Sheaf-Theoretic Representation of Quantum Measure Algebras

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Abstract

We construct a sheaf-theoretic representation of quantum probabilistic structures, in terms of covering systems of Boolean measure algebras. These systems coordinatize quantum states by means of Boolean coefficients, interpreted as Boolean localization measures. The representation is based on the existence of a pair of adjoint functors between the category of presheaves of Boolean measure algebras and the category of quantum measure algebras. The sheaf-theoretic semantic transition of quantum structures shifts their physical significance from the orthoposet axiomatization at the level of events, to the sheaf-theoretic gluing conditions at the level of Boolean localization systems.

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1 Introduction

The groundbreaking 1936 paper by von Neumann and G. Birkhoff entitled "*The Logic of Quantum Mechanics*" has introduced for the first time the notion of logic of a physical theory. For classical theories the appropriate logic is a Boolean algebra; but for quantum theories a non-Boolean logical structure is necessary, which can be an orthocomplemented lattice, or a partial Boolean algebra, or some other structure of a related form. The logic of a physical theory reflects the structure of the propositions describing the behavior of a physical system in the domain of the corresponding theory.

Naturally, the typical mathematical structure associated with logic is an ordered structure. The original quantum logical formulation of quantum theory [1, 2] depends in an essential way on the identification of propositions with projection operators on a complex Hilbert space. A non-classical, non-Boolean logical structure is effectively induced which has its origins in quantum theory. More accurately, the Hilbert space quantum logic has been initially axiomatized as a complete, atomic, orthomodular lattice. Equivalently, it could be cast isomorphic to the partial Boolean algebra of closed subspaces of the Hilbert space associated with the quantum system, or alternatively the partial Boolean algebra of projection operators of the system. On the contrary, the propositional logic of classical mechanics is Boolean logic, meaning that the class of models over which validity and associated semantic notions are defined for the propositions of classical mechanics is the class of Boolean logic structures.

The notion of logic of a physical theory essentially reflects the structure of events being observed in the context of that theory. Associated with such an events structure, there always exists a corresponding probabilistic structure, defined by means of convex sets of measures on that logic. In this sense, the probabilistic structure of a classical system is described by convex sets of probability measures on the Boolean algebra of events of the system, whereas the probabilistic structure of a quantum system is described by convex sets of probability measures on the quantum logic structure of that system. More accurately, in the case of quantum systems, if the quantum events logic is denoted by L, each quantum probability measure, called quantum state, is defined by a mapping;

$$p : L \longrightarrow [0, 1]$$

such that the following conditions are satisfied: $p(1) = 1$ and $p(x \vee y) = p(x) + p(y)$, if $x \perp y$, where, x, $y \in L$. In the Hilbert space formulation of quantum theory, L denotes the Hilbert space quantum logic, whereas a quantum state is defined by the Hilbert space inner product;

$$\langle \varphi, \mathrm{P}x\varphi \rangle$$

where; $x \in L$, φ is a normalized vector in the Hilbert space, and $\mathrm{P}x$ is the orthogonal projection operator corresponding to $x \in L$. We remind that there exists a bijective correspondence between elements of L, that is closed subspaces of the Hilbert space, and orthogonal projection operators.

In this work we will develop the idea that in quantum theory, Boolean localization measures can be understood as providing a coordinatization of a quantum probabilistic structure by establishing a principle of contextuality. More concretely, we shall argue that the covering coordinatization process induced by Boolean localization systems, being formed from families of collated compatible local Boolean measures, leads naturally to a contextual description of quantum events, and their associated quantum probabilities of a corresponding global quantum structure, with respect to local Boolean reference frames of measurement.

An intuitive flavor of this insight is provided by Kochen-Specker theorem [3], according to which the complete comprehension of a quantum mechanical system is impossible, in case that, a single system of Boolean devices is only used globally. On the other side, in every concrete measurement context, the set of events that have been actualized in this context forms a Boolean algebra. This fact motivates the assertion that a Boolean algebra in the lattice of quantum events, serves as a local reference frame, conceived in a precise category-theoretical sense, relative to which a measurement result is being coordinatized. The conceptual meaning of the proposed scheme implies that a quantum logical or quantum probabilistic structure is being construed by means of covering Boolean reference frames, regulated by our measurement localization procedures, which interlock to form a global coherent picture in a non-trivial way. Hence Boolean descriptive contexts are not abandoned once and for all, but instead are used locally, accomplishing the task of providing partial congruent relations with globally non-Boolean objects, the internal structure and functioning of which, is being hopefully recovered by the interconnecting machinery governing the local objects. In this work we propose a mathematical scheme for the implementation of the above assertion, in relation to quantum measure algebras, based on categorical and sheaf-theoretic methods [4-8]. Contextual category theoretical approaches to quantum structures have been also considered, from a different viewpoint in [9,10], and discussed in [11,12]. A remarkable conceptual affinity to the viewpoint of the present paper, although not based on categorical methods, can be found in references [13,14]. For a general mathematical and philosophical discussion of sheaves, variable sets, and related structures, the interested reader should consult reference [15]. Recently, there has also appeared in the literature a complete treatement of the dynamcal aspects of physical theories, and in particular gauge theories, along topological sheaf-theoretic lines [16, 17], as an application of the framework of Mallios's Abstract Differential Geometry [18]. Finally, it is worth mentioning that a sheaf-theoretic approach to quantum structures has been initiated independently by de Groote in a series of preperints [19-22]. In a general setting, de Groote constructs a theory of presheaves on the quantum lattice of closed subspaces of a complex Hilbert space, by transposing literally and generalizing the corresponding constructions from the lattice of open sets of a topological space to the quantum lattice. In comparison, our approach emphasizes the crucial role of Boolean localization systems in the global formation of quantum structures, and thus, shifts the focus of relevant constructions to sheaves over suitable Grothendieck topologies on a base category of Boolean subalgebras of global quantum algebras.

The development of the conceptual and technical machinery of localization systems for generating non-trivial global event and observable structures, as it has been recently demonstrated in [23, 24], effectuates a transition in the semantics of events and observables from a set-theoretic to a sheaf-theoretic one. This is a crucial semantic difference that characterizes the present approach in comparison to the vast literature on quantum measurement and quantum logic. In the following section we will attempt to motivate physically the necessity of this transition on the basis of appropriate requirements that generalized procedures of physical measurement should respect, referring to the apprehension of physical information in terms of observables.

2 Physical Motivation and General Conceptual Framework

Procedures of physical measurement presuppose, at the fundamental level, the existence of a localization process for extracting information related with the local behaviour of a physical system, and thus, discerning observable events. In a general setting, a localization process is being usually implemented physically by the preparation of suitable local reference domains for measurement of observable attributes. Subsequently, these reference domains instantiate local physical contexts for observation of events, that takes place by means of events-registering measurement devices, operating locally within these contexts. In a broad perspective, it is important to notice that registering an event, that has been observed in the context of a prepared reference domain, is not always equivalent to conferring a numerical identity to it, expressed in terms of some real value corresponding to a physical attribute. On the contrary, the latter is only a limited case of the localization process, when, in particular, it is assumed that all local reference contexts can be contracted to points, meaning that points are considered as unique measures of localization in the physical "continuum".

This is exactly the crucial assumption underlying the employment of the set-theoretic structure of the real line as a model of the physical "continuum". The semantics of the physical "continuum" in the standard interpretation of physical theories is associated with the codomain of valuation of physical attributes. Usually the notion of "continuum" is tied with the attribute of position, serving as the range of values characterizing this particular attribution. In this sense, the model adopted to represent these values is the real line, specified as a set- theoretic structure of points that are independent and possess the property of infinite distinguishability with absolute precision. The adoption of the set theoretical real line model is usually justified on the basis of arguments, stipulating that quantities admissible as measured results must be real numbers, since the resort to real numbers has the advantage of securing our empirical access to the external world. Essentially, the basic semantic assumption underlying the employment of the set theoretic structure of the real line for the modeling of the localization structure of the physical "continuum" is that real number representability constitutes our global form of observation.

The success of this localizing philosophy for classical theories is due to the association of the notion of physical "continuum" with the attribute of position and the theoretical fact that all classical observables can be determined precisely and simultaneously at the unique measure of localization of that attribute, viz., at a spatial point, parametrized by the field of real numbers. Nevertheless, the major foundational difference between classical and quantum physical systems from the perspective of the modeling scheme by observables is a consequence of a single principle that can be termed principle of simultaneous observability. According to this, in the classical description of physical systems all their observables are theoretically compatible, or else, they can be simultaneously specified in a single local measurement context. On the other side, the quantum description of physical systems is based on the assertion of incompatibility of all theoretical observables in a single local measurement context, and as a consequence quantum-theoretically the simultaneous specification of all observables is not possible. The conceptual roots of the violation of the principle of simultaneous observability in the quantum regime is tied with Heisenberg's uncertainty principle and Bohr's principle of complementarity of physical descriptions.

In this train of thought, a fruitful fundamental strategy implied by quantum theory would ideally fulfill the following objectives: Firstly, it should disassociate the physical meaning of the notion of localization from its restricted spatial connotation reference context. Secondly, it should allow the functional dependence of observables on generalized localization measures induced by the preparation of suitably structured domains of measurement, not necessarily based on the existence of an underlying set-theoretic structure of points on the real line. Regarding the implementation of this strategy, it should be essential to interpret any local observable as a relational information algebraic number-like object with respect to the corresponding local context of measurement. At a further stage, it should be necessary to establish appropriate compatibility conditions for gluing the information content of local observables globally. Mathematically, the implementation of this strategy is being precisely captured by the concept of a sheaf-theoretic fibered structure, explained in the sequel. The primary physical motivation of this paper concerns the possibility of constructing explicitly an appropriate localization process suited to quantum physical observation, along the objectives of the strategy stated above, and study in particular its consequences refering to the interpretation of quantum probabilistic structures. For this purpose, the focus is shifted from point-set to topological localization models of partially ordered global quantum event structures.

