mathrm{d}x \geq \int_\Omega d^{-\gamma-2p} V |u|^p \mathrm{d}x \tag{19}
$$

where the function V has the form

$$
V = r_0 + r_1\eta + r_2\eta^2 + r_2'\zeta + r_3\eta^3 + r_3'\eta\zeta + r_3''\theta + O(\eta^4).
$$

We compute the coefficients r_i, r_i', r_i'' by adding the respective coefficients from (13), (17) and (18). We find

$$\begin{aligned}
r_0 &= \frac{(k-\gamma-2p)(pk-k+\gamma)}{p}\lambda - (p-1)\lambda^{\frac{p}{p-1}},\\
r_1 &= \frac{(k-\gamma-2p)(pk-k+\gamma)}{p}\lambda\alpha - p\lambda^{\frac{p}{p-1}}\alpha,\\
r_2 &= \frac{pk+2p-2k+2\gamma}{2p}\alpha\lambda + \frac{(k-\gamma-2p)(pk-k+\gamma)}{p}\beta\lambda\\
&\quad -(p-1)\Big(\frac{p\beta}{p-1} + \frac{p\alpha^2}{2(p-1)^2}\Big)\lambda^{\frac{p}{p-1}}, \qquad (20)\\
r_2' &= \Big(\frac{p-1}{p} + \frac{pk-2k+2p+2\gamma}{2p}\alpha\Big)\lambda + \mu\lambda^{\frac{p}{p-1}},\\
r_3 &= \lambda\Big(-\frac{\alpha}{2} + (2\gamma+4p-k-2)\beta\Big) - (p-1)\lambda^{\frac{p}{p-1}}\Big(\frac{p\alpha\beta}{(p-1)^2} - \frac{p(p-2)\alpha^3}{6(p-1)^3}\Big),\\
r_3' &= \lambda\Big(-\frac{\alpha}{2} + (2\gamma+4p-k-2)\beta\Big) + \frac{p\alpha\mu}{p-1}\lambda^{\frac{p}{p-1}},\\
r_3'' &= -\lambda\alpha.
\end{aligned}$$

We now proceed to specify λ, α and β. We choose λ so as to optimize r_0, which yields

$$\lambda = Q^{p-1}, \quad r_0 = Q^p.$$

Then $r_1 = 0$ irrespective of the choice of α and β. We subsequently choose

$$\alpha = \frac{(p-1)(pk-2k+2p+2\gamma)}{(k-\gamma-2p)(pk-k+\gamma)},$$

which yields $r_2 = 0$ and $r_2' = B(2,\gamma) + \mu A(2,\gamma)$. Hence it remains to show that β can be chosen so that for large enough D there holds

$$r_3\eta^3 + r_3'\eta\zeta + r_3''\theta + O(\eta^4) \geq 0 \quad \text{in } \Omega. \qquad (21)$$

This is done in the following lemma and this is where condition $(*)$ is needed. □

Lemma 4 *If $\gamma \neq (3pk-8p^2-2k+6p)/(4p-2)$ or $p > (13+\sqrt{105})/4$ then there exists $\beta \in \mathbf{R}$ such that for large enough D there holds*

$$r_3\eta^3 + r_3'\eta\zeta + r_3''\theta + O(\eta^4) \geq 0 \quad \text{in } \Omega \qquad (22)$$

(here $\eta = \eta(d(x)/D)$, and similarly for ζ and θ).

Proof We claim that it is enough to find $\beta \in \mathbf{R}$ such that for large enough D we have

$$r_3 + r_3' + r_3'' > 0. \qquad (23)$$

Indeed, the fact that

$$\lim_{t\to 0+}\frac{\eta^3(t)}{X_1^3(t)} = \lim_{t\to 0+}\frac{\eta(t)\zeta(t)}{X_1^3(t)} = \lim_{t\to 0+}\frac{\theta(t)}{X_1^3(t)} = 1,$$

implies that

$$r_3\eta^3 + r_3'\eta\zeta + r_3''\theta + O(\eta^4) = \Big(r_3 + r_3'(1+o(1)) + r_3''(1+o(1))\Big)\eta^3 + O(\eta^4)$$

where $\lim o(1) = 0$ as $D \to +\infty$, uniformly in $x \in \Omega$; hence (22) follows.

To prove (23) we calculate r_3, r_3' and r_3''; from (20) we obtain

$$
\begin{aligned}
r_3 &= \Big(-\frac{\alpha Q^{p-1}}{2} + \frac{p(p-2)\alpha^3 Q^p}{6(p-1)^2} \Big) - R\beta, \\
r_3' &= \Big(-\frac{\alpha Q^{p-1}}{2} + \frac{p\mu\alpha Q^p}{p-1} \Big) + \Big(-R + \frac{p\alpha Q^p}{p-1} \Big)\beta, \\
r_3'' &= -\alpha Q^{p-1},
\end{aligned}
$$

where $R = 2(p-1)(k-\gamma-2p)Q^{p-1}/p$. We distinguish two cases.

(i) $\gamma \neq (3pk - 8p^2 - 2k + 6p)/(4p-2)$. We then observe that the coefficient of β in $r_3 + r_3' + r_3''$ is non-zero. Hence (23) is satisfied if β is either large and negative or large and positive.

