Generalized Topological Covering Systems on Quantum Events Structures

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Abstract

Homologous operational localization processes are effectuated in terms of generalized topological covering systems on structures of physical events. We study localization systems of quantum events structures by means of Gtothendieck topologies on the base category of Boolean events algebras. We show that a quantum events algebra is represented by means of a Grothendieck sheaf-theoretic fibered structure, such that the global partial order of quantum events fibers over the base category of local Boolean frames.

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1 Introduction

Physical observation presupposes, at the fundamental level, the existence of a localization process for extracting information related with the local behaviour of a physical system. On the basis of a localization process it is possible to discern observable events and assign an individuality to them. Generally, a localization process is being co-implied by the preparation of suitable local reference domains for measurement. These domains identify concretely the kind of reference loci used for observation of events. The methodology of observation is being effectuated by the functioning of events-registering measurement devices, that operate locally within the contexts of the prepared reference loci. In this general setting, it is important to notice that registering an event, that has been observed in the context of a reference locus, is not always equivalent to conferring a numerical identity to it, by means of a real value corresponding to a physical attribute. On the contrary, the latter is only a limited case of the localization process, when, in particular, it is assumed that all reference loci can be contracted to points. This is exactly the crucial assumption underlying the employment of the set-theoretic structure of the real line as a model of the physical "continuum". The semantics of the latter is associated with the codomain of valuation of physical attributes used for registering events. It is instructive to clarify that the set-theoretic model of the real line stands for a structure of points that are independent and possess the property of sharp distinguishability.

The primary motivation of this paper concerns the possibility of mathematically implementing a general localization process referring to physical observation, that is not necessarily based on the existence of an underlying structure of points on the real line. For this purpose, the focus is shifted from point-set to topological localization models of ordered global events structures. In particular, the central aim of this study targets the problem of representation of quantum logics in terms of Boolean localization systems. The notion of a localization system is the referent of a homologous operational physical procedure of observation. The latter is defined by the requirement that the reference loci used for observational purposes, together with their structural transformations, should form a mathematical category. The development of the conceptual and technical machinery of localization systems for generating non-trivial global events structures, as it will be explicitly demonstrated to be the case for quantum logics, effectuates a transition in the semantics of events from a set-theoretic to a sheaf-theoretic one. This is a crucial semantic difference that characterizes the present approach in comparison to the vast literature on quantum measurement and quantum logic.

The plan of development of the paper is the following: In Section 2, we introduce the notion of abstract localization systems and explain their functional role in a category-theoretic environment. Moreover, we motivate the use of Boolean localization systems for the generation of quantum events structures. In Section 3, we develop the construction of classical topological sheaves and explain their physical semantics in terms of fibered events structures. In Section 4, we define generalized topological covering systems by means of covering sieves on a base categorical environment of reference loci. The covering sieves are interpreted physically as generalized measures of localization of events. In Section 5, we define the categories of quantum and Boolean events algebras respectively, and furthermore, construct the functor of Boolean frames of a quantum events algebra. In Section 6, we apply the machinery of generalized topological covering systems for the analysis and generation of quantum events algebras by means of Boolean localization systems. In particular, we prove that the functor of Boolean frames of a quantum events algebra, is a sheaf for the Grothendieck topology of epimorphic families on the base category of Boolean localizing contexts. In Section 7, we formulate and prove a representation of quantum events by means of equivalence classes in a structure sheaf of Boolean coefficients, associated with local Boolean contexts of measurement. Finally, we conclude in Section 8.

2 Abstract Localization Systems

The general purpose of abstract localization systems amounts to filtering the information contained in a global structure of ordered physical events, through a concretely specified categorical environment that is determined by a homologous operational physical procedure. The latter specifies the kind of loci of variation, or equivalently, reference contexts, that are used for observation of events. These contexts play the role of generalized reference frames, such that reference to concrete events of the specified categorical kind can be made possible with respect to them. It is necessary to emphasize that the kind of loci of variation signifies exactly the concrete categorical environment employed operationally, for instance, the category of open sets, ordered by inclusion, in a topological measurement space. In this sense, localization systems can be precisely conceived as a generalization of the notion of functional dependence. In the trivial case, the only locus is a point serving as a unique idealized measure of localization, and moreover, the only kind of reference frame is the one attached to a point.

Since an underlying structure of points is not assumed for localization purposes, the functioning of a localization system in a global structure of physical events should be implemented by other means. The basic constitutive premise of our scheme is that the reference loci together with their structural transformations form a category. Thus, the localization process should be understood in terms of an action of the category of reference loci on a set-theoretic global structure of physical events. The latter, is then partitioned into sorts parameterized by the objects of the category of reference loci. In this sense, the functioning of a localization system can be represented by means of a fibered construct, understood geometrically, as a variable set over the base category of reference loci. The fibers of this construct may be thought, in analogy to the case of the action of a group on a set of points, as the "generalized orbits" of the action of the category of loci. The notion of functional dependence incorporated in this action, forces the ordered structure of physical events to fiber over the base category of reference loci. It is instructive to remark at this point, that ordered event structures have been of considerable interest in the literature of both quantum gravity and quantum computation research, and thus, the functioning of localization systems is also important, both conceptually and technically, for these disciplines as well.

From a physical perspective, the meaningful representation of a global ordered structure of events as a fibered construct, in terms of localization systems, should incorporate the requirement of uniformity. The latter may be formulated as a physical principle according to the following definition:

Principle of Uniformity: For any two events observed over the same domain of measurement, the structure of all reference frames that relate to the first cannot be distinguished in any possible way from the structure of frames relating to the second.

According to this principle, all the localized events within any particular reference locus in a localization system should be uniformly equivalent to each other. The compatibility of the localization process with the principle of uniformity, demands that the relation of (partial) order in a global set-theoretic universe of events is induced by lifting appropriately a structured family of arrows from the base category of reference loci to the fibers.

If we take together the requirements enforcing a representation of a global partially ordered structure of physical events, as a uniform fibered construct over a base category of reference loci, localization systems of the former are precisely modeled in the syntactical terms of sheaves. In this perspective, the transition in the semantics of physical events from a set-theoretic to a sheaf-theoretic one is completely justified, as it will become clear in the forthcoming Sections.

Before proceeding in the technical exposition of the mathematical structures involved it is instructive to discuss briefly an example. This refers to the case of localization on a global partial order of physical events over a base categorical environment $\mathcal{O}(X)$, consisting of open sets U, of a topological measurement space X, the arrows between them being inclusions.

In this case, the reference loci of the operational environment employed for observation, are all the open sets U of X, partially ordered by inclusion. Equivalently, the open sets inclusions $U \hookrightarrow X$ are considered as varying base reference frames of open loci over which the global partial order of events fibers. We may use the suggestive term "local observer" to refer to an events-registering device associated with a reference locus U of the base category $\mathcal{O}(X)$. Note that, the meaning of "local" is understood with respect to the topology of X. Then, a "local observer", in an measurement situation taking place over a reference locus U, individuates events by means of local real-valued observables, being continuous maps $s : U \to \mathcal{R}$. Thus, the "local observers" do not have a global perception of continuous functions $f : X \to \mathcal{R}$, but rather register events localized over the associated reference loci in terms of local observables. Of course, appropriate conditions are further needed for pasting their findings together, that as we shall explain in the sequel, are the necessary and sufficient conditions for a topological sheaf-theoretic structure. Intuitively, at this stage we notice that, in the implemented fibered construct, the viewpoint offered by a reference locus is not that of a globally defined real valued continuous function, but that of a continuously variable real number over the associated open locus. The latter is called a local observable section and is interpreted physically as a localized and accordingly individuated event.