Before embarking on a qualitative discussion of the relevance of the concept of sheaf for this endeavor, it is initially instructive to clarify that the functioning of a localization process amounts to filtering the information content of a global structure of partially ordered physical events, through a concretely specified structure of observation domains determined by a homologous operational physical procedure. The latter is defined by the requirement that the reference contexts of measurement, together with their structural transformations, should form a mathematical category. Thus, the localization process should be implemented in terms of an action of the category of reference contexts on a set-theoretic global structure of physical events. The latter, is then partitioned into sorts parameterized by the objects of the category of contexts. In this sense, the functioning of a localization process can be represented by means of a fibered construct, understood geometrically as a presheaf, or equivalently, as a variable set over the base category of contexts. The fibers of this construct may be thought, in analogy to the case of the action of a group on a set of points, as the "generalized orbits" of the action of the category of contexts. The notion of functional dependence incorporated in this action, forces the ordered structure of physical events to fiber over the base category of reference contexts. Most importantly, the presheaf fibered construct incorporates the physical requirement of uniformity of observed events. More concretely, for any two events observed over the same domain of measurement, the structure of all reference contexts that relate to the first cannot be distinguished in any possible way from the structure of contexts relating to the second. Consequently, all the events observed within any particular reference context, implementing a localization process, are uniformly equivalent to each other. Equivalently stated, the compatibility of the localization process with the physical requirement of uniformity, demands that the relation of (partial) order in a global set-theoretic universe of events is induced by lifting appropriately a structured family of arrows from the base category of reference contexts to the fibers. It is precisely that condition of compatibility being formalized by the construction of the category of elements of the corresponding presheaf.

The disassociation of the physical meaning of a localization process from its restricted spatial connotation reference context requires, first of all, the abstraction of the constitutive properties of localization in appropriate categorical terms, and then, the effectuation of these properties for the definition of localization systems of global event structures. Regarding these objectives, the sought abstraction is being implemented by means of covering devices on the base category of reference contexts, called in categorical terminology covering sieves. The constitutive properties of localization being abstracted categorically in terms of sieves, being qualified as covering ones, satisfy the following basic requirements:

[i]. The covering sieves are covariant under pullback operations, viz., they are stable under change of a base reference context. Most importantly, the stability conditions are functorial. This requirement means, in particular, that the intersection of covering sieves is also a covering sieve, for each reference context in the base category.

[ii]. The covering sieves are transitive, such that, intuitively stated, covering sieves of figures of a context in covering sieves of this context, are also covering sieves of the context themselves.

From a physical perspective, the consideration of covering sieves as generalized measures of localization of events in a global partially ordered structure of events, gives rise to localization systems of the latter. More specifically, the operation which assigns to each reference context of the base category a collection of covering sieves satisfying the closure conditions stated previously, gives rise to the notion of a Grothendieck topology on the category of contexts. The construction of a suitable Grothendieck topology on the base category of contexts is significant for the following reasons: Firstly, it elucidates precisely and unquestionably the conception of local in a categorical measurement environment, such that this conception becomes detached from its restricted spatial connotation, and thus, expressed exclusively in relational information terms. Secondly, it permits the collation of local observable information into global ones by utilization of the notion of sheaf for that Grothendieck topology. The definition of sheaf essentially expresses gluing conditions, providing the means for studying the global consequences of locally defined properties. The transition from locally defined observable information into global ones is being effectuated via a compatible family of elements over a localization system of a global event strucure. A sheaf assigns a set of elements to each reference context of a localization system, representing local observable data colected within that context. A choice of elements from these sets, one for each context, forms a compatible family if the choice respects the mappings induced by the restriction functions among contexts, and moreover, if the elements chosen agree whenever two contexts of the localization system overlap. If such a locally compatible choice induces a unique choice for a global event structure being localized, viz. a global choice, then the condition for being a sheaf is satisfied. We note that in general, there will be more locally defined or partial choices than globally defined ones, since not all partial choices need be extendible to global ones, but a compatible family of partial choices uniquely extends to a global one, or in other words, any presheaf uniquely defines a sheaf.

Having explained in detail the physical motivation, as well as, the key conceptual prerequisites and ideas underlying the modeling of localization processes for acquisition and efficient handling of observable information related with the behaviour of physical systems in a broad perspective, in the sequel, we focus our attention on the implementation of a concrete localization process of quantum probabilistic structures effectuated by Boolean localization systems of quantum measurement.

3 Categorical Probabilistic Structures

According to the category-theoretic approach to each kind of mathematical structure, there corresponds a **category** whose objects have that structure, and whose morphisms preserve it. Moreover to any natural construction on structures of one kind, yielding structures of another kind, there corresponds a **functor** from the category of the first kind to the category of the second.

A **Classical event structure** is a small category, denoted by $\mathcal{B}$, which is called the category of Boolean event algebras. Its objects are Boolean algebras of events, and its arrows are Boolean algebraic morphisms.

A **Quantum event structure** is a small category, denoted by $\mathcal{L}$, which is called the category of quantum event algebras.

Its objects, denoted by L, are quantum algebras of events, that is orthomodular σ-orthoposets. More concretely, each object L in $\mathcal{L}$, is considered as a partially ordered set of quantum events, endowed with a maximal element 1, and with an operation of orthocomplementation $[-]^* : L \longrightarrow L$, which satisfy, for all $l \in L$, the following conditions: [a] $l \leq 1$, [b] $l^{**} = l$, [c] $l \vee l^* = 1$, [d] $l \leq \acute{l} \Rightarrow \acute{l}^* \leq l^*$, [e] $l \bot \acute{l} \Rightarrow l \vee \acute{l} \in L$, [f] for $l, \acute{l} \in L, l \leq \acute{l}$ implies that l and $\acute{l}$ are compatible, where $0 := 1^*$, $l \bot \acute{l} := l \leq \acute{l}^*$, and the operations of meet $\wedge$ and join $\vee$ are defined as usually. We also recall that $l, \acute{l} \in L$ are compatible if the sublattice generated by $\{l, l^*, \acute{l}, \acute{l}^*\}$ is a Boolean algebra, namely if it is a Boolean sublattice. The σ-completeness condition, namely that the join of countable families of pairwise orthogonal events must exist, is also required in order to have a well defined theory of observables over L.

Its arrows are quantum algebraic morphisms, that is maps $L \xrightarrow{H} K$, which satisfy, for all $k \in K$ the following conditions: [a] $H(1) = 1$, [b] $H(k^*) = [H(k)]^*$, [c] $k \leq \acute{k} \Rightarrow H(k) \leq H(\acute{k})$, [d] $k \bot \acute{k} \Rightarrow H(k \vee \acute{k}) \leq H(k) \vee H(\acute{k})$.

Next we introduce the categories associated with probabilistic structures.

A **Quantum convex measure structure** is a small category, denoted by $\sum$, which is called the category of convex sets of quantum probability measures.

Its objects are the convex sets Θ of quantum states or quantum probability measures on a quantum event algebra L. Each quantum probability measure, or quantum state, is defined by a mapping;

$$p : L \rightarrow [0, 1]$$

such that the following conditions are satisfied: $p(1) = 1$ and $p(x \vee y) = p(x) + p(y)$, if $x \perp y$, where, x, $y \in L$. On each set Θ, there is defined the operation of convex mixing by means of the mappings;

$$\sigma^n : E^n \times \Theta^n \to \Theta$$

for each natural number n, such that:

$$\sigma^n(\underline{e}, \underline{p}) := \sum_i e_i p_i$$

where $\underline{e} = \langle e_1, \ldots, e_n \rangle$, is a vector of real numbers, with $e_i \geq 0$ and $\sum_i e_i = 1$, and also, $\underline{p} = \langle p_1 \ldots p_n \rangle$ is a vector of quantum states. The unique quantum state $\sum_i e_i p_i$ is called the convex mixture of $\underline{p}$. The convex mixture of $\underline{p}$, evaluated at $x \in L$, is the superposition of probabilities $\sum_i e_i p_i(x)$. For a quantum state p_i and an event $x \in L$, $p_i(x)$ denotes the probability of occurrence of x in state p_i.

The arrows in the category $\sum$ are morphisms of convex sets of probability measures, that is morphisms of sets $[]^h : \Theta \to \Phi$ which commute with the operation of convex mixing, that is;

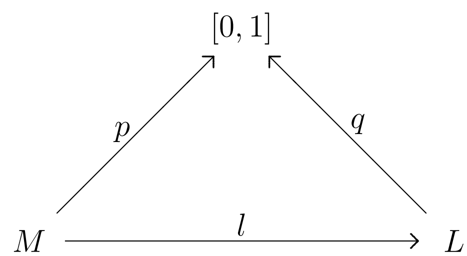
$$[\sigma^n(\underline{e}, \underline{p})]^h = \sigma^n(\underline{e}, \underline{p}^h)$$

We note that Θ and Φ are regarded as defined over the same quantum event algebra L, otherwise we have to take into account the quantum algebraic morphisms as well.

Using the information encoded in the categories of quantum event algebras $\mathcal{L}$, and quantum probabilistic structures $\sum$, it is possible to construct a new category, called the category of quantum probabilities, constructed as a category fibered in groupoids over the category of quantum event algebras $\mathcal{L}$, as follows:

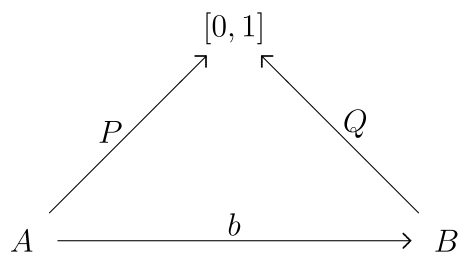
A **Quantum probabilistic structure** is a small category, denoted by $\mathcal{Q}$, which is called the category of quantum states or quantum probabilities.

