

## SHARP HEAT-KERNEL ESTIMATES FOR HIGHER-ORDER OPERATORS WITH SINGULAR COEFFICIENTS

GERASSIMOS BARBATIS

*Department of Mathematics, University of Ioannina,  
 45110 Ioannina, Greece (gbarbati@cc.uoi.gr)*

(Received 25 June 2003)

*Abstract* We obtain heat-kernel estimates for higher-order operators with measurable coefficients that can be singular or degenerate. Precise constants are given, which are sharp for small times.

*Keywords:* heat-kernels estimates; higher-order operators; singular-degenerate coefficients

2000 *Mathematics subject classification:* Primary 35K30; 47D06  
 Secondary 35K25; 35K35

### 1. Introduction

Let

$$Hf(x) = (-1)^m \sum_{\substack{|\alpha|=m \\ |\beta|=m}} D^\alpha \{a_{\alpha\beta}(x) D^\beta f(x)\}, \quad x \in \Omega \subset \mathbb{R}^N, \quad (1.1)$$

be a self-adjoint uniformly elliptic operator of order  $2m$  with measurable coefficients and subject to Dirichlet boundary conditions on  $\partial\Omega$ . It is known that if  $2m > N$ , then the associated heat semigroup  $e^{-Ht}$  has a kernel  $K(t, x, y)$  which satisfies the estimate

$$|K(t, x, y)| < c_1 t^{-N/2m} \exp \left\{ -c_2 \frac{|x-y|^{2m/(2m-1)}}{t^{1/(2m-1)}} + c_3 t \right\}$$

for some positive constants  $c_i$ . Under suitable conditions this was recently [4] sharpened to

$$|K(t, x, y)| < c_\epsilon t^{-N/2m} \exp \left\{ -(\sigma_m - cD - \epsilon) \frac{d_M(x, y)^{2m/(2m-1)}}{t^{1/(2m-1)}} + c_{\epsilon, M} t \right\}, \quad (1.2)$$

where  $\sigma_m = (2m-1)(2m)^{-2m/(2m-1)} \sin(\pi/(4m-2))$ ,  $D \geq 0$ , depends on the regularity of the coefficients and  $d_M(x, y)$  is a Finsler-type metric that is induced by the principal symbol of  $H$  and depends on the arbitrarily large parameter  $M$ ; as  $M \rightarrow \infty$ ,  $d_M(x, y)$  increases to a Finsler distance  $d(x, y)$ , but (1.2) is valid only for  $M < \infty$ . This estimate is sharp, as is seen by comparison with the small-time asymptotics for operators with

smooth coefficients obtained in [10] (see (2.12) below). In the same direction, Dungey [8] used resolvent estimates to obtain a better estimate than (1.2) for powers of second-order operators. He showed in a general framework that if the self-adjoint operator  $H$  satisfies a standard Gaussian estimate with exponential constant  $\frac{1}{4} - \epsilon$ , then the heat kernel of  $H^m$  satisfies (1.2) with  $D = 0$  and  $M = +\infty$ . For an alternative approach valid also for higher-order systems see [1]. For a comprehensive review of recent results on the spectral theory of higher-order operators with measurable coefficients see [7].

Dungey's result also applies to operators with singular or degenerate coefficients, but it does not apply when the operator is not the power of a second-order operator. A sharp heat-kernel estimate for operators of the form (1.1) with singular and/or degenerate coefficients is the main result of this paper. At the same time, for the sake of greater generality, we do not assume that  $H$  is self-adjoint.

Concerning the singularity or degeneracy of  $H$ , we assume that there is a positive function  $a(x)$  that controls in a suitable sense the behaviour of the coefficient matrix  $\{a_{\alpha\beta}\}$  and we then impose two conditions (H1) and (H2) on  $a(x)$ . The first is a weighted Sobolev inequality and the second is a weighted interpolation inequality. These conditions were introduced in [3] and led to (non-sharp) off-diagonal estimates on the heat kernel of non-uniformly elliptic self-adjoint operators. Besides conditions (H1) and (H2) we shall assume that the symbol  $A(x, \xi)$  is close—in a suitable sense—to a certain class of 'good' symbols denoted by  $\mathcal{G}_a$ . These symbols, besides satisfying (H1) and (H2), correspond to operators that are self-adjoint, their coefficients have some local regularity, and they are strongly convex in the sense of [9]. We make use of a certain stability property inherent in our approach and obtain bounds that are asymptotically sharp: they involve the exponential constant  $\sigma_m - cD$ , where  $c$  is an absolute constant and  $D$  is the distance of the symbol  $A(x, \xi)$  from the class  $\mathcal{G}_a$  in a certain weighted norm. In particular, the constant  $\sigma_m$  is obtained for symbols in  $\mathcal{G}_a$ . To the best of our knowledge such estimates are new even if the coefficients are assumed to be smooth and the symbol lies in  $\mathcal{G}_a$ .

## 2. Formulation of results

We first fix some notation. Given a multi-index  $\alpha = (\alpha_1, \dots, \alpha_N)$  we write  $\alpha! = \alpha_1! \cdots \alpha_N!$  and  $|\alpha| = \alpha_1 + \cdots + \alpha_N$ . We write  $\gamma \leq \alpha$  to indicate that  $\gamma_i \leq \alpha_i$  for all  $i$ , in which case we also set  $c_\gamma^\alpha = \alpha! / \gamma! (\alpha - \gamma)!$ . We use the standard notation  $D^\alpha$  for the differential expression  $(\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_N)^{\alpha_N}$ , and for  $k \geq 0$  we denote by  $\nabla^k f$  the vector  $(D^\alpha f)_{|\alpha|=k}$ . We denote by  $\hat{f}$  the Fourier transform of a function  $f$ ,  $\hat{f}(\xi) = (2\pi)^{-N/2} \int e^{i\xi \cdot x} f(x) dx$ . We shall denote by  $\|A\|_{p \rightarrow q}$  the norm of an operator  $A$  from  $L^p(\Omega)$  to  $L^q(\Omega)$ . The letter  $c$  will stand for a positive constant whose value may change from line to line.

Let  $\Omega$  be a domain in  $\mathbb{R}^N$ . We fix an integer  $m \geq 1$  and consider the operator

$$Hf(x) = (-1)^m \sum_{\substack{|\alpha|=m \\ |\beta|=m}} D^\alpha \{a_{\alpha\beta}(x) D^\beta f(x)\} \quad (2.1)$$

subject to Dirichlet boundary conditions on  $\partial\Omega$ ; the precise definition shall be given below. The matrix-valued function  $\{a_{\alpha\beta}\}$  is assumed to be measurable and to take its

values in the set of all complex,  $\nu \times \nu$  matrices,  $\nu$  being the number of multi-indices  $\alpha$  of length  $|\alpha| = m$ . We assume that each  $a_{\alpha\beta}$  lies in  $L^\infty_{\text{loc}}(\Omega)$ ; we do not assume  $\{a_{\alpha\beta}\}$  to be self-adjoint.

