STABILITY AND REGULARITY OF HIGHER ORDER ELLIPTIC OPERATORS WITH MEASURABLE COEFFICIENTS

G. BARBATIS

0. *Introduction*

In this paper, we investigate two problems concerning a class of higher order uniformly elliptic operators with measurable coefficients. In our first result, we prove a stability estimate for the resolvent of such an operator under perturbation of its principal coefficients in L^p spaces. Our second result is a regularity theorem concerning the smoothing properties of the resolvent, and it provides a sufficient condition under which the first result can be applied.

More precisely, we work on a bounded domain of $\mathbf{R}^N$, and consider operators of the form

$$Hf = \sum_{\substack{|\alpha| \leqslant m \\ |\beta| \leqslant m}} (-1)^{|\alpha|} D^\alpha \{a_{\alpha\beta} D^\beta f\}$$

which are uniformly elliptic and have measurable coefficients. Our first main result, Theorem 5, has as a corollary the statement that, for two such operators H_1, H_2 with corresponding coefficient matrices a_1, a_2, we have

$$\|(H_2+1)^{-1} - (H_1+1)^{-1}\|_r \leqslant c\|a_2 - a_1\|_p$$

for r and p in a certain range. Here, the first norm is the Schatten norm of index r, and the second norm is the usual L^p norm. See Corollary 12 for the details. This generalises [**3**, Theorem 9], where the second order problem is treated. The proof of the above estimate depends heavily on a regularity property of the corresponding resolvent operator. In our second theorem, Theorem 10, we prove that $(H+1)^{-1}$ has the L^p smoothing property

$$(H+1)^{-1} : L^p \to W^{m,p}$$

for p sufficiently close to 2. This is the higher order version of a result of Meyers [**9**] that was used in [**3**].

We chose as a setting for our results an abstract axiomatic one, where we consider general operators on L^2-spaces. The reason for this is not only the natural desire for generality, which may also allow for other applications, but also the fact that such an exposition allows one to see more clearly the features that are really important for the theory to work.

The paper consists essentially of two parts. In the first, we prove our stability estimate in an abstract axiomatic setting which is subject to two hypotheses,

Received 9 February 1996; revised 20 June 1996.

1991 *Mathematics Subject Classification* 35J30.

Work partially supported by the Alexander S. Onassis Public Benefit Foundation and the SERC.

Hypothesis 1 and Hypothesis 2, of which Hypothesis 2 is the more difficult to verify. In the second part, we prove our main regularity estimate, thus establishing a general condition under which Hypothesis 2 is valid.

0.1. *The setting*

Let Ω be a locally compact, second countable, Hausdorff topological space, and let dx be a finite Borel measure on Ω. Let $(T_x, \langle,\rangle_x)$ be a measurable field of r-dimensional Hilbert spaces on Ω (see **[7]** for a precise definition).

We denote by $L^p_v(\Omega)$ the Banach space of measurable p-integrable vector fields ξ equipped with the natural norm

$$\|\xi\|_p = \left\{\int_\Omega |\xi_x|^p dx\right)^{1/p} \quad 1 \leqslant p < +\infty$$

$$\|\xi\|_\infty = \operatorname*{ess\,sup}_{x\in\Omega} |\xi_x|.$$

Let

$$D : \mathscr{D} \to L^2_v(\Omega)$$

be a closed densely defined operator on $L^2(\Omega)$ with domain $\mathscr{D}$. Let $S = \{S_x\}$ be a self-adjoint measurable operator field on Ω such that

$$\lambda \leqslant S_x \leqslant \mu \quad x\in\Omega \tag{1}$$

in the sense of the Hilbert space $(T_x, \langle,\rangle_x)$, where λ and μ are positive constants independent of $x\in\Omega$.

Define the closed densely defined operator H on $L^2(\Omega)$ as

$$H = D^*SD.$$

Equivalently, H is the self-adjoint operator associated to the closed quadratic form Q given by $\mathrm{Dom}(Q) = \mathscr{D}$ and

$$Q(f) = \int_\Omega \langle SDf, Df\rangle dx \quad \text{for all } f\in\mathscr{D}.$$

We introduce the following two hypotheses on H.

HYPOTHESIS 1. H has discrete spectrum $\{\lambda_n\}$ and $\lambda_n \sim n^{1/\alpha}$ as n tends to $+\infty$, for some $\alpha > 0$.

