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REFINED GEOMETRIC L^p HARDY INEQUALITIES

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For a bounded convex domain Ω in $\mathbf{R}^N$ we prove refined Hardy inequalities that involve the Hardy potential corresponding to the distance to the boundary of Ω, the volume of Ω, as well as a finite number of sharp logarithmic corrections. We also discuss the best constant of these inequalities.

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1. Introduction

For a convex domain $\Omega \subset \mathbf{R}^N$ the Hardy inequality

$$\int_\Omega |\nabla u|^p dx \geq \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|u|^p}{d^p} dx\,, \quad d(x) = \operatorname{dist}(x, \partial\Omega) \quad u \in W_0^{1,p}(\Omega) \tag{1.1}$$

is valid. The constant $(\frac{p-1}{p})^p$ is optimal as was shown by Matskewich and Sobolevskii [11], Marcus, Mizel and Pinchover [10]. Brezis and Marcus [2] have established an improved version of (1.1) when $p = 2$: they showed that for bounded and convex Ω there holds

$$\int_\Omega |\nabla u|^2 dx \geq \frac{1}{4} \int_\Omega \frac{u^2}{d^2} dx + \frac{1}{4\operatorname{diam}^2(\Omega)} \int_\Omega u^2 dx\,, \quad u \in H_0^1(\Omega)\,. \tag{1.2}$$

The question was asked in that paper as to whether it is possible to replace $\operatorname{diam}^{-2}(\Omega)$ by $c|\Omega|^{-2/N}$, where $|\Omega|$ denotes the volume of Ω. A positive answer was given by M. and T. Hoffmann–Ostenhof and Laptev [9], who showed that

$$\int_\Omega |\nabla u|^2 dx \geq \frac{1}{4}\int_\Omega \frac{u^2}{d^2}dx + k_2\left(\frac{a_N}{|\Omega|}\right)^{\frac{2}{N}}\int_\Omega u^2 dx\,, \quad u \in H_0^1(\Omega)\,, \tag{1.3}$$

where a_N is the volume of the unit ball and $k_2 = N/4$.

In connection with this let us notice that when we take as $d(x)$ the distance from a point of Ω, say the origin, the following improved Hardy inequality was established by Brezis and Vazquez [3]

$$\int_\Omega |\nabla u|^2 dx \geq \left(\frac{N-2}{2}\right)^2\int_\Omega \frac{u^2}{|x|^2}dx + \mu_2\left(\frac{a_N}{|\Omega|}\right)^{\frac{2}{N}}\int_\Omega u^2 dx\,, \quad u \in H_0^1(\Omega)\,; \tag{1.4}$$

here $\mu_2 \simeq 5.783$ is the first eigenvalue of the Dirichlet Laplacian for the unit disk in $\mathbf{R}^2$. This constant is optimal when Ω is a ball centered at the origin, independently of the dimension $N \geq 2$, cf. [3], whereas for general Ω this constant is not optimal, cf. [8, Proposition 5.1].

An L^p -version of (1.3) was recently obtained by Tidblom [12] who showed that for convex Ω there holds

$$\int_\Omega |\nabla u|^p dx \geq \left(\frac{p-1}{p}\right)^p\int_\Omega \frac{|u|^p}{d^p}dx + k_p\left(\frac{a_N}{|\Omega|}\right)^{\frac{p}{N}}\int_\Omega |u|^p dx\,, \quad u \in W_0^{1,p}(\Omega) \tag{1.5}$$

with

$$k_p = (p-1)\left(\frac{p-1}{p}\right)^p \frac{\sqrt{\pi}\,\Gamma(\frac{N+p}{2})}{\Gamma(\frac{p+1}{2})\Gamma(\frac{N}{2})}\,. \tag{1.6}$$

For $p = 2$ this reduces to (1.3); in particular $k_2 = N/4$.

In addition to (1.3) it was shown in [9, Theorem 3.4] that if

$$X_1(t) = (1-\log t)^{-1}\,, \quad t \in (0,1)\,, \tag{1.7}$$

the following more refined improvement of (1.3) is true: for any $D \geq \mathrm{diam}(\Omega)/2$ there holds

$$\begin{aligned}\int_\Omega |\nabla u|^2 dx \geq \frac{1}{4}\int_\Omega \frac{u^2}{d^2}dx &+ \frac{1}{4}\int_\Omega \frac{u^2}{d^2}X_1^2(d/D)dx \\ &+ k_2(1-X_1(\mathrm{diam}(\Omega)/(2D))^2|\Omega|^{-2/N}\int_\Omega u^2 dx\end{aligned} \tag{1.8}$$

for all $u \in H_0^1(\Omega)$. Note that if we let $D \to \infty$ in (1.8) we regain (1.3).