In quantum theory, the global structures of quantum events are technically characterized as orthomodular orthoposets. The original quantum logical formulation of quantum theory [1, 2] depends in an essential way on the identification of events, or propositions, with projection operators on a complex Hilbert space. A non-classical, non-Boolean events structure is effectively induced which has its origins in quantum theory [3]. On the other side, in every concrete quantum measurement context, the set of events that have been actualized in this context forms a Boolean algebra. This fact motivates the assumption that a Boolean algebra in the poset of quantum events, could be interpreted appropriately as a generalized reference frame, relative to which a measurement result could be consequently coordinatized. Therefore, it seems reasonable in this case, to associate the previously described functioning of localization systems with systems of local Boolean contexts of quantum measurement. We will show in the sequel, that this task can be accomplished by defining generalized topological covering systems on the base category of Boolean contexts, that remarkably constitute Grothendieck topologies. According to the interpretation put forward, we shall obtain a well-defined notion of localized events in a global quantum structure, varying over a multiplicity of Boolean covering domains determined by the topological categorical environment they share. The mathematical scheme for the implementation of our model will be based on categorical and sheaf-theoretic concepts and methods [4-8]. Contextual category-theoretic approaches to quantum structures have been also considered, from a different viewpoint in [9,10], and discussed in [11,12]. A remarkable conceptual affinity to the viewpoint of the present paper, although not based on categorical methods, can be found in references [13,14]. According to these authors, quantization of a proposition of classical physics is equivalent to interpreting it in a Boolean extension of a set theoretical universe, where B is a complete Boolean algebra of projection operators on a Hilbert space. Finally, for a general mathematical and philosophical discussion of sheaves, variable sets, and related structures, the interested reader should consult reference [15].

3 Classical Topological Sheaves

Let us consider the category of open sets $\mathcal{O}(X)$ in a topological measurement space, partially ordered by inclusion. If $\mathcal{O}(X)^{op}$ is the opposite category of $\mathcal{O}(X)$, and **Sets** denotes the category of sets, we define:

Definition: The **functor category of presheaves** on varying reference contexts U, identified by open sets of a topological measurement space X, denoted by $\mathbf{Sets}^{\mathcal{O}(X)^{op}}$, admits the following objects-arrows description: The objects of $\mathbf{Sets}^{\mathcal{O}(X)^{op}}$ are all functors $\mathbf{P} : \mathcal{O}(X)^{op} \longrightarrow \mathbf{Sets}$, whereas, the arrows are all natural transformations between such functors.

Definition: Each object $\mathbf{P}$ in $\mathbf{Sets}^{\mathcal{O}(X)^{op}}$ is a contravariant set-valued functor on $\mathcal{O}(X)$, called a **presheaf of sets** on $\mathcal{O}(X)$.

Remark: For each base open set U of $\mathcal{O}(X)$, $\mathbf{P}(\mathrm{U})$ is a set, and for each arrow $F : V \longrightarrow U$, $\mathbf{P}(F) : \mathbf{P}(U) \longrightarrow \mathbf{P}(\mathrm{V})$ is a set-function. If $\mathbf{P}$ is a presheaf on $\mathcal{O}(X)$ and $p \in \mathbf{P}(\mathrm{U})$, the value $\mathbf{P}(F)(p)$ for an arrow $F : V \longrightarrow U$ in $\mathcal{O}(X)$ is called the restriction of p along F and is denoted by $\mathbf{P}(F)(p) := p \cdot F$.

Remark: A presheaf $\mathbf{P}$ of $\mathbf{Sets}^{\mathcal{O}(X)^{op}}$ may be understood as a right action of $\mathcal{O}(X)$ on a set. This set is partitioned into sorts parameterized by the objects of $\mathcal{O}(X)$, and has the following property: If $F : V \longrightarrow U$ is an inclusion arrow in $\mathcal{O}(X)$ and p is an element of $\mathbf{P}$ of sort U, then $p \cdot F$ is specified as an element of $\mathbf{P}$ of sort V. Such an action $\mathbf{P}$ is referred as a $\mathcal{O}(X)$-variable set.

Definition: A **natural transformation** τ from $\mathbf{P}$ to $\mathbf{Q}$ is a mapping assigning to each open locus V in $\mathcal{O}(X)$ a morphism τ_V from $\mathbf{P}(V)$ to $\mathbf{Q}(V)$ in $\mathbf{Sets}$, such that for every arrow $F : V \to U$ in $\mathcal{O}(X)$ the following diagram in $\mathbf{Sets}$ commutes;

$$\begin{tikzcd}
\mathbf{P}(U) \arrow[r, "\tau_U"] \arrow[d, "\mathbf{P}(F)"'] & \mathbf{Q}(U) \arrow[d, "\mathbf{Q}(F)"] \\
\mathbf{P}(V) \arrow[r, "\tau_V"] & \mathbf{Q}(V)
\end{tikzcd}$$

That is, for every arrow $F : V \to U$ in $\mathcal{O}(X)$ we have:

$$\mathbf{Q}(F) \circ \tau_U = \tau_V \circ \mathbf{P}(F)$$

Definition: The **category of elements of a presheaf P**, denoted by $\int(\mathbf{P}, \mathcal{O}(X))$, admits the following objects-arrows description: The objects of $\int(\mathbf{P}, \mathcal{O}(X))$ are all pairs (U, p), with U in $\mathcal{O}(X)$ and $p \in \mathbf{P}(\mathrm{U})$. The arrows of $\int(\mathbf{P}, \mathcal{O}(X))$, that is, $(\acute{U}, \acute{p}) \to (U, p)$, are those morphisms $Z : \acute{U} \to U$ in $\mathcal{O}(X)$, such that $\acute{p} = \mathbf{P}(Z)(p) := p \cdot Z$.

Remark: Notice that the arrows in $\int(\mathbf{P}, \mathcal{O}(X))$ are those morphisms $Z : \acute{U} \to U$ in the base category $\mathcal{O}(X)$, that pull a chosen element $p \in \mathbf{P}(\mathrm{U})$ back into $\acute{p} \in \mathbf{P}(\acute{U})$.

Definition: The category of elements $\int(\mathbf{P}, \mathcal{O}(X))$ of a presheaf $\mathbf{P}$, together with, the projection functor $\int_{\mathbf{P}} : \int(\mathbf{P}, \mathcal{O}(X)) \to \mathcal{O}(X)$ is called the **split discrete fibration induced by P**, where $\mathcal{O}(X)$ is the base category of the fibration.

Remark: We note that the fibers are categories in which the only arrows are identity arrows. If U is a open reference locus of $\mathcal{O}(X)$, the inverse image under $\int_{\mathbf{P}}$ of U is simply the set $\mathbf{P}(\mathrm{U})$, although its elements are written as pairs so as to form a disjoint union. The construction of the fibration induced by $\mathbf{P}$, is an instance of the general Grothendieck construction [8].

Remark: The split discrete fibration induced by $\mathbf{P}$, where $\mathcal{O}(X)$ is the base category of the fibration, provides a well-defined notion of a uniform homologous fibered structure in the following sense: Firstly, by the arrows specification defined in the category of elements of $\mathbf{P}$, any element p, determined over the reference locus U, is homologously related with any other element $\acute{p}$ over the reference locus $\acute{U}$, and so on, by variation over all the reference loci of the base category. Secondly, all the elements p of $\mathbf{P}$, of the same sort U, viz. determined over the same reference locus U, are uniformly equivalent to each other, since all the arrows in $\int(\mathbf{P}, \mathcal{O}(X))$ are induced by lifting arrows from the base $\mathcal{O}(X)$.

From a physical viewpoint, the purpose of introducing the notion of a presheaf $\mathbf{P}$ on $\mathcal{O}(X)$, in the environment of the functor category $\mathbf{Sets}^{\mathcal{O}(X)^{op}}$, is the following: We identify an element of $\mathbf{P}$ of sort U, that is $p \in \mathbf{P}(\mathrm{U})$, with an event observed by means of a physical procedure over the reference locus U, being an open set of a topological measurement space X. This identification forces the interrelations of observed events, over all reference loci of the base category $\mathcal{O}(X)$, to fulfill the requirements of a uniform and homologous fibered structure, explained in detail previously. The next crucial step of the construction, aims to the satisfaction of the following physical requirement: Since the operational specification of measurement environments assumed their existence locally, the information gathered about local events in different measurement situations should be collated together by appropriate means. Mathematically, this requirement is implemented by the methodology of sheafification or localization of the presheaf $\mathbf{P}$. In our context of enquiry, sheafification represents the process of conversion of the category of element-events of the presheaf $\mathbf{P}$ into a category of continuous real-valued functions, that is local observables, identified with the local sections of the corresponding sheaf.

Definition: A **sheaf** is an arbitrary presheaf $\mathbf{P}$ that satisfies the following condition: If $U = \bigcup_a U_a$, U_a in $\mathcal{O}(X)$, and elements $p_a \in \mathbf{P}(U_a)$, $a \in I$:index set, are such that for arbitrary a, $b \in I$, it holds:

$$p_a \mid U_{ab} = p_b \mid U_{ab}$$

where, $U_{ab} := U_a \bigcap U_b$, and the symbol $\mid$ denotes the operation of restriction on the corresponding open domain, then there exists a unique element $p \in \mathbf{P}(U)$, such that $p \mid U_a = p_a$ for each a in I. Then an element of $\mathbf{P}(U)$ is called a section of the sheaf $\mathbf{P}$ over the open domain U. The sheaf condition means that sections can be glued together over the reference loci of the base category $\mathcal{O}(X)$.