HYPOTHESIS 2. Let ϕ_n be a normalised eigenfunction for λ_n. There exists a $q_0 > 2$ such that, for all q with $2 < q < q_0$, $D\phi_n \in L^q(\Omega)$ and

$$\|D\phi_n\|_q \leqslant c_q \lambda_n^{\gamma(q)} \quad \text{for some } \gamma(q) \text{ and all } n.$$

1. *Stability estimates*

Let $H_i = D^*S_iD$, $i = 1, 2$, on $L^2(\Omega)$ be as above, satisfying Hypothesis 1 and Hypothesis 2. We define the operators $F_i = S_i^{1/2}DD^*S_i^{1/2}$, $i = 1, 2$. Thus F_i is a non-negative self-adjoint operator on $L^2_v(\Omega)$, and it is a standard result that $\mathrm{Sp}(F_i) \cup \{0\} = \mathrm{Sp}(H_i) \cup \{0\}$ and that, for $\mu \neq 0$, we have $\dim \mathrm{Ker}(F_i - \mu) = \dim \mathrm{Ker}(H_i - \mu)$.

The following is a more abstract version of a formula that has been extensively used in scattering theory **[6]**.

LEMMA 3. *Let $g(t) = t^{1/2}/(t+1)$. There exist partial isometries $U_1, U_2 : L^2 \to L^2_v$ such that*

$$(H_2+1)^{-1} - (H_1+1)^{-1} = U_2^* g(F_2) S_2^{-1/2}(S_1 - S_2) S_1^{-1/2} g(F_1) U_1.$$

Proof. Writing the identity

$$(T^*T+1)^{-1} + T^*(TT^*+1)^{-1}T = 1$$

for $T = S_1^{1/2}D$ and $T = S_2^{1/2}D$ and then subtracting, we have

$$\begin{aligned}(H_2+1)^{-1} - (H_1+1)^{-1} \\ &= -D^* S_2^{1/2}(F_2+1)^{-1} S_2^{1/2} D + D^* S_1^{1/2}(F_1+1)^{-1} S_1^{1/2} D \\ &= -D^*[(DD^* + S_2^{-1})^{-1} - (DD^* + S_1^{-1})^{-1}]D \\ &= D^*(DD^* + S_2^{-1})^{-1}(S_2^{-1} - S_1^{-1})(DD^* + S_1^{-1})^{-1}D \\ &= D^* S_2^{1/2}(F_2+1)^{-1} S_2^{1/2}(S_2^{-1} - S_1^{-1}) S_1^{1/2}(F_1+1)^{-1} S_1^{1/2} D \\ &= D^* S_2^{1/2}(F_2+1)^{-1} S_2^{-1/2}(S_1 - S_2) S_1^{-1/2}(F_1+1)^{-1} S_1^{1/2} D.\end{aligned}$$

Using polar decomposition, we write $S_i^{1/2}D = |D^* S_i^{1/2}| U_i = F_i^{1/2} U_i$, $i = 1, 2$, and the result follows.

The following arguments, together with Lemma 4, are valid for both H_1 and H_2, and we therefore omit the indices and consider a single operator H.

Let $\{\lambda_n\}$ be the eigenvalues of H, $H\phi_n = \lambda_n \phi_n$. Let $p_0 = 2q_0/(q_0 - 2)$, and let $p > p_0$ be a parameter. We make $\mathrm{Sp}(F)$ into a measure space as follows. Each eigenvalue $\lambda_n \neq 0$ with multiplicity $m(\lambda_n)$ carries a weight

$$w_{n,p} = m(\lambda_n) \lambda_n^{-1+2\gamma(2p/(p-2))}$$

where γ is the function that appears in Hypothesis 2. To 0, which may or may not be an eigenvalue, we assign the weight $w_0 = +\infty$. For $1 \leqslant s \leqslant \infty$, we then have the induced weighted l^s spaces

$$l^s(\mathrm{Sp}(F), w_{n,p}) = \left\{ G \mid G(0) = 0 \text{ and } \sum_{n=1}^{\infty} |G(\lambda_n)|^s w_{n,p} < +\infty \right\} \qquad 1 \leqslant s < \infty$$

$$l^\infty(\mathrm{Sp}(F)) = \{ G \mid \sup_n |G(\lambda_n)| < \infty \}$$

which are Banach spaces when equipped with the natural norm. Finally, for $1 \leqslant r < +\infty$, we denote by $\|A\|_{\mathscr{C}^r}$, or simply $\|A\|_r$, the Schatten norm of index r of an operator A

$$\|A\|_r = (\mathrm{tr}|A|^r)^{1/r}.$$

We have Lemma 4.