(ii) $\gamma = (3pk - 8p^2 - 2k + 6p)/(4p-2)$. We then choose $\beta = 0$ and we have

$$
Q = \frac{(k-2)^2(4p-3)}{4(2p-1)^2}, \quad \alpha Q = \frac{2(k-2)(p-1)^2}{p(2p-1)},
$$

from which follows that

$$
r_3 + r_3' + r_3'' = \frac{2(2p-1)\alpha Q^{p-1}}{3p(4p-3)}(2p^2 - 13p + 8).
$$

Since $\alpha > 0$ in this case, this is positive as $(13 + \sqrt{105})/4$ is the largest root of the polynomial $2p^2 - 13p + 8$. □

Note. Using $\phi = \lambda d^{-\gamma-2p+2}(1 + \alpha\eta + \beta\eta^2 + \beta_1\zeta)$ in order to remove $(*)$ does not work, as the coefficient of β_1 in $r_3 + r_3' + r_3''$ turns out to be zero when the corresponding coefficient of β is zero.

Lemma 5 *Let $m \in \mathbf{N}$ and $\gamma \geq 0$. Then:*

(i) *If m is even then*
   (a) $A(m,\gamma) = A(2,\gamma)A(m-2,\gamma+2p)$,
   (b) $B(m,\gamma) = A(2,\gamma)B(m-2,\gamma+2p) + A(m-2,\gamma+2p)B(2,\gamma)$.

(ii) *If m is odd then*
   (a) $A(m,\gamma) = A(1,\gamma)A(m-1,\gamma+p)$,
   (b) $B(m,\gamma) = A(1,\gamma)B(m-1,\gamma+p) + A(m-1,\gamma+p)B(1,\gamma)$.

Proof We shall only prove (i)(b), the other cases being simpler or similar. So let us assume that $m = 2r$, $r \in \mathbf{N}$. Then

$$\begin{aligned}
&A(2,\gamma)B(2r-2,\gamma+2p) + A(2r-2,\gamma+2p)B(2,\gamma)\\
&= \Big(\frac{k-\gamma-2p}{p}\Big)^p \Big(\frac{pk-k+\gamma}{p}\Big)^p \frac{p-1}{2p}\\
&\quad\times \prod_{i=0}^{r-2}\prod_{j=1}^{r-1} \Big(\frac{k-\gamma-(2r-2i)p}{p}\Big)^p \Big(\frac{pk-k+\gamma+(2r-2j)p}{p}\Big)^p\\
&\quad\times \Big\{ \sum_{i=0}^{r-2} \Big(\frac{k-\gamma-(2r-2i)p}{p}\Big)^{-2} + \sum_{j=1}^{r-1} \Big(\frac{pk-k+\gamma+(2r-2j)p}{p}\Big)^{-2} \Big\}\\
&\quad+ \prod_{i=0}^{r-2}\prod_{j=1}^{r-1} \Big(\frac{k-\gamma-(2r-2i)p}{p}\Big)^p \Big(\frac{pk-k+\gamma+(2r-2j)p}{p}\Big)^p\\
&\quad\times \frac{p-1}{2p} \Big(\frac{k-\gamma-2p}{p}\Big)^p \Big(\frac{pk-k+\gamma}{p}\Big)^p \Big\{ \Big(\frac{k-\gamma-2p}{p}\Big)^{-2} + \Big(\frac{pk-k+\gamma}{p}\Big)^{-2} \Big\}\\
&= \frac{p-1}{2p} \prod_{i=0}^{r-1}\prod_{j=1}^{r} \Big(\frac{k-\gamma-(2r-2i)p}{p}\Big)^p \Big(\frac{pk-k+\gamma+(2r-2j)p}{p}\Big)^p\\
&\quad\times \Big\{ \sum_{i=0}^{r-1} \Big(\frac{k-\gamma-(2r-2i)p}{p}\Big)^{-2} + \sum_{j=1}^{r} \Big(\frac{pk-k+\gamma+(2r-2j)p}{p}\Big)^{-2} \Big\}\\
&= A(2r,\gamma),
\end{aligned}$$

as claimed. □

Proof of Theorem 1 Before proceeding with the proof, let us make a comment on its assumptions. The proof essentially uses iteration. For example, if m is even, then we repeatedly use Proposition 3 obtaining

$$\int_\Omega \frac{|\Delta^{m/2}u|^p}{\mathrm{d}^\gamma}\mathrm{d}x \ge \int_\Omega (a_1+b_1\zeta)\frac{|\Delta^{(m-2)/2}u|^p}{d^{\gamma+2p}}dx \ge \int_\Omega (a_2+b_2\zeta)\frac{|\Delta^{(m-4)/2}u|^p}{d^{\gamma+4p}}\mathrm{d}x \ge \dots,$$

etc. Hence at the $(i+1)$th step, $0 \le i \le (m-2)/2$, we estimate the integral $\int_\Omega (a_i + b_i\zeta)d^{-(\gamma+2ip)}|\Delta^{(m-2i)/2}u|^p dx$. In applying Proposition 3, we verify that (i) $k-(\gamma+2ip)-2p > 0$ (this is satisfied since $k-\gamma-mp > 0$) and (ii) if $p \le (13+\sqrt{105})/4$, then $\gamma + 2ip \ne (3pk-8p^2-2k+6p)/(4p-2)$. This is indeed the case by the assumption of the theorem since $\gamma + jp > 3pk - 8p^2 - 2k + 6p)/(4p-2)$ for any $j \ge 2$ (recall that $\gamma \ge 0$).

We now come to the details of the proof. We shall use induction on $[(m+1)/2]$. If $[(m+1)/2] = 1$, that is $m = 1$ or $m = 2$, then (7) follows from [5, Theorem 1] or Proposition 3 respectively. We assume that the statement of the theorem is valid for $[(m+1)/2] \in \{1, 2, \dots, r-1\}$ and consider the case $[(m+1)/2] = r$. For this we distinguish two cases, depending on whether m is even or odd.

