STABILITY OF WEIGHTED LAPLACE–BELTRAMI OPERATORS UNDER L^p-PERTURBATION OF THE RIEMANNIAN METRIC

By

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Introduction

Let Ω be a bounded domain of $\mathbf{R}^N$ with Lipschitz boundary and H a second-order uniformly elliptic operator on $L^2(\Omega)$ with measurable coefficients satisfying Dirichlet boundary conditions. In [1] we established the stability of the resolvent of H in trace norms as the coefficients vary in L^p-spaces.

This paper further develops and improves the ideas used in [1], thus making possible the extension of the above results in different directions. First, we extend the range of possible applications by working on Riemannian manifolds instead of Euclidean domains. Second, we work with weighted operators and, third, we deal with more general boundary conditions. In order to prove our main result, we also prove a regularity theorem which generalizes the main theorem of [7] and, partly, of [6]. There is a straight-forward physical interpretation of our results; given a curved surface, Theorems 8 and 16 provide estimates on the effect on the heat flow produced by narrowly localised irregularities. We refer to [1] for details in the Euclidean case.

We work on a compact Riemannian manifold (M, G), which may have a boundary. For a positive measurable weight σ^2 on M, we consider weighted Laplace–Beltrami operators of the form

$$H_{G,\sigma^2} f = -\sigma^{-2}\nabla_G \cdot (\sigma^2 \nabla_G f) + f$$

satisfying mixed Dirichlet–Neumann boundary conditions (we shall use the term Laplace–Beltrami operators, despite it being usually used only for the operator $-\Delta$, without the extra constant). In our main result we obtain explicit bounds on $\|w^{-1}H^{-1}_{G_2,\sigma_2^2}w - H^{-1}_{G_1,\sigma_1^2}\|$ for various operator norms in terms of appropriate L^p-norms of $G_1 - G_2$ and of $\sigma_1 - \sigma_2$. The operator w is the natural isometry between the (different) weighted L^2-spaces where the two operators are self-adjoint and is also needed in order to obtain eigenvalue stability as a corollary.

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The paper is organised as follows: We make the initial assumption that M has no boundary and in Section 1 we prove that the resolvent H^{-1} maps L^p continuously into $W^{1,p}$ if p is close enough to 2. This generalizes the main theorem of [7] and, partly, of [6].

Using this, we prove in Section 2 our main result; quantitative estimates for the stability of the resolvent in trace norms as G and σ vary in L^p. Such estimates are immediate for $p = +\infty$. The case $p < +\infty$ is harder and is the one relevant for applications.

We work with measurable metrics and weights, and this enables us in Section 3 to generalize our theorem to the case of manifolds with boundary satisfying mixed boundary conditions. The method used is to replace the manifold with boundary by its double and then to deduce the required results from those of Section 2. As a by-product, we further extend the main theorem of [7] to this context.

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Technical setting and notation

In the first two sections of this paper M will be an N-dimensional smooth compact manifold without boundary. We shall be considering Riemannian metrics on M that belong to a class $\mathcal{R}(M)$ defined as follows: a measurable metric G lies in $\mathcal{R}(M)$ if and only if there exists a smooth metric $\tilde{G}$ on M such that

$$\lambda\tilde{G} \leq G \leq \mu\tilde{G}$$

for some positive constants λ and μ. We shall also denote by $\mathcal{W}(M)$ the class of all measurable weights σ^2 on M such that

$$\lambda\tilde{\sigma}^2 \leq \sigma^2 \leq \mu\tilde{\sigma}^2$$

for some positive smooth weight $\tilde{\sigma}^2$. By compactness, if such a $\tilde{\sigma}^2$ exists then we may as well choose $\tilde{\sigma}^2 = 1$. We do not make this particular choice since the actual values of the constants λ and μ will turn out to be important.

We shall use the same notation, G, for both the metric and its representation $G = \{g_{ij}\}$ as a matrix with respect to a given local coordinate system (x_i) on M:

$$ds^2 = \sum g_{ij}dx_idx_j.$$

We also set $g = \det(G)$. Similarly, if $\xi = \{\xi_x\}$ is a vector field on M, we shall also denote by ξ the vector field (ξ_i) in $\mathbf{R}^N$ whose components for each $x \in M$ are the components of ξ_x relative to a basis of the tangent space T_x induced by a given local coordinate system (x_i):

$$\xi = \sum_i \xi_i \frac{\partial}{\partial x_i}.$$

It will be clear from the context which meaning is intended. Finally, we shall denote by $\bar{\nabla}$ the 'Euclidean' gradient $\bar{\nabla} = (\partial/\partial x_1, \ldots, \partial/\partial x_N)$ relative to some local coordinates, while a dot will denote the Euclidean inner product, except in the symbol $\nabla\cdot$, which will stand for the divergence operator induced by G. The inner product in T_x, $x \in M$ shall be denoted by $\langle\ ,\ \rangle_x$ or simply $\langle\ ,\ \rangle$; this last symbol shall also be used to denote the inner product on various L^2-spaces.

We shall also use the same notation, $L^p(M, G, \sigma^2)$, for the space of various fields on M that are p-integrable with respect to the volume element $\sigma^2 d_G\mathrm{vol}$: functions, vector fields or operator fields. One or more of the arguments will often be dropped when the meaning is clear.

For $G \in \mathcal{R}(M)$, $\sigma^2 \in \mathcal{W}(M)$, we shall be considering operators given formally by

$$H = -\sigma^{-2}\nabla \cdot (\sigma^2 \nabla) + 1.$$

By definition, H is the self-adjoint operator on $L^2(M, G, \sigma^2)$ associated to the quadratic form Q given by

$$\mathrm{Dom}(Q) = W^{1,2}(M),$$

$$Q(f) = \int_M (|f|^2 + |\nabla f|^2)\sigma^2 d\mathrm{vol}.$$

It is well-known that such an operator has a discrete spectrum $\{\lambda_n\}$ and that

$$\lambda_n \sim n^{2/N} \qquad \text{as } n \longrightarrow +\infty. \tag{1}$$

1. $W^{1,q}$-estimates on solutions of equations

In this section we prove a regularity theorem for operators in the class defined above. The basic idea is that used in [7], but working on a manifold instead of a Euclidean domain adds several technical complications.

Let $f_0, \vec{f} \in L^1(M)$. We say that the function $u \in W^{1,1}(M)$ solves the equation

$$Hu = f_0 + \nabla \cdot (\sigma^2 \vec{f}) \tag{2}$$

if

$$Q(u, \phi) = \int_M (f_0\phi - \langle \vec{f}, \nabla\phi \rangle)\sigma^2 d\mathrm{vol}, \qquad \text{all } \phi \in C^\infty(M). \tag{3}$$

Here and below, we use the notation $Q(u,\phi)$ for the integral

$$\int_M (u\phi + \langle \nabla u, \nabla\phi\rangle)\sigma^2 d\mathrm{vol}$$

provided that it exists, without necessarily having $u,\phi \in W^{1,2}$.