LEMMA 4. *Let $p > p_0$. Let V be a measurable operator-valued map on Ω, and let G be a complex valued function on* $\mathrm{Sp}(F)$. *For any $1 \leqslant r \leqslant \infty$, $V \in L^{pr}(\Omega)$ and $G \in l^{2r}(\mathrm{Sp}(F), w_{n,p})$ imply that $VG(F) \in \mathscr{C}^{2r}(L^2(\Omega))$ and*

$$\|VG(F)\|_{2r} \leqslant c \|V\|_{pr} \|G\|_{2r}. \tag{2}$$

Here, the L^s-norm of the operator field V is defined in the natural way by $\|V\|_s = \| |V| \|_s$, where $|V|_x$ is the norm of the operator V_x acting on the Hilbert space T_x.

Proof. For $r=\infty$, expression (2) is obvious. We shall prove it for $r=1$, the general case then following by interpolation. As mentioned above, 0 may or may not belong in $\mathrm{Sp}(F)$. If it does, we can assume that $G(0)=0$, since otherwise the right-hand side of (2) is infinite. Setting

$$\psi_n = \lambda_n^{-1/2} S^{1/2} D\phi_n,$$

we observe that $\{\psi_n\}$ is a complete orthonormal system of $\mathrm{Ker}(F)^{\perp}$ and $F\psi_n = \lambda_n \psi_n$. Hence we can estimate the Hilbert–Schmidt norm of $VG(F)$:

$$\begin{aligned}
\|VG(F)\|_2^2 &= \sum_n \|VG(F)\psi_n\|_2^2 \\
&= \sum_n |G(\lambda_n)|^2 \|V\psi_n\|_2^2 \\
&\leqslant \|V\|_p^2 \sum_n |G(\lambda_n)|^2 \|\psi_n\|_{2p/(p-2)}^2 \\
&\leqslant c\|V\|_p^2 \sum_n |G(\lambda_n)|^2 \lambda_n^{-1} \|D\phi_n\|_{2p/(p-2)}^2 \\
&\leqslant c\|V\|_p^2 \sum_n |G(\lambda_n)|^2 \lambda_n^{-1+2\gamma(2p/(p-2))} \\
&= c\|V\|_p^2 \|G\|_{l^2(\mathrm{Sp}(F), w_{n,p})}^2
\end{aligned}$$

as required.

The main theorem of this section is Theorem 5.

THEOREM 5. *Let $H_1 = D^*S_1D$ and $H_2 = D^*S_2D$ satisfy Hypothesis* 1 *and Hypothesis* 2, *and set $q_0 = \min\{q_{1,0}, q_{2,0}\}$, $p_0 = 2q_0/(q_0-2)$, and $\gamma(q) = \max\{\gamma_1(q), \gamma_2(q)\}$. If p and r satisfy the following two conditions*:

(1) $p > p_0$;

(2) $r > \alpha + 2\gamma(2p/(p-2)) - 1$;

then

$$\|(H_2+1)^{-1} - (H_1+1)^{-1}\|_r \leqslant c\|S_2 - S_1\|_{pr/2}. \tag{3}$$

Proof. Without any loss of generality, we can assume that $S_2 - S_1 \geqslant 0$. If we let $p > p_0$, then it follows from Lemma 3 and Lemma 4 that

$$\begin{aligned}
\|(H_2+1)^{-1} - (H_1+1)^{-1}\|_r &\leqslant c\|(S_2-S_1)^{1/2} g(F_1)\|_{2r} \|(S_2-S_1)^{1/2} g(F_2)\|_{2r} \\
&\leqslant \|(S_2-S_1)^{1/2}\|_{pr}^2 \|g\|_{l^{2r}(\mathrm{Sp}(F_1), w_{n,p})} \|g\|_{l^{2r}(\mathrm{Sp}(F_2), w_{n,p})}
\end{aligned}$$

where, we recall, $g(t) = t^{1/2}/(t+1)$. In particular, $g(0) = 0$, and therefore $\|g\|_{2r} < \infty$ for r sufficiently large, depending on $p > P$. In fact, using Hypothesis 1, we see that

$$\|g\|_{2r} < \infty \Leftrightarrow r > \alpha + 2\gamma\left(\frac{2p}{p-2}\right) - 1.$$

This completes the proof of the theorem.