We define a quadratic form  $Q(\cdot)$  on  $C_c^\infty(\Omega)$  by

$$Q(f) = \int_{\Omega} \sum_{\substack{|\alpha|=m \\ |\beta|=m}} a_{\alpha\beta}(x) D^\alpha f(x) D^\beta \bar{f}(x) dx, \quad f \in C_c^\infty(\Omega).$$

We assume that there exists a positive weight  $a(x)$  with  $a^{\pm 1} \in L^\infty_{\text{loc}}(\Omega)$  that controls the size of the matrix  $\{a_{\alpha\beta}\}$  in the following sense: first,

$$|a_{\alpha\beta}(x)| \leq ca(x), \quad x \in \Omega, \quad (2.2)$$

for all multi-indices  $\alpha, \beta$ ; and second, the weighted Gårding inequality

$$\operatorname{Re} Q(f) \geq c \int_{\Omega} a(x) |\nabla^m f|^2 dx, \quad f \in C_c^\infty(\Omega), \quad (2.3)$$

is valid for some  $c > 0$ . We also assume the symbol version of (2.3), namely

$$\operatorname{Re} A(x, \xi) \geq ca(x) |\xi|^{2m}, \quad x \in \Omega, \quad \xi \in \mathbb{R}^N, \quad (2.4)$$

where  $A(x, \xi) := \sum a_{\alpha\beta}(x) \xi^{\alpha+\beta}$ . Relations (2.2) and (2.3) imply in particular that there exists  $\beta > 0$  such that

$$|Q(f)| \leq \beta \operatorname{Re} Q(f), \quad f \in C_c^\infty(\Omega). \quad (2.5)$$

It is easily seen that  $Q$  is closable [3]. The domain of its closure is a weighted Sobolev space that we denote by  $W_{a,0}^{m,2}(\Omega)$ . We retain the same symbol,  $Q$ , for the closure of the above form and denote by  $H$  the associated accretive operator on  $L^2(\Omega)$ , so that  $\langle Hf, f \rangle = Q(f)$ ,  $f \in \operatorname{Dom}(H)$ , and (2.1) is valid in a weak sense.

We make two hypotheses on the weight  $a$ : the first is a weighted Sobolev inequality and the second is a weighted interpolation inequality.

(H1) There exists  $s \in [N/2m, 1]$  and  $c > 0$  such that

$$\|f\|_\infty \leq c [\operatorname{Re} Q(f)]^{s/2} \|f\|_2^{1-s}, \quad f \in C_c^\infty(\Omega). \quad (2.6)$$

(H2) There exists a constant  $c$  such that

$$\int_{\Omega} a^{k/m} |\nabla^k f|^2 dx < \epsilon \int_{\Omega} a |\nabla^m f|^2 dx + c\epsilon^{-k/(m-k)} \int_{\Omega} |f|^2 dx, \quad (2.7)$$

for all  $0 < \epsilon < 1$ ,  $0 \leq k < m$  and all  $f \in C_c^\infty(\Omega)$ .

Both (H1) and (H2) are satisfied when  $H$  is uniformly elliptic, in which case the best value for the constant  $s$  is  $s = N/2m$ , showing that in the general case we cannot expect any value that is better (smaller) than  $N/2m$ ; in particular, (H1) is valid trivially with

$s = N/2m$  if  $a(x)$  is bounded away from zero. We refer to [3] for non-trivial examples for which (H1) and (H2) are satisfied; they involve suitable powers of either  $1 + |x|$  or  $\text{dist}(x, K)$ , where  $K$  is a smooth surface of lower dimension.

We note that condition (H2) implies that for any  $k, l$  with  $0 \leq k, l \leq m$ ,  $k + l < 2m$ , there exists a constant  $c$  such that

$$(1 + \lambda^{2m-k-l}) \int_{\Omega} a^{(k+l)/2m} |\nabla^k f| |\nabla^l f| dx < \epsilon \text{Re} Q(f) + c\epsilon^{-(k+l)/(2m-k-l)} (1 + \lambda^{2m}) \|f\|_2^2, \quad (2.8)$$

for all  $\epsilon \in (0, 1)$ ,  $\lambda > 0$  and all  $f \in C_c^\infty(\Omega)$ . Indeed, for  $\lambda = 1$ , (2.8) is a consequence of (H2) and the Cauchy–Schwarz inequality; the case  $\lambda < 1$  follows trivially from the case  $\lambda = 1$ ; finally, writing (2.8) for  $\lambda = 1$  and replacing  $\epsilon$  by  $\epsilon\lambda^{k+l-2m}$  we obtain the result for  $\lambda > 1$ .

Next we introduce the distance that shall be used in the heat-kernel estimates. Consider the set

$$\mathcal{E}_a = \{\phi \in C^\infty(\Omega) \cap L^\infty(\Omega) : \phi \text{ real valued and } a^{k/2m} \nabla^k \phi \in L^\infty(\Omega), 1 \leq k \leq m\}$$

and its subset (recall (2.4))

$$\begin{aligned} \mathcal{E}_{A,M} = \{\phi \in C^\infty(\Omega) \cap L^\infty(\Omega) : \text{Re} A(x, \nabla \phi(x)) \leq 1, \\ |\nabla^k \phi(x)| \leq M a(x)^{-k/2m}, 2 \leq k \leq m, \text{ a.e. } x \in \Omega\}. \end{aligned} \quad (2.9)$$

Our estimates will be expressed in terms of the distance

$$d_M(x, y) = \sup\{\phi(y) - \phi(x) : \phi \in \mathcal{E}_{A,M}\} \quad (2.10)$$

for arbitrarily large but finite  $M$ . For  $M = +\infty$  this reduces to the distance

$$d_\infty(x, y) = \sup\{\phi(y) - \phi(x) : \text{Re} A(x, \nabla \phi(x)) \leq 1, x \in \Omega\}.$$

This is a Finsler distance, induced by the (singular/degenerate) Finsler metric with length element

$$ds = ds(x, dx) = \sup_{\substack{\eta \in \mathbb{R}^N \\ \eta \neq 0}} \frac{\langle dx, \eta \rangle}{(\text{Re} A(x, \eta))^{m/2}}. \quad (2.11)$$

We refer the reader to the recent book [2] for a comprehensive introduction to Finsler geometry. The distance  $d_\infty(x, y)$  relates to the short-time off-diagonal behaviour of the heat kernel: it was shown in [10] that if  $\Omega = \mathbb{R}^N$  and  $H$  is self-adjoint, uniformly elliptic with strongly convex symbol (see (2.13)), then  $d_\infty(\cdot, \cdot)$  controls the small-time behaviour of  $K(t, x, y)$  in the sense that

$$\log t^{N/2m} K(t, x, y) = -\sigma_m \frac{d_\infty(x, y)^{2m/(2m-1)}}{t^{1/(2m-1)}} (1 + o(1)), \quad \text{as } t \rightarrow 0, \quad (2.12)$$

for  $x, y$  fixed and close enough; here and below we have

$$\sigma_m = (2m - 1)(2m)^{-2m/(2m-1)} \sin(\pi/(4m - 2)).$$

Let us now proceed with the definition of the class  $\mathcal{G}_a$ . Let the functions  $a_\gamma(\cdot)$ ,  $|\gamma| = 2m$ , be defined by requiring that

$$\sum_{\substack{|\alpha|=m \\ |\beta|=m}} a_{\alpha\beta}(x)\xi^{\alpha+\beta} = \sum_{|\gamma|=2m} c_\gamma^{2m} a_\gamma(x)\xi^\gamma, \quad x \in \Omega, \quad \xi \in \mathbb{R}^N$$

(recall that  $c_\gamma^{2m} = (2m)!/\gamma!$ ). Following [9] we say that the principal symbol  $A(x, \xi)$  of  $H$  is strongly convex if the quadratic form

$$\Gamma(x, p) = \sum_{\substack{|\alpha|=m \\ |\beta|=m}} a_{\alpha+\beta}(x) p_\alpha \bar{p}_\beta, \quad p = (p_\alpha) \in \mathbb{C}^\nu, \quad (2.13)$$

is positive semidefinite for a.e.  $x \in \Omega$ .

