TOPOS-THEORETIC RELATIVIZATION OF PHYSICAL REPRESENTABILITY AND QUANTUM GRAVITY

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Abstract

In the current debate referring to the construction of a tenable background independent theory of Quantum Gravity we introduce the notion of topos-theoretic relativization of physical representability and demonstrate its relevance concerning the merging of General Relativity and Quantum Theory. For this purpose we show explicitly that the dynamical mechanism of physical fields can be constructed by purely algebraic means, in terms of connection inducing functors and their associated curvatures, independently of any background substratum. The application of this mechanism in General Relativity is constrained by the absolute representability of the theory in the field of real numbers. The relativization of physical representability inside operationally selected topoi of sheaves forces an appropriate interpretation of the mechanism of connection functors in terms of a generalized differential geometric dynamics of the corresponding fields in the regime of these topoi. In particular, the relativization inside the topos of sheaves over commutative algebraic contexts makes possible the formulation of quantum gravitational dynamics by suitably adapting the functorial mechanism of connections inside that topos.

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1 Prologue

There exists a significant amount of current research in theoretical physics devoted to the construction of a tenable quantum theory of gravity, conceived as an extensive unifying framework of both General Relativity and Quantum Theory [1-11]. It has been generally argued that these fundamental physical theories are based on incompatible conceptual and mathematical foundations. In this sense, the task of their reconciliation in a unifying framework, that respects the constraints posed by both theories, requires a radical revision, or at least, a careful rethinking of our current understanding of the basic notions, such as the conception of spacetime, physical fields, localization, observables and dynamics.

In this communication we would like to draw attention regarding these issues from the algebraic, categorical and topos-theoretic perspective of modern mathematics [12-19], as a substitute of the set-theoretic one, and especially the impact of the consequences of this perspective, in relation to their foundational significance towards the crystallization of the basic notions constituting a theory of Quantum Gravity. An initial motivation regarding the relevance of the categorical viewpoint originates from the realization that both of our fundamental theories can be characterized in general terms as special instances of the replacement of the constant by the variable. The semantics of this transition, for both General Relativity and Quantum Theory, may be incorporated in an algebraic topos-theoretic framework, that hopefully provides the crucial pointers for the schematism of the essential concepts needed for the intelligibilty of a theory of Quantum Gravity, which respects the normative requirements of its predecessors. Epigrammatically it is instructive to remark that, in the case of General Relativity this process takes place through the rejection of the fixed kinematical structure of spacetime, by making the metric into a dynamical object determined solely by the solution of Einstein's field equations. In the case of Quantum Theory, the process of replacement of the constant by the variable, is signified by the imposition of Heisenberg's uncertainty relations, that determines the limits for simultaneous measurements of certain pairs of complementary physical properties, like position and momentum. Although this process in Quantum Theory is not immediately transparent as in General Relativity, it will eventually become clear that it is indispensable to a unifying perspective.

From a mathematical point of view, the general process of semantic transition from constant to variable structures is being effectuated by passing to appropriate topoi, where, an abstract topos is conceived as a universe of variable sets, whose variation is being considered over generalized localization domains. Thus, there exists the possibility of comprehending uniformly the difference in the distinct instances of replacement of the constant by the variable, as they are explicated in the concrete cases of General Relativity and Quantum Theory respectively, by employing different topoi, corresponding to the localization properties of observables in each theory. Of course, this strategy would be fruitful in a unifying quantum relativistic perspective, if we managed to disassociate the dependence of dynamics in the regime of each theory from any fixed background spatiotemporal reference. Equivalently stated, the dynamical mechanism should be ideally formulated functorially and purely algebraically. The benefit of such a formulation has to do with the fact that, because of its functoriality, it can be algebraically forced uniformly inside the appropriate localization topoi of the above theories. Thus, both of these theories can be treated homogenously regarding their dynamical mechanism, whereas, their difference can be traced to the distinctive localization topoi employed in each case. In particular, the functorial representation of general relativistic gravitational dynamics induces a reformulation of the issue of quantization as a problem of selection of an appropriate localization topos, in accordance with the behaviour of observables in that regime, that effectuates the difference in the semantic interpretation of the dynamical machinery corresponding to the transition from the classical to the quantum case. In this work, we initially show that such a functorial dynamical mechanism can be actually constructed using methods of categorical homological algebra. More precisely, the homological dynamical mechanism is based on the modelling of the notion of physical fields in terms of connections, which, effectuate the functorial algebraic process of infinitesimal scalars extensions, due to interactions caused by these fields. Subsequently, we explain the applicability of homological functorial dynamics to the problem of quantum gravity, according to the preceding remarks, by implementing the principle of topos-theoretic relativization of physical representability, using the technique of sheafification over appropriate localization domains.

The central focus of the categorical way of rethinking basic notions in this endeavour can be described as a shift in the emphasis of what is considered to be fundamental for the formation of structures. In the set-theoretic mode of thinking, structures of any conceivable form are defined as sets of elements endowed with appropriate relations. In the category-theoretic mode, the emphasis is placed on the transformations among the objects of a category devised to represent a certain structure by means of appropriate structural constraints on the collection of these transformations. In this sense, the notion of structure does not refer exclusively to a fixed universe of sets of predetermined elements, but acquires a variable reference. We will argue that this is an appealing feature, pertaining decisively to a revised conceptualization of the basic notions, as above, in a viable Quantum Gravity framework.

2 A Homological Schema of Functorial Field Dynamics

The basic conceptual and technical issue pertaining to the current research attempts towards the construction of a tenable Quantum Gravity theory, refers to the problem of independence of this theory from a fixed spacetime manifold substratum. In relation to this problem, we demonstrate the existence and functionality of a homological schema of modelling general relativistic dynamics functorially, constructed by means of connection inducing functors and their associated curvatures, which is, remarkably, independent of any background substratum.

2.1 Algebraic Dynamicalization and Representability

The basic defining feature of General Relativity, in contradistinction to Newtonian classical theory, as well as Special Relativity, is the abolishment of any fixed preexisting kinematical framework by means of dynamicalization of the metric tensor. This essentially means that, the geometrical relations defined on a four dimensional manifold, making it into a spacetime, become variable. Moreover, they are constituted dynamically by the gravitation field, as well as other fields from which matter can be derived, by means of Einstein's field equations, through the imposition of a compatibility requirement relating the metric tensor, which represents the spacetime geometry, with the affine connection, which represents the gravitational field. The dynamic variability of the geometrical structure on the spacetime manifold constitutes the means of dynamicalization of geometry in the descriptive terms of General Relativity, formulated in terms of the differential geometric framework on smooth manifolds. The intelligibility of the framework is enriched by the imposition of the principle of general covariance of the field equations under arbitrary coordinate transformations of the points of the manifold preserving the differential structure, identified as the group of manifold diffeomorphisms. As an immediate consequence, the points of the manifold lose any intrinsic physical meaning, in the sense that, they are not dynamically localizable entities in the theory. Most importantly, manifold points assume an indirect reference as indicators of spacetime events only after the dynamical specification of geometrical relations among them, as particular solutions of the generally covariant field equations. From an algebraic viewpoint [12-14, 21], a real differential manifold M can be recovered completely from the $\mathcal{R}$-algebra $\mathcal{C}^{\infty}(M)$ of smooth real-valued functions on it, and in particular, the points of M may be recovered from the algebra $\mathcal{C}^{\infty}(M)$ as the algebra morphisms $\mathcal{C}^{\infty}(M) \rightarrow \mathcal{R}$.