Let q' denote the conjugate exponent of q. We denote by $W^{-1,q'}$ the dual of $W^{1,q} =: W^{1,q}(M,G,\sigma^2)$, where

$$\| f\|_{W^{1,q}(M,G,\sigma)} = \int_M (| f |^q + | \nabla f |^q)^{1/q}\sigma^2 d\mathrm{vol}.$$

Using a standard argument one can prove that for $T \in W^{-1,q'}$ there exists $g = (g_0,\vec{g}) \in L^{q'}$ such that

$$\langle T,\phi\rangle = \int (g_0\phi + \langle \vec{g}, \nabla\phi\rangle)\sigma^2 d\mathrm{vol}, \quad \text{all } \phi \in W^{1,q} \tag{4}$$

and

$$\|T\|_{-1,q'} = \|g\|_{q'}. \tag{5}$$

For $1 < q < \infty$ we define the operator

$$H_q : W^{1,q} \longrightarrow W^{-1,q}$$

by

$$\langle H_q u,\phi\rangle = \int (u\phi + \langle \nabla u, \nabla\phi\rangle)\sigma^2 d\mathrm{vol}, \quad \text{all } \phi \in W^{1,q'}, \quad u \in W^{1,q}.$$

The family $\{H_q\}_q$ is self-adjoint in the sense that

$$H_q^* = H_{q'}, \quad \text{all } 1 < q < \infty. \tag{6}$$

Finally, for $1 < q < +\infty$ we set

$$\frac{1}{K_q} = \inf_{\|\psi\|_{1,q'}=1} \sup_{\|\phi\|_{1,q}=1} | Q(\psi,\phi) | .$$

For $1 < q < \infty$ we define the properties:

$(\mathrm{P}_{1,q})$ H_q is invertible with a bounded inverse.

$(\mathrm{P}_{2,q})$ For $f = (f_0,\vec{f}) \in L^q$ the equation $Hu = f_0 + \nabla\cdot(\sigma^2\vec{f})$ has a unique solution $u =: T_p f \in W^{1,q}$ and the operator $T_q : L^q \longrightarrow W^{1,q}$ is bounded.

$(P_{3,q})$ $\quad K_q < +\infty$.
We have the following

Proposition 1 *For* $2 \le q < +\infty$, *properties* $(P_{1,q})$, $(P_{2,q})$, $(P_{3,q})$, $(P_{1,q'})$ *and* $(P_{2,q'})$ *are equivalent, and if they are valid, then*

$$\|H_q^{-1}\| = \|H_{q'}^{-1}\| = \|T_q\| = \|T_{q'}\| = K_q.$$

Proof $(P_{1,q}) \iff (P_{1,q'})$. Follows directly from (6).

$(P_{1,q}) \Rightarrow (P_{2,q})$, $(P_{1,q'}) \Rightarrow (P_{2,q'})$. Let p be either q or q' and let $f \in L^p$. Define $T \in W^{-1,p}$ by

$$\langle T, \phi\rangle = \int (f_0\phi - \langle \vec{f}, \nabla\phi\rangle)\sigma^2 d\text{vol}, \qquad \text{all } \phi \in W^{1,p'}$$

so that, in particular, $\|T\|_{-1,p} \le \|f\|_p$.

Setting $u = H_p^{-1}T$ we have

$$Q(u, \phi) = \int (f_0\phi - \langle \vec{f}, \nabla\phi\rangle)\sigma^2 d\text{vol}, \qquad \text{all } \phi \in W^{1,p'},$$

that is, $Hu = f_0 + \nabla \cdot (\sigma^2\vec{f})$. This solution is unique in $W^{1,p}$ since H_p is 1-1 and if T_p is defined by

$$T_pf = u$$

it follows that $\|T_p\| \le \|H_p^{-1}\|$.

$(P_{2,q}) \Rightarrow (P_{1,q})$, $(P_{2,q'}) \Rightarrow (P_{1,q'})$. Let p be either q or q' and let $T \in W^{-1,p}$. Let $f \in L^p$ be such that

$$\langle T, \phi\rangle = \int (f_0\phi - \langle \vec{f}, \nabla\phi\rangle)\sigma^2 d\text{vol}, \qquad \text{all } \phi \in W^{1,p'}$$

and $\|T\|_{-1,p} = \|f\|_p$. Letting $u = T_pf \in W^{1,p}$ we see that $H_pu = T$, so H_p is onto, and it is 1-1 since T_p is 1-1. Moreover,

$$\|H_p^{-1}T\|_{1,p} = \|u\|_{1,p} \le \|T_p\|\|f\|_p = \|T_p\|\|T\|_{-1,p}$$

and so $\|H_p^{-1}\| \le \|T_p\|$.

$(P_{3,q}) \Rightarrow (P_{2,q'})$. Let $g \in L^{q'}$. Let $(g_k) \subset L^2$ satisfy $\|g - g_k\|_{q'} \longrightarrow 0$ and let $u_k \in W^{1,2}$ be the solution of

$$Hu_k = g_{0,k} + \nabla \cdot (\sigma^2\vec{g_k}), \quad k = 1, 2, \ldots .$$

Then

$$Q(u_k,\phi) = \int (g_{0,k}\phi - \langle \vec{g}_k, \nabla\phi\rangle)\sigma^2 d\mathrm{vol}, \qquad \text{all } \phi \in W^{1,2}. \tag{7}$$

Taking absolute values of both sides and then supremum over all $\phi \in W^{1,2}$ such that $\|\phi\|_{1,q} \leq 1$, $(\mathrm{P}_{3,q})$ implies

$$\frac{1}{K_q}\|u_k\|_{1,q'} \leq \|g_k\|_{q'}. \tag{8}$$

Hence (u_k) is Cauchy in $W^{1,q'}$. If $u \in W^{1,q'}$ is its limit, letting $k \longrightarrow +\infty$ in (7) and (8) yields

$$Hu = g_0 + \nabla\cdot(\sigma^2\vec{g})$$

and $\|u\|_{1,q'} \leq K_q\|g\|_{q'}$. The solution u is unique in $W^{1,q'}$ since, by $(\mathrm{P}_{3,q})$, the only solution in $W^{1,q'}$ of the equation $Hv = 0$ is the trivial solution.

$(\mathrm{P}_{1,q}) \Rightarrow (\mathrm{P}_{3,q'})$, $(\mathrm{P}_{1,q'}) \Rightarrow (\mathrm{P}_{3,q})$. Let p be either q or q'. Then

$$\|H_p u\|_{-1,p} \geq \frac{1}{\|H_p^{-1}\|}\|u\|_{1,p}, \qquad \text{all } u \in W^{1,p}$$

and hence

$$\sup_{\|\phi\|_{1,p'}=1} | Q(u,\phi) | \geq \frac{1}{\|H_p^{-1}\|}\|u\|_{1,p}, \qquad \text{all } u \in W^{1,p}$$

from which $(\mathrm{P}_{3,p'})$ follows. □

Let $G \in \mathcal{R}(M)$, $\sigma^2 \in \mathcal{W}(M)$ and let H be the corresponding weighted Laplace–Beltrami operator

$$H = -\sigma^{-2}\nabla\cdot(\sigma^2\nabla) + 1.$$

There exist $\tilde{G}, \tilde{\sigma}^2$ smooth such that

$$\lambda_1\tilde{G} \leq G \leq \mu_1\tilde{G}, \tag{9}$$

$$\lambda_2\tilde{\sigma}^2 \leq \sigma^2 \leq \mu_2\tilde{\sigma}^2, \tag{10}$$

for some positive constants λ_i, μ_i, $i = 1,2$.