Induced by the weight  $a(x)$  is the weighted Sobolev space

$$W_a^{m-1, \infty}(\Omega) = \{f \in W_{\text{loc}}^{m-1, \infty}(\Omega) : |\nabla^i f(x)| \leq ca(x)^{(2m-i)/2m}, \text{ a.e. } x \in \Omega, \ i \leq m-1\}. \quad (2.14)$$

**Definition 2.1.** We say that the symbol  $A(x, \xi)$  lies in  $\mathcal{G}_a$  if

- (i)  $A(x, \xi)$  is strongly convex;
- (ii)  $\{a_{\alpha\beta}\}$  is real and symmetric;
- (iii) the coefficients  $a_{\alpha\beta}$  lie in  $W_a^{m-1, \infty}(\Omega)$ .

We denote by  $D$  the distance of the coefficient matrix  $\{a_{\alpha\beta}\}$  from  $\mathcal{G}_a$  in the weighted uniform norm

$$\|f\|_{a, \infty} := \sup_{x \in \Omega} |f(x)/a(x)|;$$

that is

$$D = \inf_{\{\tilde{a}_{\alpha\beta}\}} \|\{a_{\alpha\beta}\} - \{\tilde{a}_{\alpha\beta}\}\|_{a, \infty}, \quad (2.15)$$

where the infimum is taken over all matrix-valued functions  $\{\tilde{a}_{\alpha\beta}\}$  that induce a symbol in  $\mathcal{G}_a$ . Here we have used the notation

$$\|\{b_{\alpha\beta}\}\|_{a, \infty} := \sup_{x \in \Omega} \frac{|\{b_{\alpha\beta}(x)\}|}{a(x)},$$

where, for each  $x \in \Omega$ ,  $|\{b_{\alpha\beta}(x)\}|$  denotes the norm of  $\{b_{\alpha\beta}(x)\}$  regarded as an operator on  $\mathbb{C}^\nu$ .

Our main result is as follows.

**Theorem 2.2.** *Assume that (H1) and (H2) are satisfied. Then for all  $\delta \in (0, 1)$  and all  $M$  large there exist positive constants  $c_\delta$ ,  $c_{\delta, M}$  such that*

$$|K(t, x, y)| < c_\delta t^{-s} \exp\{-(\sigma_m - cD - \delta)d_M(x, y)^{2m/(2m-1)} t^{-1/(2m-1)} + c_{\delta, M} t\} \quad (2.16)$$

for all  $x, y \in \Omega$  and  $t > 0$ ; the constant  $c$  is independent of  $x, y, t, \delta, D$  and  $M$ .

In the special case where  $H$  is uniformly elliptic and self-adjoint this estimate has already been obtained in [4].

### 3. Proof of Theorem 2.2

Given  $\phi \in \mathcal{E}_a$ , the mapping  $f \mapsto e^\phi f$  maps  $W_{a,0}^{m,2}(\Omega)$  into itself [3, Lemma 7]. Hence one can define a sesquilinear form  $Q_\phi(\cdot, \cdot)$  with domain  $W_{a,0}^{m,2}(\Omega)$  by

$$\begin{aligned} Q_\phi(f) &= Q(e^\phi f, e^{-\phi} f) \\ &= \int_{\Omega} \sum_{\substack{|\alpha|=m \\ |\beta|=m}} a_{\alpha\beta} D^\alpha(e^\phi f) D^\beta(e^{-\phi} \bar{f}) dx, \quad f \in W_{a,0}^{m,2}(\Omega). \end{aligned} \quad (3.1)$$

The associated operator is  $H_\phi = e^{-\phi} H e^\phi$  and has domain  $\text{Dom}(H_\phi) = e^{-\phi} \text{Dom}(H)$ . The form  $Q_\phi$  is a lower-order perturbation of  $Q$  (cf. (3.8)) and it is a consequence of (H2) [3, Lemma 8] that for all  $\epsilon > 0$  and  $f \in W_{a,0}^{m,2}(\Omega)$ , the following inequality holds:

$$|Q(f) - Q_\phi(f)| < \epsilon \text{Re} Q(f) + c\epsilon^{-2m+1} (1 + p(\phi))^{2m} \|f\|_2^2, \quad (3.2)$$

where we have used the seminorm

$$p(\phi) := \sup_{1 \leq k \leq m} \text{ess sup}_{x \in \Omega} a(x)^{k/2m} |\nabla^k \phi(x)|. \quad (3.3)$$

Defining  $s(\phi) = (1 + p(\phi))^{2m}$ , it follows in particular that

$$\text{Re} Q_\phi(f) \geq -cs(\phi) \|f\|_2^2, \quad f \in C_c^\infty(\Omega), \quad (3.4)$$

where  $c$  is independent of  $\phi$ , and this justifies the definition

$$-k_\phi = \inf\{\text{Re} Q_\phi(f) : f \in C_c^\infty(\Omega), \|f\|_2 = 1\}. \quad (3.5)$$

The next lemma closely follows an argument used in [5].

**Lemma 3.1.** *Assume that (H2) is satisfied. Then for any  $\phi \in \mathcal{E}_a$ , the following inequalities hold:*

- (i)  $\|e^{-H_\phi t}\|_{2 \rightarrow 2} \leq e^{k_\phi t}$ ,
- (ii)  $\|H_\phi e^{-H_\phi t}\|_{2 \rightarrow 2} \leq (c_\delta/t) e^{k_\phi t} e^{\delta s(\phi)t}$ , for all  $\delta > 0$ ,

where the constant  $c_\delta$  is independent of  $\phi \in \mathcal{E}_a$  and  $t > 0$ .

**Proof.** Part (i) is the standard energy estimate that follows by integrating

$$\frac{d}{dt} \|e^{-H_\phi t} f\|_2^2 = -2 \text{Re} \langle H_\phi e^{-H_\phi t} f, e^{-H_\phi t} f \rangle \leq 2k_\phi \|e^{-H_\phi t} f\|_2^2.$$

Now by (3.2) the following inequality holds:

$$|Q_\phi(f) - Q(f)| \leq \frac{1}{2} \text{Re} Q(f) + c' s(\phi) \|f\|_2^2, \quad f \in C_c^\infty(\Omega), \quad (3.6)$$

where  $c' > 0$  depends only on  $m$ . Hence, for any  $\epsilon \in (0, 1)$ ,

$$\begin{aligned} \operatorname{Re} Q_\phi(f) &= \epsilon \operatorname{Re} Q_\phi(f) + (1 - \epsilon) \operatorname{Re} Q_\phi(f) \\ &\geq \frac{1}{2} \epsilon \operatorname{Re} Q(f) - [c' \epsilon s(\phi) + (1 - \epsilon) k_\phi] \|f\|_2^2, \end{aligned}$$

and hence

$$\operatorname{Re}[Q(f) - Q_\phi(f)] \leq (1 - \frac{1}{2}\epsilon) \operatorname{Re} Q(f) + [c' \epsilon s(\phi) + (1 - \epsilon) k_\phi] \|f\|_2^2.$$

