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Trace estimates and invariance of the essential spectrum

By

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Abstract. We provide sufficient conditions under which the difference of the resolvents of two higher-order operators acting in $\mathbb{R}^N$ belongs to trace classes $\mathcal{C}^p$. We provide explicit estimates on the norm of the resolvent difference in terms of L^p norms of the difference of the coefficients. Such inequalities are useful in estimating the effect of localized perturbations of the coefficients.

1. Introduction. Let H and $\tilde{H}$ be second-order elliptic differential operators on $\mathbb{R}^N$. Various sufficient conditions exist under which the resolvent difference $(\tilde{H}+1)^{-1} - (H+1)^{-1}$ is compact and, subsequently, H and $\tilde{H}$ have the same essential spectrum. See [6], [7], [8], [5] and references therein. These conditions typically involve some decay of the difference $\tilde{a}_{ij} - a_{ij}$ of the respective coefficients near infinity. Analogous results were obtained recently in [5] in a very general setting which includes the case of higher-order operators on $\mathbb{R}^N$ or Laplace-Beltrami operators on manifolds.

In this note we show how an application of the Fourier transform can yield quantitative results of this type for higher-order self-adjoint operators of order $2m$ on $\mathbb{R}^N$ provided one of the operators has constant coefficients. Hence we adopt the attitude that H is a 'good', known operator and $\tilde{H}$ is a perturbed operator for which information is sought. In our main result sufficient conditions are given under which the difference $(\tilde{H}+1)^{-1} - (H+1)^{-1}$ is not only compact on $L^2(\mathbb{R}^N)$ but belongs in the Schatten class $\mathcal{C}^p(L^2(\mathbb{R}^N))$. More significantly, explicit estimates are obtained: it is shown that if the coefficient matrix $\tilde{a}$ of $\tilde{H}$ is such that $\tilde{a}^{-1/2}(\tilde{a}-a)a^{-1/2} \in L^p$ for some $p > N/m$ then

$$\|(\tilde{H}+1)^{-1} - (H+1)^{-1}\|_{\mathcal{C}^p} \leqq c\|\tilde{a}^{-1/2}(\tilde{a}-a)a^{-1/2}\|_{L^p}. \tag{1}$$

It is this estimate that is novel with respect to earlier work, for both the second- and higher-order case. As a typical application, (1) is useful in order to estimate the effect of narrowly localised impurities of the underlying medium; see the example following Theorem 3.

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Estimates of this type were obtained by the author in [1] without the assumption that one of the coefficient matrices is constant, but the discreteness of the spectrum was a fundamental hypothesis there.

The proof uses a formula for the resolvent difference (Lemma 1, used also in [5]) together with a trace estimate for a class of operators acting on vector-valued functions. It is for the latter estimate that the Fourier transform plays a crucial role.

We fix some notation. We work with compex-valued functions in $L^2(\mathbb{R}^N)$. Given a multi-index $\alpha = (\alpha_1, \ldots, \alpha_N)$ we use the standard notation D^α for the differential expression $(\partial/\partial x_1)^{\alpha_1} \ldots (\partial/\partial x_N)^{\alpha_N}$. Throughout this article we fix a positive integer m and denote by $\nu = \nu(m, N)$ be the number of multi-indices α of length $|\alpha| := \alpha_1 + \cdots + \alpha_N = m$. Given a vector $v = (v_\alpha) \in \mathbb{C}^\nu$ we denote by $v \otimes v$ the rank-one matrix $(v_\alpha \overline{v}_\beta) \in M^{\nu\times\nu}$. The summation convention over repeated indices is adopted throughout this article.