(i) m even. We first use Proposition 3 and then the induction hypothesis (and for this we note that the assumption $k-\gamma-mp > 0$ implies both $k-\gamma-2p > 0$ and

$k-(\gamma+2p)-(m-2)p>0$). We have

$$\begin{aligned}
&\int_\Omega d^{-\gamma}(1+\mu\zeta)|\Delta^{m/2}u|^p\mathrm{d}x\\
&\quad\geq A(2,\gamma)\int_\Omega d^{-\gamma-2p}|\Delta^{\frac{m-2}{2}}u|^p\mathrm{d}x+[B(2,\gamma)+A(2,\gamma)\mu]\int_\Omega d^{-\gamma-2p}\zeta|\Delta^{\frac{m-2}{2}}u|^p\mathrm{d}x\\
&\quad\geq A(2,\gamma)\left\{A(m-2,\gamma+2p)\int_\Omega d^{-\gamma-mp}|u|^p\mathrm{d}x+B(m-2,\gamma+2p)\int_\Omega d^{-\gamma-mp}\zeta|u|^p\mathrm{d}x\right\}\\
&\qquad+[B(2,\gamma)+A(2,\gamma)\mu]A(m-2,\gamma+2p)\int_\Omega d^{-\gamma-mp}\zeta|u|^p\mathrm{d}x\\
&\quad=A(2,\gamma)A(m-2,\gamma+2p)\int_\Omega d^{-\gamma-mp}|u|^p\mathrm{d}x+\\
&\qquad+\Big\{\Big[A(2,\gamma)B(m-2,\gamma+2p)+A(m-2,\gamma+2p)B(2,\gamma)\Big]\\
&\qquad+A(2,\gamma)A(m-2,\gamma+2p)\mu\Big\}\int_\Omega d^{-\gamma-mp}\zeta|u|^p\mathrm{d}x,
\end{aligned}$$

and the proof is complete if we recall Lemma 5.
(ii) m odd. The proof is similar, the only difference being that we use [5, Theorem 1] instead of Proposition 3. We omit the details. □

Remark 6 We point out that in the proofs of Proposition 3 and Theorem 1 we did not use at any point the assumption that k is the codimension of the set K. Indeed, a careful look at the two proofs shows that K can be any closed set such that $\mathrm{dist}(x,K)$ is bounded in Ω and for which the inequality $d\Delta d-k+1\geq 0$ is satisfied in $\Omega\setminus K$; the proof does not even require k to be an integer. Of course, the natural realization of this assumption is that K is smooth and $k=\mathrm{codim}(K)$.

Let us define the *inradius of* Ω *relative to* K by $\mathrm{Inr}(\Omega;K)=\sup_\Omega d(x)$. Looking at the proof of Theorem 1 we see that when D is chosen large enough, the actual requirement is that $d(x)/D$ is small uniformly in $x\in\Omega$. This, combined with the fact that $t^{-\gamma-mp}X_1^2\ldots X_i^2(t)$ has a positive minimum in $(0,1)$, leads to the following corollary of Theorem 1:

Corollary 7 *Under the conditions of Theorem 1 for any* $r\geq 0$ *there exists a constant* $c=c(m,p,k,r)>0$ *such that*

$$\begin{aligned}
&\int_\Omega d^{-\gamma}|\Delta^{m/2}u|^p\mathrm{d}x\geq A(m,\gamma)\int_\Omega d^{-\gamma-mp}|u|^p\mathrm{d}x \qquad (24)\\
&\quad+B(m,\gamma)\sum_{i=1}^r\int_\Omega d^{-\gamma-mp}X_1^2X_2^2\cdots X_i^2|u|^p\mathrm{d}x+c\,\mathrm{Inr}(\Omega;K)^{-\gamma-mp}\int_\Omega|u|^p\mathrm{d}x,
\end{aligned}$$

for all $u\in C_c^\infty(\Omega\setminus K)$.

We end this section with a proposition about the case where condition $(*)$ is not satisfied.

Proposition 8 *Suppose that all conditions of Theorem 1 except* $(*)$ *are satisfied. Then*

$$\int_\Omega d^{-\gamma}|\Delta^{m/2}u|^p \mathrm{d}x \geq A(m,\gamma)\int_\Omega d^{-\gamma-mp}|u|^p \mathrm{d}x + c_\epsilon \int_\Omega d^{-\gamma-mp+\epsilon}|u|^p \mathrm{d}x, \qquad (25)$$

for any $\epsilon > 0$ *and all* $u \in C_c^\infty(\Omega \setminus K)$.

Proof We only give a sketch of the proof. Suppose first that $m = 2$. We use (12), but this time with $\phi = \lambda d^{-\gamma-2p+2}(1+\mu d^\epsilon)$; here μ is to be determined and $\lambda = Q^{p-1}$. Arguing as in the proof of Proposition 3 we obtain

$$\int_\Omega \mathrm{d}^{-\gamma}|\Delta^{m/2}u|^p \mathrm{d}x \geq A(m,\gamma)\int_\Omega d^{-\gamma-mp}|u|^p \mathrm{d}x + \int_\Omega \tilde{V} d^{-\gamma-2p}|u|^p \mathrm{d}x, \qquad (26)$$

where

$$\tilde{V}(x) = \frac{\lambda\mu}{p}(p-1)(k-\gamma-2p+\epsilon)^2 d^\epsilon + \lambda\mu(k-\gamma-2p+\epsilon)(\gamma+2p-2-\epsilon)d^\epsilon$$
$$+(p-1)\lambda^{\frac{p}{p-1}} - (p-1)\lambda^{\frac{p}{p-1}}(1+\mu d^\epsilon)^{\frac{p}{p-1}};$$

here we have added and subtracted $(p-1)\lambda^{\frac{p}{p-1}}$ in order to create the first term in the right-hand side of (26). Using Taylor's theorem we obtain

$$\tilde{V}(x) = (c_1\lambda\mu\epsilon + O(\epsilon^2))d^\epsilon + O(d^{2\epsilon}),$$

where $c_1 = (pk - 2k + 2\gamma + 2p)/p$. The fact that $(*)$ is violated implies that $c_1 \neq 0$, and choosing μ so that $c_1\mu > 0$ completes the proof when $m = 2$. Iteration yields the result in the general case when m is even. The case where m is odd is treated similarly. □

3 Optimality of the constants

In this section we present the proof of Theorem 2. Hence we assume throughout that Ω is domain in $\mathbf{R}^N$ and K is an affine hypersurface of codimension $k \in \{1, 2, \ldots, N\}$.