In this sense, manifold points constitute the $\mathcal{R}$-spectrum of $\mathcal{C}^{\infty}(M)$, being isomorphic with the maximal ideals of that algebra. Notice that, the $\mathcal{R}$-algebra $\mathcal{C}^{\infty}(M)$ is a commutative algebra that contains the field of real numbers $\mathcal{R}$ as a distinguished subalgebra. This particular specification incorporates the physical assumption that our form of observation is being represented globally by evaluations in the field of real numbers. In the setting of General Relativity the form of observation is being coordinatized by means of a commutative unital algebra of scalar coefficients, called an algebra of observables, identified as the $\mathcal{R}$-algebra of smooth real-valued functions $\mathcal{C}^{\infty}(M)$. Hence, the background substratum of the theory remains fixed as the $\mathcal{R}$-spectrum of the coefficient algebra of scalars of that theory, and consequently, the points of the manifold M, although not dynamically localizable degrees of freedom of General Relativity, are precisely the semantic information carriers of an absolute representability principle, formulated in terms of global evaluations of the algebra of scalars in the field of real numbers. Of course, at the level of the $\mathcal{R}$-spectrum of $\mathcal{C}^{\infty}(M)$, the only observables are the smooth functions evaluated over the points of M. In physical terminology, the introduction of new observables is conceived as the result of interactions caused by the presence of a physical field, identified with the gravitational field in the context of General Relativity. Algebraically, the process of extending the form of observation with respect to the algebra of scalars we have started with, that is $\mathcal{A} = \mathcal{C}^{\infty}(M)$, due to field interactions, is described by means of a fibering, defined as an injective morphism of $\mathcal{R}$-algebras $\iota : \mathcal{A} \hookrightarrow \mathcal{B}$. Thus, the $\mathcal{R}$-algebra $\mathcal{B}$ is considered as a module over the algebra $\mathcal{A}$. A section of the fibering $\iota : \mathcal{A} \hookrightarrow \mathcal{B}$, is represented by a morphism of $\mathcal{R}$-algebras $s : \mathcal{B} \rightarrow \mathcal{A}$, left inverse to ι, that is $s \circ \iota = id_A$. The fundamental extension of scalars of the $\mathcal{R}$-algebra $\mathcal{A}$ is obtained by tensoring $\mathcal{A}$ with itself over the distinguished subalgebra of the reals, that is $\iota : \mathcal{A} \hookrightarrow \mathcal{A} \bigotimes_{\mathcal{R}} \mathcal{A}$. Trivial cases of scalars extensions, in fact isomorphic to $\mathcal{A}$, induced by the fundamental one, are obtained by tensoring $\mathcal{A}$ with $\mathcal{R}$ from both sides, that is $\iota_1 : \mathcal{A} \hookrightarrow \mathcal{A} \bigotimes_{\mathcal{R}} \mathcal{R}$, $\iota_2 : \mathcal{A} \hookrightarrow \mathcal{R} \bigotimes_{\mathcal{R}} \mathcal{A}$.

The basic idea of Riemann that has been incorporated in the context of General Relativity is that geometry should be built from the infinitesimal to the global. Geometry in this context is understood in terms of metric structures that can be defined on a differential manifold. If we adopt the algebraic viewpoint, geometry as a result of interactions, requires the extension of scalars of the algebra $\mathcal{A}$ by infinitesimal quantities, defined as a fibration:

$$d_* : \mathcal{A} \hookrightarrow \mathcal{A} \oplus \mathbf{M} \cdot \epsilon$$

$$f \mapsto f + d_*(f) \cdot \epsilon$$

where, $d_*(f) =: df$ is considered as the infinitesimal part of the extended scalar, and ϵ the infinitesimal unit obeying $\epsilon^2 = 0$ [20]. The algebra of infinitesimally extended scalars $\mathcal{A} \oplus \mathbf{M} \cdot \epsilon$ is called the algebra of dual numbers over $\mathcal{A}$ with coefficients in the $\mathcal{A}$-module $\mathbf{M}$. It is immediate to see that the algebra $\mathcal{A} \oplus \mathbf{M} \cdot \epsilon$, as an abelian group is just the direct sum $\mathcal{A} \oplus \mathbf{M}$, whereas the multiplication is defined by:

$$(f + df \cdot \epsilon) \bullet (f' + df' \cdot \epsilon) = f \cdot f' + (f \cdot df' + f' \cdot df) \cdot \epsilon$$

Note that, we further require that the composition of the augmentation $\mathcal{A} \oplus \mathbf{M} \cdot \epsilon \to \mathcal{A}$, with d_* is the identity. Equivalently, the above fibration, viz., the homomorphism of algebras $d_* : \mathcal{A} \hookrightarrow \mathcal{A} \oplus \mathbf{M} \cdot \epsilon$, can be formulated as a derivation, that is, in terms of an additive $\mathcal{R}$-linear morphism:

$$d : \mathcal{A} \to \mathbf{M}$$

$$f \mapsto df$$

that, moreover, satisfies the Leibniz rule:

$$d(f \cdot g) = f \cdot dg + g \cdot df$$

Since the formal symbols of differentials $\{df, f \in \mathcal{A}\}$, are reserved for the universal derivation, the $\mathcal{A}$-module $\mathbf{M}$ is identified as the free $\mathcal{A}$-module $\mathbf{\Omega}$ of 1-forms generated by these formal symbols, modulo the Leibniz constraint, where the scalars of the distinguished subalgebra $\mathcal{R}$, that is the real numbers, are treated as constants. Kähler observed that the free $\mathcal{A}$-module $\mathbf{\Omega}$ can be constructed explicitly from the fundamental form of scalars extension of $\mathcal{A}$, that is $\iota : \mathcal{A} \hookrightarrow \mathcal{A} \bigotimes_{\mathcal{R}} \mathcal{A}$ by considering the morphism:

$$\delta : \mathcal{A} \bigotimes_{\mathcal{R}} \mathcal{A} \to A$$

$$f_1 \otimes f_2 \mapsto f_1 \cdot f_2$$

Then by taking the kernel of this morphism of algebras, that is, the ideal:

$$\mathbf{I} = \{f_1 \otimes f_2 \in \mathcal{A} \bigotimes_{\mathcal{R}} \mathcal{A} : \delta(f_1 \otimes f_2) = 0\} \subset \mathcal{A} \bigotimes_{\mathcal{R}} \mathcal{A}$$

it can be shown that the morphism of $\mathcal{A}$-modules:

$$\Sigma : \mathbf{\Omega} \to \frac{\mathbf{I}}{\mathbf{I}^2}$$

$$df \mapsto 1 \otimes f - f \otimes 1$$

is an isomorphism.

We can prove the above isomorphism as follows: The fractional object $\frac{\mathbf{I}}{\mathbf{I}^2}$ has an $\mathcal{A}$-module structure defined by:

$$f \cdot (f_1 \otimes f_2) = (f \cdot f_1) \otimes f_2 = f_1 \otimes (f \cdot f_2)$$

for $f_1 \otimes f_2 \in \mathbf{I}$, $f \in \mathcal{A}$. We can check that the second equality is true by proving that the difference of $(f \cdot f_1) \otimes f_2$ and $f_1 \otimes (f \cdot f_2)$ belonging to $\mathbf{I}$, is actually an elememt of $\mathbf{I}^2$, viz., the equality is true modulo $\mathbf{I}^2$. So we have:

$$(f \cdot f_1) \otimes f_2 - f_1 \otimes (f \cdot f_2) = (f_1 \otimes f_2) \cdot (f \otimes 1 - 1 \otimes f)$$