Fix  $f \in L^2(\Omega)$  and  $\theta \in (-\pi/2, \pi/2)$ , and for  $\rho > 0$  set  $f_\rho = \exp(-H_\phi \rho e^{i\theta}) f$ . We then have

$$\begin{aligned} \frac{d}{d\rho} \|f_\rho\|_2^2 &= -2 \operatorname{Re}[e^{i\theta} Q_\phi(f_\rho)] \\ &= -2 \cos \theta \operatorname{Re} Q(f_\rho) + 2 \sin \theta \operatorname{Im} Q_\phi(f_\rho) + 2 \cos \theta [\operatorname{Re} Q(f_\rho) - \operatorname{Re} Q_\phi(f_\rho)] \\ &\leq -2 \cos \theta \operatorname{Re} Q(f_\rho) + 2 \sin |\theta| [(\frac{1}{2} + \beta) \operatorname{Re} Q(f_\rho) + c' s(\phi) \|f_\rho\|_2^2] \\ &\quad + 2 \cos \theta [(1 - \frac{1}{2}\epsilon) \operatorname{Re} Q(f_\rho) + [c' \epsilon s(\phi) + (1 - \epsilon) k_\phi] \|f_\rho\|_2^2] \\ &= [-\epsilon \cos \theta + (2\beta + 1) \sin |\theta|] \operatorname{Re} Q(f_\rho) \\ &\quad + [2 \cos \theta \{c' \epsilon s(\phi) + (1 - \epsilon) k_\phi\} + 2c' \sin |\theta| s(\phi)] \|f_\rho\|_2^2. \end{aligned}$$

Let  $\alpha \in (0, \pi/2)$  be such that  $\tan \alpha = \epsilon/(2\beta + 1)$ . For  $|\theta| \leq \alpha$  we then have  $-\epsilon \cos \theta + (2\beta + 1) \sin |\theta| \leq 0$  and hence

$$\begin{aligned} \frac{d}{d\rho} \|f_\rho\|_2^2 &\leq 2 \cos \theta \left[ c' \epsilon s(\phi) + (1 - \epsilon) k_\phi + s(\phi) \frac{c' \epsilon}{2\beta + 1} \right] \|f_\rho\|_2^2 \\ &\leq 2(k_\phi + 2c' \epsilon s(\phi)) \|f_\rho\|_2^2 \\ &=: 2A_\epsilon \|f_\rho\|_2^2. \end{aligned}$$

It follows that  $\|e^{-H_\phi z}\|_{2 \rightarrow 2} \leq e^{A_\epsilon |z|}$  in the sector  $|\arg z| \leq \alpha$ . We conclude that letting

$$\tau_\epsilon = \frac{A_\epsilon}{\cos \alpha}$$

we have

$$\|\exp\{-(H_\phi + \tau_\epsilon)z\}\|_{2 \rightarrow 2} \leq 1,$$

and hence [6, Lemma 2.38]

$$\|(H_\phi + \tau_\epsilon)e^{-(H_\phi + \tau_\epsilon)t}\| \leq \frac{c}{\alpha t},$$

for all  $t > 0$ . Multiplying both sides by  $e^{\tau_\epsilon t}$  and using the triangle inequality we obtain

$$\|H_\phi e^{-H_\phi t}\|_{2 \rightarrow 2} \leq \frac{c}{\alpha t} \exp\left\{\frac{k_\phi + 2c' \epsilon s(\phi)}{\cos \alpha} t\right\} + \tau_\epsilon e^{k_\phi t}.$$

This last expression can be made smaller than the right-hand side of Lemma 3.1 (ii) provided  $\epsilon$  is chosen small enough; this completes the proof.  $\square$

**Proposition 3.2.** *Assume that (H1) and (H2) are satisfied. Then for any  $\delta > 0$  there exists  $c_\delta > 0$  independent of  $\phi \in \mathcal{E}_a$  such that*

$$\|e^{-H_\phi t}\|_{1 \rightarrow \infty} \leq c_\delta t^{-s} e^{k_\phi t} e^{\delta s(\phi)t}. \quad (3.7)$$

**Proof.** Let  $f \in L^2(\Omega)$  and set  $f_t = e^{-H_\phi t} f$ ,  $t > 0$ . Using (H1) we have

$$\begin{aligned} \|f_t\|_\infty &\leq c[\operatorname{Re} Q(f_t)]^{s/2} \|f_t\|_2^{1-s} \\ &\leq c[\operatorname{Re} Q_\phi(f_t) + s(\phi)\|f_t\|_2^2]^{s/2} \|f_t\|_2^{1-s} \quad (\text{by (3.6)}) \\ &\leq c[\|H_\phi f_t\|_2 \|f_t\|_2 + s(\phi)\|f_t\|_2^2]^{s/2} \|f_t\|_2^{1-s} \\ &\leq c[(c_\epsilon/t)e^{\epsilon s(\phi)t} + s(\phi)]^{s/2} e^{k_\phi t} \|f\|_2 \quad (\text{by Lemma 3.1 (ii) and Lemma 3.1 (i)}) \\ &= ct^{-s/2} [c_\epsilon e^{\epsilon s(\phi)t} + s(\phi)t]^{s/2} e^{k_\phi t} \|f\|_2. \end{aligned}$$

Taking  $\epsilon$  to be small enough we conclude that given  $\delta > 0$  there exists  $c_\delta$  such that

$$\|e^{-H_\phi t}\|_{2 \rightarrow \infty} \leq c_\delta t^{-s/2} e^{k_\phi t} e^{\delta s(\phi)t}.$$

The same arguments are valid for  $(H_\phi)^* = (H^*)_{-\phi}$ , the constant  $k_\phi$  clearly staying the same. Hence by duality and the semigroup property, (3.7) follows.  $\square$

In order for Proposition 3.2 to be useful we need a precise upper estimate on  $k_\phi$ , which amounts to a precise lower estimate on  $\operatorname{Re} Q_\phi(\cdot)$  (cf. (3.5)). This will be established in Lemma 3.11 following a series of intermediate lemmas. Recalling that  $c_\gamma^\alpha = \alpha!/\gamma!(\alpha-\gamma)!$  it follows immediately from (3.1) that for  $\lambda > 0$ ,  $\phi \in \mathcal{E}_a$  we have

$$Q_{\lambda\phi}(f) = \int_\Omega \sum_{\substack{|\alpha|=m \\ |\beta|=m}} a_{\alpha\beta} \sum_{\substack{\gamma \leq \alpha \\ \delta \leq \beta}} c_\gamma^\alpha c_\delta^\beta P_{\gamma,\lambda\phi} P_{\delta,-\lambda\phi} D^{\alpha-\gamma} f D^{\beta-\delta} \bar{f} \, dx, \quad (3.8)$$

where

$$P_{\gamma,\lambda\phi}(x) := e^{-\lambda\phi(x)} D^\gamma [e^{\lambda\phi(x)}]$$

is a polynomial in various derivatives of  $\lambda\phi$ . Now, the induction relation  $P_{\gamma+e_j,\lambda\phi} = (\lambda\partial_j\phi + \partial_j)P_{\gamma,\lambda\phi}$  implies that  $P_{\gamma,\lambda\phi}$  has the form