The weighted $W^{1,q}$-norms $\|\cdot\|_{1,q}$ and $\|\cdot\|\tilde{}_{1,q}$ induced by G,σ^2 and $\tilde{G},\tilde{\sigma}^2$ are equivalent. In fact, defining

$$\lambda = \lambda_1^{N/2}\lambda_2, \quad \mu = \mu_1^{N/2}\mu_2$$

and

$$\gamma_{1,q} = \min\{1, \mu_1^{-1/2}\}\lambda^{1/q} \quad \text{and} \quad \gamma_{2,q} = \max\{1, \lambda_1^{-1/2}\}\mu^{1/q}$$

we easily check that

$$\gamma_{1,q}\|f\|\tilde{}_{1,q} \le \|f\|_{1,q} \le \gamma_{2,q}\|f\|\tilde{}_{1,q}, \quad \text{all } f \in W^{1,q}. \tag{11}$$

It is clear that the λ_i, μ_i can be chosen arbitrarily as long as the ratios

$$\rho_i =: \frac{\mu_i}{\lambda_i}, \qquad i = 1, 2$$

remain fixed. We set

$$\rho = \rho_1^{N/2}\rho_2$$

and make the particular choice

$$\lambda_1 = \frac{\rho\rho_1 + 1}{\rho_1(\rho + 1)}, \tag{12}$$

$$\lambda_2 = \frac{2}{\rho + 1}\left(\frac{\rho\rho_1 + 1}{\rho_1(\rho + 1)}\right)^{-N/2}, \tag{13}$$

which will be justified later.

We define the measurable operator field $S = S(x)$, $x \in M$, on the tangent bundle TM by

$$S = G^{-1}\tilde{G}. \tag{14}$$

One checks easily that this is coordinate-independent.

Lemma 2 *$S(x)$ is self-adjoint on the Hilbert space $(T_xM, \tilde{G})$ and satisfies*

$$\mu_1^{-1} \le S \le \lambda_1^{-1} \tag{15}$$

in the sense of $(T_xM, \tilde{G})$. Moreover, for $f, g \in W^{1,2}$ we have

$$\langle \nabla f, \nabla g\rangle_G = \langle S\tilde{\nabla} f, \tilde{\nabla} g\rangle_{\tilde{G}} \tag{16}$$

pointwise a.e. on M.

Proof For $\xi, \eta \in T_x M$ we have in local coordinates

$$\begin{aligned}\langle S\xi, \eta\rangle_{\tilde{G}} &= \tilde{G}S\xi \cdot \eta \\ &= \tilde{G}\xi \cdot S\eta \\ &= \langle \xi, S\eta\rangle_{\tilde{G}}.\end{aligned}$$

Inequality (15) is equivalent to the inequality

$$\mu_1^{-1}\tilde{G} \leq \tilde{G}S \leq \lambda_1^{-1}\tilde{G}$$

in the sense of $(\mathbf{R}^N, \cdot)$, and so follows directly from (9). Finally, for $f, g \in W^{1,2}$ we have

$$\begin{aligned}\langle S\tilde{\nabla} f, \tilde{\nabla} g\rangle_{\tilde{G}} &= \tilde{G}S\tilde{G}^{-1}\bar{\nabla} f \cdot \tilde{G}^{-1}\bar{\nabla} g \\ &= \tilde{G}G^{-1}\bar{\nabla} f \cdot \tilde{G}^{-1}\bar{\nabla} g \\ &= G^{-1}\bar{\nabla} f \cdot \bar{\nabla} g \\ &= \langle \nabla f, \nabla g\rangle_G. \qquad \square\end{aligned}$$

Let $\tilde{H}$ be the self-adjoint operator on $L^2(\tilde{G}, \tilde{\sigma}^2)$ induced by $\tilde{G}$ and $\tilde{\sigma}^2$ and let $\tilde{Q}$ be the associated form.

We shall need the following estimate on the difference of the quadratic forms Q and $\tilde{Q}$:

Lemma 3 *For $\phi \in W^{1,q}$, $\psi \in W^{1,q'}$ we have*

$$(17) \qquad |\, Q(\psi, \phi) - \tilde{Q}(\psi, \phi) \,| \leq \frac{\rho\rho_1 - 1}{\rho\rho_1 + 1} \|\psi\|\tilde{}_{1,q'} \|\phi\|\tilde{}_{1,q}.$$

Proof Define the function a on M by

$$a = \sigma^2 g^{1/2} / \tilde{\sigma}^2 \tilde{g}^{1/2}.$$

This is coordinate-independent and we have

$$(18) \qquad \sigma^2 d_G \mathrm{vol} = a\tilde{\sigma}^2 d_{\tilde{G}} \mathrm{vol}.$$

It follows from (16) and (18) that

$$Q(\psi, \phi) = \int (\psi\phi + \langle S\tilde{\nabla}\psi, \tilde{\nabla}\phi\rangle_{\tilde{G}}) a\tilde{\sigma}^2 d_{\tilde{G}} \mathrm{vol}$$

so that

$$\tilde{Q}(\psi, \phi) - Q(\psi, \phi) = \int \left(\psi\phi(1 - a) + \langle (1 - aS)\tilde{\nabla}\psi, \tilde{\nabla}\phi\rangle_{\tilde{G}}\right) \tilde{\sigma}^2 d_{\tilde{G}} \mathrm{vol}$$

and hence

$$(19)\qquad |\,\tilde{Q}(\psi,\phi)-Q(\psi,\phi)\,| \le \max\{\|1-a\|_\infty, \|1-aS\|_\infty\}\,\|\psi\|\tilde{}_{1,q'}\|\phi\|\tilde{}_{1,q},$$

where $\|1-aS\|_\infty$ is the L^∞-norm of the operator field $\{1-aS(x)\}_x$ acting pointwise on $(T_xM,\tilde{G})$ or, equivalently, the operator norm of the operator $1-aS$ acting on $L^2(\tilde{G},\tilde{\sigma}^2)$.

Setting

$$\lambda = \lambda_1^{N/2}\lambda_2, \qquad \mu = \mu_1^{N/2}\mu_2$$

(9) and (10) imply that

$$(20)\qquad \lambda \le a \le \mu$$

and so

$$\|1-a\|_\infty \le \max\{|\,1-\lambda\,|, |\,1-\mu\,|\}.$$

Similarly, the inequality

$$\mu_1^{-1}\lambda \le aS \le \lambda_1^{-1}\mu$$

implies

$$\|1-aS\|_\infty \le \max\{|\,1-\mu_1^{-1}\lambda\,|\,,\,|\,1-\lambda_1^{-1}\mu\}.$$

Combining the last two inequalities with (19) yields

$$|\,\tilde{Q}(\psi,\phi)-Q(\psi,\phi)\,| \le K\|\psi\|\tilde{}_{1,q'}\|\phi\|\tilde{}_{1,q}$$

where

$$K = \max\{\,|\,1-\lambda\,|\,,\,|\,1-\mu\,|\,,\,|\,1-\mu_1^{-1}\lambda\,|\,,\,|\,1-\lambda_1^{-1}\mu\,|\,\}.$$

The choice of λ_1 and λ_2 in (12) and (13) yields

$$K = \frac{\rho\rho_1-1}{\rho\rho_1+1}. \qquad \square$$

Now we can prove the main theorem of this section. It is a regularity theorem for $H = -\sigma^{-2}\nabla\cdot(\sigma^2\nabla)+1$ and generalises Theorem 1 of [7].