For the sake of simplicity we shall only consider the special case $\gamma = 0$, the proof in the general case presenting no difference whatsoever other than the additional dependence of some constants on γ. Also, for the sake of bravity we shall prove the theorem only for m even, the proof when m is odd being similar.

Hence, writing $A(2m)$ and $B(2m)$ for $A(2m, 0)$ and $B(2m, 0)$ respectively, we intend to look closely at

$$I_{2m,r-1}[u] := \int_\Omega |\Delta^m u|^p \mathrm{d}x - |A(2m)| \int_\Omega \frac{|u|^p}{d^{2mp}} \mathrm{d}x - |B(2m)| \sum_{i=1}^{r-1} \int_\Omega \frac{|u|^p}{d^{2mp}} X_1^2 \cdots X_i^2 \mathrm{d}x,$$

for particular test functions u; here and below, $X_j = X_j(d(x)/D)$ for some fixed $D \geq \sup_\Omega d(x)$. We begin by defining the polynomial

$$\alpha_m(s) = \prod_{i=0}^{m-1}(s-2i)\prod_{j=1}^{m}(s+k-2j)\,, \quad s \in \mathbf{R}\,,$$

which will play an important role in the sequel.

Lemma 9 *There holds*

$$\text{(i)} \quad |A(2m)| = |\alpha_m|^p \Big|_{s=\frac{2mp-k}{p}}, \tag{27}$$

$$\text{(ii)} \quad |B(2m)| = \frac{p-1}{2p}|\alpha_m|^{p-2}(\alpha_m'^2 - \alpha_m\alpha_m'')\Big|_{s=\frac{2mp-k}{p}}. \tag{28}$$

Proof Part (i) is easily verified. From the relation

$$\frac{\alpha_m'}{\alpha_m} = \sum_{i=0}^{m-1}(s-2i)^{-1} + \sum_{j=1}^{m}(s+k-2j)^{-1},$$

we obtain

$$\alpha_m^{-2}(\alpha_m'^2 - \alpha_m\alpha_m'') = -\Big(\frac{\alpha_m'}{\alpha_m}\Big)' = \sum_{i=0}^{m-1}(s-2i)^{-2} + \sum_{j=1}^{m}(s+k-2j)^{-2},$$

and (ii) follows. □

Let $s_0 > (2mp-k)/p$ and $s_1, \dots, s_r \in \mathbf{R}$ be fixed parameters. For $0 \le i \le j \le r$ we define

$$Y_{ij} = X_1^2 \cdots X_i^2 X_{i+1} \cdots X_j,$$

with the natural interpretations $Y_{00} = 1$, $Y_{ii} = X_1^2 \cdots X_i^2$, $Y_{0j} = X_1 \cdots X_j$. We then define the integrals

$$\begin{aligned}\Gamma_{ij} &= \int_\Omega d^{(s_0-2m)p} X_1^{ps_1} \cdots X_r^{ps_r} Y_{ij}\mathrm{d}x \\ &= \int_\Omega d^{(s_0-2m)p} X_1^{ps_1+2} \cdots X_i^{ps_i+2} X_{i+1}^{ps_{i+1}+1} \cdots X_j^{ps_j+1} X_{j+1}^{ps_{j+1}} \cdots X_r^{ps_r}\mathrm{d}x.\end{aligned}$$

Lemma 10 *Let $u(x) = d^{s_0}X_1^{s_1} \cdots X_r^{s_r}$, where $X_i = X_i(d(x)/D)$. Then*

$$I_{2m,r-1}[u] = \sum_{0\le i\le j\le r} a_{ij}\Gamma_{ij} + \int_\Omega d^{(s_0-2m)p} X_1^{ps_1} X_2^{ps_2} \cdots X_r^{ps_r} O(X_1^3)\mathrm{d}x, \tag{29}$$

where

$$\begin{aligned} a_{00} &= |\alpha_m|^p - |A(2m)|, \\ a_{0j} &= ps_j|\alpha_m|^{p-2}\alpha_m\alpha_m', \quad 1 \le j \le r, \\ a_{ii} &= \frac{ps_i}{2}|\alpha_m|^{p-2}\Big(\alpha_m\alpha_m''(s_i+1) + (p-1)\alpha_m'^2 s_i\Big) - |B(2m)|, \quad 1 \le i \le r-1, \\ a_{rr} &= \frac{ps_r}{2}|\alpha_m|^{p-2}\Big(\alpha_m\alpha_m''(s_r+1) + (p-1)\alpha_m'^2 s_r\Big), \\ a_{ij} &= \frac{ps_j}{2}|\alpha_m|^{p-2}\Big(\alpha_m\alpha_m''(2s_i+1) + 2(p-1)\alpha_m'^2 s_i\Big), \quad 1 \le i < j \le r; \end{aligned} \tag{30}$$

here and below, α_m, α_m' and α_m'' stand for $\alpha_m(s_0)$, $\alpha_m'(s_0)$ and $\alpha_m''(s_0)$, respectively.