Its objects are the quantum measure algebras $\langle M, p\rangle := {}^pM$, where M is a quantum event algebra and p is a quantum probability measure on M, defined by the measurable mapping $p : M \to [0, 1]$. The arrows in $\mathcal{Q}$, denoted by ${}^pM \longrightarrow {}^qL$, are commutative triangles, or equivalently, are those quantum logic morphisms $M \xrightarrow{l} L$ in $\mathcal{L}$, such that $p = q \circ l$ in the diagram below, is again a quantum probability measure.



Correspondingly, a **Boolean probabilistic structure** is a small category, denoted by $\mathcal{C}$, which is called the category of Boolean probability measures, or classical states.

Its objects are the Boolean measure algebras $\langle A, P\rangle := {}^PA$, where A is a Boolean event algebra and P is a Boolean probability measure on M, defined by the measurable mapping $P : A \to [0, 1]$. The arrows in $\mathcal{C}$, denoted by ${}^PA \longrightarrow {}^QB$, are commutative triangles, or equivalently, are those Boolean logic morphisms $A \xrightarrow{b} B$ in $\mathcal{B}$, such that $P = Q \circ b$ in the diagram below, is again a classical state.



4 Presheaf and Coefficients Boolean Functors

4.1 Presheaves of Boolean Probability Measures

If $\mathcal{C}^{op}$ is the opposite category of $\mathcal{C}$, then $\mathbf{Sets}^{\mathcal{C}^{op}}$ denotes the functor category of presheaves on Boolean measure algebras. Its objects are all functors $\mathbf{X} : \mathcal{C}^{op} \longrightarrow \mathbf{Sets}$, and its morphisms are all natural transformations between such functors. Each object $\mathbf{X}$ in this category is a contravariant set-valued functor on $\mathcal{C}$, called a presheaf of Boolean probability measures on $\mathcal{C}$.

A functor $\mathbf{X}$ is a structure-preserving morphism of these categories, that is it preserves composition and identities. A functor in the category $\mathbf{Sets}^{\mathcal{C}^{op}}$ can be understood as a contravariant translation of the language of $\mathcal{C}$ into that of $\mathbf{Sets}$. Given another such translation (contravariant functor) $\acute{\mathbf{X}}$ of $\mathcal{C}$ into $\mathbf{Sets}$ we need to compare them. This can be done by giving, for each object ${}^{P}A$ in $\mathcal{C}$ a transformation $\tau_{{}^{P}A} : \mathbf{X}({}^{P}A) \longrightarrow \acute{\mathbf{X}}({}^{P}A)$ which compares the two images of the object ${}^{P}A$. Not any morphism will do, however, as it would be necessary the construction to be parametric in ${}^{P}A$, rather than ad hoc. Since ${}^{P}A$ is an object in $\mathcal{C}$ while $\mathbf{X}({}^{P}A)$ is in $\mathbf{Sets}$ we cannot link them by a morphism. Rather the goal is that the transformation should respect the morphisms of $\mathcal{C}$, or in other words the interpretations of $v : {}^{P}A \longrightarrow {}^{Q}B$

by $\mathbf{X}$ and $\acute{\mathbf{X}}$ should be compatible with the transformation under τ. Then τ is a natural transformation in the category of presheaves $\mathbf{Sets}^{\mathcal{C}^{op}}$.

For each Boolean measure algebra ${}^{P}A$ of $\mathcal{C}$, $\mathbf{X}({}^{P}A)$ is a set, and for each arrow $f : {}^{Q}B \longrightarrow {}^{P}A$, $\mathbf{X}(f) : \mathbf{X}({}^{P}A) \longrightarrow \mathbf{X}({}^{Q}B)$ is a set function. If $\mathbf{X}$ is a presheaf on $\mathcal{C}$ and $x \in \mathbf{X}(\mathrm{O})$, the value $\mathbf{X}(f)(x)$ for an arrow $f : {}^{Q}B \longrightarrow {}^{P}A$ in $\mathcal{C}$ is called the restriction of x along f and is denoted by $\mathbf{X}(f)(x) = x \cdot f$.

Each object ${}^{P}A$ of $\mathcal{C}$ gives rise to a contravariant Hom-functor $y[{}^{P}A] := Hom_{\mathcal{C}}(-, {}^{P}A)$. This functor defines a presheaf on $\mathcal{C}$. Its action on an object ${}^{Q}B$ of $\mathcal{C}$ is given by

$$y[{}^{P}A]({}^{Q}B) := Hom_{\mathcal{C}}({}^{Q}B, {}^{P}A)$$

whereas its action on a morphism ${}^{R}C \xrightarrow{w} {}^{Q}B$, for $v : {}^{Q}B \longrightarrow {}^{P}A$ is given by

$$y[{}^{P}A](w) : Hom_{\mathcal{C}}({}^{Q}B, {}^{P}A) \longrightarrow Hom_{\mathcal{C}}({}^{R}C, {}^{P}A)$$

$$y[{}^{P}A](w)(v) = v \circ w$$

Furthermore y can be made into a functor from $\mathcal{C}$ to the contravariant functors on $\mathcal{C}$

$$y : \mathcal{C} \longrightarrow \mathbf{Sets}^{\mathcal{C}^{op}}$$

such that ${}^{P}A \mapsto Hom_{\mathcal{C}}(-, {}^{P}A)$. This is an embedding, called the Yoneda embedding [5] , and it is a full and faithful functor.

The functor category of presheaves on Boolean measure algebras $\mathbf{Sets}^{\mathcal{C}^{op}}$, provides an instantiation of a structure known as topos [6-8]. A topos exemplifies a well defined notion of a categorical universe of variable sets. It can be conceived as a local mathematical framework corresponding to a generalized model of set theory or as a generalized space. Moreover it provides a natural example of a many-valued truth structure, which remarkably is not ad hoc, but reflects genuine constraints of the surrounding universe.

4.2 Boolean Measure Algebras Fibrations

Since $\mathcal{C}$ is a small category, there is a set consisting of all the elements of all the sets $\mathbf{X}({}^{P}A)$, and similarly there is a set consisting of all the functions $\mathbf{X}(f)$. This observation regarding $\mathbf{X} : \mathcal{C}^{op} \longrightarrow \mathbf{Sets}$ permits us to take the disjoint union of all the sets of the form $\mathbf{X}({}^{P}A)$ for all objects ${}^{P}A$ of $\mathcal{C}$. The elements of this disjoint union can be represented as pairs $({}^{P}A, \chi)$ for all objects ${}^{P}A$ of $\mathcal{C}$ and elements $\chi \in \mathbf{X}({}^{P}A)$. Thus the disjoint union of sets is made by labelling the elements. Now we can construct a category whose set of objects is the disjoint union just mentioned. This structure is called the category of elements of the presheaf $\mathbf{X}$, denoted by $\int(\mathbf{X}, \mathcal{C})$. Its objects are all pairs $({}^{P}A, \chi)$, and its morphisms $({}^{R}C, \acute{\chi}) \longrightarrow ({}^{P}A, \chi)$ are those morphisms $u : {}^{R}C \longrightarrow {}^{P}A$ of $\mathcal{C}$ for which $\chi \cdot u = \acute{\chi}$. Projection on the second coordinate of $\int(\mathbf{X}, \mathcal{C})$, defines a functor $\int_{\mathbf{X}} : \int(\mathbf{X}, \mathcal{C}) \longrightarrow \mathcal{C}$. $\int(\mathbf{X}, \mathcal{C})$ together with the projection functor $\int_{\mathbf{X}}$ is equivalent to the discrete fibration induced by $\mathbf{X}$, and $\mathcal{C}$ is the base category of the fibration. We note that the fibration is discrete because the fibers are categories in which the only arrows are identity arrows. If ${}^{P}A$ is a Boolean measure algebra of $\mathcal{C}$, the inverse image under $\int_{\mathbf{X}}$ of ${}^{P}A$ is simply the set $\mathbf{X}({}^{P}A)$, although its elements are written as pairs so as to form a disjoint union. The instantiation of the fibration induced by $\mathbf{X}$, is an application of the general Grothendieck construction [8].

$$\begin{array}{ccc} \int(\mathbf{X}, \mathcal{C}) & & \\ \Big\downarrow{\scriptstyle \int_{\mathbf{X}}} & & \\ \mathcal{C} & \xrightarrow{\mathbf{X}} & \mathbf{Sets} \end{array}$$

The split discrete fibration induced by $\mathbf{X}$, where $\mathcal{C}$ is the base category of the fibration, provides a well-defined notion of a uniform homologous fibered structure in the following sense: Firstly, by the arrows specification defined in the category of elements of $\mathbf{X}$, any element χ, determined over the measure algebra ${}^{P}A$, is homologously related with any other element $\acute{\chi}$ over the measure algebra ${}^{R}C$, and so on, by variation over all the contexts of the base category. Secondly, all the elements χ of $\mathbf{X}$, of the same sort ${}^{P}A$, viz. determined over the same measure algebra ${}^{P}A$, are uniformly equivalent to each other, since all the arrows in $\int(\mathbf{X}, \mathcal{C})$ are induced by lifting arrows from the base category $\mathcal{C}$.

4.3 Functor of Boolean measure coefficients

We define a modeling Boolean coefficients functor, $\mathbf{M} : \mathcal{C} \longrightarrow \mathcal{Q}$, which assigns to Boolean measure algebras in $\mathcal{C}$, that instantiates a model category, the underlying quantum measure algebras from $\mathcal{Q}$, and to Boolean measurable morphisms the underlying quantum measurable morphisms. Hence $\mathbf{M}$ acts as a forgetful functor, forgetting the extra Boolean structure of $\mathcal{C}$.