Proposition: If $\mathbf{A}$ is the contravariant functor that assigns to each open locus $U \subset X$, the set of all real-valued continuous functions on U, then $\mathbf{A}$ is actually a sheaf.

Proof: First of all, it is instructive to clarify that the specification of a topology on a measurement space X is solely used for the definition of the continuous functions on X; in the present case the continuous functions from any open locus U in X to the real line $\mathcal{R}$. We notice that the continuity of each function $f : U \to \mathcal{R}$ can be determined locally. This property means that continuity respects the operation of restriction to open sets, and moreover that, continuous functions can be collated in a unique manner, as it is required for the satisfaction of the sheaf condition. More concretely;

If $f : U \to \mathcal{R}$ is a continuous function and $V \subset U$ is an open set in the topology, then the function f restricted to V is also continuous. The operation of restriction $f \mapsto f \mid V$, corresponds to a morphism of sets $\mathbf{A}(U) \longrightarrow \mathbf{A}(V)$. Moreover, if $W \subset V \subset U$ stand for three nested open sets in the topology, partially ordered by inclusion, the operation of restriction is transitive. Thus, the assignments;

$$U \mapsto \mathbf{A}(U)$$

$$\{V \hookrightarrow U\} \mapsto \{\mathbf{A}(U) \longrightarrow \mathbf{A}(V) \quad by \quad f \mapsto f \mid V\}$$

amount to the definition of a presheaf functor $\mathbf{A}$ on $\mathcal{O}(X)$, in the category $\mathbf{Sets}^{\mathcal{O}(X)^{op}}$. Furthermore, let us consider that U is covered by open sets U_a, such that $U = \bigcup_a U_a$, U_a in $\mathcal{O}(X)$, and also that, the I-indexed family of functions $f_a : U_a \to \mathcal{R}$ consists of continuous functions for all a in I. Then, due to the local determination of continuity, there is at most one continuous real-valued function $f : U \to \mathcal{R}$, with restrictions $f \mid U_a := f_a$ for all a in the index set I. Nevertheless, such a continuous function $f : U \to \mathcal{R}$ exists, if and only if, the f_a can be collated together on all the overlapping domains $U_a \bigcap U_b := U_{ab}$, such that:

$$f_a \mid U_{ab} = f_b \mid U_{ab}$$

Consequently, the presheaf of sets $\mathbf{A}$, viz. the presheaf of continuous real-valued functions on $\mathcal{O}(X)$, satisfies the sheaf condition.

Remark: Because of the fact that the presheaf $\mathbf{A}$, is actually a sheaf, we are allowed to make a further identification with a physical content: We identify an element of $\mathbf{A}$ of sort U, that is a local section of $\mathbf{A}$, with an event observed by means of a continuous physical procedure over the reference locus U. Equivalently, we represent an event observed over U, by means of a real-valued continuous function $f : U \to \mathcal{R}$, that is a local observable, such that the operations of restriction and collation as above, are being satisfied. The sheaf-theoretic qualification of a uniform and homologous fibered structure of events makes the latter also coherent in terms of compatibility of the information content it carries, under the operations of restriction and collation.

Remark: Actually, $\mathbf{A}$ is a sheaf of algebras over the field of the reals $\mathcal{R}$, because it is obvious that each set of sort U, $\mathbf{A}(U)$, is an $\mathcal{R}$-algebra under pointwise sum, product, and scalar multiple; whereas the morphisms $\mathbf{A}(U) \longrightarrow \mathbf{A}(V)$ stand for $\mathcal{R}$-linear morphisms of rings. In this algebraic setting, the sheaf condition means that the following sequence of $\mathcal{R}$-algebras of local observables is left exact;

$$0 \to \mathbf{A}(U) \to \prod_a \mathbf{A}(U_a) \to \prod_{a,b} \mathbf{A}(U_{ab})$$

Remark: It is instructive to explain the construction of the inductive limit of $\mathcal{R}$-algebras $\mathbf{A}(U)$, denoted by $\mathbf{Colim}[\mathbf{A}(U)]$, as follows:

Let us consider that x is a point of the topological measurement space X. Moreover, let K be a set consisting of open subsets of X, containing x, such that the following condition holds: For any two open reference domains U, V, containing x, there exists an open set $W \in K$, contained in the intersection domain $U \bigcap V$. We may say that K constitutes a basis for the system of open reference domains around x. We form the disjoint union of all $\mathbf{A}(U)$, denoted by;

$$\mathbf{D}(x) := \coprod_{U \in K} \mathbf{A}(U)$$

Then we can define an equivalence relation in $\mathbf{D}(x)$, by requiring that $f \sim g$, for $f \in \mathbf{A}(U)$, $g \in \mathbf{A}(V)$, provided that, they have the same restriction to a smaller open set contained in K. Then we define;

$$\mathbf{Colim}_K[\mathbf{A}(U)] := \mathbf{D}(x)/\sim_K$$

Furthermore, if we denote, the inclusion mapping of V into U by;

$$i_{V,U} : V \hookrightarrow U$$

and also, the restriction morphism of sets from U to V by;

$$\varrho_{U,V} : \mathbf{A}(U) \longrightarrow \mathbf{A}(V)$$

we can introduce well-defined notions of addition and scalar multiplication on the set $\mathbf{Colim}_K[\mathbf{A}(U)]$, making it into an $\mathcal{R}$-module, or even an $\mathcal{R}$-algebra, as follows:

$$[f_U] + [g_V] := [\varrho_{U,W}(f_U) + \varrho_{V,W}(g_V)]$$

$$\mu[g_V] := [\mu g_V]$$

where, f_U and g_V are elements in $\mathbf{A}(U)$ and $\mathbf{A}(V)$, that is, real-valued continuous functions defined over the open domains U and V respectively, and $\mu \in \mathcal{R}$. Now, if we consider that K and Λ are two bases for the system of open sets domains around $x \in X$, we can show that there are canonical isomorphisms between $\mathbf{Colim}_K[\mathbf{A}(U)]$ and $\mathbf{Colim}_\Lambda[\mathbf{A}(U)]$. In particular, we may take all the open subsets of X containing x: Indeed, we consider first the case when K is arbitrary and Λ is the set of all open subsets containing x. Then $\Lambda \supset K$ induces a morphism;

$$\mathbf{Colim}_K[\mathbf{A}(U)] \rightarrow \mathbf{Colim}_\Lambda[\mathbf{A}(U)]$$

which is an isomorphism, since whenever V is an open subset containing x, there exists an open subset U in K contained in V. Since we can repeat that procedure for all bases of the system of open sets domains around $x \in X$, the initial claim follows immediately.

Definition: The **stalk** of $\mathbf{A}$ at the point $x \in X$, denoted by $\mathbf{A}_x$, is defined as the inductive limit of $\mathcal{R}$-algebras $\mathbf{A}(U)$;

$$\mathbf{Colim}_K[\mathbf{A}(U)] := \coprod_{U \in K} \mathbf{A}(U) / \sim_K$$

where, K is a basis for the system of open reference domains around x, and $\sim_K$ denotes the equivalence relation of restriction within an open set in K.

Note that the definition is independent of the chosen basis K.

Remark: For an open reference domain W containing the point x, we obtain an $\mathcal{R}$-linear morphism of $\mathbf{A}(W)$ into the stalk at the point x;

$$i_{W,x} : \mathbf{A}(W) \rightarrow \mathbf{A}_x$$

For an element $f \in \mathbf{A}(W)$ its image $i_{W,x}(f) := f_x$ is called the **germ** of f at the point x.

Remark: The fibered structure that corresponds to the sheaf of real-valued continuous functions on a topological measurement space X is a topological **bundle** defined by the continuous mapping $\varphi : A \rightarrow X$, where;

$$A = \coprod_{x \in X} \mathbf{A}_x$$

$$\varphi^{-1}(x) = \mathbf{A}_x = \mathbf{Colim}_{\{x \in U\}}[\mathbf{A}(U)]$$

The mapping φ is locally a homeomorphism of topological spaces. The topology in A is defined as follows: for each $f \in \mathbf{A}(U)$, the set $\{f_x, x \in U\}$ is open, and moreover, an arbitrary open set is a union of sets of this form.