Proof The fact that K is affine implies that $\Delta d = (k-1)/d$ and therefore

$$\Delta(f(d)) = f''(d) + \frac{k-1}{d} f'(d), \tag{31}$$

for any smooth function f on $(0, +\infty)$. We define the functions $g, \tilde{g}$ by

$$g(x) = s_1 X_1 + s_2 X_1 X_2 + \cdots + s_r X_1 X_2 \cdots X_r, \quad \nabla g = \frac{\tilde{g}}{d} \nabla d,$$

and observe that by (8),

$$g^3(t) = O(X_1^3), \quad \tilde{g}^2(t) = O(X_1^4). \tag{32}$$

Now (31) and (32) together with a simple induction argument on m imply

$$\Delta^m u = d^{s_0 - 2m} X_1^{s_1} \cdots X_r^{s_r} \Big(\alpha_m + \alpha'_m g(d) + \frac{\alpha''_m}{2} g^2(d) + \frac{\alpha''_m}{2} \tilde{g}(d) + O(X_1^3) \Big).$$

Using Taylor's theorem we then obtain

$$\begin{aligned} |\Delta^m u|^p &= d^{(s_0 - 2m)p} X_1^{ps_1} \cdots X_r^{ps_r} \Big\{ |\alpha_m|^p + p|\alpha_m|^{p-2} \alpha_m \alpha'_m g(d) \\ &\quad + \frac{p}{2} |\alpha_m|^{p-2} \Big(\alpha_m \alpha''_m + (p-1) \alpha'^2_m \Big) g^2 + \frac{p}{2} |\alpha_m|^{p-2} \alpha_m \alpha''_m \tilde{g} + O(X_1^3) \Big\}. \end{aligned} \tag{33}$$

On the other hand we have (cf. (8))

$$\begin{aligned} &\int_\Omega d^{(s_0 - 2m)p} X_1^{ps_1} \cdots X_r^{ps_r} g \, dx = \sum_{j=1}^r s_j \Gamma_{0j}, \\ &\int_\Omega d^{(s_0 - 2m)p} X_1^{ps_1} \cdots X_r^{ps_r} g^2 dx = \sum_{i=1}^r s_i^2 \Gamma_{ii} + 2 \sum_{1 \le i < j \le r} s_i s_j \Gamma_{ij}, \\ &\int_\Omega d^{(s_0 - 2m)p} X_1^{ps_1} \cdots X_r^{ps_r} \tilde{g} \, dx = \sum_{i=1}^r s_i \Gamma_{ii} + \sum_{1 \le i < j \le r} s_j \Gamma_{ij}. \end{aligned} \tag{34}$$

The stated relation follows from (33), (34) and the fact that $I_{2m,r-1}[u] = \int_\Omega |\Delta^m u|^p dx - |A(2m)|\Gamma_{00} - |B(2m)| \sum_{i=1}^{r-1} \Gamma_{ii}$. □

Up to this point the exponents $s_0, s_1, \ldots, s_r$ where arbitrary subject only to $s_0 > (2mp - k)/p$. We now make a more specific choice, taking

$$s_0 = \frac{2mp - k + \epsilon_0}{p}, \quad s_j = \frac{-1 + \epsilon_j}{p}, \;\; 1 \le j \le r, \tag{35}$$

where $\epsilon_0, \ldots, \epsilon_r$ are small positive parameters. We consider $I_{2m,r-1}[u]$ as a function of these parameters and intend to take the limits $\epsilon_0 \searrow 0, \ldots, \epsilon_r \searrow 0$. In taking these limits we shall ignore terms that are bounded uniformly in the ϵ_i's. In order to distinguish such terms we shall need the following criterion, which is a simple consequence of (8):

$$\int_\Omega d^{-k+\epsilon_0} X_1^{1+\epsilon_1} \ldots X_r^{1+\epsilon_r} \mathrm{d}x < \infty \Longleftrightarrow \begin{cases} \quad \epsilon_0 > 0 \\ \text{or } \epsilon_0 = 0 \text{ and } \epsilon_1 > 0 \\ \text{or } \epsilon_0 = \epsilon_1 = 0 \text{ and } \epsilon_2 > 0 \\ \quad \ldots \\ \text{or } \epsilon_0 = \epsilon_1 = \cdots = \epsilon_{r-1} = 0 \text{ and } \epsilon_r > 0. \end{cases} \tag{36}$$

Also, concerning terms that diverge as the ϵ_i's tend to zero, we shall need some quantitive information on the rate of divergence as well as some mutual cancelation properties. These are collected in the following

Lemma 11 *We have*

(i) $\displaystyle\int_\Omega d^{-k+\epsilon_0} X_1^\beta \mathrm{d}x \le c_\beta \epsilon_0^{-1+\beta}, \quad \beta < 1;$

(ii) $\displaystyle\int_\Omega d^{-k} X_1 \ldots X_{i-1} X_i^{1+\epsilon_i} X_{i+1}^\beta \mathrm{d}x \le c_\beta \epsilon_i^{-1+\beta}, \qquad \beta < 1, \quad 1 \le i \le r-1;$