$$P_{\gamma,\lambda\phi}(x) = \sum_{k=1}^{|\gamma|} \lambda^k \sum c_{\gamma;\gamma_1,\dots,\gamma_k} (D^{\gamma_1}\phi) \cdots (D^{\gamma_k}\phi), \quad (3.9)$$

where the second sum is taken over all non-zero multi-indices  $\gamma_1, \dots, \gamma_k$  such that  $\gamma_1 + \dots + \gamma_k = \gamma$  and  $c_{\gamma;\gamma_1,\dots,\gamma_k}$  are constants. Hence, recalling that  $|\nabla^k \phi| \leq ca^{-k/2m}$ , we can write

$$P_{\gamma,\lambda\phi}(x) = \sum_{k=1}^{|\gamma|} \lambda^k \tilde{P}_{k,\phi}(x),$$

where  $|\tilde{P}_{k,\phi}(x)| \leq ca^{-|\gamma|/2m}$ . It follows from (3.8) that

$$Q_{\lambda\phi}(f) = \int_{\Omega} \sum_{\substack{|\alpha|=m \\ |\beta|=m}} \sum_{\substack{\gamma \leq \alpha \\ \delta \leq \beta}} \sum_{\substack{k \leq |\gamma| \\ j \leq |\delta|}} \lambda^{k+j} w_{\alpha\beta\gamma\delta k j}(x) D^{\alpha-\gamma} f D^{\beta-\delta} \bar{f} dx, \quad (3.10)$$

where  $w_{\alpha\beta\gamma\delta k j} := a_{\alpha\beta} c_{\gamma}^{\alpha} c_{\delta}^{\beta} \tilde{P}_{k,\phi} \tilde{P}_{j,-\phi}$  satisfies  $|w_{\alpha\beta\gamma\delta k j}| \leq ca^{(2m-|\gamma+\delta|)/2m}$ . Replacing  $\gamma$  and  $\delta$  by  $\alpha - \gamma$  and  $\beta - \delta$ , respectively, we conclude from (3.10) the following lemma.

**Lemma 3.3.**  $Q_{\lambda\phi}(f)$  is a linear combination of terms of the form

$$T(f) = \lambda^s \int_{\Omega} w(x) D^{\gamma} f D^{\delta} \bar{f} dx, \quad (3.11)$$

where  $|w| \leq ca^{(|\gamma+\delta|)/2m}$  on  $\Omega$  and

- (i)  $s$  is an integer with  $0 \leq s \leq 2m$ ;
- (ii)  $\gamma$  and  $\delta$  are multi-indices with  $|\gamma|, |\delta| \leq m$ ;
- (iii)  $s + |\gamma + \delta| \leq 2m$ .

**Definition 3.4.** We call the number  $s + |\gamma + \delta|$  the *essential order* of  $T$ .

Hence the essential order is an integer between 0 and  $2m$ . We denote by  $\mathcal{L}_{a,m}$  the linear space consisting of (finite) linear combinations of forms whose essential order is smaller than  $2m$ . In Lemma 3.9 we will see that terms in  $\mathcal{L}_{a,m}$  are in a sense negligible. We also point out for later use that (2.8) implies the interpolation inequality

$$|T(f)| < c\{\operatorname{Re} Q(f) + \lambda^{2m} \|f\|_2^2\}, \quad f \in W_{a,0}^{m,2}(\Omega), \quad (3.12)$$

valid for all terms  $T(\cdot)$  of essential order  $2m$ .

We have the following lemma.

**Lemma 3.5.** Given  $\phi \in \mathcal{E}_a$  and  $\lambda > 0$  define

$$Q_{1,\lambda\phi}(f) = \int_{\Omega} \sum_{\substack{|\alpha|=m \\ |\beta|=m}} \sum_{\substack{\gamma \leq \alpha \\ \delta \leq \beta}} a_{\alpha\beta} c_{\gamma}^{\alpha} c_{\delta}^{\beta} (\lambda \nabla \phi)^{\gamma} (-\lambda \nabla \phi)^{\delta} D^{\alpha-\gamma} f D^{\beta-\delta} \bar{f} dx.$$

Then the difference  $Q_{\lambda\phi}(f) - Q_{1,\lambda\phi}(f)$  lies in  $\mathcal{L}_{a,m}$ .

**Proof.** One simply has to recall (3.8) and observe from (3.9) that  $P_{\gamma,\lambda\phi}$ , considered as a polynomial in  $\lambda$ , has  $\lambda^{|\gamma|} (\nabla \phi)^{\gamma}$  as its highest-degree term.  $\square$

### 3.1. Symbols in $\mathcal{G}_a$

At this point and for the whole of this subsection we restrict our attention to operators  $H$  whose symbol belongs to  $\mathcal{G}_a$ . For  $x \in \Omega$ ,  $\xi, \eta \in \mathbb{C}^N$  and  $\zeta \in \mathbb{R}^N$  let us define

$$\begin{aligned} k_m &= [\sin(\pi/(4m-2))]^{-2m+1}, \\ A(x, \xi, \eta) &= \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x) \xi^\alpha \bar{\eta}^\beta, \\ S(x, \zeta; \xi, \eta) &= \operatorname{Re} A(x, \xi - i\zeta, \eta + i\zeta) + k_m \operatorname{Re} A(x, \zeta). \end{aligned}$$

**Lemma 3.6.** *Assume that the symbol  $A(x, \xi)$  lies in  $\mathcal{G}_a$ . Then*

$$\begin{aligned} \operatorname{Re} Q_{1, \lambda\phi}(f) + k_m \lambda^{2m} \int_{\Omega} \operatorname{Re} A(x, \nabla\phi(x)) |f|^2 dx \\ = (2\pi)^{-N} \iiint_{\Omega \times \mathbb{R}^N \times \mathbb{R}^N} S(x, \lambda\nabla\phi; \xi, \eta) e^{i(\xi-\eta)\cdot x} \hat{f}(\xi) \overline{\hat{f}(\eta)} dx d\xi d\eta \end{aligned} \quad (3.13)$$

for all  $\phi \in \mathcal{E}_a$ ,  $\lambda > 0$  and  $f \in C_c^\infty(\Omega)$ .

**Proof.** Writing

$$D^\gamma f(x) = (2\pi)^{-N/2} \int_{\mathbb{R}^N} (i\xi)^\gamma e^{i\xi\cdot x} \hat{f}(\xi) d\xi$$

we have

$$\begin{aligned} Q_{1, \lambda\phi}(f) &= (2\pi)^{-N} \iiint_{\Omega \times \mathbb{R}^N \times \mathbb{R}^N} \sum_{\substack{|\alpha|=m \\ |\beta|=m}} a_{\alpha\beta} \sum_{\substack{\gamma \leq \alpha \\ \delta \leq \beta}} c_\gamma^\alpha c_\delta^\beta (-i\lambda\nabla\phi)^\gamma (-i\lambda\nabla\phi)^\delta \\ &\quad \times \xi^{\alpha-\gamma} \eta^{\beta-\delta} e^{i(\xi-\eta)\cdot x} \hat{f}(\xi) \overline{\hat{f}(\eta)} d\xi d\eta dx \\ &= (2\pi)^{-N} \iiint_{\Omega \times \mathbb{R}^N \times \mathbb{R}^N} \sum_{\substack{|\alpha|=m \\ |\beta|=m}} a_{\alpha\beta} (\xi - i\lambda\nabla\phi)^\alpha (\eta - i\lambda\nabla\phi)^\beta \\ &\quad \times e^{i(\xi-\eta)\cdot x} \hat{f}(\xi) \overline{\hat{f}(\eta)} d\xi d\eta dx \\ &= (2\pi)^{-N} \iiint_{\Omega \times \mathbb{R}^N \times \mathbb{R}^N} A(x, \xi - i\lambda\nabla\phi(x), \eta + i\lambda\nabla\phi(x)) \\ &\quad \times e^{i(\xi-\eta)\cdot x} \hat{f}(\xi) \overline{\hat{f}(\eta)} d\xi d\eta dx. \end{aligned}$$