Remark: In the physical state of affairs, we remind that we have identified an element of $\mathbf{A}$ of sort U, that is a local section of $\mathbf{A}$, with an event f observed by means of a continuous physical procedure over the reference locus U. Then the equivalence relation, used in the definition of the stalk $\mathbf{A}_x$ at the point $x \in X$ is interpreted as follows: Two events $f \in \mathbf{A}(U)$, $g \in \mathbf{A}(V)$, induce the same contextual information at x in X, provided that, they have the same restriction to a smaller open locus contained in the basis K. Then, the stalk $\mathbf{A}_x$ is the set containing all contextual information at x, that is the set of all equivalence classes. Moreover, the image in the stalk $\mathbf{A}_x$ of an event $f \in \mathbf{A}(U)$, that is the equivalence class of this event f, is precisely the germ of f at the point x.

Remark: The sheaf of real-valued continuous functions on a topological measurement space X is an object in the functor category of sheaves $\mathbf{Sh}(X)$ on varying reference loci U, being open sets of a topological measurement space X, partially ordered by inclusion. The morphisms in $\mathbf{Sh}(X)$ are all natural transformations between sheaves. It is instructive to notice that a sheaf makes sense only if the base category of reference loci is specified, which is equivalent in our context to the determination of a topology on an underlying measurement space X. Once this is accomplished, a sheaf can be thought of as measuring the space X. The functor category of sheaves $\mathbf{Sh}(X)$, provides an exemplary case of a construct known as topos. A topos can be conceived as a local mathematical framework corresponding to a generalized model of set theory or as a generalized space [6-8].

Remark: The particular significance of the sheaf of real-valued continuous functions on X, that we have used as a uniform homologous and coherent fibered structure of local observables for modeling an "events-continuum", according to the physical requirements posed in Section 2, is due to the following isomorphism [6]: The sheaf of continuous real-valued functions on X, is isomorphic to the object of Dedekind real numbers in the topos of sheaves $\mathbf{Sh}(X)$. The aforementioned isomorphism validates the physical intuition of considering a local observable as a continuously variable real number over its locus of definition.

4 Generalized Topological Covering Systems

Until now, it has become evident that the sheaf-theoretic fibered model of a globally partially ordered structure of physical events is not based on an underlying structure of points. On the contrary, the fundamental entities are the base reference contexts, identified previously with the open sets of a topological measurement space X. The basic intuition behind their functioning is related with the expectation that the reference domains of the base category, in that fibered construct, serve the purpose of generalizing the notion of localization of events. In this sense, the unique measure of localization of the set-theoretical model, being a point, is substituted by a variety of localization measures, instantiated by the open sets of the base category ordered by inclusion. In the latter context, a point-localization measure, is identified precisely with the ultrafilter of open set domains containing the point. This identification permits the conception of other filters, being formed by the base reference contexts, as generalized measures of localization. The meaningful association of filters with generalized localization measures in a global structure of physical events has to meet certain requirements, that remarkably have a sound physical basis, as it will become clear in the sequel, and leads to the notion of generalized topological covering systems. It is significant, that once the notion of a topological covering system has been crystallized, the sheaf-theoretic fibered model of a global events structure can be defined explicitly in these descriptive terms.

Generalized topological covering systems are being effectuated by means of systems of covering devices on the base category of reference contexts, called in categorical terminology covering sieves. Firstly, we shall explain the general notion of sieves, and afterwards, we shall specialize our exposition to the notion of covering sieves. Our presentation applies to any small category $\mathcal{B}$, consisting of base reference categorical objects B, with structure preserving morphisms between them, as arrows. Of course, in the classical topological case of the previous section, $\mathcal{B}$ is tautosemous with $\mathcal{O}(X)$ and the reference contexts B are tautosemous with the open sets U of X, partially ordered by inclusion. As a preamble for the discussion of quantum events structures, it is instructive to point that $\mathcal{B}$ can be thought as a base category of Boolean contexts of quantum measurement.

Definition: A B-**sieve** with respect to a reference context B in $\mathcal{B}$, is a family S of $\mathcal{B}$-morphisms with codomain B, such that if $C \to B$ belongs to S and $D \to C$ is any $\mathcal{B}$-morphism, then the composite $D \to C \to B$ belongs to S.

Remark: We may think of a B-sieve as a right B-ideal. We notice that, in the case of $\mathcal{O}(X)$, since $\mathcal{O}(X)$-morphisms are inclusions of open loci, a right U-ideal is tautosemous with a downwards closed U-subset.

Proposition: A B-sieve is equivalent to a subfunctor $\mathbf{S} \hookrightarrow \mathbf{y}[B]$ in $\mathbf{Sets}^{\mathcal{B}^{op}}$, where $\mathbf{y}[B] := Hom_{\mathcal{B}}(-, B)$, denotes the contravariant representable functor of the reference locus B in $\mathcal{B}$.

Proof: Given a B-sieve S, we define:

$$\mathbf{S}(C) = \{g/g : C \to B, g \in S\} \subseteq \mathbf{y}[B](C)$$

This definition yields a functor $\mathbf{S}$ in $\mathbf{Sets}^{\mathcal{B}^{op}}$, which is obviously a subfunctor of $\mathbf{y}[B]$. Conversely, given a subfunctor $\mathbf{S} \hookrightarrow \mathbf{y}[B]$ in $\mathbf{Sets}^{\mathcal{B}^{op}}$, the set:

$$S = \{g/g : C \to B, g \in \mathbf{S}(C)\}$$

for some reference loci C in $\mathcal{B}$, is a B-sieve. Thus, epigramatically, we state:

$\langle$ B-sieve: S $\rangle$=$\langle$ Subfunctor of $\mathbf{y}[B]$: $\mathbf{S} \hookrightarrow \mathbf{y}[B]$ $\rangle$

Remark: We notice that if S is a B-sieve and $h : C \to B$ is any arrow to the locus B, then:

$$h^*(S) = \{f/cod(f) = C, (h \circ f) \in S\}$$

is a C-sieve, called the pullback of S along h, where, $cod(f)$ denotes the codomain of f. Consequently, we may define a presheaf functor $\mathbf{\Omega}$ in $\mathbf{Sets}^{\mathcal{B}^{op}}$, such that its action on locoi B in $\mathcal{B}$, is given by:

$$\mathbf{\Omega}(B) = \{S/S : B - sieve\}$$

and on arrows $h : C \to B$, by $h^*(-) : \mathbf{\Omega}(B) \to \mathbf{\Omega}(C)$, given by:

$$h^*(S) = \{f/cod(f) = C, (h \circ f) \in S\}$$

We notice that for a context B in $\mathcal{B}$, the set of all arrows into B, is a B-sieve, called the maximal sieve on B, and denoted by, $t(B) := t_B$.

At a next stage of development, the key conceptual issue we have to settle for our purposes, is the following: How is it possible to restrict $\mathbf{\Omega}(B)$, that is the set of B-sieves for each reference context B in $\mathcal{B}$, such that each B-sieve of the restricted set can acquire the interpretation of a covering B-sieve, with respect to a generalized topological covering system. Equivalently stated, we wish to impose the satisfaction of appropriate conditions on the set of B-sieves for each context B in $\mathcal{B}$, such that, the subset of B-sieves obtained, denoted by $\mathbf{\Omega}_\chi(B)$, implement the partial order relation between events. In this sense, the B-sieves of $\mathbf{\Omega}_\chi(B)$, for each locus B in $\mathcal{B}$, to be thought as generalized topological covering B-sieves, can be legitimately used for the definition of a localization scheme in a global partially ordered structure of events. The appropriate physical requirements for our purposes are the following:

[1]. According to the principle of uniformity, the partial order relation in a structure of events should be implemented by an appropriate relational property of reference contexts B in the base category $\mathcal{B}$. In this sense, an arrow $C \to B$, such that C, B are contexts in $\mathcal{B}$, is interpreted as a figure of B, and thus, B is interpreted as an extension of C in $\mathcal{B}$. It is a natural requirement that the set of all figures of B should belong in $\mathbf{\Omega}_\chi(B)$ for each context B in $\mathcal{B}$.

[2]. The covering sieves should be stable under pullback operations, and most importantly, the stability conditions should be expressed functorially. This requirement means, in particular, that the intersection of covering sieves should also be a covering sieve, for each reference context B, in the base category $\mathcal{B}$.