Theorem 4 *There exists q_0, $2 < q_0 < \infty$ such that properties $(\mathrm{P}_{i,q})$ are valid for all $q_0' < q < q_0$.*

Proof By standard regularity theorems for elliptic differential operators with smooth coefficients (see, for example, Theorem 4.6 of [8]) $\tilde{H}$ satisfies properties $(\mathrm{P}_{i,q})$ for all $1 < q < \infty$ and so

$$\sup_{\|\phi\|_{1,q}} |\, \tilde{Q}(\phi, \psi) \,| \geq \frac{1}{\tilde{K}_q} \|\psi\|_{1,q'}, \quad \text{all } \psi \in W^{1,q'} \tag{21}$$

for some $\tilde{K}_q < +\infty$. From Proposition 1 it follows that $\log \tilde{K}_q$ is a convex function of $1/q \in (1, +\infty)$ and satisfies $\tilde{K}_q = \tilde{K}_{q'}$. Hence, $\tilde{K}_q$ has a minimum at $q = 2$ and one easily sees that $\tilde{K}_2 = 1$. Moreover, assuming that we do not have $K_q \equiv 1$ for all $1 < p < +\infty$ (in which case everything works much better), convexity implies that $\tilde{K}_q$ converges to $+\infty$ as q tends to 1 or $+\infty$.

For $\psi \in W^{1,q'}$ we have

$$\begin{aligned}
\sup_{\|\phi\|_{1,q}=1} |\, Q(\psi, \phi) \,| &\geq \gamma_{2,q}^{-1} \sup_{\|\phi\|_{1,q}=1} |\, Q(\psi, \phi) \,| \\
&\geq \gamma_{2,q}^{-1} \left(\sup_{\|\phi\|_{1,q}=1} |\, \tilde{Q}(\psi, \phi) \,| - \sup_{\|\phi\|_{1,q}=1} |\, \tilde{Q}(\psi, \phi) - Q(\phi, \psi) \,| \right) \\
&\geq \gamma_{2,q}^{-1} \left(\frac{1}{\tilde{K}_q} \|\psi\|_{1,q'} - \frac{\rho\rho_1 - 1}{\rho\rho_1 + 1} \|\psi\|_{1,q'} \right)
\end{aligned}$$

and so

$$\frac{1}{K_q} \geq \frac{1}{\Lambda_q} =: \gamma_{2,q}^{-1} \gamma_{2,q'}^{-1} \left(\frac{1}{\tilde{K}_q} - \frac{\rho\rho_1 - 1}{\rho\rho_1 + 1} \right). \tag{22}$$

From the comments we made on $\tilde{K}_q$ it follows that there exists a unique q_0, $2 < q_0 < +\infty$, such that

$$\tilde{K}_{q_0} = \frac{\rho\rho_1 + 1}{\rho\rho_1 - 1}$$

and thus

$$\frac{1}{\tilde{K}_q} - \frac{\rho\rho_1 - 1}{\rho\rho_1 + 1} > 0$$

for all q such that $q_0' < q < q_0$. Hence H satisfies property $(\mathrm{P}_{3,q})$ of Proposition 1 for all such q, as required. □

Note : It is clear from this proof that q_0 depends only on $\tilde{G}, \tilde{\sigma}^2$ and the 'ellipticity ratios' $\rho_i, i = 1, 2$. Moreover, for fixed $\tilde{G}, \tilde{\sigma}^2$, the dependence on $\rho_i, i = 1, 2$ is such that

$$q_0 \longrightarrow +\infty \quad \text{as } \rho \longrightarrow 1,$$

$$q_0 \longrightarrow 2 \quad \text{as } \rho \longrightarrow +\infty.$$

Corollary 5 *Let H be as in Theorem 4 and let ϕ_n be a normalized eigenfuction of H corresponding to the nth eigenvalue λ_n. There exists $2 < q_0 < +\infty$ such that for any $q < q_0$ we have*

$$\|\nabla\phi_n\|_q \leq c\lambda_n^{\gamma} \tag{23}$$

provided

$$\gamma > \gamma_0 =: 1 + \frac{N}{4} - \frac{N}{2q} - \frac{q_0 - q}{q(q_0 - 2)}. \tag{24}$$

Proof From the theorem, there exists $2 < q_0 < +\infty$ such that

$$\|\nabla\phi_n\|_q \leq c\lambda_n\|\phi_n\|_q, \qquad \text{all } q < q_0.$$

Now, it is a well known result (see [3, p. 22]) that the eigenfunctions of H lie in L^q for all $2 \leq q \leq +\infty$ and in fact

$$\|\phi_n\|_q \leq c\lambda_n^{N(q-2)/4q}$$

for some constant c independent of $n \in \mathbf{N}$. Hence

$$\|\nabla\phi_n\|_q \leq c\lambda_n^{1+N/4-N/2q}, \qquad \text{all } q < q_0. \tag{25}$$

This is not a good estimate for q close to 2. Interpolation between (25) and

$$\|\nabla\phi_n\|_2 = \lambda_n^{1/2}$$

yields the result. □

2. Perturbation of the metric and weight

Perturbation of the metric

Let $G \in \mathcal{R}(M)$ and $\sigma^2 \in \mathcal{W}(M)$. Let $S = S(x)$ be a positive measurable operator field bounded away from zero and infinity and set

$$F = \nabla\nabla^*, \quad F_S = S^{1/2}\nabla\nabla^* S^{1/2}.$$

The following lemma is the Riemannian manifold version of a well known result that has been useful in problems such as spectral asymptotics [2, Lemma 8.1] and scattering theory [4].