(iii) $\displaystyle\epsilon_0^2 \Gamma_{00} - 2\epsilon_0 \sum_{j=i+1}^r (1-\epsilon_j)\Gamma_{0j} = \sum_{i=1}^r (\epsilon_i - \epsilon_i^2)\Gamma_{ii} - \sum_{1 \le i < j \le r} (1-\epsilon_j)(1-2\epsilon_i)\Gamma_{ij} + O(1),$

where the O(1) is uniform in $\epsilon_0, \ldots, \epsilon_r$*;*

(iv) *let* $i \ge 0$ *and (if* $i \ge 1$*) assume that* $\epsilon_0 = \cdots = \epsilon_{i-1} = 0$*. Then*

$$\epsilon_i \Gamma_{ii} = \sum_{j=i+1}^r (1-\epsilon_j)\Gamma_{ij} + O(1),$$

where the O(1) is uniform in $\epsilon_i, \ldots, \epsilon_r$*.*

Proof Parts (i) and (ii) are proved using the coarea formula and [3, Lemma 9]. Parts (iii) and (iv) are proved by integrating by parts; see [4], pages 181 and 184 respectively for the detailed proof. □

Remark 12 We are now in position to prove Theorem 2, but before proceeding some comments are necessary. The proof of the theorem is local: we fix a point $x_0 \in \Omega \cap K$ and work entirely in a small ball $B(x_0, \delta)$ using a cut-off function ϕ. The sequence of functions that is used is then given by

$$u(x) = \phi(x) \mathrm{d}(x)^{\frac{2mp-k+\epsilon_0}{p}} X_1(\mathrm{d}(x)/D)^{\frac{-1+\epsilon_1}{p}} \cdots X_r(d(x)/D)^{\frac{-1+\epsilon_r}{p}}, \quad (\epsilon_0, \ldots, \epsilon_r > 0)$$

and, as already mentioned, we take the successive limits $\epsilon_0 \searrow 0, \ldots, \epsilon_r \searrow 0$; in taking this limits, we work modulo terms that are bounded uniformly in the remaining ϵ_i's. Such terms are *any* terms that contain derivatives of ϕ. Hence, for the sake of simplicity and bravity, we shall completely drop ϕ from the ensuing coomputations; see also the remark in [7, p 521] or the proof of [5, Theorem 4].

Proof of Theorem 2 We consider the function

$$u(x) = d^{\frac{2mp-k+\epsilon_0}{p}} X_1^{\frac{-1+\epsilon_1}{p}} \cdots X_r^{\frac{-1+\epsilon_r}{p}}, \tag{37}$$

where $\epsilon_0, \dots, \epsilon_r$ are small and positive. A standard argument shows that u lies in the appropriate Sobolev space. We have seen that

$$I_{2m,r-1}[u] = \sum_{0 \le i \le j \le r} a_{ij} \Gamma_{ij} + O(1), \tag{38}$$

where the coefficients a_{ij} are given by (30) and the s_i's are related to the ϵ_i's by (35).

We let $\epsilon_0 \searrow 0$ in (29). It follows from (36) that all Γ_{ij}'s with $i \ge 1$ have finite limits. As for the remaining terms, applying Lemma 11 with $\beta = -3/2$ (for $j = 0$) and with $\beta = -1/2$ (for $j \ge 1$) we obtain respectively

$$\Gamma_{00} \le c\epsilon_0^{-\frac{5}{2}}, \quad \Gamma_{0j} \le c\epsilon_0^{-\frac{3}{2}}, \tag{39}$$

where in both cases $c > 0$ is independent of all the ϵ_i's. Now, we think of the quantities a_{0j} of Lemma 10 as functions of ϵ_0 and consider $\epsilon_1, \dots, \epsilon_r$ as small positive parameters. Using Taylor's theorem we shall expand the coefficient a_{0j} of Γ_{0j}, $j = 0$ (resp. $j \ge 1$) in powers of ϵ_0 and (39) shows that we can discard powers with exponent ≥ 3 (resp. ≥ 2). We shall compute the remaining ones and for this we define

$$\hat{a}_{0j}(\epsilon_0) := a_{0j}(s_0) = a_{0j}((2mp - k + \epsilon_0)/p)$$

and denote by $A_{k,0j}$ the coefficient of ϵ_0^k in $\hat{a}_{0j}(\epsilon_0)$. We then have from Lemma 10:

- Constant term in a_{00}: We have

$$\begin{aligned} \hat{a}_{00}(\epsilon_0) &= |\alpha_m(s_0)|^p - |A(2m)| \\ &= \left| \prod_{i=0}^{m-1} \Big(\frac{(2m-2i)p - k + \epsilon_0}{p} \Big) \prod_{j=1}^{m} \Big(\frac{(2m-2j)p + kp - k + \epsilon_0}{p} \Big) \right|^p - |A(2m)| \end{aligned} \tag{40}$$

  and therefore, using (27), $A_{0,00} = \hat{a}_{00}(0) = |\alpha_m(s_0)|^p \Big|_{\epsilon_0=0} - |A(2m)| = 0$.
- *Coefficient of* ϵ_0 *in* a_{00}*:* Differentiating (40) we obtain

$$\hat{a}'_{00}(\epsilon_0) = \frac{1}{p} a'_{00}(s_0) = |a_m(s_0)|^{p-2} a_m(s_0) a'_m(s_0) \tag{41}$$

  and therefore the coefficient is

$$A_{1,00} = \hat{a}'_{00}(0) = |a_m(s_0)|^{p-2} a_m(s_0) a'_m(s_0) \Big|_{\epsilon_0=0}.$$

- *Coefficient of* ϵ_0^2 *in* a_{00}*:* We have from (41)

$$A_{2,00} = \frac{\hat{a}''_{00}(s_0)}{2} \Big|_{\epsilon_0=0} = \frac{1}{2p} \Big(|a_m(s_0)|^{p-2} a_m(s_0) a'_m(s_0) \Big)' \Big|_{\epsilon_0=0}.$$