REMARK 6. Theorem 5 immediately yields eigenvalue stability. Let $\{\mu_n(K)\}$ be the singular values of a compact operator K. Using the formula

$$\left(\sum_n |\mu_n(B)-\mu_n(A)|^r\right)^{1/r} \leqslant \|B-A\|_r \tag{4}$$

which is valid for any compact operators A and B (see [**14**]), we obtain

$$\left(\sum_{n=1}^{\infty} |\lambda_{n,1}-\lambda_{n,2}|^r n^{-2r/\alpha}\right)^{1/r} \leqslant c\|a_2-a_1\|_{pr/2}$$

for any p and r satisfying conditions (1) and (2) of Theorem 5.

EXAMPLE 7. Let $U \subset \Omega$, and suppose that S_1 and S_2 coincide outside U. Then (3) implies that

$$\|(H_2+1)^{-1}-(H_1+1)^{-1}\|_r \leqslant c|U|^{2/pr}.$$

Now, in applications, Ω is typically a compact Riemannian manifold, and D^*SD is a partial differential operator related to some physical phenomenon, the operator S containing information about relevant properties of the underlying manifold such as conductivity, elasticity, etc. Hence, by taking $|U|$ to be small, we can estimate the effect of narrowly localised irregularities on the solution of the evolution equation

$$\frac{\partial u}{\partial t} = -Hu.$$

For example, and subject to the verification of Hypothesis 2, for $m = 1$, we can estimate the effect on the heat flow produced by localised impurities of different conductivity, while, for $m = 2$, we can estimate the effect on the vibrations of an elastic shell produced by small regions with different elasticity properties.

For such applications, Hypothesis 2 is typically the one that is most difficult to verify. The most common method for proving such estimates depends upon appropriate smoothing properties being proved for the corresponding resolvent operator. In the next section, we prove such a property under quite general conditions in the case in which H is a higher order elliptic operator with measurable coefficients acting on a bounded Euclidean domain. In the case of a second order uniformly elliptic operator, where $D = \nabla$ and

$$H = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left\{a_{ij}(x)\frac{\partial}{\partial x_j}\right\},$$

there are a number of known results, which we now quote. The second and third results were proved for $\Omega = \mathbf{R}^N$, but they are also true for bounded domains under suitable regularity assumptions.

(1) (See [**8, 9**].) If Ω is a bounded Euclidean domain with C^1 boundary, $\mathscr{D} = W_0^{1,2}(\Omega)$ and $D = \nabla$, then Hypothesis 1 is satisfied with $\alpha = N/2$, and Hypothesis 2 is also satisfied with

$$\gamma(q) = \frac{1}{4}\left(N+2-\frac{2N}{q}\right).$$

We then regain the main theorem of [**3**].

(2) (See [**10**].) If $\nabla a_{ij} \in L^N$, then $(H+1)^{-1}: L^2 \to W^{2,2}$.

(3) (See [**11**].) If the coefficients a_{ij} are continuous, then $(H+1)^{-1}: L^p \to W^{1,p}$ for all $1 < p < \infty$.

For a result of this type concerning non-self-adjoint operators, see [**1**, Proposition 3.1]. See also [**12**] for similar results in the case of unbounded coefficients. Finally, see [**4**] for similar estimates for Laplace–Beltrami operators on compact Riemannian manifolds satisfying mixed Dirichlet–Neumann boundary conditions.

2. *Regularity estimates*

We prove a theorem that provides a general condition under which Hypothesis 2 is valid.

Let Ω and the spaces $L^p_v(\Omega)$ be as above, and define the 'mixed' Banach spaces

$$L^p_m(\Omega) = L^p(\Omega) \oplus L^p_v(\Omega) \quad 1 < p < +\infty$$

equipped with the norm

$$\|(f_0, \mathbf{f})\|_p = \{\|f_0\|_p^p + \|\mathbf{f}\|_p^p\}^{1/p}.$$

Let $\mathscr{S}$ be a function space on Ω contained in $\bigcap_{p>1} L^p(\Omega)$ and dense in each $L^p(\Omega)$, $1 < p < +\infty$. Let D be a map with domain $\mathscr{S}$, such that the following are true.

(1) $Df \in \bigcap_{p>1} L^p_v$ for all $f \in \mathscr{S}$.

(2) For each $1 < p < +\infty$, D is closable as an operator from $\mathscr{S} \subset L^p(\Omega)$ into $L^p_v(\Omega)$.