The first factor of the above product of elements belongs to $\mathbf{I}$ by assumption, whereas the second factor also belongs to $\mathbf{I}$, since we have that:

$$\delta(f \otimes 1 - 1 \otimes f) = 0$$

Hence the product of elements above belongs to $\mathbf{I} \cdot \mathbf{I} = \mathbf{I}^2$. Consequently, we can define a morphism of $\mathcal{A}$-modules:

$$\Sigma : \mathbf{\Omega} \to \frac{\mathbf{I}}{\mathbf{I}^2}$$

$$df \mapsto 1 \otimes f - f \otimes 1$$

Now, we construct the inverse of that morphism as follows: The $\mathcal{A}$-module $\mathbf{\Omega}$ can be made an ideal in the algebra of dual numbers over $\mathcal{A}$, viz., $\mathcal{A} \oplus \mathbf{\Omega} \cdot \epsilon$. Moreover, we can define the morphism of algebras:

$$\mathcal{A} \times \mathcal{A} \to \mathcal{A} \oplus \mathbf{\Omega} \cdot \epsilon$$

$$(f_1, f_2) \mapsto f_1 \cdot f_2 + f_1 \cdot df_2 \epsilon$$

This is an $\mathcal{R}$-bilinear morphism of algebras, and thus, it gives rise to a morphism of algebras:

$$\Theta : \mathcal{A} \otimes_{\mathcal{R}} \mathcal{A} \to \mathcal{A} \oplus \mathbf{\Omega} \cdot \epsilon$$

Then, by definition we have that $\Theta(\mathbf{I}) \subset \mathbf{\Omega}$, and also, $\Theta(\mathbf{I}^2) = 0$. Hence, there is obviously induced a morphism of $\mathcal{A}$-modules:

$$\mathbf{\Omega} \leftarrow \frac{\mathbf{I}}{\mathbf{I}^2}$$

which is the inverse of Σ. Consequently, we conclude that:

$$\mathbf{\Omega} \cong \frac{\mathbf{I}}{\mathbf{I}^2}$$

Thus the free $\mathcal{A}$-module $\mathbf{\Omega}$ of 1-forms is isomorphic with the free $\mathcal{A}$-module $\frac{\mathbf{I}}{\mathbf{I}^2}$ of Kähler differentials of the algebra of scalars $\mathcal{A}$ over $\mathcal{R}$, conceived as a distinguished ideal in the algebra of infinitesimally extended scalars $\mathcal{A} \oplus \mathbf{\Omega} \cdot \epsilon$, due to interaction, according to the following split short exact sequence:

$$\mathbf{\Omega} \hookrightarrow \mathcal{A} \oplus \mathbf{\Omega} \cdot \epsilon \twoheadrightarrow \mathcal{A}$$

or equivalently formulated as:

$$0 \to \mathbf{\Omega}_A \to \mathcal{A} \bigotimes_{\mathcal{R}} \mathcal{A} \to \mathcal{A}$$

By dualizing, we obtain the dual $\mathcal{A}$-module of $\mathbf{\Omega}$, that is $\mathbf{\Xi} := Hom(\mathbf{\Omega}, \mathcal{A})$. Thus we have at our disposal, expressed in terms of infinitesimal scalars extension of the algebra $\mathcal{A}$, semantically intertwined with the generation of geometry as a result of interaction, new types of observables related with the incorporation of differentials and their duals, called vectors. Let us now explain the functionality of geometry, as related with the infinitesimally extended rings of scalars defined above, in the context of General Relativity. As we have argued before, the absolute representability principle of this theory, necessitates that our form of observation is tautosemous with real numbers representability. This means that all types of observables should possess uniquely defined dual types of observables, such that their representability can be made possible my means of real numbers. This is exactly the role of a geometry induced by a metric. Concretely, a metric structure assigns a unique dual to each observable, by effectuating an isomorphism between the $\mathcal{A}$-modules $\mathbf{\Omega}$ and $\mathbf{\Xi} = Hom(\mathbf{\Omega}, \mathcal{A})$, that is:

$$g : \mathbf{\Omega} \simeq \mathbf{\Xi}$$

$$df \mapsto v_f := g(df)$$

Thus the functional role of a metric geometry forces the observation of extended scalars, by means of representability in the field of real numbers, and is reciprocally conceived as the result of interactions causing infinitesimal variations in the scalars of the $\mathcal{R}$-algebra $\mathcal{A}$.

Before proceeding further, it is instructive at this point to clarify the meaning of a universal derivation, playing a paradigmatic role in the construction of extended algebras of scalars as above, in appropriate category-theoretic terms as follows [20]: The covariant functor of left $\mathcal{A}$-modules valued derivations of $\mathcal{A}$:

$$\overleftarrow{\nabla}_{\mathcal{A}}(-) : \mathcal{M}^{(\mathcal{A})} \to \mathcal{M}^{(\mathcal{A})}$$

is being representable by the left $\mathcal{A}$-module of 1-forms $\mathbf{\Omega}^1(\mathcal{A})$ in the category of left $\mathcal{A}$-modules $\mathcal{M}^{(\mathcal{A})}$, according to the isomorphism:

$$\overleftarrow{\nabla}_{\mathcal{A}}(N) \cong Hom_{\mathcal{A}}(\mathbf{\Omega}^1(\mathcal{A}), N)$$

Thus, $\mathbf{\Omega}^1(\mathcal{A})$ is characterized categorically as a universal object in $\mathcal{M}^{(\mathcal{A})}$, and the derivation:

$$d : \mathcal{A} \to \mathbf{\Omega}^1(\mathcal{A})$$

as the universal derivation [20]. Furthermore, we can define algebraically, for each $n \in N$, $n \geq 2$, the n-fold exterior product:

$$\mathbf{\Omega}^n(\mathcal{A}) = \bigwedge^n \mathbf{\Omega}^1(\mathcal{A})$$

where $\mathbf{\Omega}(\mathcal{A}) := \mathbf{\Omega}^1(\mathcal{A})$, $\mathcal{A} := \mathbf{\Omega}^{\mathbf{0}}(\mathcal{A})$, and finally show analogously that the left $\mathcal{A}$-modules of n-forms $\mathbf{\Omega}^n(\mathcal{A})$ in $\mathcal{M}^{(\mathcal{A})}$ are representable objects in

$\mathcal{M}^{(\mathcal{A})}$ of the covariant functor of left $\mathcal{A}$-modules valued n-derivations of $\mathcal{A}$, denoted by $\overleftarrow{\nabla}^{n}{}_{\mathcal{A}}(-) : \mathcal{M}^{(\mathcal{A})} \to \mathcal{M}^{(\mathcal{A})}$. We conclude that, all infinitesimally extended algebras of scalars that have been constructed from $\mathcal{A}$ by fibrations, presented equivalently as derivations, are representable as left $\mathcal{A}$-modules of n-forms $\mathbf{\Omega}^n(\mathcal{A})$ in the category of left $\mathcal{A}$-modules $\mathcal{M}^{(\mathcal{A})}$.

We emphasize that the intelligibility of the algebraic schema is based on the conception that infinitesimal variations in the scalars of $\mathcal{A}$, are caused by interactions, meaning that they are being effectuated by the presence of a physical field, identified as the gravitational field in the context of General Relativity. Thus, it is necessary to establish a purely algebraic representation of the notion of physical field and explain the functional role it assumes for the interpretation of the theory. The key idea for this purpose amounts to expressing the process of scalars extension in functorial terms, and by anticipation identify the functor of infinitesimal scalars extension due to interaction with the physical field that causes it [20]. Regarding the first step of this strategy we clarify that the general process of scalars extension from an algebra $\mathcal{S}$ to an algebra $\mathcal{T}$ is represented functorially by means of the functor of scalars extension [12], from $\mathcal{S}$ to $\mathcal{T}$ as follows:

$$\mathbf{F} : \mathcal{M}^{(\mathcal{S})} \to \mathcal{M}^{(\mathcal{T})}$$

$$\mathbf{E} \mapsto \mathbf{T} \bigotimes\nolimits_{\mathcal{S}} \mathbf{E}$$

The second step involves the application of the functorial algebraic procedure for the case admitting the identifications:

$$\mathcal{S} = \mathcal{A}$$

$$\mathcal{T} = [\mathcal{A} \oplus \mathbf{\Omega}^1(\mathcal{A}) \cdot \epsilon]$$

corresponding to infinitesimal scalars extension. Consequently, the physical field as the causal agent of interactions admits a purely algebraic description as the functor of infinitesimal scalars extension, called a connection-inducing functor:

$$\widehat{\nabla} : \mathcal{M}^{(\mathcal{A})} \to \mathcal{M}^{(\mathcal{A} \oplus \mathbf{\Omega}^1(\mathcal{A}) \cdot \epsilon)}$$

$$\mathbf{E} \mapsto [\mathcal{A} \oplus \mathbf{\Omega}^1(\mathcal{A}) \cdot \epsilon] \bigotimes_{\mathcal{A}} \mathbf{E}$$