  Concerning $a_{0j}, j \ge 1$, we have $\hat{a}_{0j}(\epsilon_0) = ps_j |\alpha_m(s_0)|^{p-2} \alpha_m(s_0) \alpha'_m(s_0)$ and therefore

$$\hat{a}'_{0j}(\epsilon_0) = s_j \Big(|\alpha_m(s_0)|^{p-2} \alpha_m(s_0) \alpha'_m(s_0) \Big)'.$$

  Hence:
- *Constant term in* a_{0j}*,* $j \ge 1$*:* This is

$$A_{0,0j} = \hat{a}_{0j}(0) = ps_j |\alpha_m(s_0)|^{p-2} \alpha_m(s_0) \alpha'_m(s_0) \Big|_{\epsilon_0=0}$$

– *Coefficient of ϵ_0 in a_{0j}:* This is

$$A_{1,0j} = \hat{a}'_{0j}(0) = s_j\Big(|\alpha_m(s_0)|^{p-2}\alpha_m(s_0)\alpha'_m(s_0)\Big)'\Big|_{\epsilon_0=0}.$$

Now, we observe that $A_{0,0j} = ps_jA_{1,00} = (\epsilon_j - 1)A_{1,00}$. Hence (iv) of Lemma 11 implies that

$$A_{1,00}\epsilon_0\Gamma_{00} + \sum_{j=1}^{r} A_{0,0j}\Gamma_{0j} = O(1) \tag{42}$$

uniformly in $\epsilon_1, \ldots, \epsilon_r$. Similarly, we observe that $A_{1,0j} = 2ps_jA_{2,00} = 2(-1+\epsilon_j)A_{2,00}$. Hence, by (iii) of Lemma 11, the remaining 'bad' terms when combined give

$$\begin{aligned}
&A_{2,00}\epsilon_0^2\Gamma_{00} + \epsilon_0\sum_{j=1}^{r} A_{1,0j}\Gamma_{0j}\\
&\quad = A_{2,00}\Big(\epsilon_0^2\Gamma_{00} - 2\epsilon_0\sum_{j=1}^{r}(1-\epsilon_j)\Gamma_{0j}\Big)\\
&\quad = A_{2,00}\Big(\sum_{i=1}^{r}(\epsilon_i - \epsilon_i^2)\Gamma_{ii} - \sum_{1\le i<j\le r}(1-\epsilon_j)(1-2\epsilon_i)\Gamma_{ij}\Big) + O(1),
\end{aligned} \tag{43}$$

uniformly in $\epsilon_1, \ldots, \epsilon_r$. Note that the right-hand side of (43) has a finite limit as $\epsilon_0 \searrow 0$. From (38), (42) and (43) we conclude that, after letting $\epsilon_0 \searrow 0$, we are left with

$$\begin{aligned}
I_{2m,r-1}[u] &= \sum_{i=1}^{r}\Big(a_{ii} + A_{2,00}(\epsilon_i - \epsilon_i^2)\Big)\Gamma_{ii}\\
&\quad + \sum_{1\le i<j\le r}\Big(a_{ij} - A_{2,00}(1-\epsilon_j)(1-2\epsilon_i)\Big)\Gamma_{ij} + O(1)\\
&=: \sum_{i=1}^{r} b_{ii}\Gamma_{ii} + \sum_{1\le i<j\le r} b_{ij}\Gamma_{ij} + O(1), \qquad (\epsilon_0 = 0),
\end{aligned} \tag{44}$$

where the $O(1)$ is uniform in $\epsilon_1, \ldots, \epsilon_r$.

We next let $\epsilon_1 \searrow 0$ in (44). It follows from (36) that all the Γ_{ij}'s have finite limits, except those with $i = 1$ which diverge to $+\infty$. For the latter we have

$$\Gamma_{11} \le c\epsilon_1^{-\frac{5}{2}}, \quad \Gamma_{1j} \le c\epsilon_1^{-\frac{3}{2}}, \quad j \ge 2,$$

by (ii) of Lemma 11 with $\beta = -3/2$ and $\beta = -1/2$ respectively; in both cases the constant c is independent of $\epsilon_2, \ldots, \epsilon_r$. We think of the coefficients b_{1j} as functions – indeed, polynomials – of ϵ_1 and we expand these in powers of ϵ_1. The estimates above on Γ_{1j} imply that only the terms $1, \epsilon_1$ and ϵ_1^2 (resp. 1 and ϵ_1) give contributions for Γ_{11} (resp. Γ_{1j}, $j \ge 2$) that do not vanish as $\epsilon_1 \searrow 0$. We shall compute the coefficients of these terms. Our starting point are the relations (cf. (44))

$$\begin{aligned}
b_{11}(\epsilon_1) &= a_{11}(s_0) + A_{2,00}(\epsilon_1 - \epsilon_1^2)\\
&= \frac{\epsilon_1 - 1}{2p}|\alpha_m|^{p-2}\Big(\alpha_m\alpha''_m(p-1+\epsilon_1) + (p-1)(\epsilon_1 - 1)\alpha'^2_m\Big)\\
&\quad + A_{2,00}(\epsilon_1 - \epsilon_1^2) - |B(2m)|
\end{aligned} \tag{45}$$

and, for $j \geq 2$,

$$\begin{aligned} b_{1j}(\epsilon_1) &= a_{1j}(s_0) - A_{2,00}(1-\epsilon_j)(1-2\epsilon_1) \\ &= (\epsilon_j - 1)\left\{ \frac{|\alpha_m|^{p-2}}{2p}\Big[\alpha_m\alpha_m''(2\epsilon_1 + p - 2) + 2(p-1)\alpha_m'^2(\epsilon_1 - 1)\Big] \right. \\ &\quad \left. + A_{2,00}(1-2\epsilon_1)\right\}. \end{aligned} \tag{46}$$

Hence, denoting by $B_{k,1j}$ the coefficient of ϵ_1^k in b_{1j}, $j \geq 1$, we have:

– *Constant term in b_{11}:* This is

$$\begin{aligned} B_{0,11} &= b_{11}(0) \\ &= a_{11}(s_0)\Big|_{\epsilon_1=0} \\ &= -\frac{1}{2p}|\alpha_m|^{p-2}\Big(\alpha_m\alpha_m''(p-1) - (p-1)\alpha_m'^2\Big) - |B(2m)| \\ &= 0, \end{aligned}$$

by (28).