Improved Hardy inequalities have recently been used in many contexts and in particular in the study of the existence and asymptotic behavior of solutions of the heat equation with singular potential. See for instance the work of Cabré and Martel [4] as well as Vázquez and Zuazua [13]. Moreover, Davies in [5] showed that the Hardy inequality implies stability of the eigenvalues of the Dirichlet Laplacian under perturbation of the boundary; the precise rate of convergence was shown to depend on the coefficient of u^2/d^2 (which, in general, is not $1/4$ if Ω is not convex). This is useful in the numerical computation of eigenvalues.

In this work we derive more refined improved Hardy inequalities. Before stating our results we introduce some notation. With $X_1(t)$ as in (1.7) we define recursively

$$X_k(t) = X_1(X_{k-1}(t)), \quad k = 2, 3, \ldots, t \in (0, 1). \tag{1.9}$$

These are iterated logarithmic functions that vanish at an increasingly low rate at $t = 0$. Let us fix $k \geq 1$ and set

$$a = \begin{cases} 0, & \text{if} \quad 1 < p \leq 2, \\ \dfrac{(p-2)k}{3(p-1)} > 0, & \text{if} \quad p > 2, \end{cases} \tag{1.10}$$

and

$$\eta(t) = \sum_{i=1}^{k} X_1(t) \cdots X_i(t),$$

whereas for $k = 0$ we set $\eta = 0$. For $D \geq \operatorname{diam}(\Omega)/2$ we also set

$$\eta_D = \eta\left(\frac{\operatorname{diam}(\Omega)}{2D}\right).$$

Then our main result reads:

Theorem 1.1. *Assume that Ω is convex and bounded. Let $k \geq 0$ be a fixed integer. Then, there exists $D_0 = D_0(k, p, \operatorname{diam}(\Omega)) \geq \operatorname{diam}(\Omega)/2$ such that for $D \geq D_0$ there holds*

$$\begin{aligned} &\int_\Omega |\nabla u|^p dx \\ &\quad \geq \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|u|^p}{d^p} dx + \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1} \sum_{i=1}^{k} \int_\Omega \frac{|u|^p}{d^p} X_1^2(d/D) \cdots X_i^2(d/D) dx \\ &\qquad + k_p (1 - \eta_D - a\eta_D^2)^{\frac{p}{p-1}} \left(\frac{a_N}{|\Omega|}\right)^{\frac{p}{N}} \int_\Omega |u|^p dx, \end{aligned} \tag{1.11}$$

for all $u \in W_0^{1,p}(\Omega)$. When $p = 2$ we can take as D_0 the unique solution of $\eta_{D_0} = 1$.

Note that if we let $D \to +\infty$ in (1.11) we recover (1.5). Also, for $p = 2$ and $k = 1$ we recover (1.8). Moreover, the terms in the series are sharp: it was shown in [1, Theorem A] that for each $k \geq 1$ the relation

$$\begin{aligned} &\int_\Omega |\nabla u|^p dx - \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|u|^p}{d^p} dx + \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1} \sum_{i=1}^{k-1} \int_\Omega \frac{|u|^p}{d^p} X_1^2 \cdots X_i^2 dx \\ &\qquad \geq c \int_\Omega \frac{|u|^p}{d^p} X_1^2 \cdots X_k^\gamma dx \end{aligned} \tag{1.12}$$

is not valid for $\gamma < 2$; In addition, the best constant c in (1.12) when $\gamma = 2$ is equal to $\frac{1}{2}(\frac{p-1}{p})^{p-1}$, for any $k = 1, 2, \ldots$.

A natural question is whether the constants appearing in (1.5) or (1.11) are optimal. Working towards this we consider the simplest case (1.3) (corresponding to $p = 2$, $k = 0$). Let $\Omega = B$, be the unit ball in $\mathbf{R}^N$, and denote by C_N the best constant of (1.3), that is

$$C_N = \inf_{u \in H_0^1(B)} \frac{\int_B |\nabla u|^2 dx - \frac{1}{4}\int_B \frac{u^2}{d^2} dx}{\int_B u^2 dx}. \tag{1.13}$$

We then show that in this case the constant $k_2 = \frac{N}{4}$ appearing in (1.3) is far from being optimal. In particular we have:

Theorem 1.2. *For $N = 3$, $C_3 = \mu_2$, whereas for any $N \geq 3$ there holds*:

$$C_N \geq \mu_2 + \frac{(N-1)(N-3)}{4}, \tag{1.14}$$

where $\mu_2 \simeq 5.783$ is the best constant of inequality (1.4).