In this sense, the vectors of the left $\mathcal{A}$-module $\mathbf{E}$ are being infinitesimally extended into vectors of the left $(\mathcal{A} \oplus \mathbf{\Omega}^1(\mathcal{A}) \cdot \epsilon)$-module $[\mathcal{A} \oplus \mathbf{\Omega}^1(\mathcal{A}) \cdot \epsilon] \bigotimes_{\mathcal{A}} \mathbf{E}$. Notice that these two kinds of vectors are being defined over different algebras. Hence, in order to compare them we have to pull the infinitesimally extended ones back to the initial algebra of scalars, viz., the $\mathcal{R}$-algebra $\mathcal{A}$. Algebraically this process is implemented by restricting the left $(\mathcal{A} \oplus \mathbf{\Omega}^1(\mathcal{A}) \cdot \epsilon)$-module $[\mathcal{A} \oplus \mathbf{\Omega}^1(\mathcal{A}) \cdot \epsilon] \bigotimes_{\mathcal{A}} \mathbf{E}$ to the $\mathcal{R}$-algebra $\mathcal{A}$. If we perform this base change we obtain the left $\mathcal{A}$-module $\mathbf{E} \oplus [\mathbf{\Omega}^1(\mathcal{A}) \bigotimes_{\mathcal{A}} \mathbf{E}] \cdot \epsilon$. Thus, the effect of the action of the physical field on the vectors of the left $\mathcal{A}$-module $\mathbf{E}$ can be expressed by means of the following comparison morphism of left $\mathcal{A}$-modules:

$$\nabla^{\star}{}_{\mathbf{E}} : \mathbf{E} \to \mathbf{E} \oplus [\mathbf{\Omega}^1(\mathcal{A}) \bigotimes_{\mathcal{A}} \mathbf{E}] \cdot \epsilon$$

Equivalently, the irreducible amount of information incorporated in the comparison morphism can be now expressed as a connection on $\mathbf{E}$. The latter is defined algebraically as an $\mathcal{R}$-linear morphism of $\mathcal{A}$-modules [21]:

$$\nabla_{\mathbf{E}} : \mathbf{E} \to \mathbf{\Omega}^1(\mathcal{A}) \bigotimes_{\mathcal{A}} \mathbf{E} = \mathbf{E} \bigotimes_{\mathcal{A}} \mathbf{\Omega}^1(\mathcal{A}) = \mathbf{\Omega}^1(\mathbf{E})$$

such that the following Leibniz type constraint is satisfied:

$$\nabla_{\mathbf{E}}(f \cdot v) = f \cdot \nabla_{\mathbf{E}}(v) + df \otimes v$$

for all $f \in \mathcal{A}$, $v \in \mathbf{E}$.

In the context of General Relativity, the absolute representability principle over the field of real numbers, necessitates as we have explained above the existence of uniquely defined duals of observables. Thus, the gravitational field is identified with a linear connection on the $\mathcal{A}$-module $\mathbf{\Xi} = Hom(\mathbf{\Omega}^1, \mathcal{A})$, being isomorphic with $\mathbf{\Omega}^1$, by means of a metric:

$$g : \mathbf{\Omega}^1 \simeq \mathbf{\Xi} = \mathbf{\Omega}^{1^*}$$

Consequently, the gravitational field may be represented by the pair $(\Xi, \nabla_{\mathbf{\Xi}})$. The metric compatibility of the connection required by the theory is simply expressed as:

$$\nabla_{Hom_{\mathcal{A}}(\mathbf{\Xi}, \mathbf{\Xi}^*)}(g) = 0$$

It is instructive to emphasize that the functorial conception of physical fields in general, according to the proposed schema, based on the notion of causal agents of infinitesimal scalars extension, does not depend on any restrictive representability principle, like the absolute representability principle over the real numbers, imposed by General Relativity. Consequently, the meaning of functoriality implies covariance with respect to representability, and thus, covariance with respect to generalized geometric realizations. In the same vein of ideas, the reader has already noticed that all the algebraic arguments refer, on purpose, to a general observables algebra $\mathcal{A}$, that has been identified with the $\mathcal{R}$-algebra $\mathcal{C}^\infty(M)$ in the model case of General Relativity. Of course, the functorial mechanism of understanding the notion of interaction, should not depend on the observables algebras used for the particular manifestations of it, thus, the only actual requirement for the intelligibility of functoriality of interactions by means of physical fields rests on the algebra-theoretic specification of what we characterize structures of observables. Put differently, the functorial coordinatization of the universal mechanism of encoding physical interactions in terms of observables, by means of causal agents, namely physical fields effectuating infinitesimal scalars extension, should respect the algebra-theoretic structure.

2.2 The Algebraic De Rham Complex and Field Equations

The next stage of development of the algebraic schema of comprehending the mechanism of dynamics involves the satisfaction of appropriate global constraints, that impose consistency requirements referring to the transition from the infinitesimal to the global. For this purpose it is necessary to employ the methodology of homological algebra. We start by reminding the algebraic construction, for each $n \in N$, $n \geq 2$, of the n-fold exterior product as follows: $\mathbf{\Omega}^n(\mathcal{A}) = \bigwedge^n \mathbf{\Omega}^1(\mathcal{A})$ where $\mathbf{\Omega}(\mathcal{A}) := \mathbf{\Omega}^1(\mathcal{A})$, $\mathcal{A} := \mathbf{\Omega}^{\mathbf{0}}(\mathcal{A})$. We notice that there exists an $\mathcal{R}$-linear morphism:

$$d^n : \mathbf{\Omega}^n(\mathcal{A}) \rightarrow \mathbf{\Omega}^{n+1}(\mathcal{A})$$

for all $n \geq 0$, such that $d^0 = d$. Let $\omega \in \mathbf{\Omega}^n(\mathcal{A})$, then ω has the form:

$$\omega = \sum f_i(dl_{i1} \bigwedge \ldots \bigwedge dl_{in})$$

with f_i, l_{ij}, $\in \mathcal{A}$ for all integers i, j. Further, we define:

$$d^n(\omega) = \sum df_i \bigwedge dl_{i1} \bigwedge \ldots \bigwedge dl_{in}$$

Then, we can easily see that the resulting sequence of $\mathcal{R}$-linear morphisms;

$$\mathcal{A} \rightarrow \mathbf{\Omega}^1(\mathcal{A}) \rightarrow \ldots \rightarrow \mathbf{\Omega}^n(\mathcal{A}) \rightarrow \ldots$$

is a complex of $\mathcal{R}$-vector spaces, called the algebraic de Rham complex of $\mathcal{A}$. The notion of complex means that the composition of two consequtive $\mathcal{R}$-linear morphisms vanishes, that is $d^{n+1} \circ d^n = 0$, simplified symbolically as:

$$d^{\mathbf{2}} = 0$$

If we assume that $(\mathbf{E}, \nabla_{\mathbf{E}})$ is an interaction field, defined by a connection $\nabla_{\mathbf{E}}$ on the $\mathcal{A}$-module $\mathbf{E}$, then $\nabla_{\mathbf{E}}$ induces a sequence of $\mathcal{R}$-linear morphisms:

$$\mathbf{E} \rightarrow \mathbf{\Omega}^1(\mathcal{A}) \bigotimes_{\mathcal{A}} \mathbf{E} \rightarrow \ldots \rightarrow \mathbf{\Omega}^n(\mathcal{A}) \bigotimes_{\mathcal{A}} \mathbf{E} \rightarrow \ldots$$

or equivalently:

$$\mathbf{E} \to \mathbf{\Omega}^1(\mathbf{E}) \to \ldots \to \mathbf{\Omega}^n(\mathbf{E}) \to \ldots$$

where the morphism:

$$\nabla^n : \mathbf{\Omega}^n(\mathcal{A}) \bigotimes_{\mathcal{A}} \mathbf{E} \to \mathbf{\Omega}^{n+1}(\mathcal{A}) \bigotimes_{\mathcal{A}} \mathbf{E}$$

is given by the formula:

$$\nabla^n(\omega \otimes v) = d^n(\omega) \otimes v + (-1)^n \omega \wedge \nabla(v)$$

for all $\omega \in \mathbf{\Omega}^n(\mathcal{A})$, $v \in \mathbf{E}$. It is immediate to see that $\nabla^0 = \nabla_{\mathbf{E}}$. Let us denote by:

$$\mathbf{R}_{\nabla} : \mathbf{E} \to \mathbf{\Omega}^2(\mathcal{A}) \bigotimes_{\mathcal{A}} \mathbf{E} = \mathbf{\Omega}^2(\mathbf{E})$$

the composition $\nabla^1 \circ \nabla^0$. We see that $\mathbf{R}_{\nabla}$ is actually an $\mathcal{A}$-linear morphism, that is $\mathcal{A}$-covariant, and is called the curvature of the connection $\nabla_{\mathbf{E}}$. We note that, for the case of the gravitational field $(\Xi, \nabla_{\mathbf{\Xi}})$, in the context of General Relativity, $\mathbf{R}_{\nabla}$ is tautosemous with the Riemannian curvature of the spacetime manifold. We notice that, the latter sequence of $\mathcal{R}$-linear morphisms, is actually a complex of $\mathcal{R}$-vector spaces if and only if:

$$\mathbf{R}_{\nabla} = 0$$

We say that the connection $\nabla_{\mathbf{E}}$ is integrable if $\mathbf{R}_{\nabla} = 0$, and we refer to the above complex as the de Rham complex of the integrable connection $\nabla_{\mathbf{E}}$ on $\mathbf{E}$ in that case. It is also usual to call a connection $\nabla_{\mathbf{E}}$ flat if $\mathbf{R}_{\nabla} = 0$. A flat connection defines a maximally undisturbed process of dynamical variation caused by the corresponding physical field. In this sense, a non-vanishing curvature signifies the existence of disturbances from the maximally symmetric state of that variation. These disturbances can be cohomologically identified as obstructions to deformation caused by physical sources. In that case, the algebraic de Rham complex of the algebra of scalars $\mathcal{A}$ is not acyclic, viz. it has non-trivial cohomology groups. These groups measure the obstructions caused by sources and are responsible for a non-vanishing curvature of the connection. In the case of General Relativity, these disturbances are associated with the presence of matter distributions, being incorporated in the specification of the energy-momentum tensor. Taking into account the requirement of absolute representability over the real numbers, and thus considering the relevant evaluation trace operator by means of the metric, we arrive at Einstein's field equations, which in the absence of matter sources read:

$$\mathcal{R}(\nabla_{\Xi}) = 0$$

where $\mathcal{R}(\nabla_{\Xi})$ denotes the relevant Ricci scalar curvature.

3 Topos-Theoretic Relativization and Localization of Functorial Dynamics

3.1 Conceptual Setting

The central focus of the studies pertaining to the formulation of a tenable theory of Quantum Gravity resolves around the issue of background manifold independence. In Section 2, we have constructed a general functorial framework of modelling field dynamics using categorical and homological algebraic concepts and techniques. In particular, we have applied this framework in the case of General Relativity recovering the classical gravitational dynamics. The significance of the proposed functorial schema of dynamics, in relation to a viable topos-theoretic approach to Quantum Gravity, lies on the fact that, the coordinatization of the universal mechanism of encoding physical interactions in terms of observables, by means of causal agents, viz., physical fields effectuating infinitesimal scalars extension, should respect only the algebra-theoretic structure of observables. Consequently, it is not dependent on the codomain of representability of the observables, being thus, only subordinate to the algebra-theoretic characterization of their structures. In particular, it not constrained at all by the absolute representability principle over the field of real numbers, imposed by classical General Relativity, a byproduct of which is the fixed background manifold construct of that theory.

In this perspective, the absolute representability principle of classical General Relativity in terms of real numbers, may be relativized without affecting the functionality of the algebraic dynamical mechanism. Consequently, it is possible to describe the dynamics of gravitational interactions in generalized localization environments, instantiated by suitable topoi. The latter are understood in the sense of categories of sheaves, defined over a base category of reference localization contexts, with respect to some suitable Grothendieck topology. From a physical viewpoint, the construction of a sheaf of observables constitutes the natural outcome of a well-defined localization process. Generally speaking, a localization process is being implemented in terms of an action of some category of reference contexts on a set-theoretic global algebra of observables. The latter, is then partitioned into sorts parameterized by the objects of the category of contexts. In this manner, the functioning of a localization process can be represented by means of a fibered construct, understood geometrically as a presheaf, or equivalently, as a variable set (algebra) over the base category of contexts. The fibers of this construct may be thought, in analogy to the case of the action of a group on a set of points, as the "generalized orbits" of the action of the category of contexts. The notion of functional dependence incorporated in this action, forces the global algebraic structure of observables to fiber over the base category of reference contexts.

At a further stage of development of these ideas, the disassociation of the physical meaning of a localization process from its usual classical spatiotemporal connotation, requires, first of all, the abstraction of the constitutive properties of localization in appropriate categorical terms, and then, the effectuation of these properties for the definition of localization systems of global observable structures. Regarding these objectives, the sought abstraction is being implemented by means of covering devices on the base category of reference contexts, called in categorical terminology covering sieves. The constitutive properties of localization being abstracted categorically in terms of sieves, being qualified as covering ones, satisfy the following basic requirements:

[i]. The covering sieves are covariant under pullback operations, viz., they are stable under change of a base reference context. Most importantly, the stability conditions are functorial. This requirement means, in particular, that the intersection of covering sieves is also a covering sieve, for each reference context in the base category.

[ii]. The covering sieves are transitive, such that, intuitively stated, covering sieves of subcontexts of a context in covering sieves of this context, are also covering sieves of the context themselves.

From a physical perspective, the consideration of covering sieves as generalized measures of localization of observables within a global observable structure, gives rise to localization systems of the latter. More specifically, the operation which assigns to each local reference context of the base category a collection of covering sieves satisfying the closure conditions stated previously, gives rise to the notion of a Grothendieck topology on the category of contexts. The construction of a suitable Grothendieck topology on the base category of contexts is significant for the following reasons: Firstly, it elucidates precisely and unquestionably the conception of local in a categorical measurement environment, such that, this conception becomes detached from its usual spatiotemporal connotation, and thus, expressed exclusively in relational information terms. Secondly, it permits the collation of local observable information into global ones by utilization of the notion of sheaf for that Grothendieck topology. The definition of sheaf essentially expresses gluing conditions, providing the means for studying the global consequences of locally defined properties. The transition from locally defined observable information into global ones is being effectuated via a compatible family of elements over a localization system of a global observable strucure. A sheaf assigns a set of elements to each reference context of a localization system, representing local observable data colected within that context. A choice of elements from these sets, one for each context, forms a compatible family if the choice respects the mappings induced by the restriction functions among contexts, and moreover, if the elements chosen agree whenever two contexts of the localization system overlap. If such a locally compatible choice induces a unique choice for a global observable structure being localized, viz. a global choice, then the condition for being a sheaf is satisfied. We note that, in general, there will be more locally defined or partial choices than globally defined ones, since not all partial choices need be extendible to global ones, but a compatible family of partial choices uniquely extends to a global one, or in other words, any presheaf uniquely defines a sheaf.