– *Coefficient of ϵ_1 in b_{11}:* From (45) we obtain

$$b_{11}'(\epsilon_1) = \frac{|\alpha_m|^{p-2}}{2p}\left\{\alpha_m\alpha_m''(2\epsilon_1 + p - 2) + (p-1)\alpha_m'^2(2\epsilon_1 - 2)\right\} + A_{2,00}(1-2\epsilon_1) \tag{47}$$

and therefore the coefficient is

$$B_{1,11} = b_{11}'(0) = \frac{|\alpha_m|^{p-2}}{2p}\left\{(p-2)\alpha_m\alpha_m'' - 2(p-1)\alpha_m'^2\right\} + A_{2,00}.$$

– *Coefficient of ϵ_1^2 in b_{11}:* From (47),

$$B_{2,11} = \frac{1}{2}b_{11}''(0) = \frac{|\alpha_m|^{p-2}}{2p}\left\{\alpha_m\alpha_m'' + (p-1)\alpha_m'^2\right\} - A_{2,00} = 0.$$

– *Constant term in b_{1j}, $j \geq 2$:* This is

$$B_{0,1j} = b_{1j}(0) = (\epsilon_j - 1)\frac{|\alpha_m|^{p-2}}{2p}\Big[(p-2)\alpha_m\alpha_m'' - 2(p-1)\alpha_m'^2\Big] + A_{2,00}.$$

– *Coefficient of ϵ_1 in b_{1j}, $j \geq 2$:* From (46),

$$B_{1,1j} = b_{1j}'(0) = (\epsilon_j - 1)\left\{\frac{|\alpha_m|^{p-2}}{p}(\alpha_m\alpha_m'' + (p-1)\alpha_m'^2) - 2A_{2,00}\right\} = 0.$$

We obsrerve that $B_{0,1j} = (\epsilon_j - 1)B_{1,11}$, $j \geq 2$. Hence (iv) of Lemma 11 gives

$$\epsilon_1 B_{1,11}\Gamma_{11} + \sum_{j=2}^{r} B_{0,1j}\Gamma_{1j} = O(1), \tag{48}$$

uniformly in $\epsilon_2, \ldots, \epsilon_r$. Combining (44) and (48) we conclude that after letting $\epsilon_1 \searrow 0$ we are left with

$$I_{2m,r-1}[u] = \sum_{2 \leq i \leq j \leq r} b_{ij}\Gamma_{ij} + O(1), \qquad (\epsilon_0 = \epsilon_1 = 0), \tag{49}$$

uniformly in $\epsilon_2, \dots, \epsilon_r$. Note that we have the same coefficients b_{ij} as in (44), unlike the case where the limit $\epsilon_0 \searrow 0$ was taken, where we passed from the coefficients a_{ij} to the coefficients b_{ij}.

We proceed in this way. At the ith step we denote by $B_{k,ij}$ the coefficient of ϵ_i^k in b_{ij}, $j \geq i$, and observe that

$$B_{0,ij} = (\epsilon_j - 1)B_{1,ii}, \quad B_{2,ii} = B_{1,ij} = 0, \quad j \geq i+1.$$

Hence (iv) of Lemma iv implies the cancelation (modulo uniformly bounded terms) of all terms that, separetely, diverge as $\epsilon_i \searrow 0$. Eventually, after letting $\epsilon_{r-1} \searrow 0$, we arrive at

$$I_{2m,r-1}[u] = b_{rr}\Gamma_{rr} + O(1), \quad (\epsilon_0 = \epsilon_1 = \cdots = \epsilon_{r-1} = 0). \tag{50}$$

Since

$$\int_\Omega \frac{|u|^p}{d^{2mp}} X_1^2 \cdots X_r^2 \mathrm{d}x = \Gamma_{rr}$$

and $\lim_{\epsilon_r \searrow 0} \Gamma_{rr} = +\infty$ (cf (11)) we conclude that

$$\begin{aligned}
\inf_{C_c^\infty(\Omega\setminus K)} \frac{I_{2m,r-1}[v]}{\int_\Omega \frac{|v|^p}{d^{2mp}} X_1^2 \cdots X_r^2 \mathrm{d}x} &\leq \lim_{\epsilon_r \searrow 0} \frac{b_{rr}\Gamma_{rr} + O(1)}{\Gamma_{rr}} \\
&= \lim_{\epsilon_r \searrow 0} b_{rr} \\
&= \lim_{\epsilon_r \searrow 0} a_{rr} \\
&= \lim_{\epsilon_r \searrow 0} \frac{ps_r}{2} |\alpha_m|^{p-2} \Big(\alpha_m \alpha_m''(s_r + 1) + (p-1)\alpha_m'^2 s_r \Big) \\
&= \frac{p-1}{2p} |\alpha_m|^{p-2} (\alpha_m'^2 - \alpha_m \alpha_m'') \\
\text{(by (28))} \qquad &= |B(2m)|.
\end{aligned}$$

This proves part (ii) of the theorem. Part (i) follows by slightly modifying the above argument; we omit the details. □

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