This last integral has the form  $\int_{\Omega} q[g] dx$ , where, for fixed  $x \in \Omega$ ,

$$\begin{aligned} g(\xi) &= e^{i\xi\cdot x} \hat{f}(\xi), \\ q[g] &= \int_{\mathbb{R}^N \times \mathbb{R}^N} p(\xi, \eta) g(\xi) \overline{g(\eta)} d\xi d\eta, \\ p(\xi, \eta) &= A(x, \xi - i\lambda\nabla\phi(x), \eta + i\lambda\nabla\phi(x)). \end{aligned}$$

Since the matrix  $\{a_{\alpha\beta}\}$  is symmetric we have  $p(\xi, \eta) = p(\eta, \xi)$  and therefore

$$\overline{q[g]} = \int_{\mathbb{R}^N \times \mathbb{R}^N} \overline{p(\xi, \eta)g(\xi)g(\eta)} \, d\xi \, d\eta.$$

Hence

$$\operatorname{Re} q[g] = \int_{\mathbb{R}^N \times \mathbb{R}^N} \operatorname{Re} p(\xi, \eta) \, d\xi \, d\eta$$

and integration over  $x \in \Omega$  yields

$$\begin{aligned} & \operatorname{Re} Q_{1, \lambda\phi}(f) + k_m \int_{\Omega} \operatorname{Re} A(x, \lambda\nabla\phi(x)) |f|^2 \, dx \\ &= (2\pi)^{-N} \iiint_{\Omega \times \mathbb{R}^N \times \mathbb{R}^N} \operatorname{Re}[A(x, \xi - i\lambda\nabla\phi(x), \eta + i\lambda\nabla\phi(x)) + k_m A(x, \lambda\nabla\phi)] \\ & \quad \times e^{i(\xi-\eta)\cdot x} \hat{f}(\xi) \overline{\hat{f}(\eta)} \, d\xi \, d\eta \, dx \\ &= (2\pi)^{-N} \iiint_{\Omega \times \mathbb{R}^N \times \mathbb{R}^N} S(x, \lambda\nabla\phi; \xi, \eta) e^{i(\xi-\eta)\cdot x} \hat{f}(\xi) \overline{\hat{f}(\eta)} \, d\xi \, d\eta \, dx. \end{aligned}$$

□

We now proceed to estimate the triple integral on the right-hand side of (3.13). It is shown in [9, Theorem 2.1] that there exist positive numbers  $w_0, \dots, w_{m-2}$  such that

$$S(x, \zeta; \xi, \xi) = \sum_{s=0}^{m-2} w_s \Gamma(x, p_{\xi, \zeta}^{(s)}), \quad x \in \Omega\zeta, \quad \xi \in \mathbb{R}^N, \quad (3.14)$$

where  $\Gamma(x, \cdot)$  is the quadratic form associated with the principal symbol of  $H$  (cf. (2.13)) and  $p_{\xi, \zeta}^{(s)}$  is the vector in  $\mathbb{R}^\nu$  defined for fixed  $\xi, \zeta \in \mathbb{R}^N$  by requiring that

$$\sum_{|\alpha|=m} p_{\xi, \zeta, \alpha}^{(s)} a^\alpha = (\sin \theta_m)^{-s-2} (\xi \cdot a)^{m-s-2} (\zeta \cdot a)^s \{(\sin \theta_m)^2 (\xi \cdot a)^2 - (\cos \theta_m)^2 (\zeta \cdot a)^2\} \quad (3.15)$$

for all  $a \in \mathbb{R}^N$ ; here  $\theta_m = \pi/(4m-2)$ . To simplify the notation let us define the sesquilinear forms  $\mathbf{\Gamma}(x, \cdot, \cdot)$  on  $\mathbb{C}^{m-1} \otimes \mathbb{C}^\nu \simeq \mathbb{C}^{\nu(m-1)}$  by

$$\mathbf{\Gamma}(x, u, v) = \sum_{s=0}^{m-2} w_s \Gamma(x, u^{(s)}, v^{(s)}) = \sum_{s=0}^{m-2} \sum_{\substack{|\alpha|=m \\ |\beta|=m}} w_s a_{\alpha+\beta}(x) u_\alpha^{(s)} \overline{v_\beta^{(s)}}$$

for all  $u = (u_\alpha^{(s)})$ ,  $v = (v_\beta^{(s)}) \in \mathbb{C}^{\nu(m-1)}$ . Then  $\mathbf{\Gamma}$  is positive semi-definite by the strong convexity of  $A(x, \xi)$ . To handle the above expressions we introduce two auxiliary elliptic differential forms  $S_{\lambda\phi}$  and  $\Gamma_{\lambda\phi}$  on  $L^2(\Omega)$ . They have common domain  $W_{a,0}^{m,2}(\Omega)$  and are given by

$$S_{\lambda\phi}(f) = (2\pi)^{-N} \iiint_{\Omega \times \mathbb{R}^N \times \mathbb{R}^N} S(x, \lambda\nabla\phi; \xi, \eta) e^{i(\xi-\eta)\cdot x} \hat{f}(\xi) \overline{\hat{f}(\eta)} \, d\xi \, d\eta \, dx, \quad (3.16)$$

$$\Gamma_{\lambda\phi}(f) = (2\pi)^{-N} \iiint_{\Omega \times \mathbb{R}^N \times \mathbb{R}^N} \mathbf{\Gamma}(x, p_{\xi, \lambda\nabla\phi}, p_{\eta, \lambda\nabla\phi}) e^{i(\xi-\eta)\cdot x} \hat{f}(\xi) \overline{\hat{f}(\eta)} \, d\xi \, d\eta \, dx, \quad (3.17)$$

where

$$p_{\xi, \lambda \nabla \phi} = (p_{\xi, \lambda \nabla \phi, \alpha})_{0 \leq s \leq m-2}^{|\alpha|=m} \in \mathbb{C}^{\nu(m-1)}$$

is defined by (3.15).