Equivalently the Boolean coefficients functor can be characterized as, $\mathbf{M} : \mathcal{B} \longrightarrow \mathcal{L}$, which assigns to Boolean event algebras in $\mathcal{B}$ the underlying quantum event algebras from $\mathcal{L}$, and to Boolean morphisms the underlying quantum algebraic morphisms, such that the following diagram commutes:

$$
\begin{array}{ccc}
 & [0,1] & \\
 {\scriptstyle P}\nearrow & & \nwarrow{\scriptstyle p} \\
\mathbf{M}(B) & \xrightarrow{[\psi_B]} & L
\end{array}
$$

5 Adjoint Functorial Relation

We consider the category of quantum measure algebras $\mathcal{Q}$ and the modelling functor $\mathbf{M}$, and we define the functor $\mathbf{R}$ from $\mathcal{Q}$ to the topos of presheaves on Boolean measure algebras $\mathbf{Sets}^{\mathcal{C}^{op}}$, given by;

$$\mathbf{R}({}^{p}L) : {}^{P}A \mapsto Hom_{\mathcal{Q}}(\mathbf{M}({}^{P}A), {}^{p}L)$$

A natural transformation τ in the topos of presheaves on Boolean measure algebras $\mathbf{Sets}^{\mathcal{C}^{op}}$ between $\mathbf{X}$ and $\mathbf{R}({}^{p}L)$, $\tau : \mathbf{X} \longrightarrow \mathbf{R}({}^{p}L)$ is a family $\tau_{{}^{P}A}$ indexed by Boolean measure algebras ${}^{P}A$ of $\mathcal{C}$ for which each $\tau_{{}^{P}A}$ is a map

$$\tau_{{}^{P}A} : \mathbf{X}({}^{P}A) \rightarrow Hom_{\mathcal{Q}}(\mathbf{M}({}^{P}A), {}^{p}L)$$

of sets, such that the diagram of sets below, commutes for each Boolean morphism $u : {}^{R}C \rightarrow {}^{P}A$ of $\mathcal{C}$.

$$
\begin{array}{ccc}
\mathbf{X}({}^{P}A) & \xrightarrow{\tau_{{}^{P}A}} & Hom_{\mathcal{Q}}(\mathbf{M}({}^{P}A), {}^{p}L) \\
{\scriptstyle \mathbf{X}(u)}\downarrow & & \downarrow{\scriptstyle {}^{*}\mathbf{M}(u)} \\
\mathbf{X}({}^{R}C) & \xrightarrow{\tau_{{}^{R}C}} & Hom_{\mathcal{Q}}(\mathbf{M}({}^{R}C), {}^{p}L)
\end{array}
$$

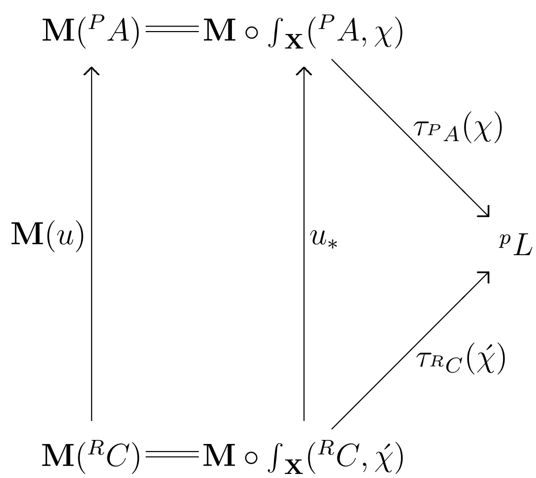
If we make use of the category of elements of the Boolean measure algebras-variable set X, being an object in the topos of presheaves $\mathbf{Sets}^{\mathcal{C}^{op}}$, then the map $\tau_{^PA}$, defined above, can be characterized as:

$$\tau_{^PA} : (^PA, \chi) \to Hom_{\mathcal{Q}}(\mathbf{M} \circ \int_{\mathbf{X}} (^PA, \chi), {}^pL)$$

Equivalently such a τ can be seen as a family of arrows of $\mathcal{Q}$ which is being indexed by objects $(^PA, \chi)$ of the category of elements of the presheaf of Boolean measure algebras $\mathbf{X}$, namely

$$\{\tau_{^PA}(\chi) : \mathbf{M}(^PA) \to {}^pL\}_{(^PA, \chi)}$$

From the perspective of the category of elements of $\mathbf{X}$, the condition of the commutativity of the above diagram is equivalent with the condition that for each Boolean morphism $u : {}^RC \to {}^PA$ of $\mathcal{C}$, the following diagram commutes.



From the diagram above, we conclude that the arrows $\tau_{^PA}(\chi)$ form a cocone from the functor $\mathbf{M} \circ \int_{\mathbf{X}}$ to the quantum measure algebra ${}^pL$. Making use of the definition of the colimit, we conclude that each such cocone emerges by the composition of the colimiting cocone with a unique arrow from the colimit $\mathbf{LX}$ to the quantum measure algebra object ${}^{p}L$. In other words, there is a bijection which is natural in $\mathbf{X}$ and ${}^{p}L$

$$Nat(\mathbf{X}, \mathbf{R}({}^{p}L)) \cong Hom_{\mathcal{Q}}(\mathbf{LX}, {}^{p}L)$$

From the above bijection we are driven to the conclusion that the functor $\mathbf{R}$ from $\mathcal{Q}$ to the topos of presheaves $\mathbf{Sets}^{\mathcal{C}^{op}}$, given by;

$$\mathbf{R}({}^{p}L) : {}^{P}A \mapsto Hom_{\mathcal{Q}}(\mathbf{M}({}^{P}A), {}^{p}L)$$

has a left adjoint $\mathbf{L} : \mathbf{Sets}^{\mathcal{C}^{op}} \to \mathcal{Q}$, which is defined for each presheaf of Boolean measure algebras $\mathbf{X}$ in $\mathbf{Sets}^{\mathcal{C}^{op}}$ as the colimit

$$\mathbf{L}(\mathbf{X}) = Colim\{\int(\mathbf{X}, \mathcal{C}) \xrightarrow{\int \mathbf{X}} \mathcal{C} \xrightarrow{\mathbf{M}} \mathcal{Q}\}$$

Consequently there is a **pair of adjoint functors** $\mathbf{L} \dashv \mathbf{R}$ as follows:

$$\mathbf{L} : \mathbf{Sets}^{\mathcal{C}^{op}} \leftrightarrows \mathcal{Q} : \mathbf{R}$$

Thus we have constructed an adjunction which consists of the functors $\mathbf{L}$ and $\mathbf{R}$, called left and right adjoints with respect to each other respectively, as well as the natural bijection:

$$\begin{array}{ccc} Nat(\mathbf{X}, \mathbf{R}({}^{p}L)) & \xrightarrow{\mathbf{r}} & Hom_{\mathcal{Q}}(\mathbf{LX}, {}^{p}L) \\ \| & & \| \\ Nat(\mathbf{X}, \mathbf{R}({}^{p}L)) & \xleftarrow{\mathbf{l}} & Hom_{\mathcal{Q}}(\mathbf{LX}, {}^{p}L) \end{array}$$

$$Nat(\mathbf{X}, \mathbf{R}({}^{p}L)) \cong Hom_{\mathcal{Q}}(\mathbf{L}\mathbf{X}, {}^{p}L)$$

In the adjunction described above, between the topos of presheaves of Boolean measure algebras and the category of quantum measure algebras, the map **r** is called the right adjoint operator and the map **l** the left adjoint operator.

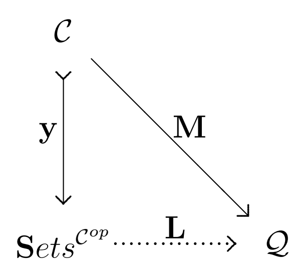
If in the bijection defining the adjunction we use as **X** the representable presheaf of the topos of Boolean measure algebras $\mathbf{y}[{}^{P}A]$, it takes the form:

$$Nat(\mathbf{y}[{}^{P}A], \mathbf{R}({}^{p}L)) \cong Hom_{\mathcal{Q}}(\mathbf{L}\mathbf{y}[{}^{P}A], {}^{p}L)$$

We note that when $\mathbf{X} = \mathbf{y}[{}^{P}A]$ is representable, then the corresponding category of elements $\int(\mathbf{y}[{}^{P}A], \mathcal{C})$ has a terminal object, namely the element $1 : {}^{P}A \longrightarrow {}^{P}A$ of $\mathbf{y}[{}^{P}A]({}^{P}A)$. Therefore the colimit of the composite $\mathbf{M} \circ \int_{\mathbf{y}[{}^{P}A]}$ is going to be just the value of $\mathbf{M} \circ \int_{\mathbf{y}[{}^{P}A]}$ on the terminal object. Thus we have

$$\mathbf{L}\mathbf{y}[{}^{P}A]({}^{P}A) \cong \mathbf{M} \circ \int_{\mathbf{y}[{}^{P}A]} ({}^{P}A, 1) = \mathbf{M}({}^{P}A)$$

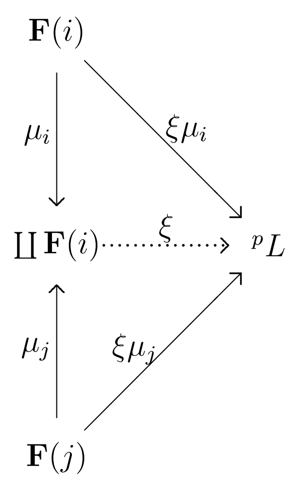
Thus we can characterize $\mathbf{M}({}^{P}A)$ as the colimit of the representable presheaf on the category of Boolean measure algebras.