We denote by D_p the corresponding closure, and set $W^p = \mathrm{Dom}(D_p)$. In particular, W^p is the completion of $\mathscr{S}$ with respect to the norm

$$\|f\|_{W^p} = \{\|f\|_p^p + \|Df\|_p^p\}^{1/p}.$$

For the sake of simplicity, we usually write Df rather than $D_p f$, even when $f \notin \mathscr{S}$.

Now let S be a measurable operator field on Ω satisfying (1). For $u \in W^p$ and $\phi \in W^{p'}$, we set

$$Q(u, \phi) = \int_\Omega (u\phi + \langle SDu, D\phi \rangle) dx.$$

We also denote by $W^{-p'}$ the Banach space dual of W^p. We have Lemma 8.

LEMMA 8. *Let $T \in W^{-p'}$. There exists an $f = (f_0, \mathbf{f}) \in L^{p'}_m(\Omega)$ such that*

$$\langle T, \phi \rangle = \int_\Omega (f_0 \phi + \langle \mathbf{f}, D\phi \rangle) dx \quad \text{for all } \phi \in W^p$$

and $\|T\| = \|f\|_{p'}$.

Proof. Let

$$R \subset L^p_m = \{\phi \oplus D_p \phi \mid \phi \in W^p\}$$

and define the functional $\tilde{T}$ on R by

$$\langle \tilde{T}, \phi \oplus D_p \phi \rangle = \langle T, \phi \rangle \quad \text{for all } \phi \in W^p.$$

We readily see that $\|T\| = \|\tilde{T}\|$. By the Hahn–Banach Theorem, there exists an extension T_0 of $\tilde{T}$ on the whole of $L^p_m(\Omega)$ such that $\|T_0\| = \|\tilde{T}\|$. It is then standard that there exists an $f \in L^{p'}_m(\Omega)$ such that

$$\langle T_0, g\rangle = \int_\Omega (f_0 g_0 + \langle \mathbf{f}, \mathbf{g}\rangle) dx \quad \text{for all } g = (g_0, \mathbf{g}) \in L^p_m$$

and that $\|T_0\| = \|f\|_{p'}$. Hence

$$\langle T, \phi\rangle = \int_\Omega (f_0 \phi + \langle \mathbf{f}, D\phi\rangle) dx \quad \text{for all } \phi \in W^p$$

and $\|T\| = \|f\|_{p'}$ as required.

For $1 < q < +\infty$, we define the operator

$$H_q \colon W^q \to W^{-q}$$

by

$$\langle H_q u, \phi\rangle = Q(u, \phi) \quad \text{for all } u \in W^q, \phi \in W^{q'}$$

and the property $(\mathrm{P}_{1,q})$ as H_q has a bounded inverse.

Let $f = (f_0, \mathbf{f}) \in \bigcup_{p>1} L^1_m$. We say that the function $u \in \bigcup_{p>1} W^p$ solves the equation

$$Hu = f_0 + D^*\mathbf{f} \tag{5}$$

if

$$Q(u, \phi) = \int_\Omega (f_0 \phi + \langle \mathbf{f}, D\phi\rangle) dx \quad \text{for all } \phi \in \mathscr{S}$$

and we define the property $(\mathrm{P}_{2,q})$ as for $f \in L^q_m(\Omega)$, equation (5) has a unique solution $u =: T_q f \in W^q$, and the operator $T_q \colon L^q_m \to W^q$ is bounded.

Finally, we define the number

$$\frac{1}{K_q} = \inf_{\|u\|_{W^{q'}} = 1} \sup_{\|\phi\|_{W^q} = 1} |Q(u, \phi)|$$

and the property $(\mathrm{P}_{3,q})$ as $K_q < +\infty$.

Proposition 9. *For $2 \leqslant q < +\infty$, properties $(\mathrm{P}_{1,q}), (\mathrm{P}_{2,q}), (\mathrm{P}_{3,q}), (\mathrm{P}_{1,q'})$ and $(\mathrm{P}_{2,q'})$ are equivalent, and, if they are valid, then*

$$\|H_q^{-1}\| = \|H_{q'}^{-1}\| = \|T_q\| = \|T_{q'}\| = K_q.$$

Proof. $(\mathrm{P}_{1,q}) \Leftrightarrow (\mathrm{P}_{1,q'})$: This follows from the fact that $H_q^* = H_{q'}$.