Lemma 6 *There exist partial isometries $U, V : L^2_{G,\sigma^2} \longrightarrow L^2_{G,\sigma^2}$ such that*

$$\begin{aligned}(26)\qquad &(\nabla^* S\nabla + 1)^{-1} - (\nabla^*\nabla + 1)^{-1} \\ &= V^* F_S^{1/2}(F_S + 1)^{-1} S^{-1/2}(1 - S) F^{1/2}(F + 1)^{-1} U.\end{aligned}$$

Proof It is not difficult to see that if $T : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ is closed and densely defined, then $(T^*T + 1)^{-1} + T^*(TT^* + 1)^{-1}T = 1$ as an operator equality on $\mathrm{Dom}(T)$. Hence

$$(\nabla^*\nabla + 1)^{-1} + \nabla^*(\nabla\nabla^* + 1)^{-1}\nabla = 1,$$

$$(\nabla^* S\nabla)^{-1} + \nabla^* S^{1/2}(S^{1/2}\nabla\nabla^* S^{1/2} + 1)^{-1} S^{1/2}\nabla = 1,$$

and therefore

$$\begin{aligned}(\nabla^* S\nabla + 1)^{-1} - (\nabla^*\nabla + 1)^{-1} &= -\nabla^*\left[(\nabla\nabla^* + S^{-1})^{-1} - (\nabla\nabla^* + 1)^{-1}\right]\nabla \\ &= \nabla^*(\nabla\nabla^* + S^{-1})^{-1}(S^{-1} - 1)(\nabla\nabla^* + 1)^{-1}\nabla \\ &= \nabla^* S^{1/2}(F_S + 1)^{-1} S^{1/2}(S - 1)(F + 1)^{-1}\nabla.\end{aligned}$$

Using polar decomposition we can write

$$\nabla = \mid \nabla \mid U = F^{1/2}\, U \quad \text{and} \quad S^{1/2}\nabla = \mid S^{1/2}\nabla \mid V = F_S^{1/2} V$$

and (26) then follows. □

We think of $\mathrm{Sp}(F)$ as a measure space with each eigenvalue $\lambda_n \neq 0$ carrying a weight $\lambda_n^s \times m(\lambda_n)$, where m is the multiplicity of the eigenvalue and s is a parameter satisfying $s > -1 + 2\gamma_0$, γ_0 being as in Corollary 5. To the zero eigenvalue we assign measure equal to $+\infty$ reflecting the fact that $\mathrm{Ker}(F)$ is infinite dimensional. Associated to this discrete measure space are the corresponding l^q

spaces, $1 \leq q \leq \infty$, defined by

$$l^q = \left\{ g : \mathrm{Sp}(F) \longrightarrow \mathbf{R} \mid g(0) = 0, \quad \sum_n \mid g(\lambda_n) \mid^q \lambda_n^s < \infty \right\}, \quad 1 \leq q < \infty$$

and

$$l^\infty = \left\{ g : \mathrm{Sp}(F) \longrightarrow \mathbf{R} \mid \sup_n \mid g(\lambda_n) \mid < \infty \right\}.$$

For any function g on $\mathrm{Sp}(F)$ we define the operator $g(F)$ using the spectral theorem.

Denoting by $\|A\|_r$, $1 \leq r \leq +\infty$, the C^r-norm (Schatten norm) of an operator A,

$$\|A\|_r = (\mathrm{tr} \mid A \mid^r)^{1/r},$$

so that, in particular, $\|A\|_\infty$ is the operator norm, we have the following

Lemma 7 *Let T be a measurable operator field on M and g : $\mathrm{Sp}(F) \longrightarrow \mathbf{R}$. There exists $p_0 < +\infty$ such that for $p > p_0$ and $1 \leq r \leq +\infty$, then*

$$\|Tg(F)\|_{2r} \leq c\|T\|_{pr}\|g\|_{2r}. \tag{27}$$

Proof For $r = +\infty$, (27) is trivial. We shall prove it for $r = 1$, the general case then being obtained by standard interpolation arguments. We may assume that $g(0) = 0$ since otherwise the RHS of (27) is infinite.

Let $\{\phi_n\}$ denote a complete orthonormal system of eigenfunctions of L. Defining

$$\psi_n = \lambda_n^{-1/2} \nabla \phi_n, \qquad n \in \mathbf{N}$$

one easily checks that $\{\psi_n\}$ is an orthonormal system of vector fields satisfying $F\psi_n = \lambda_n \psi_n$ and spanning $\mathrm{Ker}(F)^\perp$. In fact, it is a known theorem (see [4]) that if

$$A : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$$

is closed and densely defined, then

$$\mathrm{Sp}(A^*A) \cup \{0\} = \mathrm{Sp}(AA^*) \cup \{0\}.$$

Hence, since $g(0) = 0$, we can get an estimate for the Hilbert–Schmidt norm of $Tg(F)$:

$$\|Tg(F)\|_2^2 = \sum_n \|Tg(F)\psi_n\|_2^2$$
$$\leq \|T\|_p^2 \sum_n |\, g(\lambda_n)\,|^2\, \lambda_n^{-1} \|\nabla\phi_n\|_q^2$$

where $q = 2p/(p-2)$. From (23) we have

$$\|Tg(F)\|_2^2 \leq c\|T\|_p^2 \sum_n |\, g(\lambda_n)\,|^2\, \lambda_n^{-1+2\gamma}, \quad \text{all } \gamma > \gamma_0, \quad p > p_0 \tag{28}$$

where $p_0 = 2q_0/(q_0-2)$, q_0 being as in Corollary 5. Since $s > -1 + 2\gamma_0$, this concludes the proof. □

Note that the result is also valid if we replace F by F_S, since F_S is of the same form as F for some metric $\hat{G} \in \mathcal{R}(M)$ and weight $\hat{\sigma}^2 \in \mathcal{W}(M)$.

We can now prove the main theorem of this section. Suppose that we have one single weight σ^2 and two different metrics G_1, G_2 and let

$$\nabla_i : L^2(G_i, \sigma^2) \longrightarrow L^2(G_i, \sigma^2)$$

denote the weak gradient induced by $G_i, i = 1, 2$, so that

$$H_i = \nabla_i^{(*)_i} \nabla_i, \quad i = 1, 2$$

in an obvious notation. Define the unitary operator

$$u : L^2(G_1, \sigma^2) \longrightarrow L^2(G_2, \sigma^2)$$

to be multiplication by the function

$$u = \frac{g_1^{1/4}}{g_2^{1/4}}.$$

We also define the measurable operator field $S = S(x)$ by

$$S = u^{-2} G_2^{-1} G_1. \tag{29}$$

As in Lemma 2, we see that this definition is coordinate independent, that $S(x)$ is self-adjoint on (T_xM, G_1) for all $x \in M$ and hence that S is a self-adjoint operator on $L^2(G_1, \sigma^2)$. Moreover we have

$$\nabla_2^{(*)_2} \nabla_2 = u^2 \nabla_1^{(*)_1} S \nabla_1. \tag{30}$$

Theorem 8 *There exists p_0, $2 < p_0 < +\infty$, such that if*

$$\text{(i) } p > p_0 \quad \textit{and} \quad \text{(ii) } r > \frac{N}{2} + \frac{N + p_0}{p}$$

then

$$\|u^{-1}R_2u - R_1\|_r \le c\|S - 1\|_{pr/2}. \tag{31}$$

Proof Denoting by $R_i, i = 1, 2$, the two resolvents, $R_i = H_i^{-1}$, we have

$$R_2 = (u^2\nabla_1^{(*)_1}S\nabla_1 + 1)^{-1}$$

and so

$$u^{-1}R_2u = u^{-1}(\nabla_1^{(*)_1}S\nabla_1 + u^{-2})^{-1}u^{-1}.$$

Hence

$$\begin{aligned} u^{-1}R_2u - R_1 =& u^{-1}(\nabla_1^{(*)_1}S\nabla_1 + u^{-2})^{-1}u^{-1} - (\nabla_1^{(*)_1}\nabla_1 + 1)^{-1} \\ =& u^{-1}(\nabla_1^{(*)_1}S\nabla_1 + u^{-2})^{-1}u^{-1} - (\nabla_1^{(*)_1}S\nabla_1 + u^{-2})^{-1}u^{-1} \\ &+ (\nabla_1^{(*)_1}S\nabla_1 + u^{-2})^{-1}u^{-1} - (\nabla_1^{(*)_1}S\nabla_1 + u^{-2})^{-1} \\ &+ (\nabla_1^{(*)_1}S\nabla_1 + u^{-2})^{-1} - (\nabla_1^{(*)_1}S\nabla_1 + 1)^{-1} \\ &+ (\nabla_1^{(*)_1}S\nabla_1 + 1)^{-1} - (\nabla_1^{(*)_1}\nabla_1 + 1)^{-1} \\ =:& A_1 + A_2 + A_3 + B. \end{aligned}$$