[3]. Finally, it would be desirable to impose: (i) a transitivity requirement

on the specification of the covering sieves, such that, intuitively stated, covering sieves of figures of a context in covering sieves of this context, should also be covering sieves of the context themselves, and (ii) a requirement of common refinement of covering sieves.

If we take into account the above requirements we can define a generalized topological covering system in the environment of $\mathcal{B}$ as follows:

Definition: A **generalized topological covering system** on $\mathcal{B}$ is an operation $\mathbf{J}$, which assigns to each reference context B in $\mathcal{B}$, a collection $\mathbf{J}(B)$ of B-sieves, called covering B-sieves, such that the following three conditions are satisfied:

[1]. For every reference context B in $\mathcal{B}$, the maximal B-sieve $\{g : cod(g) = B\}$ belongs to $\mathbf{J}(B)$ (maximality condition).

[2]. If S belongs to $\mathbf{J}(B)$ and $h : C \to B$ is a figure of B, then $h^*(S) = \{f : C \to B, (h \circ f) \in S\}$ belongs to $\mathbf{J}(C)$ (stability condition).

[3]. If S belongs to $\mathbf{J}(B)$, and if for each figure $h : C_h \to B$ in S, there is a sieve R_h belonging to $\mathbf{J}(C_h)$, then the set of all composites $h \circ g$, with $h \in S$, and $g \in R_h$, belongs to $\mathbf{J}(B)$ (transitivity condition).

Remark: As a consequence of the definition above, we can easily check that any two B-covering sieves have a common refinement, that is: if S, R

belong to $\mathbf{J}(B)$, then $S \cap R$ belongs to $\mathbf{J}(B)$.

Remark: A generalized topological covering system on $\mathcal{B}$ satisfying the physical requirements, posed previously, is tautosemous, in categorical terminology, with the notion of a Grothendieck topology on $\mathcal{B}$.

Remark: As a first application we may consider the partially ordered set of open subsets of a topological measurement space X, viewed as the base category of open reference domains, $\mathcal{O}(X)$. Then we specify that S is a covering U-sieve if and only if U is contained in the union of open sets in S. The above specification fulfills the requirements of covering sieves posed above, and consequently, defines a topological covering system on $\mathcal{O}(X)$.

Remark: Obviously a topological covering system $\mathbf{J}$ exists as a presheaf functor $\mathbf{\Omega}_\chi$ in $\mathbf{Sets}^{\mathcal{B}^{op}}$, such that: by acting on contexts B in $\mathcal{B}$, $\mathbf{J}$ gives the set of all covering B-sieves, denoted by $\mathbf{\Omega}_\chi(B)$, whereas by acting on figures $h : C \to B$, it gives a morphism $h^*(-) : \mathbf{\Omega}_\chi(B) \to \mathbf{\Omega}_\chi(C)$, expressed as: $h^*(S) = \{f/cod(f) = C, (h \circ f) \in S\}$, for $S \in \mathbf{\Omega}_\chi(B)$.

Definition: A small category $\mathcal{B}$ together with a Grothendieck topology $\mathbf{J}$, is called a site, denoted by, $(\mathcal{B}, \mathbf{J})$.

Definition: A sheaf on a site $(\mathcal{B}, \mathbf{J})$ is a contravariant functor $\mathbf{P} : \mathcal{B}^{op} \to$ **Sets**, satisfying an equalizer condition, expressed, in terms of covering B-sieves S, as in the following diagram in **Sets**:

$$\prod_{f \circ g \in S} \mathbf{P}(dom(g)) \;\leftleftarrows\; \prod_{f \in S} \mathbf{P}(dom(f)) \xleftarrow{\quad e \quad} \mathbf{P}(B)$$

Remark: If the above diagram is an equalizer for a particular covering sieve S, we obtain that $\mathbf{P}$ satisfies the sheaf condition with respect to the covering sieve S. The theoretical advantage of the above relies on the fact that it provides a description of sheaves entirely in terms of objects of the category of presheaves.

From a physical perspective, the consideration of covering sieves as generalized measures of localization of events in a global partially ordered structure of events, together with the requirements posed for the formation of topological covering systems, elucidates the sheaf-theoretic fibered model of local real-valued observables established previously. In the following sections, we will apply the machinery of generalized topological covering systems for the analysis of global structures of quantum events.

5 Quantum Events Algebras and Functors of Boolean Frames

Definition: A **quantum events structure** is a small cocomplete category, denoted by $\mathcal{L}$, which is called the category of quantum events algebras.

The objects of $\mathcal{L}$, denoted by L, are quantum events algebras, that is orthomodular orthoposets of events, defined as follows:

Definition: A **quantum events algebra** L in $\mathcal{L}$, is a partially ordered set of quantum events, endowed with a maximal element 1, and with an operation of orthocomplementation $[-]^* : L \longrightarrow L$, which satisfy, for all $l \in L$, the following conditions: [a] $l \leq 1$, [b] $l^{**} = l$, [c] $l \vee l^* = 1$, [d] $l \leq \acute{l} \Rightarrow \acute{l}^* \leq l^*$, [e] $l \bot \acute{l} \Rightarrow l \vee \acute{l} \in L$, [f] for $l, \acute{l} \in L, l \leq \acute{l}$ implies that l and $\acute{l}$ are compatible, where $0 := 1^*$, $l \bot \acute{l} := l \leq \acute{l}^*$, and the operations of meet $\wedge$ and join $\vee$ are defined as usually.

Remark: We recall that $l, \acute{l} \in L$ are compatible if the sublattice generated by $\{l, l^*, \acute{l}, \acute{l}^*\}$ is a Boolean algebra, namely if it is a Boolean sublattice.

The arrows of $\mathcal{L}$ are quantum algebraic homomorphisms, defined as follows:

Definition: A **quantum algebraic homomorphism** in $\mathcal{L}$ is a morphism $K \xrightarrow{H} L$, which satisfies, for all $k \in K$, the following conditions: [a] $H(1) = 1$, [b] $H(k^*) = [H(k)]^*$, [c] $k \leq \acute{k} \Rightarrow H(k) \leq H(\acute{k})$, [d] $k \perp \acute{k} \Rightarrow H(k \vee \acute{k}) \leq H(k) \vee H(\acute{k})$.

Definition: A **Boolean events structure** is a small cocomplete category, denoted by $\mathcal{B}$, which is called the category of Boolean events algebras. Its objects are Boolean algebras of events and its arrows are the corresponding Boolean algebraic morphisms.

The conceptual basis of the attempt to define a generalized topological covering system, in terms of Boolean reference contexts, for a global quantum events structure $\mathcal{L}$, is the expectation that it is possible to analyze a quantum events algebra L, by means of structure preserving maps $B \longrightarrow L$, with local Boolean algebras B in $\mathcal{B}$, as their domains. Put differently, we expect to coordinatize the events information contained in a quantum events algebra L in $\mathcal{L}$, by means of families of local Boolean reference frames. The latter are understood as morphisms $B \longrightarrow L$, having as their domains, locally defined Boolean events algebras B in $\mathcal{B}$, corresponding to typical quantum measurement situations. Any single map, from a Boolean coordinates domain into a quantum events algebra, is not enough for a complete determination of the latter's information content, and hence, it contains only a limited amount of information about it. More concretely, it includes only the amount of information related to a Boolean reference context, and thus, it is inevitably constrained to represent the abstractions associated it. This problem may be tackled, only if, we employ many structure preserving maps from the coordinatizing local Boolean contexts to a quantum events algebra simultaneously, so as to cover it completely. Of course, it is desirable to consider the minimum number of such maps, which is specified by the requirement of distinguishability of the elements of the quantum events algebra. In turn, the information available about each map of the specified covering system by Boolean frames, may be used to determine the global quantum events algebra itself. In order to accomplish this task, we consider that the category of Boolean contexts $\mathcal{B}$ is a generating subcategory of $\mathcal{L}$, such that, the set of all arrows $w : B_i \to L$, I: index set, constitute an epimorphic family. Equivalently stated, the set of objects $\{B_i / i \in I\}$, in $\mathcal{B}$, where, I: index set, generate $\mathcal{L}$, in the sense that;

$$B_i \xrightarrow{w_i} L \underset{u}{\overset{v}{\rightrightarrows}} K$$

the identity $v \circ w_i = u \circ w_i$, for every arrow $w_i : B_i \to L$, and every B_i, implies that $v = u$.