According to the strategy of relativization of physical representability with respect to generalized localization environments, instantiated by suitable categories of sheaves of observables, the problem of quantization of gravity is equivalent to forcing the algebraic general relativistic dynamical mechanism of the gravitational connection functorial morphism inside an appropriate topos, being capable of incorporating the localization properties of observables in the quantum regime. The only cost to be paid for this topos-theoretic relativization is the rejection of the fixed background manifold structure of the classical theory. This is actually not a cost at all, since it would permit the intelligibility of the field equations over geometric realizations that include manifold singularities and other pathologies, without affecting the algebraic mechanism of dynamics.

Equivalently stated, the requirement of background manifold independence of Quantum Gravity can be attained, by rejecting the absolute representability of the classical theory over the real numbers, and thus, the fixed spacetime manifold substratum, while keeping at the same time, the homological machinery of functorial dynamics. In the current Section, we argue that the abolishment of the above absolute representability requirement of the classical theory, paving the way towards Quantum Relativity, can be achieved by effectuating a process of topos-theoretic relativization of physical representability suitable for the modelling of quantum phenomena. This process is based on the technique of sheafification of observables algebras and constitutes the necessary conceptual and technical apparatus for a merging of General Relativity and Quantum Theory along the topos-theoretic lines advocated in this work.

3.2 Topological Sheafification of Field Dynamics and Abstract Differential Geometry

Initially, it is important to clarify the conception of relativity of representability in appropriate category-theoretic terms. The absolute representability principle is based on the set-theoretic conception of the real line, as a set of infinitely distinguished points coordinatized by means of the field of real numbers. Expressed categorically, this is equivalent to the interpretation of the algebraic structure of the reals inside the absolute universe of **Sets**, or more precisely inside the topos of constant **Sets**. It is also well known that algebraic structures and mechanisms can admit a variable reference, formulated in category-theoretic jargon in terms of arrows only specifications, inside any suitable topos of discourse [13, 15]. A general topos can be conceived as a manifestation of a universe of variable sets [17, 18]. For example the topos of sheaves of sets $\mathbf{Shv}(X)$ over the category of open sets of an abstract topological space X, ordered by inclusion, is understood as a categorical universe of varying sets over the opens of the topology covering

X. The relativization of physical representability with respect to the topos of sheaves $\mathbf{Shv}(X)$, amounts to the relativization of both the notion and the algebraic structure of the real numbers inside this topos [22]. Regarding the notion of real numbers inside the topos $\mathbf{Shv}(X)$, this is equivalent to the notion of continuously variable real numbers over the open reference domains of X, or else, equivalent to the notion of real-valued continuous functions on X, when interpreted respectively inside the topos of **Sets** [17, 18] . Regarding the algebraic structure of the reals inside the topos $\mathbf{Shv}(X)$, they form only an algebra in this topos, which is tautosemous with the sheaf of commutative $\mathcal{R}$-algebras of continuous real-valued functions on X, where $\mathcal{R}$ corresponds in that case to the constant sheaf of real numbers over X.

Let us discuss briefly from a physical viewpoint, the meaning of relativization of representability with respect to the internal reals of the topos of sheaves $\mathbf{Shv}(X)$ of $\mathcal{R}$-algebras of observables $\mathcal{A}$ over the category of opens of X. Inside this topos, it is assumed that for every open U in X, $\mathcal{A}(U)$ is a commutative, unital $\mathcal{R}$-algebra of continuous local sections of the sheaf of $\mathcal{R}$-algebras $\mathcal{A}$. In particular, the algebra of reals in this topos consists of continuous real-valued local sections localized sheaf-theoretically over the opens of X. Thus, the semantics of the codomain of valuation of observables is transformed from a set-theoretic to a sheaf-theoretic one. More concretely, it is obvious that inside the topos **Sets** the unique localization measure of observables is a point of the $\mathcal{R}$-spectrum of the corresponding algebra of scalars, which is assigned a numerical identity. In contradistinction, inside the topos $\mathbf{Shv}(X)$, the former is substituted by a variety of localization measures, dependent only on the open sets in the topology of X. In the latter context, a point-localization measure, is identified precisely with the ultrafilter of all opens containing the point. This identification permits the conception of other filters being formed by admissible operations between opens as generalized measures of localization of observables. Furthermore, the relativization of representability in $\mathbf{Shv}(X)$ is physically significant, because the operational specification of measurement environments exists only locally and the underlying assumption is that the information gathered about local observables in different measurement situations can be collated together by appropriate means, a process that is precisely formalized by the notion of sheaf. Conclusively, we assert that, localization schemes referring to observables may not depend exclusively on the existence of points, and thus, should not be tautosemous with the practice of conferring a numerical identity to them. Thus, the relativization of representability with respect to the internal reals of the topos of sheaves $\mathbf{Shv}(X)$, amounts to the substitution of point localization measures, represented numerically, with localization measures fibering over the base category of open reference loci, represented respectively by local sections in the sheaf of internal reals [22].

The main purpose of the discussion of relativized representability inside the topos $\mathbf{Shv}(X)$, is again to focus attention in the fact that the purely algebraic functorial dynamical mechanism of connections, depending only on the algebra theoretic structure, still holds inside that topos. The interpretation of the mechanism in $\mathbf{Shv}(X)$ has been accomplished by the development of Abstract Differential Geometry (ADG) [21, 26-27]. In particular, ADG generalizes the differential geometric mechanism of smooth manifolds, by explicitly demonstrating that most of the usual differential geometric constructions can be carried out by purely sheaf-theoretic means without any use of any sort of C^{∞}-smoothness or any of the conventional calculus that goes with it. Thus, it permits the legitimate use of any appropriate $\mathcal{R}$-algebra sheaf of observables localized over a topological environment, even Rosinger's singular algebra sheaf of generalized functions, without loosing the differential mechanism, prior believed to be solely associated with smooth manifolds [23-29].

The operational machinery of ADG is essentially based on the exactness of the following abstract de Rham complex, interpreted inside the topos of sheaves $\mathbf{Shv}(X)$:

$$\mathbf{0} \rightarrow \mathcal{R} \rightarrow \mathcal{A} \rightarrow \mathbf{\Omega}^1(\mathcal{A}) \rightarrow \ldots \rightarrow \mathbf{\Omega}^n(\mathcal{A}) \rightarrow \ldots$$

It is instructive to note that the exactness of the complex above, for the classical case, where, $\mathcal{A}$ stands for a smooth $\mathcal{R}$-algebra sheaf of observables on X, constitutes an expression of the lemma of Poincaré, according to which, every closed differential form on X is exact at least locally in X. ADG's power of abstracting and generalizing the classical calculus on smooth manifolds basically lies in the possibility of assuming other more general coordinates sheaves, that is, more general commutative coefficient structure sheaves $\mathcal{A}$, while at the same time retaining, via the exactness of the abstract de Rham complex, as above, the mechanism of differentials, instantiated, in the first place, in the case of classical differential geometry on smooth manifolds. Thus, any cohomologically appropriate sheaf of algebras $\mathcal{A}$ may be regarded as a coordinates sheaf capable of providing a differential geometric mechanism, independently of any manifold concept, analogous, however, to the one supported by smooth manifolds.