The L^p-norm of a matrix valued function $V = (V_{\alpha\beta}(x)) : \mathbb{R}^N \to M^{\nu\times\nu}(\mathbb{C})$ is defined in the standard way,

$$\|V\|_p = \left(\int_{\mathbb{R}^N} |V(x)|^p dx \right)^{1/p},$$

where $|V(x)|$ denotes the norm of the matrix $V(x)$ regarded as an operator on $\mathbb{C}^\nu$; the L^∞-norm is defined similarly. Such a potential V induces a multiplication operator on $L^2(\mathbb{R}^N)^\nu$, also denoted by V, with domain $\mathrm{Dom}(V) = \{u \in L^2(\mathbb{R}^N)^\nu : Vu \in L^2(\mathbb{R}^N)^\nu\}$, where

$$(Vu)_\alpha(x) = V_{\alpha\beta}(x) u_\beta(x), x \in \mathbb{R}^N, \quad (u_\beta) \in L^2(\mathbb{R}^N)^\nu.$$

If $V \in L^\infty$ then V is a bounded operator and the two norms coincide.

We shall consider operators of order $2m$ acting on $L^2(\mathbb{R}^N)$ and given formally by

$$Hu(x) = (-1)^m \sum_{\substack{|\alpha|=m \\ |\beta|=m}} D^\alpha\{a_{\alpha\beta}(x) D^\beta u(x)\}, \quad x \in \mathbb{R}^N. \tag{2}$$

We assume that the complex matrix-valued function $a = (a_{\alpha\beta}(x))$ is self-adjoint and positive definite for a.e. $x \in \mathbb{R}^N$ and, moreover, that $a, a^{-1} \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$. To define the operator H we first define the quadratic form

$$Q(u) = \int_\Omega \sum_{\substack{|\alpha|=m \\ |\beta|=m}} a_{\alpha\beta}(x) D^\alpha u(x) D^\beta \bar{u}(x)\, dx,$$

on $C_c^\infty(\mathbb{R}^N)$ and *assume* that Q is closable; we also denote by Q its closure. The operator H is then defined as the self-adjoint operator associated with its closure. There are various sufficient conditions for the closability of Q, for which we refer to [3], [5], [8], [9] and references therein.

2. Main results. We shall initially consider uniformly elliptic operators with $a, a^{-1} \in L^{\infty}(\mathbb{R}^N)$ and we shall drop this assumption in Theorem 4. So let H be as above, with $a, a^{-1} \in L^{\infty}(\mathbb{R}^N)$; this implies in particular that the domain $\mathrm{Dom}(Q)$ coincides with the Sobolev space $H^m(\mathbb{R}^N)$. We define the (closed, densely defined) operator $D_m : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)^\nu$,

$$\mathrm{Dom}(D_m) = H^m(\mathbb{R}^N),\ D_m u = (D^\alpha u).$$

We also denote by $b = (b_{\alpha\beta}(x))$ the square root of the matrix $a = (a_{\alpha\beta}(x))$ and define $T = bD_m$ so that $\mathrm{Dom}(T) = H^m(\mathbb{R}^N)$ and

$$H = T^*T.$$

We finally define the self-adjoint operator

$$F : L^2(\mathbb{R}^N)^\nu \to L^2(\mathbb{R}^N)^\nu,\ F = TT^*. \tag{3}$$

with $\mathrm{Dom}(F) = \{v = (v_\alpha) \in \mathrm{Dom}(T^*) : T^*v \in \mathrm{Dom}(T)\}$.

Suppose now that we have two such operators H and $\tilde{H}$. Keeping the above notation and using tildes in an obvious way we have:

Lemma 1. *There exist partial isometries $U, \tilde{U} : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)^\nu$ such that*

$$(\tilde{H}+1)^{-1} - (H+1)^{-1} = \tilde{U}^*\tilde{F}^{1/2} \cdot(\tilde{F}+1)^{-1}\tilde{a}^{-1/2}(a-\tilde{a})a^{-1/2}(F+1)^{-1}F^{1/2}U.$$

P r o o f. We write the identity [4, p. 271]

$$(S^*S+1)^{-1} + S^*(SS^*+1)^{-1}S = I$$

first for $S = T$, then for $S = \tilde{T}$ and subtract the two relations; we obtain