It is remarkable that when Ω is a ball and $N = 3$ inequalities (1.3) and (1.4) have the same best constant. For any $N \geq 3$ the lower bound (1.14) on C_N improves the estimate $C_N \geq k_2 = \frac{N}{4}$.

To prove Theorem 1.1 we combine a vector field approach (cf. [1]) along with ideas of [9] or [12]. It is worth noting that the "mean distance" method of Davies (cf. [5, 6]) plays an essential role. For Theorem 1.2 after restricting to radial functions we use a suitable change of variables.

2. Preliminary Inequalities

In this section we will prove some auxiliary one-dimensional inequalities. Throughout this section $b \leq \frac{\mathrm{diam}(\Omega)}{2}$ is a fixed positive constant. We have the following

Lemma 2.1. *Let $\rho(t) = \min\{t, 2b - t\}$. For any function $g \in C^1((0, b])$ there holds*

$$\text{(i)} \quad \int_0^{2b} |u'(t)|^p dt \geq \int_0^{2b} \{g'(\rho(t)) - (p-1)|g(\rho(t))|^{\frac{p}{p-1}}\}|u(t)|^p dt - 2g(b)|u(b)|^p,$$

$$\text{(ii)} \quad \int_0^{2b} |u'(t)|^p dt \geq \int_0^{2b} \{g'(\rho(t)) - (p-1)|g(\rho(t)) - g(b)|^{\frac{p}{p-1}}\}|u(t)|^p dt \tag{2.1}$$

for all $u \in C_c^\infty(0, 2b)$.

Proof. We first prove (i). For $u \in C_c^\infty(0, 2b)$ we have

$$\begin{aligned} \int_0^b g'(t)|u(t)|^p dt &= g(b)|u(b)|^p - p\int_0^b g(t)|u|^{p-2}u'u\,dt \\ &\leq g(b)|u(b)|^p + p\left(\int_0^b |u'|^p dt\right)^{\frac{1}{p}}\left(\int_0^b |g|^{\frac{p}{p-1}}|u|^p dt\right)^{\frac{p-1}{p}} \\ &\leq g(b)|u(b)|^p + \int_0^b |u'|^p dt + (p-1)\int_0^b |g|^{\frac{p}{p-1}}|u|^p dt, \end{aligned}$$

hence

$$\int_0^b |u'(t)|^p dt \geq \int_0^b \{g'(t) - (p-1)|g(t)|^{\frac{p}{p-1}}\}|u(t)|^p dt - g(b)|u(b)|^p .$$

A similar argument on $(b, 2b)$ gives

$$\int_b^{2b} |u'(t)|^p dt \geq \int_b^{2b} \{g'(2b-t) - (p-1)|g(2b-t)|^{\frac{p}{p-1}}\}|u|^p dt - g(b)|u(b)|^p$$

and (i) follows by adding up the last two inequalities.

Part (ii) follows immediately from (i) by using the function $g(x) - g(b)$ in the place of $g(x)$. $\square$

In order to apply the above lemma we fix a positive integer k and define the functions

$$\eta(t) = \sum_{i=1}^{k} X_1(t) \cdots X_i(t) ,$$

$$B(t) = \sum_{i=1}^{k} X_1^2(t) \cdots X_i^2(t) , \quad t \in (0,1) ,$$

where the X_i's are given by (1.9). It is easy to check that both η and B are increasing functions of t with $\eta(0^+) = B(0^+) = 0$ and $\eta(1^-) = B(1^-) = k$. We also note that

$$\frac{1}{k}\eta^2(t) \leq B(t) \leq \eta^2(t) , \quad t \in (0,1) . \tag{2.2}$$

For $0 < b \leq \frac{\mathrm{diam}(\Omega)}{2} \leq D$ we define the following functions of $s \in (0, b)$:

$$\begin{aligned} g(s) &= -\left(\frac{p-1}{p}\right)^{p-1} s^{-(p-1)}(1 - \eta(s/D) - a\eta^2(s/D)) \\ A(s) &= g'(s) - (p-1)|g(s) - g(b)|^{\frac{p}{p-1}} - \left(\frac{p-1}{p}\right)^p s^{-p} - \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1} s^{-p} B(s/D) . \end{aligned} \tag{2.3}$$

Recall that a is defined in (1.10). We then have the following

Lemma 2.2. *There exists $D_0 = D_0(k, p, \mathrm{diam}(\Omega)) \geq \frac{\mathrm{diam}(\Omega)}{2}$, such that for all $D \geq D_0$ there holds*:

(i) $1 - \eta\left(\frac{\mathrm{diam}(\Omega)}{2D}\right) - a\eta^2\left(\frac{\mathrm{diam}(\Omega)}{2D}\right) \geq 0,$

(ii) $g'(s) - \left(\frac{p-1}{p}\right)^p s^{-p} - \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1} s^{-p} B(s/D) \geq (p-1)|g(s)|^{\frac{p}{p-1}},$

(iii) *$A(s)$ is a decreasing function of $s \in (0, b)$.*

For $p=2$, (ii) *becomes equality. Also, for* $p=2$, *we can take as* D_0 *the unique solution of* $1=\eta(\frac{\mathrm{diam}(\Omega)}{2D_0})$.

Proof. A straightforward calculation shows that

$$\frac{d}{ds}\eta(s/D)=\frac{1}{s}\left[\frac{B(s/D)}{2}+\frac{\eta^2(s/D)}{2}\right]. \tag{2.4}$$

Setting $\Gamma(t)=tB'(t)$ we also have

$$\frac{d}{ds}B(s/D)=\frac{1}{s}\Gamma(s/D)>0\,; \tag{2.5}$$

the positivity follows from the fact that $B(t)$ is an increasing function of t.

Since $\eta(t)$ is an increasing function of t with $\eta(0)=0$, (i) is immediate.

We shall henceforth omit the argument s/D from η, B, Γ in the subsequent formulas. We next prove (ii). For $p=2$ an easy calculation shows that (ii) becomes equality. For $p\neq 2$ the left hand side of (ii) is equal to

$$\begin{aligned}
&g'(s)-\left(\frac{p-1}{p}\right)^p s^{-p}-\frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1}s^{-p}B(s/D)\\
&\quad=\left(\frac{p-1}{p}\right)^p(p-1)s^{-p}\left[1-\frac{p\eta}{p-1}+\left(\frac{p}{2(p-1)^2}-\frac{ap}{p-1}\right)\eta^2\right.\\
&\qquad\left.+\frac{ap}{(p-1)^2}\eta^3+\frac{ap}{(p-1)^2}\eta B\right].
\end{aligned} \tag{2.6}$$

On the other hand, taking the Taylor expansion of $(1-t)^{\frac{p}{p-1}}$ about $t=0$, we see that the right hand side of (ii) is written as (for η small)

$$\begin{aligned}
&\left(\frac{p-1}{p}\right)^p(p-1)s^{-p}(1-\eta-a\eta^2)^{\frac{p}{p-1}}\\
&\quad=\left(\frac{p-1}{p}\right)^p(p-1)s^{-p}\left[1-\frac{p\eta}{p-1}-\frac{ap}{p-1}\eta^2+\frac{p\eta^2}{2(p-1)^2}+\frac{pa\eta^3}{(p-1)^2}\right.\\
&\qquad\left.+\frac{p(p-2)\eta^3}{6(p-1)^3}+O(\eta^4)\right].
\end{aligned} \tag{2.7}$$

Comparing (2.6) and (2.7) we see that the corresponding right-hand sides agree to order $O(\eta^2)$. Recalling (2.2) and the choice of a (cf. (1.10)) we see that the cubic term in (2.6) is larger than the cubic term of (2.7). Hence (ii) is true provided η is small enough, which amounts to D_0 being large enough.

We now prove (iii). Note that (ii) implies that g' is positive in $(0,b)$ if D_0 is large enough. Hence for $s\in(0,b)$ we have

$$
\begin{aligned}
A'(s) &= g''(s) + p[g(b)-g(s)]^{\frac{1}{p-1}} g'(s) + \left(\frac{p-1}{p}\right)^{p-1}(p-1)s^{-p-1} \\
&\quad + \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1} p s^{-p-1} B - \frac{1}{2}\left(\frac{p-1}{p}\right) s^{-p-1}\Gamma \\
&\le g''(s) + p|g(s)|^{\frac{1}{p-1}} g'(s) + \left(\frac{p-1}{p}\right)^{p-1}(p-1)s^{-p-1} \\
&\quad + \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1} p s^{-p-1} B - \frac{1}{2}\left(\frac{p-1}{p}\right) s^{-p-1}\Gamma\,. \qquad (2.8)
\end{aligned}
$$