Variation of Boolean frames over all contexts of the subcategory of $\mathcal{L}$, consisting of Boolean event algebras, produces the functor of Boolean frames of L, restricted to the subcategory of Boolean coordinatizing contexts, identified with $\mathcal{B}$. The functor of Boolean frames of a quantum events algebra L is made, then, an object in the category of presheaves $\mathbf{Sets}^{\mathcal{B}^{op}}$, representing L in the environment of the topos of presheaves over the category of Boolean contexts. This methodology will prove to be successful, if it could be possible to establish an isomorphic representation of L, in terms of the information being carried by its Boolean frames $B_i \to L$, associated with measurement situations, collated together by appropriate means. In more detail, we have the following:

Definition: The **representation functor** of a quantum events structure $\mathcal{L}$ into the category of presheaves of Boolean events algebras $\mathbf{Sets}^{\mathcal{B}^{op}}$, is given by:

$$\mathbf{\Upsilon} : \mathcal{L} \to \mathbf{Sets}^{\mathcal{B}^{op}}$$

Definition: The **functor of Boolean frames** of a quantum events algebra L in $\mathcal{L}$, is the image of the representation functor $\mathbf{\Upsilon}$, evaluated at L, into the category of presheaves of Boolean events algebras $\mathbf{Sets}^{\mathcal{B}^{op}}$, that is:

$$\mathbf{\Upsilon}(L) := \mathbf{\Upsilon}_L : \mathcal{B}^{op} \longrightarrow \mathbf{Sets}$$

Remark: The representation functor of $\mathcal{L}$, is completely determined by the action of the functor of Boolean frames, for each quantum event algebra L

in $\mathcal{L}$, on the objects and arrows of the category $\mathcal{B}$, specified as follows: Its action on an object B in $\mathcal{B}$ is given by

$$\mathbf{\Upsilon}(L)(B) := \mathbf{\Upsilon}_L(B) = Hom_{\mathcal{L}}(B, L)$$

whereas its action on a morphism $D \xrightarrow{x} B$ in $\mathcal{B}$, for $v : B \longrightarrow L$ is given by

$$\mathbf{\Upsilon}(L)(x) : Hom_{\mathcal{L}}(B, L) \longrightarrow Hom_{\mathcal{L}}(D, L)$$

$$\mathbf{\Upsilon}(L)(x)(v) = v \circ x$$

Remark: Notice that the functor of Boolean frames of a quantum events algebra L in $\mathcal{L}$, is a presheaf $\mathbf{\Upsilon}(L) := \mathbf{\Upsilon}_L : \mathcal{B}^{op} \longrightarrow \mathbf{Sets}$. Thus, we can legitimately consider the category of elements $\int(\mathbf{\Upsilon}_L, \mathcal{B})$, together with, the projection functor $\int_{\mathbf{\Upsilon}_L} : \int(\mathbf{\Upsilon}_L, \mathcal{B}) \rightarrow \mathcal{B}$, viz. the split discrete fibration induced by the funcor of Boolean frames of L, where $\mathcal{B}$ is the base category of the fibration. Hence, the functor of Boolean frames of a quantum events algebra, induces a uniform and homologous fibered representation of quantum events in terms of Boolean reference frames.

At this stage of development, two further important issues have to be properly settled:

The first issue is concerned with the physical requirement of making the established fibered representation of quantum events also coherent. Put differently, this issue poses the problem of defining an appropriate topological covering system $\mathbf{J}$ on the base category $\mathcal{B}$, such that:

[i]. The Boolean reference frames acquire the semantics of local frames with respect to that Grothendieck topology $\mathbf{J}$ on $\mathcal{B}$, and

[ii]. The functor of Boolean frames for a quantum events algebra L in $\mathcal{L}$ becomes a sheaf on the site $(\mathcal{B}, \mathbf{J})$ for that $\mathbf{J}$.

The second issue, is concerned with the physical requirement of preservation of the whole information content of a quantum events algebra, under the action of the category of Boolean contexts endowed with a topological covering system $\mathbf{J}$, as above. Equivalently stated, this issue poses the problem of constructing a representation of quantum events in terms of equivalence classes of Boolean decoding coefficients, in local Boolean reference frames within $\mathbf{J}$, such that:

[i]. A quantum events algebra L becomes isomorphic with the colimit taken in the category of elements of the sheaf functor of Boolean frames of L, and consequently;

[ii]. The whole information encoded in a quantum events algebra can be faithfully captured, and completely reconstructed, by the information structure of these equivalence classes of contextual, locally decoding, Boolean coefficients.

These two issues will be dealt with in the forthcoming sections correspondingly.

6 Topological Covering Systems on Boolean Contexts and Sheaves

Regarding the first issue posed previously, we will show that the functor $\mathbf{\Upsilon} : \mathcal{L} \to \mathbf{Sets}^{\mathcal{B}^{op}}$, transforms quantum events algebras L in $\mathcal{L}$ not just into presheaves $\mathbf{\Upsilon}_L$ in $\mathbf{Sets}^{\mathcal{B}^{op}}$, but into sheaves for a suitable Grothendieck topology $\mathbf{J}$ on the category of Boolean reference contexts $\mathcal{B}$, such that the functor of Boolean frames for a quantum events algebra L in $\mathcal{L}$ is a sheaf on the site $(\mathcal{B}, \mathbf{J})$ for a suitable $\mathbf{J}$. For this purpose we define:

Definition: A B-sieve S on a Boolean reference context B in $\mathcal{B}$ is called a covering sieve of B, if all the arrows $s : C \to B$ belonging to the sieve S, taken together, form an epimorphic family in $\mathcal{L}$. This requirement may be, equivalently, expressed in terms of a map

$$G_S : \coprod_{(s:C \to B) \in S} C \to B$$

being an epi in $\mathcal{L}$.

Proposition: The specification of covering sieves on Boolean contexts B in $\mathcal{B}$, in terms of epimorphic families of arrows in $\mathcal{L}$, does indeed, define a generalized topological covering system $\mathbf{J}$ on $\mathcal{B}$.

Proof: First of all, we notice that the maximal sieve, on each Boolean context B, includes the identity $B \to B$, and thus, it is a covering sieve. Next, the transitivity property of the depicted covering sieves is obvious. It remains to demonstrate that the covering sieves remain stable under pullback. For this purpose we consider the pullback of such a covering sieve S on B along any arrow $h : B' \to B$ in $\mathcal{B}$, according to the following diagram;

$$\begin{array}{ccc} \coprod_{s \in S} C \times_B \acute{B} & \longrightarrow & \acute{B} \\ \downarrow & & \downarrow h \\ \coprod_{s \in S} C & \xrightarrow{G} & B \end{array}$$

The Boolean algebras B in $\mathcal{B}$, generate the category of quantum event algebras $\mathcal{L}$, hence, there exists for each arrow $s : D \to B$ in S, an epimorphic family of arrows $\coprod_{s \in S} [B]^s \to D \times_B \acute{B}$, or equivalently, $\{[B]^s{}_j \to D \times_B \acute{B}\}_j$, with each domain $[B]^s$ a Boolean algebra. Consequently, the collection of all the composites:

$$[B]^s{}_j \to D \times_B \acute{B} \to \acute{B}$$

for all $s : D \to B$ in S, and all indices j together form an epimorphic family in $\mathcal{L}$, that is contained in the sieve $h^*(S)$, being the pullback of S along $h : \acute{B} \to B$. Therefore, the sieve $h^*(S)$ is a covering sieve. Thus, the operation $\mathbf{J}$, which assigns to each Boolean reference context B in $\mathcal{B}$, a collection $\mathbf{J}(B)$ of covering B-sieves, being epimorphic families of arrows in $\mathcal{L}$, constitutes a generalized topological covering system $\mathbf{J}$ on $\mathcal{B}$.

Proposition: [i]. The presheaf functor of Boolean frames $\mathbf{\Upsilon}_L = Hom_{\mathcal{L}}(-, L)$ in $\mathbf{Sets}^{\mathcal{B}^{op}}$, satisfies the sheaf-theoretic condition for a covering sieve S, being an epimorphic family of arrows in $\mathcal{L}$.

[ii]. The functor of Boolean frames $\mathbf{\Upsilon}_L$ is a sheaf for the Grothendieck topology $\mathbf{J}$, defined by covering sieves of epimorphic families on the category of Boolean reference contexts, as above.