$$\begin{aligned}
&(\tilde{H}+1)^{-1} - (H+1)^{-1}\\
&\quad = -D_m^*\tilde{a}^{1/2}(\tilde{F}+1)^{-1}\tilde{a}^{1/2}D_m + D_m^*a^{1/2}(F+1)^{-1}a^{1/2}D_m\\
&\quad = -D_m^*\{(D_mD_m^* + \tilde{a}^{-1})^{-1} - (D_mD_m^* + a^{-1})^{-1}\}D_m\\
&\quad = D_m^*(D_mD_m^* + \tilde{a}^{-1})^{-1}(\tilde{a}^{-1} - a^{-1})(D_mD_m^* + a^{-1})^{-1}D_m\\
&\quad = D_m^*\tilde{a}^{1/2}(\tilde{F}+1)^{-1}\tilde{a}^{1/2}(\tilde{a}^{-1} - a^{-1})a^{1/2}(F+1)^{-1}a^{1/2}D_m\\
&\quad = D_m^*\tilde{a}^{1/2}(\tilde{F}+1)^{-1}\tilde{a}^{-1/2}(a-\tilde{a})a^{-1/2}(F+1)^{-1}a^{1/2}D_m\,.
\end{aligned}$$

Using polar decomposition, we write

$$a^{1/2}D_m = F^{1/2}U, \quad \tilde{a}^{1/2}D_m = \tilde{F}^{1/2}\tilde{U},$$

where $U, \tilde{U} : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)^\nu$ are partial isometries; this completes the proof. □

R e m a r k. It is an immediate consequence of Lemma 1 that

$$\|(\tilde{H}+1)^{-1}-(H+1)^{-1}\| \leqq \frac{1}{4}\|\tilde{a}^{-1/2}(\tilde{a}-a)a^{-1/2}\|_{\infty}.$$

This of course does not contain any information on possible compactness of the resolvent difference; nor is it any useful in the context of the example following Theorem 3. In what follows we shall concentrate with the case where $\tilde{a}^{-1/2}(\tilde{a}-a)a^{-1/2} \in L^p(\mathbb{R}^N)$ for some $p < \infty$.

To proceed we define the weighted L^p spaces

$$L^p(\mathbb{R}_+, t^{\frac{N-2m}{2m}}dt) = \left\{ g : \mathbb{R}_+ \to \mathbb{R} \ : \ \int\limits_0^\infty |g(t)|^p t^{\frac{N-2m}{2m}} dt < +\infty \right\}, \quad 1 \leqq p < \infty,$$

equipped with the natural norm, which for simplicity we denote by $\|\cdot\|_p^*$. The next lemma is a vector-valued version of [10, Theorem 4.1]; we note that it does not follow directly from that result. Although the proof is very similar to that of [10], we present it for the sake of completeness.

Lemma 2. *Assume that the operator H has constant coefficients. Let $V = (V_{\alpha\beta}(x))$ be a matrix-valued function and let $g : [0, +\infty) \to \mathbb{R}$ be a bounded continuous function with $g(0) = 0$. If for some $p \in [2, +\infty)$ there holds $V \in L^p(\mathbb{R}^N)$ and $g \in L^p(\mathbb{R}_+, t^{\frac{N-2m}{2m}}dt)$, then $Vg(F) \in \mathcal{C}^p(L^2(\mathbb{R}^N)^\nu)$ and moreover*

$$\|Vg(F)\|_{\mathcal{C}^p} \leqq c^{1/p}\|V\|_{L^p}\|g\|_p^*. \tag{4}$$

for a constant $c = c(H)$.

P r o o f. We may assume that both V and g have compact supports since the general case will then follow by approximation and an application of the dominated convergence theorem for trace ideals. Also, it is enough to establish (4) for $p = 2$ and $p = \infty$ since the intermediate cases will then follow by interpolation [10, Theorem 2.9].

Let us denote by $\mathcal{F}$ the Fourier transform regarded as a unitary operator on $L^2(\mathbb{R}^N)$; we use the same symbol for the unitary operator induced component-wise on $L^2(\mathbb{R}^N)^\nu$. The fact that the differential operator F (cf. (3)) has constant coefficients implies that F is unitarily equivalent via $\mathcal{F}$ to a multiplication operator on $L^2(\mathbb{R}^N)^\nu$. More precisely, for $\xi \in \mathbb{R}^N$ (the variable in the Fourier space) let us define the vector

$$(B(\xi))_\alpha = b_{\alpha\gamma}\xi^\gamma.$$

We then have $\mathcal{F}F\mathcal{F}^{-1} = B(\xi) \otimes B(\xi)$, that is

$$(\mathcal{F}F\mathcal{F}^{-1}v)_\alpha(\xi) = (B(\xi) \otimes B(\xi))_{\alpha\beta}v_\beta(\xi) = b_{\alpha\gamma}\overline{b}_{\beta\delta}\xi^{\gamma+\delta}v_\beta(\xi).$$