**Lemma 3.7.** *Assume that the symbol  $A(x, \xi)$  lies in  $\mathcal{G}_a$ . Then the form  $S_{\lambda \phi}(\cdot) - \Gamma_{\lambda \phi}(\cdot)$  lies in  $\mathcal{L}_{a,m}$ .*

**Proof.** It follows from (3.14) that  $S_{\lambda \phi}$  and  $\Gamma_{\lambda \phi}$  have integral kernels which are polynomials of  $\xi$  and  $\eta$  and whose values coincide for  $\xi = \eta$ . Using the inverse Fourier transform this implies that the difference  $S_{\lambda \phi}(f) - \Gamma_{\lambda \phi}(f)$  is a linear combination of terms of the form

$$T(f) = \lambda^s \int_{\Omega} w(x) [D^{\gamma+\kappa} f D^{\delta} \bar{f} - (-1)^{\kappa} D^{\gamma} f D^{\delta+\kappa} \bar{f}] dx, \quad (3.18)$$

where  $w$  is some function and  $\kappa$  is a multi-index of length  $|\kappa| \leq m-1$ . In fact, recalling (3.13) and the definition of  $Q_{1, \lambda \phi}$  we see that  $w = a_{\alpha\beta}(\nabla \phi)^{\mu}$ , where  $|\mu| = s$  and  $\gamma + \delta + \kappa + \mu = \alpha + \beta$ . Since  $a_{\alpha\beta} \in W_a^{m-1, \infty}(\Omega) \subset W_{\text{loc}}^{m-1, \infty}(\Omega)$ , we can integrate by parts  $|\kappa|$  times and use Leibnitz's rule to obtain

$$T(f) = (-1)^{|\kappa|} \lambda^s \sum_{0 < \kappa_1 \leq \kappa} c_{\kappa_1}^{\kappa} \int_{\Omega} D^{\kappa_1} w D^{\gamma} f D^{\delta+\kappa-\kappa_1} \bar{f} dx. \quad (3.19)$$

We estimate  $D^{\kappa_1} w$ : clearly,

$$|D^{\kappa_1} (a_{\alpha\beta}(\nabla \phi)^{\mu})| \leq c \sum_{i=0}^{|\kappa_1|} |\nabla^{|\kappa_1|-i} a_{\alpha\beta}| |\nabla^i (\nabla \phi)^{\mu}| \quad \text{in } \Omega.$$

Recalling the definition of  $\mathcal{E}_{A,M}$  it is easily seen that  $|\nabla^i (\nabla \phi)^{\mu}| \leq c a^{-(|\mu|+i)/2m}$ ; recalling also from (2.14) the definition of the space  $W_a^{m-1, \infty}(\Omega)$  where the  $a_{\alpha\beta}$  lie we conclude that

$$|D^{\kappa_1} (a_{\alpha\beta}(\nabla \phi)^{\mu})| \leq c_M a(x)^{(2m-|\kappa_1+\mu|)/2m} = c_M a^{(|\gamma+\delta+\kappa-\kappa_1|)/2m}.$$

Hence (3.19) implies that  $T$  has essential order  $s + |\gamma + \delta + \kappa - \kappa_1| < 2m$ , as required.  $\square$

**Proposition 3.8.** *Let  $A(x, \xi) \in \mathcal{G}_a$ . Then for any  $\phi \in \mathcal{E}_a$ ,  $\lambda > 0$  and all  $f \in C_c^{\infty}(\Omega)$ , the following inequality holds:*

$$\text{Re } Q_{\lambda \phi}(f) \geq -k_m \lambda^{2m} \text{Re} \int_{\Omega} A(x, \nabla \phi(x)) |f|^2 dx + T(f), \quad (3.20)$$

where  $T(\cdot) \in \mathcal{L}_{a,m}$ .

**Proof.** Combining Lemmas 3.5, 3.6 and 3.7 we have

$$\text{Re } Q_{\lambda \phi}(f) + k_m \int_{\Omega} \text{Re } A(x, \lambda \nabla \phi(x)) |f|^2 dx = \Gamma_{\lambda \phi}(f) + T(f), \quad (3.21)$$

for a form  $T(\cdot) \in \mathcal{L}_{a,m}$ . Now let

$$u(x) = \int_{\mathbb{R}^N} p_{\xi, \lambda \nabla \phi} e^{i\xi \cdot x} \hat{f}(\xi) \, d\xi$$

(a  $\mathbb{C}^{\nu(m-1)}$ -valued integral defined component wise); it follows immediately from definition (3.17) that

$$\Gamma_{\lambda \phi}(f) = \int_{\Omega} \mathbf{\Gamma}(x, u(x), u(x)) \, dx, \quad (3.22)$$

and hence  $\Gamma_{\lambda \phi}(\cdot)$  is non-negative by the strong convexity of  $A(x, \xi)$ .  $\square$

### 3.2. The general case

We now remove the assumption  $A \in \mathcal{G}_a$  and return to the general setting described in §2. We recall that the quantity  $D$  measures the distance of  $A$  from  $\mathcal{G}_a$  and has been defined in (2.15).

**Lemma 3.9.** *Let  $T \in \mathcal{L}_{a,m}$ . Then for any  $\epsilon \in (0, 1)$  the following inequality holds for all  $\lambda > 0$  and  $f \in C_c^\infty(\Omega)$ :*

$$|T(f)| < \epsilon \{ \operatorname{Re} Q(f) + \lambda^{2m} \|f\|_2^2 \} + c_\epsilon \|f\|_2^2. \quad (3.23)$$

**Proof.** By definition,  $T(f)$  is a finite linear combination of expressions of the form

$$I(f) = \lambda^s \int_{\Omega} w(x) D^\gamma f(x) D^\delta \bar{f}(x) \, dx,$$

where  $|w(x)| \leq ca(x)^{|\gamma+\delta|/2m}$  and  $s + |\gamma + \delta| \leq 2m - 1$ . Setting  $\mu^{2m-|\gamma+\delta|} = \lambda^s$  and recalling (2.8) we have

$$\begin{aligned} |I(f)| &\leq c\mu^{2m-|\gamma+\delta|} \int_{\Omega} a(x)^{|\gamma+\delta|/2m} |D^\gamma f| |D^\delta f| \, dx \\ &\leq \epsilon \operatorname{Re} Q(f) + c\epsilon^{-2m+1} (1 + \mu^{2m}) \|f\|_2^2 \\ &\leq \epsilon \operatorname{Re} Q(f) + c\epsilon^{-2m+1} (1 + \lambda^{2m-1}) \|f\|_2^2 \\ &\leq \epsilon \{ \operatorname{Re} Q(f) + \lambda^{2m} \|f\|_2^2 \} + c\epsilon^{-4m^2+1} \|f\|_2^2. \end{aligned}$$

$\square$

**Remark 3.10.** It is seen from the proof that the size of the constant  $c_\epsilon$  in (3.23) depends only on  $\epsilon > 0$  and the (finite) quantity  $\max_I \sup\{|w(x)|a(x)^{-|\gamma+\delta|/2m}\}$ , where the maximum is taken over all forms  $I(\cdot)$  that make up  $T(\cdot)$ . In particular, when we restrict our attention to functions  $\phi \in \mathcal{E}_{A,M}$  we obtain a constant  $c_\epsilon = c_{\epsilon,M}$  which is otherwise independent of  $\phi$ .

**Lemma 3.11.** *For any  $\phi \in \mathcal{E}_{A,M}$ ,  $\lambda > 0$  and  $\epsilon > 0$  the following inequality holds:*

$$\operatorname{Re} Q_{\lambda \phi}(f) \geq -\{(k_m + cD + \epsilon)\lambda^{2m} + c_{\epsilon,M}\} \|f\|_2^2, \quad f \in C_c^\infty(\Omega), \quad (3.24)$$

where the constant  $c$  is independent of  $D$ ,  $M$ ,  $\epsilon$ ,  $\lambda$  and  $\phi$  and the constant  $c_{\epsilon,M}$  is independent of  $D$ ,  $\lambda$  and  $\phi$ .