Let $g_0(t) = t^{1/2}/(t + 1)$. From Lemmas 6 and 7 it follows that there exists p_0, $2 < p_0 < +\infty$, such that

$$\begin{aligned} \|B\|_r &\le \|(S - 1)^{1/2}g_0(F)\|_{2r}\|S^{-1/2}(S - 1)^{1/2}g_0(F_S)\|_{2r} \\ &\le \|g_0\|_{2r}^2\|(S - 1)^{1/2}\|_{pr}\|S^{-1/2}(S - 1)^{1/2}\|_{pr} \\ &\le c\|g_0\|_{2r}^2\|S - 1\|_{pr/2} \end{aligned}$$

for all $p > p_0$ and $1 \le r \le +\infty$.

We need to know the range of r for which $g_0 \in l^{2r}$. Recalling (1) we have

$$\begin{aligned} \|g_0\|_{2r} < +\infty \quad &\Longleftrightarrow \quad \sum_n \lambda_n^{-r+s} < +\infty \\ &\Longleftrightarrow \quad \sum_n n^{\frac{2}{N}(-r+s)} < +\infty \\ &\Longleftrightarrow \quad s < r - \frac{N}{2}. \end{aligned}$$

So, recalling from (28) that $s = -1 + 2\gamma$, we need γ to satisfy

$$\gamma < \frac{1}{2}\Big(r + 1 - \frac{N}{2}\Big)$$

and, since γ must also satisfy (24), we conclude that p and r must be such that

$$1 + \frac{N}{4} - \frac{N}{2q} - \frac{q_0 - q}{q(q_0 - 2)} < \frac{1}{2}\Big(r + 1 - \frac{N}{2}\Big)$$

where $q = 2p/(p-2)$. This is equivalent to (ii).

The terms A_1, A_2 and A_3 are much easier to estimate. For example, we have

$$A_1 = (u^{-1} - 1)H_2^{-1}u$$

and hence for any $1 \le r \le +\infty$

$$\|A_1\|_r \le c\|(u-1)g_1(H_2)\|_r \tag{32}$$

where $g_1(t) = t^{-1}$.

This can be estimated in terms of some L^p-norm of $u - 1$ using a similar argument to that in Lemma 7. In fact, everything works much better. The operator involved is now of the form $\nabla^*\nabla$ (in $L^2(G_2, \sigma^2)$) rather than $\nabla\nabla^*$, so what we need is L^q-estimates for some $q > 2$ on the eigenfunctions of the Laplace–Beltrami operator rather than on the gradient of the eigenfunctions. Moreover, the function $g_1(t)$ that appears in (32) decays at infinity faster than the function $g_0(t)$. Hence, for p and r as in the statement of the Theorem we certainly have

$$\|A_1\|_r \le c\|u - 1\|_{pr/2}.$$

Note that the fact that $g_1(t)$ diverges near $t = 0$ is not a problem since $0 \notin \mathrm{Sp}(H)$.

The same argument can be used in order to estimate $\|A_2\|_r$, while for $\|A_3\|_r$ we write

$$A_3 = (\nabla_1^{(*)_1} S\nabla_1 + u^{-2})^{-1}(1 - u^{-2})(\nabla_1^{(*)_1} S\nabla_1 + 1)^{-1}$$

and then proceed in the same way. The result follows if we note that for any two $N \times N$ matrices E and F we have

$$|\det E - \det F| \le c\|E - F\|,$$

where the constant c depends on $\|E\|$ and $\|F\|$. □

Perturbation of the weight

Suppose now that we have one single metric $G \in \dot{\mathcal{R}}(M)$ and two different weights $\sigma_1, \sigma_2 \in \mathcal{W}(M)$. Let $H_i, i = 1, 2,$ denote the corresponding Laplace–Beltrami operators so that

$$H_i = \nabla^{(*)_i}\nabla$$

in an obvious notation.

Defining the unitary operator

$$v : L^2(G, \sigma_1^2) \longrightarrow L^2(G, \sigma_2^2)$$

to be multiplication by the function

$$v = \sigma_1 \sigma_2^{-1}$$

one sees easily that

$$\nabla^{(*)_2} = v^2 \nabla^{(*)_1} v^{-2}.$$

Hence, setting $R_i = H_i^{-1}$, $i = 1, 2$, we have

$$\begin{aligned}
v^{-1}R_2 v - R_1 =&(v\nabla^{(*)_1}v^{-2}\nabla v + 1)^{-1} - ((\nabla^{(*)_1}\nabla + 1)^{-1} \\
=&v^{-1}(\nabla^{(*)_1}v^{-2}\nabla + v^{-2})^{-1}v^{-1} - (\nabla^{(*)_1}\nabla + 1)^{-1} \\
&+ (\nabla^{(*)_1}v^{-2}\nabla + v^{-2})^{-1}v^{-1} - (\nabla^{(*)_1}v^{-2}\nabla + v^{-2})^{-1} \\
&+ (\nabla^{(*)_1}v^{-2}\nabla + v^{-2})^{-1} - (\nabla^{(*)_1}v^{-2}\nabla + 1)^{-1} \\
&+ (\nabla^{(*)_1}v^{-2}\nabla + 1)^{-1} - (\nabla^{(*)_1}\nabla + 1)^{-1} \\
=&:A_1' + A_2' + A_3' + B'.
\end{aligned}$$

This expression can be handled in exactly the same way as the corresponding one in Theorem 8 and we conclude that

Corollary 9 *If p and r are as in Theorem* 8, *then*

$$\|v^{-1}R_2 v - R_1\|_r \leq c_{v,p,r}\|\sigma_2^2 - \sigma_1^2\|_{pr/2}. \tag{33}$$

Remark 10 It follows from (31) and a simple interpolation argument that for any $q > N/2$ there exists large enough t such that

$$\|u^{-1}R_2 u - R_1\|_q \leq c\|S - 1\|_s^{st}$$

for any $s \geq 1$.

Remark 11 Note that if the metrics G_i and the weights σ_i, $i = 1, 2$, are smooth, then the corresponding Laplace–Beltrami operators satisfy property $(P_{2,q})$ for any $1 < q < +\infty$ by standard elliptic regularity theorems, and hence the constant p_0 in Theorem 8 and Corollary 9 can be taken to be equal to 2.