Using Taylor's expansion we have

$$
\begin{aligned}
|g(s)|^{\frac{1}{p-1}} &= \frac{p-1}{p} s^{-1}(1-\eta-a\eta^2)^{\frac{1}{p-1}} \\
&= \frac{p-1}{p} s^{-1}\left\{1 - \frac{1}{p-1}\eta - \left[\frac{a}{p-1} + \frac{p-2}{2(p-1)^2}\right]\eta^2 - \right. \\
&\quad \left. - \left[\frac{(p-2)a}{(p-1)^2} - \frac{(p-2)(3-2p)}{6(p-1)^3}\right]\eta^3 + O(\eta^4)\right\}. \qquad (2.9)
\end{aligned}
$$

From (2.4), (2.5), (2.8) and (2.9) we obtain

$$
A'(s) \le (p-1)^2\left(\frac{p-1}{p}\right)^{p-1} s^{-p-1}\left\{\frac{p(p-2)}{6(p-1)^3}\eta^3 - \frac{ap}{(p-1)^2}\eta B + O(\eta^4)\right\}. \qquad (2.10)
$$

From this and the fact that

$$
\frac{1}{k}\eta^2 \le B \le \eta^2\,, \quad s \in (0,b)
$$

we end up with

$$
A'(s) \le p\left(\frac{p-1}{p}\right)^{p-1} s^{-p-1}\eta^3\left\{\frac{p-2}{6(p-1)} - \frac{a}{k} + O(\eta)\right\}. \qquad (2.11)
$$

To conclude the proof we distinguish various cases:

(a) $1 < p < 2$. Then $a = 0$ and it follows from (2.11) that $A'(s) < 0$ in $(0,b)$, provided D_0 is chosen large enough.
(b) $p = 2$. Again $a = 0$. A straightforward calculation shows that the right hand side of (2.8) is identically equal to zero. The only restriction here comes from (i), whence the choice of D_0.
(c) $p > 2$. Now $a = \frac{(p-2)k}{3(p-1)}$ and the result follows again from (2.11).

This completes the proof. □

3. The Hardy Inequality

Throughout the rest of the paper we assume that $\Omega \subset \mathbf{R}^N$ is convex and set $d(x) = \operatorname{dist}(x, \partial\Omega)$.

Following [9], for $\omega \in S^{N-1}$ and $x \in \Omega$ we define the following functions with values in $(0, +\infty]$:

$$\begin{aligned}
\tau_\omega(x) &= \inf\{s > 0 | x + s\omega \notin \Omega\}\,,\\
\rho_\omega(x) &= \min\{\tau_\omega(x), \tau_{-\omega}(x)\}\,,\\
b_\omega(x) &= \frac{1}{2}(\tau_\omega(x) + \tau_{-\omega}(x))\,.
\end{aligned}$$

We denote by $dS(\omega)$ the standard measure on S^{N-1} normalized so that the total measure is one. Let $K_p > 0$ be defined by

$$\int_{S^{N-1}} |v \cdot \omega|^p dS(\omega) = K_p |v|^p\,, \quad \forall\, v \in \mathbf{R}^N\,. \tag{3.1}$$

The constant K_p can be computed in terms of k_p (cf (1.6)) and is given by $K_p = (p-1)(\frac{p-1}{p})^p k_p^{-1}$. We have the following

Lemma 3.1. *Assume that Ω is convex. Then for all $x \in \Omega$ there holds*

$$\int_{S^{N-1}} \rho_\omega^{-p}(x) dS(\omega) \geq K_p d(x)^{-p}\,. \tag{3.2}$$

Proof. Let $y \in \partial\Omega$ be such that $|y - x| = d(x)$ and let P_y be the supporting hyper-plane through y which is orthogonal to $y - x$. We define the half-sphere

$$S^+ = \{\omega \in S^{N-1} | \omega \cdot (y - x) > 0\}$$

and for $\omega \in S^+$ define $\sigma_\omega(x) > 0$ by requiring that $x + \sigma_\omega(x)\omega \in P_y$, so that

$$\omega \cdot \frac{y - x}{|y - x|} = \frac{|y - x|}{\sigma_\omega(x)}\,.$$

The convexity of Ω implies that $\tau_\omega(x) \leq \sigma_\omega(x)$ and hence

$$\begin{aligned}
\int_{S^{N-1}} \frac{1}{\rho_\omega(x)^p} dS(\omega) &\geq 2 \int_{S^+} \frac{1}{\tau_\omega(x)^p} dS(\omega)\\
&\geq 2 \int_{S^+} \frac{1}{\sigma_\omega(x)^p} dS(\omega)\\
&= \frac{2}{d(x)^{2p}} \int_{S^+} |(y - x) \cdot \omega|^p dS(\omega)\\
&= \frac{K_p}{d(x)^p}\,,
\end{aligned}$$

as required. $\square$

We now give the proof of Theorem 1.1.