Proof: [i]. We initially construct the representation of covering sieves within the category $\mathcal{B}$ of Boolean reference contexts B. Firstly, we observe that for an object C of $\mathcal{B}$, and for a covering sieve in the Grothendieck topology on $\mathcal{B}$, the map;

$$G_S : \coprod_{(s:C \to B) \in S} C \to B$$

where G_S is an epi in $\mathcal{L}$, can be, equivalently, represented as the coequalizer of its kernel pair, or else, as the pullback of G_S along itself, according to the diagram;

$$\begin{array}{ccc} \coprod_{\acute{s}}\acute{D}\times_C\coprod_s D & \longrightarrow & \coprod_s D \\ \downarrow & & \downarrow G_S \\ \coprod_{\acute{s}}\acute{D} & \xrightarrow{G_S} & C \end{array}$$

Furthermore, using the fact that pullbacks in $\mathcal{L}$ preserve coproducts, the epimorhic family associated with a covering sieve on C, admits the following coequalizer presentation;

$$\coprod_{\acute{s},s}\acute{D}\times_C D \quad \overset{q_1}{\underset{q_2}{\rightrightarrows}} \quad \coprod_s D \xrightarrow{G} C$$

Moreover, since the the category $\mathcal{B}$ is a generating subcategory of $\mathcal{L}$, for each pair of arrows $s : D \to C$ and $\acute{s} : \acute{D} \to C$, in the covering sieve S on the Boolean algebra C, there exists an epimorphic family $\{B \to \acute{D}\times_C D\}$, such that each domain B is a Boolean algebra in $\mathcal{B}$. Consequently, each element of the epimorphic family associated with a covering sieve S on a Boolean algebra C is represented by a commutative diagram in $\mathcal{B}$, of the following form;

$$\begin{array}{ccc} B & \xrightarrow{l} & D \\ k\downarrow & & \downarrow s \\ \acute{D} & \xrightarrow{\acute{s}} & C \end{array}$$

Next, we may compose the representation of epimorphic families by commutative squares in $\mathcal{B}$, obtained previously, with the coequalizer presentation of the same epimorphic families. The composition results in a new coequalizer diagram in $\mathcal{B}$, of the following form;

$$\coprod_B B \quad \overset{y_1}{\underset{y_2}{\rightrightarrows}} \quad \coprod_s D \xrightarrow{G} C$$

where, the first coproduct is indexed by all B in the commutative diagrams in $\mathcal{B}$, representing elements of epimorphic families.

Now, for each quantum events algebra L in $\mathcal{L}$, we consider the functor of Boolean frames $\mathbf{\Upsilon}_L = Hom_{\mathcal{L}}(-, L)$ in $\mathbf{Sets}^{\mathcal{B}^{op}}$. If we apply this representable functor to the latter coequalizer diagram, we obtain an equalizer diagram in **Sets**, as follows;

$$\prod_B Hom_{\mathcal{L}}(B, L) \quad \leftleftarrows \quad \prod_{s \in S} Hom_{\mathcal{L}}(D, L) \longleftarrow \quad Hom_{\mathcal{L}}(C, L)$$

where, the first product is indexed by all Boolean contexts B in the commutative diagrams in $\mathcal{B}$, representing elements of epimorphic families. The equalizer in **Sets**, thus obtained, proves explicitly that the functor of Boolean frames $\mathbf{\Upsilon}_L = Hom_{\mathcal{L}}(-, L)$ in $\mathbf{Sets}^{\mathcal{B}^{op}}$, satisfies the sheaf-theoretic condition for the covering sieve S.

[ii]. Clearly, the above equalizer condition holds for every covering sieve S, that belongs to the Grothendieck topology $\mathbf{J}$, defined by covering sieves

of epimorphic families on the category $\mathcal{B}$ of Boolean reference contexts. By rephrasing the above, we conclude that the functor of Boolean frames for a quantum events algebra L, that is $\mathbf{\Upsilon}_L$, is actually a sheaf, for the generalized topological covering system of epimorphic families, defined on the category of Boolean contexts.

Remark: The propositions proved above, settle completely the first issue posed previously, in the following sense: The Boolean reference frames B acquire the semantics of local frames with respect to the Grothendieck topology $\mathbf{J}$ on $\mathcal{B}$, defined in terms of covering sieves of epimorphic families of arrows in $\mathcal{L}$. Moreover, the functor of Boolean frames for a quantum events algebra L becomes a sheaf on the site $(\mathcal{B}, \mathbf{J})$ for that $\mathbf{J}$. Thus, the split discrete fibration $\int_{\mathbf{\Upsilon}_L} : \int(\mathbf{\Upsilon}_L, \mathcal{B}) \to \mathcal{B}$, induced by the sheaf of Boolean frames of L on the site $(\mathcal{B}, \mathbf{J})$, forces a uniform, homologous, and also, coherent fibered representation of quantum events in terms of local Boolean reference frames.

7 Boolean Equivalence Classes Representations of Quantum Events

Regarding the second issue posed previously, we will show explicitly that a quantum events algebra L can be represented isomorphically by means of a colimit taken in the category of elements of the sheaf of Boolean reference frames of L. For this purpose, we define:

Definition: A sieve on a quantum events algebra L defines a covering sieve by Boolean frames, such that, the domains of the quantum algebraic morphisms belonging to the sieve are Boolean contexts in the generating subcategory $\mathcal{B}$, if all these morphisms in the sieve collectively define an epimorphism;

$$T : \coprod_{(E \in [\mathcal{B}]_0, \psi_E : E \to L)} E \to L$$

where, $[\mathcal{B}]_0$ denotes the set of Boolean reference contexts of the base category $\mathcal{B}$.

Remark: From the physical point of view, covering sieves of quantum events algebras by Boolean frames, are called, equivalently, **Boolean localization systems** of quantum events algebras. These localization systems filter the information of a global partially ordered quantum events algebra through Boolean contexts, associated with measurement situations of observables.

Proposition: A covering sieve of a quantum events algebra L by Boolean frames, viz. a Boolean localization system of L, induces an isomorphism:

$$L \cong \mathbf{Colim}\{\int(\mathbf{\Upsilon}_L, \mathcal{B}) \longrightarrow \mathcal{B}\}$$

where, $\mathbf{\Upsilon}_L$ is the sheaf of Boolean frames of L on the site $(\mathcal{B}, \mathbf{J})$, for $\mathbf{J}$ the Grothendieck topology of epimorphic families, and $\int(\mathbf{\Upsilon}_L, \mathcal{B})$ is the associated fibered category of Boolean frames of L.

Proof: If we make use of same arguments as in the proof of the second proposition of the previous Section, we obtain that the epimorphism

$$T : \coprod_{(E \in [\mathcal{B}]_0, \psi_E : E \to L)} E \to L$$

that is, a covering sieve of a quantum events algebra L by Boolean frames, can be represented in the form of a coequalizer diagram in $\mathcal{L}$ as follows;

$$\coprod_{\nu} B \quad \overset{y_1}{\underset{y_2}{\rightrightarrows}} \quad \coprod_{(E \in [\mathcal{B}]_0, \psi_E : E \to L)} E \xrightarrow{\quad T \quad} L$$

where, the first coproduct is indexed by all ν, representing commutative diagrams in $\mathcal{L}$, of the following form;

$$\begin{tikzcd}
B \arrow[r, "l"] \arrow[d, "k"] & E \arrow[d, "\psi_E"] \\
\acute{E} \arrow[r, "\psi_{\acute{E}}"] & L
\end{tikzcd}$$

where B, E, $\acute{E}$ are Boolean contexts in the generating subcategory $\mathcal{B}$ of $\mathcal{L}$.

Now, we may consider a covering sieve of quantum event algebra L by Boolean frames, consisting of quantum algebraic morphisms $T_{(E, \psi_E)}$, such that, taken together constitute collectively an epimorphic family in $\mathcal{L}$. We observe that the condition;

$$T \circ y_1 = T \circ y_2$$

is equivalent to the condition;

$$T_{(E,\psi_E)} \circ l = T_{(\acute{E},\psi_{\acute{E}})} \circ k$$

for each commutative square ν of the form above.