$(\mathrm{P}_{1,q}) \Rightarrow (\mathrm{P}_{2,q}), (\mathrm{P}_{1,q'}) \Rightarrow (\mathrm{P}_{2,q'})$: Let p be either q or q', and let $f \in L^p$. Define $T \in W^{-p}$ by

$$\langle T, \phi\rangle = \int (f_0 \phi + \langle \mathbf{f}, D\phi\rangle) dx \quad \text{for all } \phi \in W^{p'}$$

so that, in particular, $\|T\|_{W^{-p}} \leqslant \|f\|_p$.

Setting $u = H_p^{-1} T$, we have

$$Q(u, \phi) = \int (f_0 \phi + \langle \mathbf{f}, D\phi \rangle) dx \quad \text{for all } \phi \in W^{p'}$$

that is, $Hu = f_0 + D^* \mathbf{f}$. This solution is unique in W^p since H_p is 1-1. Defining the operator T_p by

$$T_p f = u,$$

it follows that $\|T_p\| \leqslant \|H_p^{-1}\|$.

$(\mathrm{P}_{2,q}) \Rightarrow (\mathrm{P}_{1,q}), (\mathrm{P}_{2,q'}) \Rightarrow (\mathrm{P}_{1,q'})$: Let p be either q or q', and let $T \in W^{-p}$. Let $f = (f_0, \mathbf{f}) \in L_m^p$ be such that

$$\langle T, \phi \rangle = \int (f_0 \phi + \langle \mathbf{f}, D\phi \rangle) dx \quad \text{for all } \phi \in W^{p'}$$

and $\|T\|_{W^{-p}} = \|f\|_p$. Letting $u = T_p f \in W^p$, we have $H_p u = T$, and so H_p is onto, and it is 1-1 since T_p is 1-1. Moreover,

$$\|H_p^{-1} T\|_{W^p} = \|u\|_{W^p} \leqslant \|T_p\| \, \|f\|_p = \|T_p\| \, \|T\|_{W^{-p}}$$

and so $\|H_p^{-1}\| \leqslant \|T_p\|$.

$(\mathrm{P}_{3,q}) \Rightarrow (\mathrm{P}_{2,q'})$: Let $g \in L^{q'}$. Let $(g_k) \subset L^2$ satisfy $\|g - g_k\|_{q'} \to 0$, and let $u_k \in W^2$ be the solution of

$$Hu_k = g_{0,k} + D^* \mathbf{g}_k \quad k = 1, 2, \ldots$$

Then

$$Q(u_k, \phi) = \int (g_{0,k} \phi + \langle \mathbf{g}_k, D\phi \rangle) dx \quad \text{for all } \phi \in W^2. \tag{6}$$

If absolute values are taken of both sides and then supremum over all $\phi \in W^2$ such that $\|\phi\|_q \leqslant 1$, then $(\mathrm{P}_{3,q})$ implies that

$$\frac{1}{K_q} \|u_k\|_{1,q'} \leqslant \|g_k\|_{q'}. \tag{7}$$

Hence (u_k) is Cauchy in $W^{q'}$. If $u \in W^{q'}$ is its limit, then, letting k tend to $+\infty$ in (6) and (7), we have

$$Hu = g_0 + D^* \mathbf{g}$$

and $\|u\|_{W^{q'}} \leqslant K_q \|g\|_{q'}$. The solution u is unique in $W^{1,q'}$ since, by $(\mathrm{P}_{3,q})$, the only solution in $W^{q'}$ of the equation $Hv = 0$ is the trivial solution.

$(\mathrm{P}_{1,q}) \Rightarrow (\mathrm{P}_{3,q'}), (\mathrm{P}_{1,q'}) \Rightarrow (\mathrm{P}_{3,q})$: Let p be either q or q'. Then

$$\|H_p u\|_{W^{-p}} \geqslant \frac{1}{\|H_p^{-1}\|} \|u\|_{W^p} \quad \text{for all } u \in W^p$$

and hence

$$\sup_{\|\phi\|_{W^{p'}} = 1} |Q(u, \phi)| \geqslant \frac{1}{\|H_p^{-1}\|} \|u\|_{W^p} \quad \text{for all } u \in W^p$$

from which $(\mathrm{P}_{3,p'})$ follows.