This implies that

$$\mathcal{F}F^k\mathcal{F}^{-1} = |B(\xi)|^{2k-2}B(\xi) \otimes B(\xi), \quad k = 1, 2, \dots ,$$

and hence

$$\mathcal{F}p(F)\mathcal{F}^{-1} = p(0)I + [p(|B(\xi)|^2) - p(0)I]|B(\xi)|^{-2}B(\xi) \otimes B(\xi)$$

for any polynomial $p(\cdot)$. Direct computation then shows that if $p(\cdot)$ does not vanish on $[0, \infty)$ then

$$\mathcal{F}p^{-1}(F)\mathcal{F}^{-1} = p^{-1}(0)I + [p^{-1}(|B(\xi)|^2) - p^{-1}(0)I] \cdot|B(\xi)|^{-2}B(\xi) \otimes B(\xi).$$

Compactifying $[0, \infty)$ we obtain by an application of the Stone-Weierstrass theorem that

$$\mathcal{F}g(F)\mathcal{F}^{-1} = g(|B(\xi)|^2)|B(\xi)|^{-2}B(\xi) \otimes B(\xi) =: (Lg)(\xi)\,, \quad \xi \in \mathbb{R}^N,$$

for any continuous function g on $[0, +\infty)$ with $g(0) = 0$. We note that L is a linear map from the space of all such g to the space of matrix-valued functions and Lg is a multiplication operator in $L^2(\mathbb{R}^N)^\nu$.

It follows that $g(F)$ has an integral kernel depending only on $x - y$, $k_g = k_g(x - y)$, where

$$k_g(x) = (2\pi)^{-N}(\mathcal{F}^{-1}Lg)(x)\,, \qquad x \in \mathbb{R}^N\,;$$

hence $Vg(F)$ has the integral kernel $V(x)k_g(x - y)$. It follows that $Vg(F)$ is a Hilbert-Schmidt operator with Hilbert-Schmidt norm given by

$$\begin{aligned} \|Vg(F)\|^2_{\mathcal{C}^2} &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |V(x)k_g(x - y)|^2 dx\, dy \\ &\leqq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |V(x)|^2|k_g(x - y)|^2 dx\, dy \\ &= \|V\|^2_{L^2}\|k_g\|^2_2 \\ &= (2\pi)^{-2N}\|V\|^2_{L^2}\|Lg\|^2_2. \end{aligned} \tag{5}$$

Using the homogeneity of the symbol $A(\xi)$ of H we obtain by an application of the coarea formula,

$$\begin{aligned} \|Lg\|^2_2 &= \int_{\mathbb{R}^N} g^2(|B(\xi)|^2)|B(\xi)|^{-4}|B(\xi) \otimes B(\xi)|^2 d\xi \\ &= \int_{\mathbb{R}^N} g^2(|B(\xi)|^2) d\xi \\ &\leqq c(H)\int_0^\infty g^2(t)t^{\frac{N-2m}{2m}}\, dt \\ &= c(H)\|g\|^{*2}_2. \end{aligned} \tag{6}$$

Combining (5) and (6) we conclude that

$$\|Vg(F)\|^2_{\mathcal{C}^2} \leqq (2\pi)^{-2N} c(H) \|V\|^2_{L^2} \|g\|^{*2}_2 .$$

We also have

$$\|Vg(F)\|_{\mathcal{C}^\infty} \leqq \|V\|_{L^\infty} \|g\|_\infty;$$

here we note that the compactness of $Vg(F)$ follows from our assumption on the supports of V, g, which, by our argument above, implies that V and g are both L^2 and hence that $Vg(F)$ is Hilbert-Schmidt. This completes the proof of the lemma. □