Proof of Theorem 1.1. We fix a direction $\omega \in S^{N-1}$ and let Ω_ω be the orthogonal projection of Ω on the hyper-plane perpendicular to ω. For each $z \in \Omega_\omega$ we apply Lemma 2.1 on the segment defined by z and ω and we then integrate over $z \in \Omega_\omega$. We conclude that for any $u \in C_c^\infty(\Omega)$ there holds

$$\int_\Omega |\nabla u \cdot \omega|^p dx \geq \int_\Omega \left\{ g'(\rho_\omega(x)) - (p-1)|g(\rho_\omega(x)) - g(b_\omega(x))|^{\frac{p}{p-1}} \right\} |u|^p dx \,.$$

Integrating over $\omega \in S^{N-1}$ and recalling definition (3.1) we obtain

$$\int_\Omega |\nabla u|^p dx \geq K_p^{-1} \int_\Omega \int_{S^{N-1}} \Big\{ g'(\rho_\omega(x)) - (p-1)|g(\rho_\omega(x)) - g(b_\omega(x))|^{\frac{p}{p-1}} \Big\} dS(\omega) |u|^p dx \,. \tag{3.3}$$

Now, let us choose g as in (2.3). Since Ω is bounded, Lemma 2.2 implies the existence of a $D_0 > 0$ such that for $D \geq D_0$, each of the functions

$$A_{\omega,x}(s) := g'(s) - (p-1)|g(s) - g(b_\omega(x))|^{\frac{p}{p-1}} - \left(\frac{p-1}{p}\right)^p s^{-p} - \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1} s^{-p} B(s/D)$$

— defined for $s \in (0, b_\omega(x))$ — is a decreasing function of $s \in (0, b_\omega(x))$. In particular $A_{\omega,x}(\rho_\omega(x)) \geq A_{\omega,x}(b_\omega(x))$, i.e.

$$g'(\rho_\omega(x)) - (p-1)|g(\rho_\omega(x)) - g(b_\omega(x))|^{\frac{p}{p-1}} \geq \left(\frac{p-1}{p}\right)^p \rho_\omega(x)^{-p} + \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1} \rho_\omega(x)^{-p} B(\rho_\omega(x)/D) + A_{\omega,x}(b_\omega(x)) \,.$$

Hence (3.3) yields

$$\begin{aligned}\int_\Omega |\nabla u|^p dx \geq K_p^{-1} \int_\Omega \int_{S^{N-1}} \Bigg\{ &\left(\frac{p-1}{p}\right)^p \rho_\omega(x)^{-p} + \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1} \\ &\times \rho_\omega(x)^{-p} B(\rho_\omega(x)/D) + g'(b_\omega(x)) - \left(\frac{p-1}{p}\right)^p b_\omega(x)^{-p} \\ &- \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1} b_\omega(x)^{-p} B(b_\omega(x)/D) \Bigg\} dS(\omega) |u|^p dx \,.\end{aligned} \tag{3.4}$$

We first estimate the first two terms of (3.4). For each $x \in \Omega$ and $\omega \in S^{N-1}$ there holds $B(\rho_\omega(x)/D) \geq B(d(x)/D)$, and Lemma 3.1 yields

$$\begin{aligned}K_p^{-1} \int_{S^{N-1}} &\left\{ \left(\frac{p-1}{p}\right)^p \rho_\omega(x)^{-p} + \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1} \rho_\omega(x)^{-p} B(\rho_\omega(x)/D) \right\} dS(\omega) \\ &\geq \left(\frac{p-1}{p}\right)^p d(x)^{-p} + \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1} d(x)^{-p} B(d(x)/D) \,,\end{aligned} \tag{3.5}$$

for all $x \in \Omega$. The remaining three terms in the right-hand side of (3.4) are estimated using Lemma 2.2(ii)

$$\begin{aligned} g'(b_\omega(x)) - \left(\frac{p-1}{p}\right)^p b_\omega(x)^{-p} - \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1} b_\omega(x)^{-p} B(b_\omega(x)/D) \\ \geq (p-1)|g(b_\omega(x))|^{\frac{p}{p-1}} \\ \geq \left(\frac{p-1}{p}\right)^p (p-1)(1-\eta_D - a\eta_D^2)^{\frac{p}{p-1}} b_\omega(x)^{-p} \,. \end{aligned}$$