We are now in the position to prove the regularity theorem for the operator $H = D^* S D + 1$. It depends on a regularity assumption of the same type on the operator $\tilde{H} =: D^* D + 1$. The proof is similar to that in [**9**]. We recall that λ and μ are such that

$$\lambda \leqslant S_x \leqslant \mu \quad \text{for all } x \in \Omega.$$

THEOREM 10. *Assume that there exists a* $\tilde{q}_0$, $2 < \tilde{q}_0 \leqslant +\infty$, *such that the operator* $\tilde{H} = D^*D + 1$ *satisfies properties* $(P_{i,q})$ *for all* $\tilde{q}_0' < q < \tilde{q}_0$. *Then there exists a* q_0 *with* $2 < q_0 < \tilde{q}_0$ *depending only on D and the constants* λ, μ, *such that H satisfies properties* $(P_{i,q})$ *for all* $q_0' < q < q_0$.

Hence, if we have some L^q-estimates for $q > 2$ on the eigenfunctions $\{\phi_n\}$ of $H = D^*SD$, Hypothesis 2 is valid.

Proof. Without any loss of generality, we assume that $\mu = 1$. It is standard that the norm $\|\tilde{H}_q^{-1}\|$ is a logarithmically convex function of $1/q \in (1/\tilde{q}_0, 1/\tilde{q}_0')$, and, since it is symmetric with respect to $q \leftrightarrow q'$, it has a minimum at $q = 2$. This minimum is equal to 1; to see this, let $f \in L_m^2$ and $u = T_2 f \in W^2$. Then

$$\tilde{Q}(u, \phi) = \int (f_0 \phi + \langle \mathbf{f}, D\phi \rangle) dx \quad \text{for all } u \in W^2$$

and, taking the supremum over all $\phi \in W^2$ such that $\|\phi\|_{W^2} \leqslant 1$, we have $\|H_2^{-1}\| \leqslant 1$. Moreover, choosing $f \in L_m^2$ of the form $f = (\psi, D\psi)$ for $\psi \in W^2$, we have $\|H_2^{-1} f\|_{W^2} = \|\psi\|_{W^2} = \|f\|_2$. Hence $\|H_2^{-1}\| = 1$.

Now let $\tilde{Q}$ be the form associated with the operator $\tilde{H}$. For any $\psi \in W^{q'}$, we have

$$\begin{aligned} \sup_{\|\phi\|_{W^q} = 1} |Q(u, \phi)| &\geqslant \sup_{\|\phi\|_{W^q} = 1} |\tilde{Q}(u, \phi)| \\ &\quad - \sup_{\|\phi\|_{W^q} = 1} |\tilde{Q}(u, \phi) - Q(u, \phi)| \\ &\geqslant \left(\frac{1}{\tilde{K}_q} - (1 - \lambda) \right) \|u\|_{W^{q'}} \end{aligned}$$

and therefore the norm K_q satisfies

$$\frac{1}{K_q} \geqslant \frac{1}{\tilde{K}_q} - 1 + \lambda.$$

Hence, if $2 < q_0 < \tilde{q}_0$ is chosen so that $\tilde{K}_{q_0} = (1 - \lambda)^{-1}$, then K_q is smaller than $+\infty$ for all $q_0' < q < q_0$, as required.

EXAMPLE 11. Let Ω be a bounded Euclidean domain in $\mathbf{R}^N$ with smooth boundary, $\mathscr{D} = W_0^{m,2}(\Omega)$, and A be a set of multi-indices $\alpha = (\alpha_1, \ldots, \alpha_N)$ such that the constant coefficient operator D^*D, where

$$Df = (D^\alpha f)_{\alpha \in A} =: ((\partial / \partial x_1)^{\alpha_1} \ldots (\partial / \partial x_N)^{\alpha_N} f)_{\alpha \in A},$$

is uniformly elliptic of order $2m$. Let $a = \{a_{\alpha\beta}(x)\}_{\alpha, \beta \in A}$ be a measurable matrix such that

$$\lambda \leqslant a(x) \leqslant \mu$$

on Ω, and let $H = D^*aD$. Thus H is a self-adjoint uniformly elliptic operator of order $2m$, and it is given formally by

$$Hf = \sum_{\alpha, \beta \in A} (-1)^{|\alpha|} D^\alpha \{a_{\alpha\beta}(x) D^\beta f\}.$$

Moreover, it satisfies the hypotheses of Theorem 10, since D^*D satisfies properties $(\mathrm{P}_{i,q})$ by [**13**, Theorem 4.6].