6 Tensor Product Representation of the Colimit

The content of the adjunction between the topos of presheaves of Boolean measure algebras and the category of quantum measure algebras can be analyzed if we make use of the categorical construction of the colimit defined above, as a coequalizer of a coproduct. We consider the colimit of any functor $\mathbf{F} : \mathcal{I} \longrightarrow \mathcal{Q}$ from some index category $\mathcal{I}$ to $\mathcal{Q}$. Let $\mu_i : \mathbf{F}(i) \rightarrow \amalg_i \mathbf{F}(i)$, $i \in I$, be the injections into the coproduct. A morphism from this coproduct, $\xi : \amalg_i \mathbf{F}(i) \rightarrow {}^{p}L$, is determined uniquely by the set of its components $\xi_i = \xi\mu_i$. These components ξ_i are going to form a cocone over $\mathbf{F}$ to the quantum measure algebra vertex ${}^{p}L$ only when for all arrows $v : i \longrightarrow j$ of the index category $\mathcal{I}$ the following conditions are satisfied:

$$(\xi\mu_j)\mathbf{F}(v) = \xi\mu_i$$



So we consider all $\mathbf{F}(domv)$ for all arrows v with its injections ν_v and obtain their coproduct $\amalg_{v:i \rightarrow j} \mathbf{F}(domv)$. Next we construct two arrows ζ and

η, defined in terms of the injections ν_v and μ_i, for each $v : i \longrightarrow j$ by the conditions

$$\zeta \nu_v = \mu_i$$

$$\eta \nu_v = \mu_j \mathbf{F}(v)$$

as well as their coequalizer ξ;

$$\begin{array}{ccccc}
\mathbf{F}(domv) & & \mathbf{F}(i) & & \\
\downarrow \mu_v & & \downarrow \mu_i & \overset{\xi\mu_i}{\searrow} & \\
\coprod_{v:i\rightarrow j} \mathbf{F}(domv) & \underset{\eta}{\overset{\zeta}{\rightrightarrows}} & \coprod \mathbf{F}(i) & \overset{\xi}{\cdots\rightarrow} & {}^{p}L
\end{array}$$

The coequalizer condition $\xi\zeta = \xi\eta$ tells us that the arrows $\xi\mu_i$ form a cocone over $\mathbf{F}$ to the quantum measure algebra vertex ${}^{p}L$. We further note that since ξ is the coequalizer of the arrows ζ and η this cocone is the colimiting cocone for the functor $\mathbf{F} : I \rightarrow \mathcal{Q}$ from some index category $\mathcal{I}$ to $\mathcal{Q}$. Hence the colimit of the functor $\mathbf{F}$ can be constructed as a coequalizer of coproduct according to the diagram below:

$$\coprod_{v:i\rightarrow j} \mathbf{F}(domv) \underset{\eta}{\overset{\zeta}{\rightrightarrows}} \coprod \mathbf{F}(i) \overset{\xi}{\longrightarrow} Colim\mathbf{F}$$

In the case considered the index category is the category of elements of the presheaf of Boolean measure algebras $\mathbf{X}$ and the functor $\mathbf{M} \circ G_{\mathbf{X}}$ plays the role of the functor $\mathbf{F} : \mathcal{I} \longrightarrow \mathcal{Q}$. In the diagram above the second coproduct is over all the objects $({}^{P}A, \chi)$ with $\chi \in \mathbf{X}({}^{P}A)$ of the category of elements, while the first coproduct is over all the maps $v : ({}^{R}C, \acute{\chi}) \longrightarrow ({}^{P}A, \chi)$

of that category, so that $v : {}^RC \longrightarrow {}^PA$ and the condition $\chi v = \acute{\chi}$ is satisfied. We conclude that the colimit $\mathbf{L}_M(\mathbf{X})$ can be equivalently presented as the coequalizer below:

$$\coprod_{v:{}^RC\rightarrow {}^PA}\mathbf{M}({}^RC) \underset{\eta}{\overset{\zeta}{\rightrightarrows}} \coprod_{({}^PA,\chi)}\mathbf{M}({}^PA) \xrightarrow{\xi} \mathbf{X}\otimes_{\mathcal{C}}\mathbf{M}$$

The coequalizer presentation of the colimit shows that the Hom-functor $\mathbf{R}$ has a left adjoint which can be characterized categorically as the tensor product $-\otimes_{\mathcal{C}}\mathbf{M}$.

In order to clarify the above observation, we forget for the moment that the discussion concerns the category of quantum measure algebras $\mathcal{Q}$, and we consider instead the category **Sets**. Then the coproduct $\amalg_p\mathbf{M}({}^PA)$ is a coproduct of sets, which is equivalent to the product $\mathbf{X}({}^PA)\times\mathbf{M}({}^PA)$ for ${}^PA\in\mathcal{C}$. The coequalizer is thus the definition of the tensor product $\mathbf{X}\otimes_{\mathcal{C}}\mathbf{M}$ of the set valued functors:

$$\mathbf{X} : \mathcal{C}^{op} \longrightarrow \mathbf{Sets}, \qquad \mathbf{M} : \mathcal{C} \longrightarrow \mathbf{Sets}$$

$$\coprod_{{}^PA,{}^RC}\mathbf{X}({}^PA)\times Hom({}^RC,{}^PA)\times\mathbf{M}({}^RC) \underset{\eta}{\overset{\zeta}{\rightrightarrows}} \coprod_{{}^PA}\mathbf{X}({}^PA)\times\mathbf{M}({}^PA) \xrightarrow{\xi} \mathbf{X}\otimes_{\mathcal{C}}\mathbf{M}$$

According to the diagram above for elements $\chi\in\mathbf{X}({}^PA)$, $v : {}^RC \rightarrow {}^PA$ and $\acute{y}\in\mathbf{M}({}^RC)$ the following equations hold:

$$\zeta(\chi, v, \acute{y}) = (\chi v, \acute{y}), \qquad \eta(\chi, v, \acute{y}) = (\chi, v\acute{y})$$

symmetric in $\mathbf{X}$ and $\mathbf{M}$. Hence the elements of the set $\mathbf{X}\otimes_{\mathcal{C}}\mathbf{M}$ are all of the form $\xi(\chi, y)$. This element can be written as

$$\xi(\chi, y) = \chi\otimes y, \quad \chi\in\mathbf{X}({}^PA), y\in\mathbf{M}({}^PA)$$

Thus if we take into account the definitions of ζ and η above, we obtain:

$$\chi v \otimes \acute{y} = \chi \otimes v\acute{y}$$

Furthermore if we define the arrows

$$k_{^P A} : \mathbf{X} \otimes_{\mathcal{C}} \mathbf{M} \longrightarrow {}^p L, \qquad l_{^P A} : \mathbf{X}(^P A) \longrightarrow Hom_{\mathcal{Q}}(\mathbf{M}(^P A), {}^p L)$$

they are related under the fundamental adjunction by

$$k_{^P A}(\chi, y) = l_{^P A}(\chi)(y), \qquad {}^P A \in \mathcal{C}, \chi \in \mathbf{X}(^P A), y \in \mathbf{M}(^P A)$$

Here we consider k as a function on $\amalg_{^P A} \mathbf{X}(^P A) \times \mathbf{M}(^P A)$ with components $k_{^P A} : \mathbf{X}(^P A) \times \mathbf{M}(^P A) \longrightarrow {}^p L$ satisfying

$$k_{^R C}(\chi v, y) = k_{^P A}(\chi, v y)$$

in agreement with the equivalence relation defined above.

Now we replace the category **Sets** by the category of quantum measure algebras $\mathcal{Q}$ under study. The element y in the set $\mathbf{M}(^P A)$ is replaced by a generalized element $y : \mathbf{M}(^R C) \rightarrow \mathbf{M}(^P A)$ from some modeling object $\mathbf{M}(^R C)$ of $\mathcal{Q}$. Then we consider k as a function $\amalg_{(^P A, \chi)} \mathbf{M}(^P A) \longrightarrow {}^p L$ with components $k_{(^P A, \chi)} : \mathbf{M}(^P A) \rightarrow {}^p L$ for each $\chi \in \mathbf{X}(^P A)$, that for all arrows $v : {}^R C \longrightarrow {}^P A$ satisfy

$$k_{(^R C, \chi v)} = k_{(^P A, \chi)} \circ \mathbf{M}(v)$$

Then the condition defining the bijection holding by virtue of the fundamental adjunction is given by

$$k_{(^P A, \chi)} \circ y = l_{^P A}(\chi) \circ y : \mathbf{M}(^R C) \rightarrow {}^p L$$

This argument, being natural in the object $\mathbf{M}({}^{R}C)$, is determined by setting $\mathbf{M}({}^{R}C) = \mathbf{A}({}^{P}A)$ with y being the identity map. Hence the bijection takes the form $k_{({}^{P}A,\chi)} = l_{{}^{P}A}(\chi)$, where $k : \amalg_{({}^{P}A,\chi)}\mathbf{M}({}^{P}A) \longrightarrow {}^{p}L$, and $l_{{}^{P}A} : \mathbf{X}({}^{P}A) \longrightarrow Hom_{\mathcal{Q}}(\mathbf{M}({}^{P}A), {}^{p}L)$.

7 System Of Localizations For Quantum Measure Algebras

The notion of a system of localizations for a quantum measure algebra, which will be defined in the sequel, is conceptually based on the expectation that a quantum measure algebra ${}^{p}L$ in $\mathcal{Q}$ is possible to be comprehended by means of certain structure preserving maps $\mathbf{M}({}^{P}A) \longrightarrow {}^{p}L$ with local or modeling objects Boolean measure algebras ${}^{P}A$ in $\mathcal{C}$ as their domains. It is obvious, that any single map from any modeling Boolean measure algebra to a quantum measure algebra, is not adequate to determine it entirely, and hence, it contains only a fraction of the total information content included in it. This problem may be tackled, only if, we employ many appropriate structure preserving maps from the modeling Boolean measure algebras to a quantum measure algebra simultaneously, so as to cover it completely. In turn the information available about each map of the specified kind may be used to determine the quantum measure algebra itself. In this case we conceive the family of such maps as the generator of a system of localizations for a quantum measure algebra. The notion of local is characterized using a notion of topology on $\mathcal{C}$, the axioms of which express closure conditions on the collection of modeling algebras of Boolean coefficient probability measures.