Combining this with (3.4) and (3.5) we obtain

$$\begin{aligned} \int_\Omega |\nabla u|^p dx \geq \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|u|^p}{d^p} dx + \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1} \int_\Omega \frac{|u|^p}{d^p} B(d/D) \\ + K_p^{-1}\left(\frac{p-1}{p}\right)^p (p-1)(1-\eta_D - a\eta_D^2)^{\frac{p}{p-1}} \\ \times \int_\Omega \int_{S^{N-1}} \frac{1}{b_\omega(x)^p} dS(\omega)|u|^p dx \,. \end{aligned} \tag{3.6}$$

We estimate the last integral using a variation of an argument of [9]. Elementary analysis shows that $\min_{t>0}(1+t^N)/(1+t)^N = 2^{-(N-1)}$ and therefore for $x \in \Omega$

$$\begin{aligned} 2^{-\frac{(N-1)p}{N+p}} &\leq \int_{S^{N-1}} \frac{(\tau_\omega(x)^N + \tau_{-\omega}(x)^N)^{p/(N+p)}}{(\tau_\omega(x) + \tau_{-\omega}(x))^{Np/(N+p)}} dS(\omega) \\ &\leq \left(\int_{S^{N-1}} (\tau_\omega^N + \tau_{-\omega}^N) dS(\omega)\right)^{\frac{p}{N+p}} \left(\int_{S^{N-1}} \frac{1}{(\tau_\omega + \tau_{-\omega})^p} dS(\omega)\right)^{\frac{N}{N+p}} \\ &= 2^{\frac{p-pN}{N+p}} \left(\int_{S^{N-1}} \tau_\omega^N dS(\omega)\right)^{\frac{p}{N+p}} \left(\int_{S^{N-1}} \frac{1}{b_\omega^p} dS(\omega)\right)^{\frac{N}{N+p}} , \end{aligned}$$

that is

$$\int_{S^{N-1}} \frac{1}{b_\omega(x)^p} dS(\omega) \geq \left(\int_{S^{N-1}} \tau_\omega(x)^N dS(\omega)\right)^{-p/N} . \tag{3.7}$$

The convexity of Ω implies $a_N \int_{S^{N-1}} \tau_\omega(x)^N dS(\omega) = |\Omega|$. Hence the proof is concluded by combining (3.6) and (3.7). □

4. On the Best Constant for $p = 2$

In this section we will prove Theorem 1.2. We recall that C_N is the best constant of inequality (1.3), in case Ω is the unit ball, defined by:

$$C_N = \inf_{u \in H_0^1(B)} \frac{\int_B |\nabla u|^2 dx - \frac{1}{4}\int_B \frac{u^2}{d^2} dx}{\int_B u^2 dx} \,. \tag{4.1}$$

We first establish

Lemma 4.1. *The infimum in* (4.1) *remains the same if it is taken over all radially symmetric functions* $u = u(r) \in H_0^1(B)$.

Proof. Let us denote by $\tilde{C}_N$ the infimum over radial functions. Clearly $\tilde{C}_N \geq C_N$. Suppose now that $u \in H_0^1(B)$ and let

$$u(x) = u_0(r) + \sum_{m=1}^{\infty} f_m(\sigma) u_m(r)\,, \quad r = |x|\,,$$

be its decomposition into spherical harmonics; here u_m are radially symmetric functions in $H_0^1(B)$ and f_m are orthonormal in $L^2(S^{N-1})$ eigenfunctions of the Laplace–Beltrami operator on $\{|x| = 1\}$, with corresponding eigenvalues $c_m = m(N-2+m)$, $m \geq 1$. It is easily seen that

$$\int_B |\nabla u|^2 dx = \int_B \left(|\nabla u_0|^2 dx + \sum_{m=1}^{\infty} \int_B (|\nabla u_m|^2 + \frac{c_m}{|x|^2} u_m^2 \right) dx\,, \tag{4.2}$$

and hence

$$\begin{aligned}
\int_B (|\nabla u|^2 - \frac{u^2}{d^2}) dx &= \int_B \left\{ |\nabla u_0|^2 - \frac{u_0^2}{4(1-|x|)^2} \right\} dx \\
&\quad + \sum_{m=1}^{\infty} \int_B \left\{ (|\nabla u_m|^2 + \left(\frac{c_m}{|x|^2} - \frac{1}{4(1-|x|)^2} \right) u_m^2 \right\} dx \\
&\geq \tilde{C}_N \int_B u_0^2 dx + \tilde{C}_N \sum_{m=1}^{\infty} \int_B u_m^2 dx \\
&= \tilde{C}_N \int_B u^2 dx\,.
\end{aligned}$$