3. Manifolds with boundary

In this section we extend our main result to cover the case of manifolds with boundary and of operators satisfying mixed Dirichlet–Neumann boundary conditions. The technique that we use, involving the double of a manifold, is quite well known and has been used for quite different purposes. See for example [6] and [3]. However, we do include proofs for the sake of completeness.

Let X be an N-dimensional compact smooth manifold with smooth boundary ∂X. Let A be an open, possibly disconnected submanifold of ∂X with smooth boundary ∂A (we do not exclude the case $A = \partial X$). We define M' to be the manifold

$$M' = X \times \{-1, 1\}$$

and M the smooth manifold resulting from M' by glueing $\bar{A} \times \{-1\}$ with $\bar{A} \times \{+1\}$. So, M is a smooth manifold with a smooth boundary ∂M that degenerates to the empty set when $A = \partial X$, in which case M is the double $\hat{X}$ of X.

Let

$$X^{\pm} = X \times \{\pm 1\}$$

so that

$$M = X^+ \cup X^-.$$

Any function $f \in L^1_{loc}(M)$ shall also be written as

$$f = [f_+, f_-]$$

where

$$f_{\pm} = f\,|_{X^{\pm}}\,.$$

We define the class $\mathcal{R}(X)$ (resp. $\mathcal{W}(X)$) to consist of all metrics G (resp. weights σ^2) such that the induced metric (resp. weight) on the double $\hat{X}$ of X lie in $\mathcal{R}(\hat{X})$ (resp. $\mathcal{W}(\hat{X})$). It is easy to see that this is equivalent to saying that G (resp. σ^2) is comparable to another metric $\tilde{G}$ (resp. weight $\tilde{\sigma}^2$) that lies in $C^\infty(\bar{X})$.

Finally, let

$$F = \{f \in C^\infty(\bar{X}) \;\; | f = 0 \text{ on } A\}$$

and let V_q be the closure of F in $W^{1,q}(X)$ for some (and hence any) $G \in \mathcal{R}(X)$. When $q = 2$ we shall simply write V.

Elliptic regularity

We shall need the following standard result:

Lemma 12 *Let $f, g \in W^{1,q}(X)$. Then $[f, g] \in W^{1,q}(M)$ if and only if $f - g \in V_q$.*

Proof We omit the proof which is quite straight-forward.

Let $f = (f_0, \vec{f}) \in L^1(X)$. Extending our earlier definition, we say that the function $u \in W^{1,1}(X)$ solves the equation

$$H_A u = f_0 + \nabla \cdot (\sigma^2 \vec{f}) \tag{34}$$

if

$$Q(u, \phi) = \int_X (f_0\phi - \langle \vec{f}, \nabla\phi \rangle)\sigma^2 d\text{vol},$$

for all $\phi \in C^\infty(\bar{X})$ that vanish in a neighbourhood of A.

Now we can generilize Theorem 4 in this context.

Theorem 13 *There exists $2 < q_0 < +\infty$ such that for any q with $q_0' < q < q_0$ and any $f = (f_0, \vec{f}) \in L^q(X)$ the equation*

$$H_A u = f_0 + \nabla \cdot (\sigma^2 \vec{f}) \tag{35}$$

has a unique solution $u \in V_q$ and

$$\|u\|_{1,q} \leq c\|f\|_q. \tag{36}$$

Proof Let H'' be the induced operator on the double $\hat{M}$ of M. Let $q_0 > 2$ be the constant for H'' whose existance is guaranteed by Theorem 4. Let $q_0' < q < q_0$ and $f = (f_0, \vec{f}) \in L^q(X)$.

Using again signs to distinguish between different copies of the same manifold, we have the natural identifications

$$L^q(\hat{M}) = L^q(M_+) \oplus L^q(M_-) \tag{37}$$

$$= L^q(X_{++}) \oplus L^q(X_{+-}) \oplus L^q(X_{-+}) \oplus L^q(X_{--}). \tag{38}$$

Under (38) define

$$\hat{f} = 0 \oplus f \oplus 0 \oplus f \in L^q(\hat{X}).$$

So, there exists $\hat{u} \in W^{1,q}(\hat{M})$ such that

$$H''\hat{u} = \hat{f}_0 + \nabla \cdot (\sigma^2 \hat{\vec{f}}),$$

that is

$$\int_{\hat{M}}(\hat{u}\hat{\phi}+\langle\nabla\hat{u},\nabla\hat{\phi}\rangle)\sigma^2 d\mathrm{vol}=\int_{\hat{M}}(\hat{f}_0\hat{\phi}-\langle\hat{\vec{f}},\nabla\hat{\phi}\rangle)\sigma^2 d\mathrm{vol} \tag{39}$$

for all $\hat{\phi}\in C^\infty(\hat{M})$, and hence for all $\hat{\phi}\in W^{1,q'}(\hat{M})$.

We readily see that $\hat{u}$ is of the form

$$\hat{u}=0\oplus u\oplus 0\oplus u$$

for some $u\in W^{1,q'}(X)$ and, additionally, Lemma 12 implies that $u\in V_q$. Now, for any $\phi\in W^{1,q'}(X)$ we have

$$0\oplus\phi\oplus 0\oplus\phi\in W^{1,q'}(\hat{M})$$

again by Lemma 12 and for such a $\hat{\phi}$ (39) yields

$$\int_X(u\phi+\langle\nabla u,\nabla\phi\rangle)\sigma^2 d\mathrm{vol}=\int_X(f_0\phi-\langle\vec{f},\nabla\phi\rangle)\sigma^2 d\mathrm{vol}.$$

Hence $u\in V_q$ solves (35) as required. □

Remark 14 It is possible to replace the smoothness condition on the boundary by a weaker one. Let X be a smooth manifold. Let G be a measurable metric on X and let $\bar{X}$ be the completion of X with respect to G. Assume that there exists a quasi-isometry π from $\bar{X}$ to a Riemannian manifold with boundary X' of the type we discussed in this section. Then everything we have proved for X' is also valid for $(\bar{X},G)$, the various constants depending on the induced ones on X' and on the quasi-isometry constants of π. This applies mainly to manifolds that have a Lipschitz boundary, in a sense similar to that of [6] and [1].

Resolvent stability

Lemma 12 is also the main ingredient in order to extend Theorem 8 in the present context. Let the operator $U:L^2(M)\longrightarrow L^2(M)$ be defined by

$$U[f_+,f_-]=[f_-,f_+],\qquad \text{all} f\in L^2(M). \tag{40}$$

U is unitary on $L^2(M,\sigma^2)$ and satisfies $U^2=I$. Moreover, the spaces $C_c^\infty(M)$, $W^{1,2}(M)$ and $W_0^{1,2}(M)$ are invariant under the action of U and the operator ∇, with domain either $W^{1,2}(M)$ or $W_0^{1,2}(M)$, commutes with U.