Applications of ADG include the reformulation of Gauge theories in sheaf-theoretic terms [26, 27], as well as, the evasion of the problem of manifold singularities appearing in the context of General Relativity [28-31]. Related with the first issue, ADG has modeled Yang-Mills fields in terms of appropriate pairs $(\mathbf{E}, \mathcal{D}_{\mathbf{E}})$, where $\mathbf{E}$ are vector sheaves whose sections have been identified with the states of the corresponding particles, and $\mathcal{D}_{\mathbf{E}}$ are connections that act on the corresponding states causing interactions by means of the respective fields they refer to. Related with the second issue, ADG has replaced the $\mathcal{R}$-algebra $\mathcal{C}^{\infty}(M)$ of smooth real-valued functions on a differential manifold with a sheaf of $\mathcal{R}$-algebras that incorporates the singularities of the manifold in terms of appropriate ideals, allowing the formulation of Einstein's equations in a covariant form with respect to the generalized scalars of that sheaf of $\mathcal{R}$-algebras.

3.3 Quantization as a Grothendieck Topos-Relativized Representability and Quantum Sheafification of Dynamics

The basic defining feature of Quantum Theory according to the Bohrian interpretation [32-34], in contradistinction to all classical theories, is the realization that physical observables are not definitely or sharply valued as long as a measurement has not taken place, meaning both, the precise specification of a concrete experimental context, and also, the registration of a value of a measured quantity in that context by means of an apparatus. Furthermore, Heisenberg's uncertainty relations, determine the limits for simultaneous measurements of certain pairs of complementary physical properties, like position and momentum. In a well-defined sense, the uncertainty relations may be interpreted as measures of the valuation vagueness associated with the simultaneous determination of all physical observables in the same experimental context. In all classical theories, the valuation algebra is fixed once and for all to be the division algebra of real numbers $\mathcal{R}$, reflecting the fact that values admissible as measured results must be real numbers, irrespective of the measurement context and simultaneously for all physical observables.

The resolution of valuation vagueness in Quantum Theory can be algebraically comprehended through the notion of relativization of representability of the valuation algebra with respect to commutative algebraic contexts that correspond to locally prepared measurement environments [35, 36]. Only after such a relativization the eigenvalue equations formulated in the context of such a measurement environment yield numbers corresponding to measurement outcomes. At a logical level commutative contexts of measurement correspond to Boolean algebras, identified as subalgebras of a quantum observables algebra. In the general case, commutative algebraic contexts are identified with commutative $\mathcal{K}$-algebras, where $\mathcal{K} = \mathcal{Z}_2, \mathcal{R}, \mathcal{C}$, which may be thought as subalgebras of a non-commutative algebra of quantum observables, represented irreducibly as an algebra of operators on a Hilbert space of quantum states.

If we consider the relativization of representability of the valuation algebra in Quantum Theory seriously, this implies that the proper topos to apply the functorial dynamical mechanism caused by quantum fields, is the topos of sheaves of algebras over the base category of commutative algebraic contexts, denoted by $\mathbf{Shv}(\mathcal{A}_{\mathcal{C}})$, where $\mathcal{A}_{\mathcal{C}}$ is the base category of commutative $\mathcal{K}$-algebras [20, 22, 37-39]. Equivalently, this means that representability in Quantum Theory should be relativized with respect to the internal reals of the topos $\mathbf{Shv}(\mathcal{A}_{\mathcal{C}})$. We mention that the interpretational aspects of the proposed topos-theoretic relativization of physical representability in relation to the truth-values structures of quantum logics have been discussed extensively in [40].

In the current context of enquiry, this admissible topos-theoretic framework of representability elaborates the interpretation of the algebraic functorial mechanism of connections inside the topos $\mathbf{Shv}(\mathcal{A}_{\mathcal{C}})$, thus allowing the conception of interactions caused by quantum fields and, in particular the notion of quantum gravitational field. In this sense, Quantum Gravity should be properly a theory constructed inside the topos $\mathbf{Shv}(\mathcal{A}_{\mathcal{C}})$, formulated by means of adaptation of the functorial mechanism of infinitesimal scalars extension in the regime of this topos. It is instructive to explain the meaning of internal reals inside this topos. In analogy with the case of internal reals inside the topos $\mathbf{Shv}(X)$, where the valuation algebra of real numbers has been relativized with respect to the base category of open sets of an abstract topological space, in the topos $\mathbf{Shv}(\mathcal{A}_{\mathcal{C}})$ the valuation algebra is relativized with respect to the base category of commutative subalgebras of the algebra of quantum observables. Thus, similarly with the former case, it admits a description as a sheaf of continuous local sections over the category of base reference loci of variation $\mathcal{A_C}$. Of course, in the quantum case the notion of topology with respect to which continuity may be conceived refers to an appropriate Grothendieck topology [17, 19, 20, 22], formulated in terms of covers on the base category $\mathcal{A_C}$, with respect to which sheaves of algebras may be defined appropriately. A Grothendieck topology suitable for this purpose can be explicitly constructed as a covering system S of epimorphic families on the base category of commutative contexts, defined by the requirement that the morphism

$$G_S : \coprod_{\{s: {A_C}' \to A_C\} \in S} {A_C}' \to A_C$$

where, A_C, ${A_C}'$ in $\mathcal{A}_C$, is an epimorphism in $\mathcal{A}_Q$. This topological specification incorporates the functorial notion of generalized localization schemes, appropriately adapted for probing the quantum regime of observable structure [20, 22, 37].

The research initiative based on the principle of relativized representability inside the topos $\mathbf{Shv}(\mathcal{A_C})$, as the proper universe of discourse for constructing a categorical theory of covariant Quantum Gravitational Dynamics is in the phase of intense development, while a nucleus of basic ideas, methods and results related with this program have been already communicated [20]. According to this schema, the representation of quantum observables algebras A_Q in the category $\mathcal{A_Q}$ in terms of sheaves over commutative arithmetics A_C in $\mathcal{A_C}$ for the Grothendieck topology of epimorphic families on $\mathcal{A_C}$, is based on the existence of the adjunctive correspondence $\mathbf{L} \dashv \mathbf{F}$ as follows:

$$\mathbf{L} : \mathbf{Sets}^{{\mathcal{A_C}}^{op}} \leftrightarrows \mathcal{A_Q} : \mathbf{F}$$

which says that the Grothendieck functor of points of a quantum observables algebra restricted to commutative arithmetics defined by:

$$\mathbf{F}(A_Q) : A_C \mapsto Hom_{\mathcal{A_Q}}(\mathbf{M}(A_C), A_Q)$$

has a left adjoint:

$$\mathbf{L} : \mathbf{Sets}^{\mathcal{A}_C{}^{op}} \rightarrow \mathcal{A}_Q$$

which is defined for each presheaf $\mathbf{P}$ in $\mathbf{Sets}^{\mathcal{A}_C{}^{op}}$ as the colimit:

$$\mathbf{L}(\mathbf{P}) = \mathbf{P} \otimes_{\mathcal{A}_C} \mathbf{M}$$

where $\mathbf{M}$ is a coordinatization functor, viz.:

$$\mathbf{M} : \mathcal{A}_C \rightarrow \mathcal{A}_Q$$

which assigns to commutative observables algebras in $\mathcal{A}_C$ the underlying quantum algebras from $\mathcal{A}_Q$. Equivalently, there exists a bijection, natural in $\mathbf{P}$ and A_Q as follows:

$$Nat(\mathbf{P}, \mathbf{F}(A_Q)) \cong Hom_{\mathcal{A}_Q}(\mathbf{LP}, A_Q)$$

The counit of the adjunction:

$$\epsilon_{A_Q} : \mathbf{LF}(A_Q) \rightarrow A_Q$$

defined by the composite endofunctor:

$$\mathbf{G} := \mathbf{LF} : \mathcal{A}_Q \rightarrow \mathcal{A}_Q$$

constitutes intuitively the first step of a functorial free resolution of a quantum observables algebra A_Q in $\mathcal{A}_Q$. Actually, by iterating the endofunctor $\mathbf{G}$, we may extend ϵ_{A_Q} to a free simplicial resolution of A_Q. In this setting, we may now apply Kähler's methodology in order to obtain the object of quantum differential 1-forms, by means of the following split short exact sequence:

$$0 \rightarrow \Omega_{A_Q} \rightarrow \mathbf{G}A_Q \rightarrow A_Q$$

or equivalently,

$$0 \rightarrow \Omega_{A_Q} \rightarrow \mathbf{F}(A_Q) \otimes_{\mathcal{A}_C} \mathbf{M} \rightarrow A_Q$$