7.1 The Notion of Grothendieck Topology on $\mathcal{C}$

We start our discussion by explicating the notion of a topology on the category of Boolean measure algebras $\mathcal{C}$. A topology on $\mathcal{C}$ is a system of arrows $\mathbf{\Lambda}$, where for each object ${}^{P}A$ there is a set $\mathbf{\Lambda}({}^{P}A)$ that contains indexed families of $\mathcal{C}$-morphisms,

$$\mathbf{\Lambda}({}^{P}A) = \{\psi_i : {}^{R}C_i \to {}^{P}A, i \in I\}$$

that is, Boolean homomorphisms to ${}^{P}A$, such that certain appropriate conditions are satisfied.

The notion of a topology on the category of Boolean measure algebras $\mathcal{C}$ is a categorical generalization of a system of set-theoretical covers on a topology $\mathbf{T}$, where a cover for $U \in \mathbf{T}$ is a set $\{U_i : U_i \in \mathbf{T}, \mathbf{i} \in \mathbf{I}\}$ such that $\cup U_i = U$. The generalization is achieved by noting that the topology ordered by inclusion is a poset category and that any cover corresponds to a collection of inclusion arrows $U_i \to U$. Given this fact, any family of arrows contained in $\mathbf{\Lambda}({}^{P}A)$ of a topology is a cover as well.

The specification of a categorial or Grothendieck topology on the category of Boolean measure algebras takes place through the introduction of appropriate covering devices, called covering sieves. For an object ${}^{P}A$ in $\mathcal{C}$, a ${}^{P}A$-sieve is a family ϱ of $\mathcal{C}$-morphisms with codomain ${}^{P}A$, such that if ${}^{R}C \to {}^{P}A$ belongs to ϱ and ${}^{Q}D \to {}^{R}C$ is any $\mathcal{C}$-morphism , then the composite ${}^{Q}D \to {}^{R}C \to {}^{P}A$ belongs to ϱ.

A Grothendieck topology on the category of Boolean measure algebras

$\mathcal{C}$, is a system J of sets, $J({}^PA)$ for each ${}^PA$ in $\mathcal{C}$, where each $J({}^PA)$ consists of a set of ${}^PA$-sieves, (called the covering sieves), that satisfy the following conditions:

1. For any ${}^PA$ in $\mathcal{C}$ the maximal sieve $\{g : cod(g) = {}^PA\}$ belongs to $J({}^PA)$ (maximality condition).

2. If ϱ belongs to $J({}^PA)$ and $f : {}^RC \to {}^PA$ is a $\mathcal{C}$-morphism, then $f^*(\varrho) = \{h : {}^RC \to {}^PA, f \cdot h \in \varrho\}$ belongs to $J({}^RC)$ (stability condition).

3. If ϱ belongs to $J({}^PA)$ and S is a sieve on ${}^RC$, where for each $f : {}^RC \to {}^PA$ belonging to ϱ, we have $f^*(S)$ in $J({}^RC)$, then S belongs to $J({}^PA)$ (transitivity condition).

The small category $\mathcal{C}$ together with a Grothendieck topology $\mathbf{J}$, is called a Boolean measure algebras site.

7.2 The Grothendieck Topology of Epimorphic Families

We consider $\mathcal{C}$ as a model category, whose set of objects $\{{}^{P_i}A_i : i \in I\}$, I: index set, generate $\mathcal{Q}$, in the sense that,

$$\mathbf{M}({}^{P_i}A_i) \xrightarrow{w_i} {}^pL \underset{u}{\overset{v}{\rightrightarrows}} {}^tK$$

the identity $v \circ w_i = u \circ w_i$, for every arrow $w_i : \mathbf{M}({}^{P_i}A_i) \to {}^pL$, and every ${}^{P_i}A_i$, implies that $v = u$. Equivalently we can say that the set of all arrows $w_i : \mathbf{M}({}^{P_i}A_i) \to {}^pL$, constitute an epimorphic family.

The consideration that $\mathcal{C}$ is a generating model category of $\mathcal{Q}$ points exactly to the depiction of the appropriate Grothendieck topology on $\mathcal{C}$.

We assert that a sieve S on a Boolean measure algebra ${}^{P}A$ in $\mathcal{C}$ is to be a covering sieve of ${}^{P}A$, when the arrows $s : {}^{R}C \to {}^{P}A$ belonging to the sieve S together form an epimorphic family in $\mathcal{Q}$. This requirement may be equivalently expressed in terms of a map

$$\Phi_S : \coprod_{(s:{}^{R}C \to {}^{P}A) \in S} {}^{R}C \to {}^{P}A$$

being an epi in $\mathcal{Q}$.

We will show that the choice of covering sieves on Boolean measure algebras ${}^{P}A$ in $\mathcal{C}$, as being epimorphic families in $\mathcal{Q}$, does indeed define a Grothendieck topology on $\mathcal{C}$.

First of all we notice that the maximal sieve on each Boolean measure algebra ${}^{P}A$, includes the identity ${}^{P}A \to {}^{P}A$, thus it is a covering sieve. Next, the transitivity property of the depicted covering sieves is obvious. It remains to demonstrate that the covering sieves remain stable under pullback. For this purpose we consider the pullback of such a covering sieve S on ${}^{P}A$ along any arrow $h : {}^{Q}D \to {}^{P}A$ in $\mathcal{C}$

$$\begin{array}{ccc}
\coprod_{s \in S} {}^{R}C \times_{{}^{P}A} {}^{Q}D & \longrightarrow & {}^{Q}D \\
\Big\downarrow & & \Big\downarrow h \\
\coprod_{s \in S} {}^{R}C & \xrightarrow{\Phi} & {}^{P}A
\end{array}$$

The Boolean measure algebras ${}^{P}A$ in $\mathcal{C}$ generate the category of quantum measure algebras $\mathcal{Q}$, hence, there exists for each arrow $s : {}^{R}C \to {}^{P}A$ in S, an epimorphic family of arrows $\coprod [{}^{T}E]^{s} \to {}^{R}C \times_{{}^{P}A} {}^{Q}D$, or equivalently $\{[{}^{T}E]^{s}{}_{j} \to {}^{R}C \times_{{}^{P}A} {}^{Q}D\}_{j}$, with each domain $[{}^{T}E]^{s}$ a Boolean measure algebra.

Consequently the collection of all the composites:

$$[{}^{T}E]^{s}{}_{j} \to {}^{R}C \times_{{}^{P}A} {}^{Q}D \to {}^{Q}D$$

for all $s : {}^{R}C \to {}^{P}A$ in S, and all indices j together form an epimorphic family in $\mathcal{Q}$, that is contained in the sieve $h^{*}(S)$, being the pullback of S along $h : {}^{Q}D \to {}^{P}A$. Therefore the sieve $h^{*}(S)$ is a covering sieve.

7.3 Covering Sieves as Localization Systems

If we consider a quantum measure algebra ${}^{p}L$, and all quantum algebraic morphisms of the form $\psi_{{}^{P}A} : \mathbf{M}({}^{P}A) \longrightarrow {}^{p}L$, with domains ${}^{P}A$, in the generating model category of Boolean measure algebras $\mathcal{C}$, then the family of all these maps $\psi_{{}^{P}A}$, constitute an epimorphism:

$$S : \coprod_{({}^{P}A \in \mathcal{C}, \psi_{{}^{P}A} : \mathbf{M}({}^{P}A) \to {}^{p}L)} \mathbf{M}({}^{P}A) \to {}^{p}L$$

We say that a sieve on a quantum measure algebra defines a covering sieve by objects of its generating model category $\mathcal{C}$, when the quantum algebraic morphisms belonging to the sieve define the preceding epimorphism.

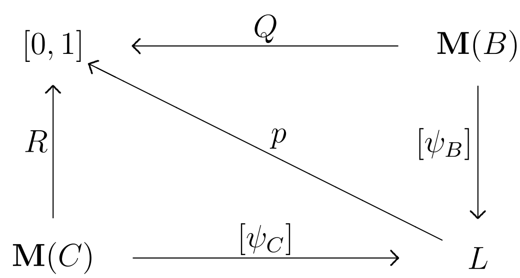
From a physical perspective covering sieves by Boolean measure algebras, are equivalent with Boolean localization systems of quantum measure algebras. These localization systems filter the information of a quantum measure algebra through Boolean domains, associated with procedures of localization in measurement environments. We will discuss localizations systems in detail, in order to unravel the physical meaning of the requirements underlying the notion of Grothendieck topology, and subsequently, the notion of covering sieves defined previously. It is instructive to begin with the notion of a system of prelocalizations for a quantum measure algebra.

A **system of prelocalizations** for a quantum measure algebra ${}^{p}L$ in $\mathcal{Q}$ is a subfunctor of the Hom-functor $\mathbf{R}({}^{p}L)$ of the form $\mathbf{S} : \mathcal{C}^{op} \rightarrow \mathbf{Sets}$, namely for all ${}^{P}A$ in $\mathcal{C}$ it satisfies $\mathbf{S}({}^{P}A) \subseteq [\mathbf{R}({}^{p}L)]({}^{P}A)$. Hence a system of prelocalizations for quantum measure algebra ${}^{p}L$ in $\mathcal{Q}$ is an ideal $\mathbf{S}({}^{P}A)$ of quantum algebraic morphisms of the form

$$\psi_{{}^{P}A} : \mathbf{M}({}^{P}A) \longrightarrow {}^{p}L, \qquad {}^{P}A \in \mathcal{C}$$

such that $\{\psi_{{}^{P}A} : \mathbf{M}({}^{P}A) \longrightarrow {}^{p}L$ in $\mathbf{S}({}^{P}A)$, and $\mathbf{A}(v) : \mathbf{M}({}^{R}C) \rightarrow \mathbf{M}({}^{P}A)$ in $\mathcal{Q}$ for $v : {}^{R}C \rightarrow {}^{P}A$ in $\mathcal{C}$, implies $\psi_{{}^{P}A} \circ \mathbf{M}(v) : \mathbf{M}({}^{R}C) \longrightarrow \mathcal{Q}$ in $\mathbf{S}({}^{P}A)\}$.