Furthermore, the coequalizer condition $T \circ y_1 = T \circ y_2$, implies that for every Boolean contexts morphism $u : \acute{E} \to E$, with B, E objects of $\mathcal{B}$ and $\psi_E : E \to L$, the diagram of the form ν below;

$$\begin{array}{ccc} \acute{E} & \xrightarrow{u} & E \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle \psi_E} \\ \acute{E} & \xrightarrow{\psi_E \circ u} & L \end{array}$$

commutes and provides the condition

$$T_{(E,\psi_E)} \circ u = T_{(\acute{E},\psi_E \circ u)}$$

Next, we define the following set:

$$\mathbf{G}(\boldsymbol{\Upsilon}_L) = \{(\psi_E, q)/(\psi_E : E \longrightarrow L) \in [\int(\boldsymbol{\Upsilon}_{\mathbf{L}}, \mathcal{B})]_0, q \in E\}$$

We notice that, if there exists a Boolean contexts morphism, $u : \acute{E} \to E$, or equivalently, $u : \psi_{\acute{E}} \to \psi_E$, such that: $u(\acute{q}) = q$ and $\psi_E \circ u = \psi_{\acute{E}}$ according to the diagram above, where, $[\mathbf{\Upsilon}_L(u)](\psi_E) := \psi_E \circ u$, then, we may define a transitive and reflexive relation $\Re$ on the set $\mathbf{G}(\mathbf{\Upsilon}_L)$. Of course, the inverse also holds true. Then, we notice that;

$$(\psi_E \circ u, q)\Re(\psi_E, u(\acute{q}))$$

for any $u : \acute{E} \to E$ in the category $\mathcal{B}$. The next step is to make this relation also symmetric by postulating that for ζ, η in $\mathbf{G}(\mathbf{\Upsilon}_L)$, where, ζ, η denote pairs in the above set, we have:

$$\zeta \sim \eta$$

if and only if, $\zeta\Re\eta$ or $\eta\Re\zeta$. Finally, by considering a sequence ξ_1, ξ_2, $\ldots$,ξ_k of elements of the set $\mathbf{G}(\mathbf{\Upsilon}_L)$, and also, ζ, η such that:

$$\zeta \sim \xi_1 \sim \xi_2 \sim \ldots \sim \xi_{k-1} \sim \xi_k \sim \eta$$

we may define an equivalence relation on the set $\mathbf{L}(\mathbf{\Upsilon}(L))$ as follows:

$$\zeta \bowtie \eta := \zeta \sim \xi_1 \sim \xi_2 \sim \ldots \sim \xi_{k-1} \sim \xi_k \sim \eta$$

Then, for each $\zeta \in \mathbf{G}(\mathbf{\Upsilon}_L)$, we define the quantum at ζ as follows:

$$Q_\zeta = \{\iota \in \mathbf{G}(\mathbf{\Upsilon}_L) : \zeta \bowtie \iota\}$$

Finally, we define:

$$\mathbf{G}(\mathbf{\Upsilon}_L)/\bowtie := \{Q_\zeta : \zeta = (\psi_E, q) \in \mathbf{G}(\mathbf{\Upsilon}_L)\}$$

and use the notation $Q_\zeta = \|(\psi_E, q)\|$. The set $[\mathbf{G}(\boldsymbol{\Upsilon}_L)/\bowtie]$ is identified categorically as the colimit in the category of Boolean frames of the sheaf functor $\boldsymbol{\Upsilon}_L$ for the Grothendieck topology $\mathbf{J}$ of epimorphic families;

$$[\mathbf{G}(\boldsymbol{\Upsilon}_L)/\bowtie] = \mathbf{Colim}\{\int(\boldsymbol{\Upsilon}_L, \mathcal{B}) \longrightarrow \mathcal{B} \hookrightarrow \mathcal{L}\}$$

The above set is naturally endowed with a quantum events algebra structure, if we are careful to notice that:

[1]. The orthocomplementation is defined as: $Q_\zeta^* = \|(\psi_E, q)\|^* = \|(\psi_E, q^*)\|$.

[2]. The unit element is defined as: $\mathbf{1} = \|(\psi_E, 1)\|$.

[3]. The partial order structure on the set $[\mathbf{G}(\boldsymbol{\Upsilon}_L)/\bowtie]$ is defined as follows: $\|(\psi_E, q)\| \preceq \|(\psi_C, r)\|$, if and only if, $d_1 \preceq d_2$ where, we have made the following identifications: $\|(\psi_E, q)\| = \|(\psi_D, d_1)\|$, and also, $\|(\psi_C, r)\| = \|(\psi_D, d_2)\|$, with $d_1,\ d_2 \in D$, according to the pullback diagram of events algebras:

$$\begin{array}{ccc} D & \xrightarrow{\beta} & E \\ \downarrow{\scriptstyle\gamma} & & \downarrow \\ C & \longrightarrow & L \end{array}$$

such that; $\beta(d_1) = q$, $\gamma(d_2) = r$. The rest of the requirements, such that, $[\mathbf{G}(\boldsymbol{\Upsilon}_L)/\bowtie]$ actually carries the structure of a quantum events algebra are obvious. Thus, we conclude that the epimorphism;

$$T : \coprod_{(E \in [\mathcal{B}]_0, \psi_E : E \to L)} E \to L$$

which determines a covering sieve of a quantum events algebra L by Boolean frames, viz. a Boolean localization system of L, induces an isomorphism:

$$L \cong \mathbf{G}(\boldsymbol{\Upsilon}_L) / \bowtie$$

of quantum events algebras in $\mathcal{L}$, or equivalently;

$$L \cong \mathbf{Colim}\{\int(\boldsymbol{\Upsilon}_L, \mathcal{B}) \longrightarrow \mathcal{B}\}$$

where, $\boldsymbol{\Upsilon}_L$ denotes the sheaf of Boolean frames of L for the Grothendieck topology $\mathbf{J}$ of epimorphic families, and $\int(\boldsymbol{\Upsilon}_L, \mathcal{B})$ denotes the associated fibered category of Boolean frames of L.

Remark: The proposition proved above, settles completely the second issue posed previously, in the following sense: Firstly, we have constructed explicitly a representation of quantum events as equivalence classes of Boolean coefficients in local Boolean reference frames, with respect to $\mathbf{J}$. Secondly, we have shown that a quantum events algebra L is represented isomorphically by the colimit taken in the category of elements of the sheaf functor of Boolean frames of L. Thus, we may state conclusively that: We have forced a uniform, homologous, and coherent fibered representation of quantum events with respect to local Boolean reference frames, inducing effectively an isomorphism between quantum events algebras and colimits of Boolean localization systems.

8 Conclusions

In this paper we have proposed a sheaf-theoretic interpretation scheme of quantum events algebras, taking into account Boolean localization processes in the quantum regime of observable structure. The latter have been effectuated by means of generalized topological covering systems on a base category of Boolean reference contexts. Thus, the focus has been shifted from point-set to topological localization models of a globally partially ordered quantum events algebra. Effectively, this shift induces a transition in the semantics of quantum events from a set-theoretic to a sheaf-theoretic one.

The sheaf-theoretic semantic transition of quantum events has been forced by means of an explicitly constructed uniform, homologous, and coherent fibered representation of quantum events with respect to local Boolean reference frames for the Grothendieck topology of epimorphic families. According to this representation, quantum events have been conceptualized as equivalence classes of local Boolean coordinates with respect to those reference frames. Subsequently, it has been constructed an isomorphic representation of quantum events algebras with colimits taken in the categories of elements of sheaves of Boolean frames.

The physical significance of this representation lies on the fact that the whole information content of a quantum events algebra is preserved by the action of some covering system, if and only if that system forms a Boolean localization system. Hence, the significance of a quantum events algebra is shifted from the orthoposet axiomatization at the level of events, to the sheaf-theoretic gluing conditions at the level of Boolean localization systems. Eventually, the former axiomatization is fully and faithfully recaptured at the level of equivalence classes in these localization systems.

The physical content of the sheaf-theoretic representation of quantum events algebras can be formulated in terms of a functoriality property. According to this, the information content of a quantum events algebra is covariant under the groupoid of gluing isomorphisms between overlapping local Boolean reference frames, along their intersections, in a Boolean localization system, preserving the quantum algebraic structure.

The covering process induced by Boolean localization systems leads naturally to a contextual description of quantum events, in a global quantum events algebra L, with respect to local Boolean reference frames of measurement. It is instructive to note that, each Boolean context corresponds to a Boolean algebra of events actualized locally in a quantum measurement situation. The equivalence classes of Boolean coefficients, with respect to local Boolean contexts in covering sieves of L, represent quantum events in L . Consequently, the information content of a quantum events algebra L is being generated, and also, isomorphically represented, by the information that its structure preserving maps carry. Notice, that the latter information is equivalent with the conditions incorporated in the sheaf-theoretic specification of the functor of Boolean frames. Most significantly, the functioning of this functor in terms of Boolean localization systems of L, accomplish the representation of quantum events as equivalence classes in a structure sheaf of Boolean coefficients associated with local contexts of measurement.

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