Let H be the operator

$$H=-\sigma^{-2}\nabla\cdot(\sigma^2\nabla)+1$$

on M satisfying Neumann boundary conditions. One easily then checks that $\mathrm{Dom}(H)$ is invariant under U and in fact

$$HU = UH. \tag{41}$$

Every function on M can be uniquely expressed as a sum of an even and an odd function. In particular, defining the spaces

$$L^q_e(M) = \{[f, f] \mid f \in L^q(X)\},$$

$$L^q_{odd}(M) = \{[f, -f] \mid f \in L^q(X)\}$$

we have the decomposition

$$L^q(M) = L^q_e(M) \oplus L^q_{odd}(M), \qquad \text{all } 1 < q < +\infty. \tag{42}$$

It follows from (41) that $L^2_e(M)$ and $L^2_{odd}(M)$ are invariant under the action of H, and so under the decomposition (42) (for $q = 2$) we can write

$$H = H_e \oplus H_{odd} \tag{43}$$

where H_e (resp. H_{odd}) is a self-adjoint operator on $L^2_e(M)$ (resp. $L^2_{odd}(M)$).

Let H_N be the operator on $L^2(X, \sigma^2)$ given formally by

$$H_N = -\sigma^{-2}\nabla \cdot (\sigma^2 \nabla) + 1$$

and satisfying Neumann boundary conditions.

We have the following

Proposition 15 *Identifying $L^2_e(M)$ and $L^2_{odd}(M)$ with $L^2(X)$ in the natural way, we have*

$$H_e = H_N, \tag{44}$$

$$H_{odd} = H_A. \tag{45}$$

Proof We shall only prove (45), (44) being proved in a similar way.

Suppose $f \in \mathrm{Dom}(H_{odd})$. So $[f, -f] \in \mathrm{Dom}(H)$ which means $[f, -f] \in W^{1,2}(M)$ and there exists $h \in L^2(X)$ such that

$$\int_M \langle \nabla[f, -f], \nabla\phi \rangle \sigma^2 d\mathrm{vol} = \int_M [h, -h]\phi\sigma^2 d\mathrm{vol}, \qquad \text{all } \phi \in W^{1,2}(M). \tag{46}$$

The fact that $[-f, f] \in W^{1,2}(M)$ implies $f \in V$ by the lemma. Moreover, for any $\psi \in V$ we have $[\psi, -\psi] \in W^{1,2}(M)$ and hence

$$\int_X \langle \nabla f, \nabla \psi \rangle \sigma^2 d\mathrm{vol} = \int_X h\psi\sigma^2 d\mathrm{vol}$$

by (46). Hence $f \in \mathrm{Dom}(H_A)$ and $H_A f = h$.

Conversely, let $f \in \mathrm{Dom}(H_A)$. So $f \in V$ and there exists an $h \in L^2(X)$ such that

$$\int_X \langle \nabla f, \nabla \phi \rangle \sigma^2 dx = \int_X h\phi\sigma^2, \qquad \text{all } \phi \in V.$$

It follows that $[f, -f] \in W^{1,2}(M)$. If ψ is arbitrary in $W^{1,2}(M)$, then $\psi_+ - \psi_- \in V$, so

$$\int_X \langle \nabla f, \nabla(\psi_+ - \psi_-) \rangle \sigma^2 d\mathrm{vol} = \int_X h(\psi_+ - \psi_-)\sigma^2 d\mathrm{vol}$$

and hence

$$\int_M \langle \nabla [f, -f], \nabla \psi \rangle \sigma^2 d\mathrm{vol} = \int_M [h, -h]\psi\sigma^2 d\mathrm{vol}$$

which implies that $f \in \mathrm{Dom} H_{odd}$ and $H_{odd} f = h$. □

Suppose now that we have two metrics and two weights $G_i \in \mathcal{R}(X)$, $\sigma_i^2 \in \mathcal{W}(X)$, $i = 1, 2$ and let $H_{A,i}$, $i = 1, 2$, be the corresponding weighted Laplace–Beltrami operators

$$H_{A,i} = -\sigma_i^2 \nabla_i \cdot (\sigma_i^2 \nabla_i) + 1, \qquad \mathrm{Dom}(H_{A,i}^{1/2}) = V.$$

Let $R_{A,i} = H_{A,i}^{-1}$, $i = 1, 2$ be the two resolvents and S be as in Theorem 8. Multiplication by the function

$$w = \sigma_1 g_1^{1/4} / \sigma_2 g_2^{1/4}$$

is the natural isometry from $L^2_{G_1,\sigma_1^2}$ onto $L^2_{G_2,\sigma_2^2}$. From Theorem 8, Corollary 9 and Proposition 15 we deduce the following

Theorem 16 *There exists p_0, $2 < p_0 < +\infty$ such that if*

$$\text{(i) } p > p_0 \quad \textit{and} \quad \text{(ii) } r > \frac{N}{2} + \frac{N + p_0}{p}$$

then

$$\|w^{-1} R_{A,2} w - R_{A,1}\|_r \leq c \left(\|S - 1\|_{pr/2} + \|\sigma_2 - \sigma_1\|_{pr/2} \right). \tag{47}$$

Proof For $i = 1, 2$ let

$$H'_{N,i} \; (H_{D,i'})$$

denote the operators induced on M by $H_{A,i}$ and subject to Neumann (resp. Dirichlet) boundary conditions. Moreover, let H_i'' be the corresponding operators induced on the double $\hat{M}$ of M.

Under the natural identifications

$$\begin{aligned} L^2(\hat{M}) &\simeq L^2(M_+) \oplus L^2(M_-) \\ &\simeq L^2(X_{++}) \oplus L^2(X_{+-}) \oplus L^2(M_-) \end{aligned} \tag{48}$$

we have for $i = 1, 2$

$$\begin{aligned} H_i'' &= H'_{N,i} \oplus H'_{D,i} \\ &= H_{N,i} \oplus H_{A,i} \oplus H'_{D,i} \end{aligned}$$

and hence

$$\|w^{-1}R_{A,2}w - w^{-1}R_{A,1}w\|_r \leq \|w^{-1}R_2''w - R_1''\|_r \tag{49}$$

where $R_i'' = (H_i'')^{-1}, i = 1, 2$. Since $\hat{M}$ is a smooth compact manifold with no boundary, we can apply Theorem 8 and Corollary 9 and the result follows. □

Remark 17 If we make no assumptions about the regularity of the boundary ∂X, then there exists a variation of Theorem 13 that involves local rather than global Sobolev estimates; this is Theorem 2 of [7]. It can be used to prove the estimates of Theorem 16, but only under the additional assumption that the weights and metrics coincide near the boundary ∂X. In fact, we believe that this assumption is strong enough to allow one to generalise those estimates in the case where the metrics and/or the weights are singular or degenerate near the boundary.

Remark 18 Let

$$A, B : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$$

be two compact operators and let $\{\mu_n(A)\}, \{\mu_n(B)\}$ be their singular values. It is known (see [8], p. 20) that for any r, $1 \leq r \leq \infty$,

$$\left(\sum_n \mid \mu_n(B) - \mu_n(A) \mid^r\right)^{1/r} \leq \|B - A\|_r. \tag{50}$$

The fact that we estimate the norm of $U^{-1}R_2U - R_1$ for some unitary operator U rather than that of $R_2 - R_1$ means that when we apply (50) the operators A and B are self-adjoint and positive and hence that their singular values coincide with their eigenvalues. Hence, our main results also yield eigenvalue stability. Moreover, resolvent stability implies stability of the corresponding spectral projections and, hence, eigenspace stability.

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