The introduction of the notion of a system of prelocalizations is forced on the basis of operational physical arguments. According to Kochen-Specker theorem it not possible to understand completely a quantum mechanical system with the use of a single system of Boolean devices. On the other side, in every concrete experimental context, the set of events that have been actualized in this context forms a Boolean algebra. In the light of this we can say that any Boolean domain object $(B, [\psi_B] : \mathbf{M}(B) \longrightarrow L)$ in a system of prelocalizations for a quantum event algebra L, making the diagram below commutative, corresponds to a set of Boolean classical events that become actualized in the experimental context of B. These Boolean domains play the role of localizing devices in a quantum event structure, that are induced by measurement situations. The above observation is equivalent to the statement that a measurement-induced Boolean algebra serves as a reference frame, in a topos-theoretical environment, relative to which a measurement result is being coordinatized. Correspondingly, by commutativity of the diagram below, we obtain naturally the notion of coordinatizing Boolean measure algebras in a system of prelocalizations for a quantum measure algebra over a quantum event algebra L. The same notion suggests an effective way of comprehending quantum theory in a contextual perspective, pointing to a relativity principle of a topos-theoretical origin. Concretely it supports the assertion that the quantum world is the universe of varying Boolean reference frames, which interconnect to form a coherent picture in a non-trivial way.

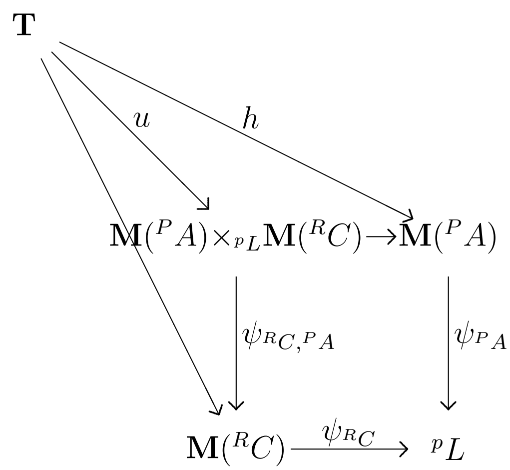


Adopting the aforementioned perspective on quantum measure algebraic structures, the operation of the Hom-functor $\mathbf{R}({}^pL)$ is equivalent to depicting an ideal of morphisms which are to play the role of local coverings of a quantum measure algebra by modeling objects. The notion of a system of prelocalizations formalizes an intuitive idea, according to which, if we sent many coordinatizing Boolean measure algebras into the quantum measure algebra homomorphically, then we would expect that these modeling objects would prove to be enough for the complete determination of the quantum measure algebra. If we consider a geometrical viewpoint, we may legitimately characterize metaphorically the maps $\psi_{^PA} : \mathbf{M}({}^PA) \longrightarrow {}^pL$, where

${}^{P}A$ in $\mathcal{C}$, in a system of prelocalizations for quantum measure algebra ${}^{p}L$ as Boolean measure algebra charts. Correspondingly the modeling Boolean domain objects $(A, [\psi_A] : \mathbf{M}(A) \longrightarrow L)$ in a system of prelocalizations for a quantum event algebra, making the diagram above commutative, may be characterized as measurement charts. Subsequently, their domains A may be called Boolean coefficient domains induced by measurement, the elements of A measured local Boolean coefficients, and the elements of L quantum events, (or quantum propositions in a logical interpretation), coordinatized by Boolean coefficients. Finally, the Boolean morphisms $v : D \longrightarrow A$ in $\mathcal{B}$ play the equivalent role of transition maps.

Under these intuitive identifications, we say that a family of Boolean measure algebra charts $\psi_{{}^{P}A} : \mathbf{M}({}^{P}A) \longrightarrow {}^{p}L$, ${}^{P}A$ in $\mathcal{C}$, (or correspondingly a family of Boolean measurement charts $[\psi_A] : \mathbf{M}(A) \longrightarrow L$ making the diagram above commutative), is the generator of the system of prelocalization $\mathbf{S}$ iff this system is the smallest among all that contains that family. It is evident that a quantum measure algebra, and correspondingly the quantum event algebra over which it is defined, can have many systems of measurement prelocalizations, that, remarkably, form an ordered structure. More specifically, systems of prelocalization constitute a partially ordered set under inclusion. Furthermore, the intersection of any number of systems of prelocalization is again a system of prelocalizations. We emphasize that the minimal system is the empty one, namely $\mathbf{S}({}^{P}A) = \emptyset$, for all ${}^{P}A$ in $\mathcal{C}$, whereas the maximal system is the Hom-functor $\mathbf{R}({}^{p}L)$ itself, or equivalently, all quantum algebraic morphisms $\psi_{{}^{P}A} : \mathbf{M}({}^{P}A) \longrightarrow {}^{p}L$, for all ${}^{P}A$ in $\mathcal{C}$.

The transition from a system of prelocalizations to a system of localizations for a quantum measure algebra, can be effected under the restriction that, certain compatibility conditions have to be satisfied on the overlap of the modeling Boolean coefficients domains covering the quantum measure algebra under investigation. In order to accomplish this we use a pullback diagram in $\mathcal{Q}$ as follows:



The pullback of the Boolean charts $\psi_{{}^{P}A} : \mathbf{M}({}^{P}A) \longrightarrow {}^{p}L$, ${}^{P}A$ in $\mathcal{C}$, and $\psi_{{}^{R}C} : \mathbf{M}({}^{R}C) \longrightarrow {}^{p}L$, ${}^{R}C \in \mathcal{C}$ with common codomain the quantum measure algebra ${}^{p}L$, consists of the object $\mathbf{M}({}^{P}A) \times_{{}^{p}L} \mathbf{M}({}^{R}C)$ and two arrows $\psi_{{}^{P}A{}^{R}C}$ and $\psi_{\ {}^{R}C{}^{P}A}$, called projections, as shown in the above diagram. The square commutes and for any object T and arrows h and g that make the outer square commute, there is a unique $u : T \longrightarrow \mathbf{M}({}^{P}A) \times_{{}^{p}L} \mathbf{M}({}^{R}C)$ that makes the whole diagram commutative. Hence we obtain the condition:

$$\psi_{{}^{R}C} \circ g = \psi_{{}^{P}A} \circ h$$

The pullback of the Boolean measure algebra charts $\psi_{{}^{P}A} : \mathbf{A}({}^{P}A) \longrightarrow {}^{p}L$, ${}^{P}A$ in $\mathcal{C}$, and $\psi_{{}^{R}C} : \mathbf{M}({}^{R}C) \longrightarrow {}^{p}L$, ${}^{R}C$ in $\mathcal{C}$, is equivalently characterized as their fiber product, because $\mathbf{M}({}^{P}A)\times_{{}^{P}L}\mathbf{M}({}^{R}C)$ is not the whole product $\mathbf{A}({}^{P}A)\times\mathbf{M}({}^{R}C)$ but the product taken fiber by fiber. We notice that if $\psi_{{}^{P}A}$ and $\psi_{{}^{R}C}$ are injective, then their pullback is isomorphic with the intersection $\mathbf{M}({}^{P}A)\cap\mathbf{M}({}^{P'}A)$. Then we can define the pasting map, which is an isomorphism, as follows:

$$\Omega_{{}^{P}A,{}^{R}C}:\psi_{{}^{R}C{}^{P}A}(\mathbf{M}({}^{P}A)\times_{{}^{P}L}\mathbf{M}({}^{R}C))\longrightarrow\psi_{{}^{P}A{}^{R}C}(\mathbf{M}({}^{P}A)\times_{{}^{P}L}\mathbf{M}({}^{R}C))$$

by putting

$$\Omega_{{}^{P}A,{}^{R}C}=\psi_{{}^{P}A{}^{R}C}\circ{\psi_{{}^{R}C{}^{P}A}}^{-1}$$

Then we have the following cocycle conditions:

$$\Omega_{{}^{P}A,{}^{P}A}=1_{{}^{P}A}\qquad\qquad 1_{{}^{P}A}:=id_{{}^{P}A}$$

$$\Omega_{{}^{P}A,{}^{R}C}\circ\Omega_{{}^{R}C,{}^{T}E}=\Omega_{{}^{P}A,{}^{T}E}\qquad if\quad \mathbf{M}({}^{P}A)\cap\mathbf{M}({}^{R}C)\cap\mathbf{M}({}^{T}E)\neq 0$$

$$\Omega_{{}^{P}A,{}^{R}C}=\Omega^{-1}{}_{{}^{R}C,{}^{P}A}\qquad if\quad \mathbf{M}({}^{P}A)\cap\mathbf{M}({}^{R}C)\neq 0$$

The pasting map assures that the mapping $\psi_{{}^{R}C{}^{P}A}(\mathbf{M}({}^{P}A)\times_{{}^{P}L}\mathbf{M}({}^{R}C))$, and also, $\psi_{{}^{P}A{}^{R}C}(\mathbf{M}({}^{P}A)\times_{{}^{P}L}\mathbf{M}({}^{R}C))$ are going to cover the same part of the quantum measure algebra in a compatible way. It is obvious that the above compatibility conditions are translated immediately to corresponding compatibility conditions concerning Boolean measurement charts on the quantum event structure.

Given a system of prelocalizations for a quantum measure algebra ${}^{P}L$ in $\mathcal{Q}$, and correspondingly for the quantum event algebra over which it is defined, we call it a **system of localizations** iff the above compatibility conditions are satisfied and moreover the quantum algebraic structure is preserved.