According to the above, we obtain that:

$$\Omega_{A_Q} = \frac{J}{J^2}$$

where $J = \mathbf{Ker}(\epsilon_{A_Q})$ denotes the kernel of the counit of the adjunction. Subsequently, we may apply the algebraic construction, for each $n \in N$, $n \geq 2$, of the n-fold exterior product $\Omega^n{}_{A_Q} = \bigwedge^n \Omega^1{}_{A_Q}$. Thus, we may now set up the algebraic de Rham complex of A_Q as follows:

$$A_Q \to \Omega_{A_Q} \to \ldots \to \Omega^n{}_{A_Q} \to \ldots$$

At a next stage, we notice that the functor of points of a quantum observables algebra restricted to commutative arithmetics $\mathbf{F}(A_Q)$ is left exact, because it is the right adjoint functor of the established adjunction. Thus, it preserves the short exact sequence defining the object of quantum differential 1-forms, in the following form:

$$0 \to \mathbf{F}(\Omega_{A_Q}) \to \mathbf{F}(\mathbf{G}(A_Q)) \to \mathbf{F}(A_Q)$$

Hence, we immediately obtain that:

$$\mathbf{F}(\Omega_{A_Q}) = \frac{Z}{Z^2}$$

where $Z = \mathbf{Ker}(\mathbf{F}(\epsilon_{A_Q}))$. Then, in analogy with the general algebraic situation, interpreted inside the proper universe that the functor of points of a quantum observables algebra assumes existence, viz., the topos $\mathbf{Sets}^{\mathcal{A}_C{}^{op}}$, we introduce the notion of an interaction field, termed quantum field, by means of the functorial pair $\big(\mathbf{F}(A_Q) := Hom_{\mathcal{A}_Q}(\mathbf{M}(-), A_Q), \nabla_{\mathbf{F}(A_Q)}\big)$, where the quantum connection $\nabla_{\mathbf{F}(A_Q)}$ is defined as the following natural transformation:

$$\nabla_{\mathbf{F}(A_Q)} : \mathbf{F}(A_Q) \to \mathbf{F}(\Omega_{A_Q})$$

Thus, the quantum connection $\nabla_{\mathbf{F}(A_Q)}$ induces a sequence of functorial morphisms, or equivalently, natural transformations as follows:

$$\mathbf{F}(A_Q) \to \mathbf{F}(\Omega_{A_Q}) \to \ldots \to \mathbf{F}(\Omega^n{}_{A_Q}) \to \ldots$$

Let us denote by:

$$\mathbf{R}_{\nabla} : \mathbf{F}(A_Q) \rightarrow \mathbf{F}({\Omega^2}_{A_Q})$$

the composition $\nabla^1 \circ \nabla^0$ in the obvious notation, where $\nabla^0 := \nabla_{\mathbf{F}(A_Q)}$, which we call the curvature of the quantum connection $\nabla_{\mathbf{F}(A_Q)}$. The latter sequence of functorial morphisms, is actually a complex if and only if $\mathbf{R}_{\nabla} = 0$. We say that the quantum connection $\nabla_{\mathbf{F}(A_Q)}$ is integrable or flat if $\mathbf{R}_{\nabla} = 0$, referring to the above complex as the functorial de Rham complex of the integrable connection $\nabla_{\mathbf{F}(A_Q)}$ in that case. The vanishing of the curvature of the quantum connection, that is:

$$\mathbf{R}_{\nabla} = 0$$

can be used as a means of transcription of Einstein's equations in the quantum regime, that is inside the topos $\mathbf{Shv}(\mathcal{A}_{\mathcal{C}})$ of sheaves of algebras over the base category of commutative algebraic contexts, in the absence of cohomological obstructions. We may explain the curvature of the quantum connection as the effect of non-trivial interlocking of commutative arithmetics, in some underlying diagram of a quantum observables algebras being formed by such localizing commutative arithmetics. The non-trivial gluing of commutative arithmetics in localization systems of a quantum algebra is caused by topological obstructions, that in turn, are being co-implied by acyclicity of the algebraic de Rham complex of A_Q. Intuitively, a non-vanishing curvature is the non-local attribute detected by an observer employing a commutative arithmetic in a discretely topologized categorical environment, in the attempt to understand the quantum localization properties, after having introduced a potential (quantum gravitational connection) in order to account for the latter by means of a differential geometric mechanism [20]. Thus, the physical meaning of curvature is associated with the apparent existence of non-local correlations from the restricted spatial perspective of disjoint classical commutative arithmetics A_C. It is instructive to make clear that, in the present schema, the notion of curvature does not refer to an underlying background manifold, since such a structure has not been required at all in the development of the differential geometric mechanism, according to functorial homological algebraic methods.

4 Epilogue

Conclusively, it is worthwhile to emphasize that discussions of background manifold independence pertaining the current research focus in Quantum Gravity, should take at face value the fact that the fixed manifold construct in General Relativity is just the byproduct of fixing physical representabilty in terms of real numbers. Moreover, it is completely independent of the possibility of formulating dynamics, since the latter can be developed precisely along purely algebraic lines, that is, by means of functorial connections. Hence the usual analytic differential geometric framework of smooth manifolds, needed for the formulation of General Relativity, is just a special coordinatization of the universal functorial mechanism of infinitesimal scalars extension, and thus should be substituted appropriately, in case a merging with Quantum Theory is being sought. The substitution is guided by the principle of relativized representability with respect to a suitable topos.

The important physical issue incorporated in the idea of relativizing physical representability with respect to appropriate topos-theoretic universes of discourse, as has been explained previously, has to do with a novel conception of physical localization schemes, that, in particular, seem to be indispensable for an accurate comprehension of the quantum regime of observable structure. More concretely, in classical theories localization has been conceived by means of metrical properties on a pre-existing smooth set-theoretic spacetime manifold. In contradistinction, quantum localization should be understood categorically and algebraically, viz., purely in functorial terms of relational information content between quantum arithmetics (algebras of quantum observables) and diagrams of commutative ones, without any supporting notion of smooth metrical backround manifold. In this sense, the resolution focus should be shifted from point-set to topological localization models of quantum observables structures, that effectively, induce a transition in the semantics of observables from a set-theoretic to a sheaf-theoretic one. Subsequently, that semantic transition effectuates the conceptual replacement of the classical metrical ruler of localization on a smooth background manifold, with a sheaf-cohomological ruler of algebraic categorical localization in a Grothendieck topos, that captures the relational information of observables in the quantum regime, filtered through diagrams of local commutative arithmetics (local reference frames in the topos constituting the functor of points of a quantum arithmetic). Thus, the dynamical properies of quantum arithmetics are being properly addressed to the global topos-theoretic dynamics generated by categorical diagrams of local commutative ones, giving rise to complexes of sheaves of observables for a suitably defined Grothendieck topology consisting of epimorphic families of coverings by local commutative arithmetics.

In particular, the application of the principle of relativized representability on the problem of merging General Relativity with Quantum Theory, forces the topos of sheaves of commutative observables algebras $\mathbf{Shv}(\mathcal{A}_{\mathcal{C}})$ as the proper universe of discourse for Quantum Gravity, requiring a functorial adaptation of the algebraic mechanism of connections inside that topos, and subsequently, an interpretation of Quantum Gravitational dynamics sheaf cohomologically with respect to the non-trivial localization schemes of observables in the quantum regime.

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