**Proof.** Let  $\tilde{A} \in \mathcal{G}_a$  be such that  $\|A - \tilde{A}\|_{a,\infty} \leq 2D$ . It follows from (3.12) that

$$\begin{aligned} |\operatorname{Re} \tilde{Q}_{\lambda\phi}(f) - \operatorname{Re} Q_{\lambda\phi}(f)| &< cD\{\operatorname{Re} Q(f) + \lambda^{2m}\|f\|_2^2\}, \\ \left| \lambda^{2m} \int_{\Omega} [A(x, \nabla\phi(x)) - \tilde{A}(x, \nabla\phi(x))] dx \right| &< cD\{\operatorname{Re} Q(f) + \lambda^{2m}\|f\|_2^2\}. \end{aligned}$$

Combining these relations with (3.20)—as applied to the operator  $\tilde{H}$ —we obtain

$$\operatorname{Re} Q_{\lambda\phi}(f) \geq -k_m \lambda^{2m} \int_{\Omega} \operatorname{Re} A(x, \nabla\phi(x)) |f|^2 dx - cD\{\operatorname{Re} Q(f) + \lambda^{2m}\|f\|_2^2\} + T(f).$$

We have  $\operatorname{Re} A(x, \nabla\phi(x)) \leq 1$  and therefore (allowing  $c$  to change from line to line and  $\epsilon$  to rescale)

$$\begin{aligned} \operatorname{Re} Q_{\lambda\phi}(f) &\geq -k_m \lambda^{2m} \|f\|_2^2 - cD\{\operatorname{Re} Q(f) + \lambda^{2m}\|f\|_2^2\} + T(f) \\ &\geq -k_m \lambda^{2m} \|f\|_2^2 - (cD + \epsilon)\{\operatorname{Re} Q(f) + \lambda^{2m}\|f\|_2^2\} - c_{\epsilon,M} \|f\|_2^2 \quad (\text{by (3.23)}) \\ &\geq -k_m \lambda^{2m} \|f\|_2^2 - (cD + \epsilon)\{\operatorname{Re} Q_{\lambda\phi}(f) + \lambda^{2m}\|f\|_2^2\} - c_{\epsilon,M} \|f\|_2^2 \quad (\text{by (3.6)}). \end{aligned}$$

Now, either  $\operatorname{Re} Q_{\lambda\phi}(f)$  is positive, in which case (3.24) is true, or it is not, in which case it can be discarded from the right-hand side of the last inequality. This completes the proof.  $\square$

**Proof of Theorem 2.2.** The rest of the proof is standard. Combining Proposition 3.2 with (3.24) and using the relation  $K_{\lambda\phi}(t, x, y) = e^{-\lambda\phi(x)} K(t, x, y) e^{-\lambda\phi(y)}$  we obtain

$$|K(t, x, y)| < c_{\delta} t^{-s} \exp\{\lambda[\phi(y) - \phi(x)] + [(k_m + cD + \delta)\lambda^{2m} + c_{\delta,M}]t\}.$$

Optimizing over  $\phi \in \mathcal{E}_{A,M}$  introduces  $d_M(x, y)$  and choosing

$$\lambda = \left( \frac{d_M(x, y)}{2mk_m t} \right)^{1/(2m-1)}$$

we obtain

$$-\lambda d_M(x, y) + k_m \lambda^{2m} t = -\sigma_m \frac{d_M(x, y)^{2m/(2m-1)}}{t^{1/(2m-1)}},$$

which completes the proof.  $\square$

**Remark 3.12.** It is shown in [4] that the term  $cD$  cannot be eliminated from (3.24). Thus for it to be removed from Theorem 2.2 an essentially different approach is needed—if indeed the term is removable at all.

**Remark 3.13.** We point out that the above method can also work for operators of the form  $H + W$ , where  $W$  is a lower-order perturbation of  $H$ . It is clear that the estimate of Theorem 2.2 is valid for  $H + W$  provided  $W_{\lambda\phi}$  can be estimated by

$$|W_{\lambda\phi}(f)| < \epsilon\{\operatorname{Re} Q(f) + \lambda^{2m}\|f\|_2^2\} + c_{\epsilon}\|f\|_2^2$$

for all  $\phi \in \mathcal{E}_a$  and  $\lambda > 0$  and any  $\epsilon > 0$ . Such estimates can be obtained by means of weighted Hardy- and Sobolev-type inequalities. We do not elaborate on this and prove a theorem for zero-order real perturbations.

**Proposition 3.14.** *Let  $V = V_+ - V_-$ , where  $V_+ \in L^1_{\text{loc}}(\Omega)$  and  $V_- \in L^1(\Omega)$  are, respectively, the positive and negative parts of the real-valued potential  $V$ . Then the heat kernel of  $H + V$  satisfies the estimate of Theorem 2.2.*

**Proof.** We have

$$\begin{aligned} \int_{\Omega} V_- |f|^2 &\leq \|V_-\|_1 \|f\|_{\infty}^2 \\ &\leq c \|V_-\|_1 [\text{Re } Q(f)]^s \|f\|_2^{2-2s} \quad (\text{by (H1)}) \\ &\leq \epsilon \text{Re } Q(f) + c_{\epsilon, V} \|f\|_2^2 \end{aligned}$$

(hence  $H + V$  is defined with the same form domain as for  $H + V_+$ ). Moreover,

$$\text{Re}(H + V)_{\lambda\phi} = \text{Re } H_{\lambda\phi} + V \geq \text{Re } H_{\lambda\phi} - V_-.$$

Hence the estimate of Lemma 3.11 is also valid for  $H + V$  and the rest of the argument goes through.  $\square$

**Acknowledgements.** I thank the referee for useful comments.

## References

1. P. AUSCHER AND M. QAFSAOUI, Equivalence between regularity theorems and heat kernel estimates for higher order elliptic operators and systems under divergence form, *J. Funct. Analysis* **177** (2000), 310–364.
2. D. BAO, S.-S. CHERN AND Z. SHEN, *An introduction to Riemann–Finsler geometry* (Springer, 2000).
3. G. BARBATUS, Spectral theory of singular elliptic operators with measurable coefficients, *J. Funct. Analysis* **155** (1998), 125–152.
4. G. BARBATUS, Explicit estimates on the fundamental solution of higher-order parabolic equations with measurable coefficients, *J. Diff. Eqns* **174** (2001), 442–463.
5. G. BARBATUS AND E. B. DAVIES, Sharp bounds on heat kernels of higher order uniformly elliptic operators, *J. Operat. Theory* **36** (1996), 179–198.
6. E. B. DAVIES, *One parameter semigroups* (Academic, 1980).
7. E. B. DAVIES,  $L^p$  spectral theory of higher-order elliptic differential operators, *Bull. Lond. Math. Soc.* **29** (1997), 513–546.
8. N. DUNGEY, Sharp constants in higher-order heat kernel bounds, *Bull. Austral. Math. Soc.* **61** (2000), 189–200.
9. M. A. EVGRAFOV AND M. M. POSTNIKOV, Asymptotic behavior of Green’s functions for parabolic and elliptic equations with constant coefficients, *Mat. USSR Sb.* **11** (1970), 1–24.
10. K. TINTAREV, Short time asymptotics for fundamental solutions of higher order parabolic equations, *Commun. PDEs* **7** (1982), 371–391.