This implies $C_N \geq \tilde{C}_N$ and Lemma 4.1 is proved. □

Proof of Theorem 1.2. By Lemma 4.1 we restrict attention to radially symmetric functions. Let $u = u(r) \in C_c^\infty(B)$ be a radial function and define v by

$$u(r) = r^{-\frac{N-1}{2}} (1-r)^{1/2} v(r)\,, \quad r \in (0,1)\,.$$

Then $v(0) = v(1) = 0$. We compute

$$\begin{aligned}
\frac{1}{N a_N} \int_B |\nabla u|^2 dx &= \int_0^1 (u')^2 r^{N-1} dr \\
&= \int_0^1 (1-r) \left(-\frac{(N-1)v}{2r} - \frac{v}{2(1-r)} + v' \right)^2 dr\,.
\end{aligned}$$

Using integration by parts for the terms involving $vv' = (v^2)'/2$ we conclude after some simple calculations that

$$\begin{aligned}
&\frac{1}{Na_N}\left(\int_B |\nabla u|^2 dx - \frac{1}{4}\int_B \frac{u^2}{d^2}dx\right) \\
&\quad = \int_0^1 (1-r)(v')^2 dr + \frac{(N-1)(N-3)}{4}\int_0^1 \frac{1-r}{r^2}v^2 dr \\
&\quad \geq \int_0^1 (1-r)(v')^2 dr + \frac{(N-1)(N-3)}{4}\int_0^1 (1-r)v^2 dr\,.
\end{aligned}$$

But (cf. [3, Sec. 4])

$$\inf_{v(0)=v(1)=0} \frac{\int_0^1 (1-r)(v')^2 dr}{\int_0^1 (1-r)v^2 dr} = \inf_{v(0)=v(1)=0} \frac{\int_0^1 r(v')^2 dr}{\int_0^1 rv^2 dr} = \mu_2\,,$$

and estimate (1.14) of Theorem 1.2 follows.

To prove that $C_3 = \mu_2$, let us define

$$u_\epsilon(r) = r^{-1}(1-r)^{\frac{1}{2}+\epsilon} w(1-r)\,, \quad r \in (0,1)\,,$$

where $\epsilon > 0$ and $w(|x|)$ is the first eigenfunction of the Dirichlet Laplacian for the unit disk in $\mathbf{R}^2$. Then

$$u'_\epsilon(r) = -r^{-1}(1-r)^{\frac{1}{2}+\epsilon}\left\{\frac{w}{r} + \left(\frac{1}{2}+\epsilon\right)\frac{w}{1-r} + w'\right\}$$

and hence $u_\epsilon \in H^1_0(B)$ and

$$\begin{aligned}
&\int_0^1 (u'_\epsilon)^2 r^{N-1} dr \\
&\quad = \int_0^1 (1-r)^{1+2\epsilon}\left\{\frac{w^2}{r^2} + \left(\frac{1}{2}+\epsilon\right)^2 \frac{w^2}{(1-r)^2} + (w')^2\right. \\
&\qquad \left. + (1+2\epsilon)\frac{w^2}{r(1-r)} + \frac{2ww'}{r} + \frac{(1+2\epsilon)ww'}{1-r}\right\} dr \quad (\text{where } w = w(1-r))\,.
\end{aligned}$$

To handle the terms containing ww' we integrate by parts: the boundary terms are equal to zero and making the change of variables $s = 1-r$ we eventually obtain

$$\int_0^1 \left((u'_\epsilon(r))^2 - \frac{1}{4}\frac{u_\epsilon^2(r)}{(1-r)^2}\right) r^2 dr = \int_0^1 s^{1+2\epsilon}\left((w'(s))^2 - \epsilon^2 \frac{w^2(s)}{s^2}\right) ds\,.$$

Now, there holds

$$\epsilon^2 \int_0^1 s^{-1+2\epsilon} w^2 ds \longrightarrow 0\,, \quad \text{as } \epsilon \to 0\,,$$

hence

$$\begin{aligned}
\lim_{\epsilon\to 0}\frac{\int_B(|\nabla u_\epsilon|^2-\frac{u_\epsilon^2}{4d^2})dx}{\int_B u_\epsilon^2 dx} &= \lim_{\epsilon\to 0}\frac{\int_0^1 (w')^2 s^{1+2\epsilon}ds}{\int_0^1 w^2 s^{1+2\epsilon}ds}\\
&= \frac{\int_0^1 (w')^2 s\, ds}{\int_0^1 w^2 s\, ds}\\
&= \mu_2\,.
\end{aligned}$$

It follows that $\tilde{C}_3 \le \mu_2$; in view of (1.14) and Lemma 4.1 we conclude that $C_3 = \mu_2$. □

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