Theorem 3. *Let H and $\tilde{H}$ be uniformly elliptic self-adjoint operators of order $2m$ and let a and $\tilde{a}$ be the respective coefficient matrices. Assume that H has constant coefficients. Then for any $p \in (N/m, \infty)$ there exists a positive constant $c = c(p, H)$ such that*

$$\|(\tilde{H}+1)^{-1} - (H+1)^{-1}\|_{\mathcal{C}^p} \leqq c \|\tilde{a}^{-1/2}(\tilde{a}-a)a^{-1/2}\|_{L^p} .$$

Proof. Setting $g(t) = t^{1/2}(t+1)^{-1}$ and using Lemmas 1 and 2 we obtain

$$\begin{aligned}\|(\tilde{H}+1)^{-1} - (H+1)^{-1}\|_{\mathcal{C}^p} &= \|g(\tilde{F})\tilde{a}^{-1/2}(\tilde{a}-a)a^{-1/2}g(F)\|_{\mathcal{C}^p} \\ &\leqq \frac{1}{2}\|\tilde{a}^{-1/2}(\tilde{a}-a)a^{-1/2}g(F)\|_{\mathcal{C}^p} \\ &\leqq \frac{1}{2}c^{1/p}\|\tilde{a}^{-1/2}(\tilde{a}-a)a^{-1/2}\|_{L^p}\|g\|^*_p.\end{aligned}$$

The proof is completed if we note that $\|g\|^*_p < \infty$ if and only if $p > N/m$. □

Example. Suppose that H is an operator with constant coefficients $a = \{a_{\alpha\beta}\}$ as above describing some physical phenomenon and assume that the presence of some localized impurities on a set U of finite volume yields a new coefficient matrix,

$$\tilde{a}(x) = \begin{cases} a + b(x), & x \in U, \\ a, & x \notin U, \end{cases}$$

where $b \in L^\infty(U)$. We then have

$$\|\tilde{a} - a\|_p = \|b\|_{L^p(U)} \leqq \|b\|_{L^\infty(U)} |U|^{1/p}.$$

Hence we have a precise estimate on the effect of the given impurity in terms of the volume $|U|$.

Finally, at the cost of having at the left-hand side the operator norm instead of a $\mathcal{C}^p$ norm, we drop the uniform ellipticity assumption on $\tilde{H}$.

Theorem 4. *Let H and $\tilde{H}$ be self-adjoint elliptic operators of order $2m$ and let a and $\tilde{a}$ be the respective coefficient matrices. Assume that H has constant coefficients. If $\tilde{a}^{-1/2}(\tilde{a}-a) \in L^p$ for some $p \in (N/m, \infty)$ then there exists a positive constant $c = c(p, H)$ such that*

$$\|(\tilde{H}+1)^{-1} - (H+1)^{-1}\| \leqq c\|\tilde{a}^{-1/2}(\tilde{a}-a)a^{-1/2}\|_{L^p}. \tag{7}$$

P r o o f. Using the diagonalization of $\tilde{a}(x)$ we define for each $x \in \mathbb{R}^N$ the matrices

$$\tilde{a}_n(x) = \max\{1/n, \min\{\tilde{a}(x), n\}\}, n = 1, 2, \dots .$$

The corresponding operators $\tilde{H}_n$ are then uniformly elliptic and Theorem 3 implies that for $p > N/m$ there exists $c = c(p, H)$ such that

$$\|(\tilde{H}_n + 1)^{-1} - (H + 1)^{-1}\| \leqq c\|\tilde{a}_n^{-1/2}(\tilde{a}_n - a)a^{1/2}\|_{L^p} , \tag{8}$$

where the constant c is independent of $n \in \mathbb{N}$. Now, it follows from [2, p. 118] and [3, Theorem 1.2.3] that $(\tilde{H}_n + 1)^{-1} \to (\tilde{H} + 1)^{-1}$ strongly as $n \to +\infty$. This together with (8) and Lebesgue's dominated convergence theorem yields (7). □

R e m a r k. A version of Lemma 2 for operators with variable coefficients would extend our results to the case where both H and $\tilde{H}$ have variable coefficients. This is an open problem.

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