Let $I = I(N,m)$ be the interval of all $q \geqslant 2$ such that the Sobolev imbedding $W_0^{m,2}(\Omega) \subset L^q(\Omega)$ is valid, and set

$$q_0 = \sup\{q \mid q \in I \text{ and } (\mathrm{P}_{i,q}) \text{ are valid}\} \in (2, \infty].$$

For any q, $2 < q < q_0$, we then have

$$\begin{aligned}\|D\phi_n\|_q &\leqslant c\lambda_n \|\phi_n\|_q \\ &\leqslant c\lambda_n \|\phi_n\|_{m,2} \\ &\leqslant c\lambda_n \|D\phi_n\|_2 \\ &\leqslant c\lambda_n^{3/2}\end{aligned}$$

and interpolation with $\|D\phi_n\|_2 \leqslant \lambda_n^{1/2}$ yields

$$\|D\phi_n\|_q \leqslant c\lambda_n^{\gamma}$$

for all $2 < q < q_0$ and all

$$\gamma > \frac{1}{2} + \frac{q_0(q-2)}{q(q_0-2)}.$$

Since Hypothesis 1 is also satisfied with $\alpha = N/2m$, for H_1 and H_2 as above, we have Corollary 12 of Theorem 5.

Corollary 12. *There exists a $p_0 < +\infty$ such that, if $p > p_0$ and $r > (N/2m) + 2p_0/p$, then*

$$\|(H_2+1)^{-1} - (H_1+1)^{-1}\|_r \leqslant c\|a_2 - a_1\|_{pr/2}. \tag{8}$$

Proof. We only need to take $p_0 = 2q_0/(q_0-2)$, recall that $p = 2q/(q-2)$, and observe that Theorem 5(2) becomes

$$\begin{aligned} r &> \frac{N}{2m} + 2\left(\frac{1}{2} + \frac{q_o(q-2)}{q(q_0-2)}\right) - 1 \\ &= \frac{N}{2m} + \frac{2p_0}{p}.\end{aligned}$$

Note that the conclusion of Theorem 5 is still valid if Hypothesis 2 is weakened to

$$\|D\phi_n\|_q \leqslant c_q \lambda_n^s \quad \text{for all } 2 < q < q_0,\ s > \gamma(q).$$

Acknowledgements. I thank E. B. Davies for many useful discussions. The material in this paper has been extracted from the author's Ph.D. thesis [**2**].

References

1. P. Auscher, A. McIntosh and P. Tchamitchian, 'Heat kernels of second order complex elliptic operators and applications', *J. Funct. Anal.* 152 (1998) 22–73.
2. G. Barbatis, 'On the stability of second order elliptic operators', Ph.D. Thesis, King's College London, 1994.
3. G. Barbatis, 'Spectral stability under L^p-perturbation of the second-order coefficients', *J. Differential Equations* 124 (1996) 302–323.
4. G. Barbatis, 'Stability of weighted Laplace–Beltrami operators under L^p-perturbation of the Riemannian metric', *J. Anal. Math.* 68 (1996) 253–276.
5. E. B. Davies, *Heat kernels and spectral theory* (Cambridge University Press, 1989).
6. P. Deift, 'Applications of a commutation formula', *Duke Math. J.* 45 (1978), 267–309.
7. J. Dixmier, *Les algèbres d'opérateurs dans l'espace Hilbertien* (Gauthier–Villars, Paris, 1957).

8. K. Gröger, 'A $W^{1,p}$-estimate for solutions to mixed boundary value problems for second order elliptic differential equations', *Math. Ann.* 283 (1989) 679–687.

9. N. G. Meyers, 'An L^p-estimate for the gradient of solutions of second order elliptic divergence equations', *Ann. Scuola Norm. Sup. Pisa* 17 (1963) 189–206.

10. C. Miranda, 'Sul problem misto per le equazioni lineari ellitiche', *Ann. Mat. Pura Appl.* (4) 39 (1955) 279–303.

11. Y. A. Semenov, 'Smoothness of generalized solutions of the equation $(\lambda - \sum_{i,j} \nabla_i a_{ij} \nabla_j)u = f$ with continuous coefficients', *Math. USSR Sb.* 46 (1983) (in English).

12. Y. A. Semenov, 'Smoothness of generalized solutions of the equation $\hat{H}u = f$ and essential selfadjointness of the operator $\hat{H} = -\sum_{i,j} \nabla_i a_{ij} \nabla_j + V$ with measurable coefficients', *Math. USSR Sb.* 55 (1986) 309–333 (in English).

13. C. G. Simader, 'On Dirichlet's boundary value problem', Lecture Notes in Mathematics 268 (Springer, 1972).

14. B. Simon, *Trace ideals and their applications* (Cambridge University Press, 1979).

6 Irodotou Street
Halandri
15231 Athens
Greece