We assert that the above compatibility conditions provide the necessary relations for understanding a system of localizations for a quantum measure algebra as a structure sheaf or sheaf of Boolean coefficients, consisting of local Boolean measure algebras. This is related to the observation that systems of localizations are actually subfunctors of the representable Hom-functor $\mathbf{R}({}^{p}L)$ of the form $\mathbf{S} : \mathcal{C}^{op} \to \mathbf{Sets}$, namely for all ${}^{P}A$ in $\mathcal{C}$ satisfy $\mathbf{S}({}^{P}A) \subseteq [\mathbf{R}({}^{p}L)]({}^{P}A)$. In this sense the pullback compatibility conditions express gluing relations on overlaps of Boolean measure algebra charts and convert a presheaf subfunctor of the Hom-functor into a sheaf for the Grothendieck topology specified. The concept of sheaf expresses exactly the pasting conditions that local Boolean coefficients algebras have to satisfy, namely, the way by which local data can be collated together into global ones. We stress the point that the transition from locally defined properties to global consequences happens via a compatible family of elements over a cover of the global object. A cover, or equivalently a localization system of the global, object, being a quantum measure algebra structure in the present scheme, can be viewed as providing a decomposition of that object into simpler modeling objects.

The comprehension of a localization system as a sheaf of Boolean coefficients permits the conception of a quantum measure algebra (or of its associated quantum event algebra) as a generalized Boolean manifold, obtained by pasting the $\psi_{{}^{R}C\,{}^{P}A}(\mathbf{M}({}^{P}A) \times_{{}^{p}L} \mathbf{M}({}^{R}C))$ and $\psi_{{}^{P}A\,{}^{P}\acute{A}}(\mathbf{M}({}^{P}A) \times_{{}^{p}L} \mathbf{M}({}^{R}C))$ covers together by the transition functions $\Omega_{{}^{P}A,{}^{R}C}$.

More specifically, the equivalence relations in the category of elements of such a structure sheaf, represented by a Boolean system of Boolean probabilities coefficients, have to be taken into account according to the analysis of the adjoint relation presented in Section 6. Equivalence relations of this form, give rise to congruences in the structure sheaf of Boolean coefficients, which are expressed categorically as a colimit in the category of elements of such a structure sheaf. In this perspective the generalized manifold, which represents categorically a quantum measure algebra, is understood as a colimit in a sheaf of Boolean coefficients, that contains compatible families of modeling Boolean measure algebras. It is instructive to emphasize that the organization of Boolean coordinatizing objects in localization systems takes the form of interconnection of these modeling objects through the categorical construction of colimit, the latter being the means to comprehend an object of complex structure (quantum measure algebra) from simpler coefficient objects (Boolean measure algebras).

The above ideas provide the basis for the formulation of a sheaf-theoretic representation theorem concerning quantum measure algebras as we shall present in the following Section.

8 Representation of Quantum Measure Algebras

8.1 Unit and Counit of the Adjoint Relation

We focus again our attention in the fundamental adjoint relation established, and investigate the unit and the counit of it. For any presheaf $\mathbf{X}$ in the topos $\mathbf{Sets}^{\mathcal{C}^{op}}$, the **unit** $\delta_{\mathbf{X}} : \mathbf{X} \longrightarrow Hom_{\mathcal{Q}}(\mathbf{M}(\_), \mathbf{X} \otimes_{\mathcal{C}} \mathbf{M})$ has components:

$$\delta_{\mathbf{X}}({}^{P}A) : \mathbf{X}({}^{P}A) \longrightarrow Hom_{\mathcal{Q}}(\mathbf{M}({}^{P}A), \mathbf{X} \otimes_{\mathcal{C}} \mathbf{M})$$

for each Boolean measure algebra ${}^{P}A$ in $\mathcal{C}$.

If we make use of the representable presheaf $y[{}^{P}A]$ we obtain

$$\delta_{\mathbf{y}[{}^{P}A]} : \mathbf{y}[{}^{P}A] \to Hom_{\mathcal{Q}}(\mathbf{M}(\_), \mathbf{y}[{}^{P}A] \otimes_{\mathcal{C}} \mathbf{M})$$

Hence for each object ${}^{P}A$ of $\mathcal{C}$ the unit, in the case considered, corresponds to a map

$$\mathbf{M}({}^{P}A) \to \mathbf{y}[{}^{P}A] \otimes_{\mathcal{C}} \mathbf{M}$$

But since

$$\mathbf{y}[{}^{P}A] \otimes_{\mathcal{C}} \mathbf{M} \cong \mathbf{M}({}^{P}A)$$

the unit for the representable presheaf of Boolean measure algebras is clearly an isomorphism. By the preceding discussion we conclude that the following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\quad \mathbf{A} \quad} & \mathcal{Q} \\
\downarrow{\scriptstyle \mathbf{y}} & \nearrow{\scriptstyle [-]\otimes_{\mathcal{Q}}\mathbf{M}} & \\
\mathbf{S}ets^{\mathcal{C}^{op}} & &
\end{array}
$$

Thus the unit of the fundamental adjunction referring to the representable presheaf of the category of Boolean measure algebras provides a quantum algebraic morphism, $\mathbf{M}({}^{P}A)\longrightarrow \mathbf{y}[{}^{P}A]\otimes_{\mathcal{C}}\mathbf{M}$, which is an isomorphism.

On the other side, for each quantum measure algebra ${}^{p}L$ in $\mathcal{Q}$ the **counit** is defined as follows:

$$\epsilon_{{}^{p}L} : Hom_{\mathcal{Q}}(\mathbf{M}(\_), {}^{p}L)\otimes_{\mathcal{C}}\mathbf{M}\longrightarrow {}^{p}L$$

The counit corresponds to the vertical map in the diagram below:

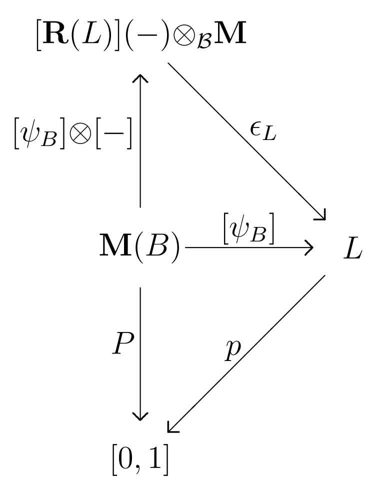
$$
\begin{array}{ccccc}
\coprod_{v:{}^{P}\acute{A}\to {}^{P}A}\mathbf{M}({}^{P}\acute{A}) & \overset{{}^{R}C}{\underset{\eta}{\rightrightarrows}} & \coprod_{({}^{P}A,p)}\mathbf{M}({}^{P}A) & \longrightarrow & [\mathbf{R}({}^{p}L)](-)\otimes_{\mathcal{C}}\mathbf{M} \\
 & & & \searrow & \vdots\,{\scriptstyle \epsilon} \\
 & & & & {}^{p}L
\end{array}
$$

8.2 Boolean Representation

The sheaf-theoretic representation of a quantum measure algebra in terms of Boolean measure localization systems, is formulated in terms of the following proposition, effectuated by means of the vertical counit map in the preceding diagram:

The representation of a quantum measure algebra ${}^{p}L$ in $\mathcal{Q}$, in terms of a coordinatization system of Boolean measure algebras localizations $\mathbf{S}$, consisting of Boolean probability coefficients, is full and faithful, if and only if the counit of the established adjoint relation, restricted to that system, is an isomorphism, that is, structure-preserving, 1-1 and onto.

It is easy to see that the counit of the adjunction, restricted to a system of Boolean measure algebras localizations is a quantum algebraic isomorphism, iff the right adjoint functor is full and faithful, or equivalently, iff the cocone from the functor $\mathbf{M} \circ \int_{\mathbf{R}({}^pL)}$ to the quantum measure algebra ${}^pL$ is universal for each ${}^pL$ in $\mathcal{Q}$. In the latter case we characterize the Boolean measure coefficients functor $\mathbf{M} : \mathcal{C} \longrightarrow \mathcal{Q}$, a proper modeling functor. As a consequence if we consider as $\mathcal{B}$ the category of Boolean subalgebras of a quantum event algebra L of ordinary quantum Mechanics, that is an orthomodular σ-orthoposet of orthogonal projections of a Hilbert space, together with a proper modeling inclusion functor $\mathbf{M} : \mathcal{B} \longrightarrow \mathcal{L}$, such that the diagram below commutes, the counit of the established adjunction restricted to a system of Boolean localizations is an isomorphism.



$$\epsilon_L : \mathbf{R}(L) \otimes_{\mathcal{B}} \mathbf{M} \rightarrow^{\cong} L$$

such that;

$$[\psi_B] = \epsilon_L \circ ([\psi_B] \otimes \_)$$

or in the notation of elements equivalently:

$$\epsilon_L([\psi_B] \otimes a) = [\psi_B](a), \qquad a \in \mathbf{M}(B)$$

where $p([\psi_B](a)) = (P(a))$, for all $[\psi_B] : \mathbf{M}(B) \longrightarrow L$ according to the commutative diagram above.

9 Conclusions

The primary physical motivation of this paper has been the implementation and explicit construction of an appropriate localization process suited to quantum physical observation, and in particular, the study of its consequences refering to the interpretation of quantum probabilistic structures. The crucial ideas and techniques related with the objective of interpreting quantum measure algebras sheaf-theoretically in the topos-theoretic environment of Grothendieck sites, are based on extension and elaboration of previous works of the author, communicated, both conceptually and technically, in the literature [23-27]. The defining characteristic of the topos-theoretic perspective enunciated by the author in this endeavor has been the change of resolution focus form point-set to variable topological localization models of quantum algebraic structures, that effectively, induce a transition in the semantics of global quantum event observable and probability algebras from a set-theoretic to a sheaf-theoretic one. The significance and semantic differentiation of this work in relation to the foundations of quantum theory can be cast in the form of the following statements:

1. Conceptually, the physical meaning of the notion of localization is being disassociated from its restricted spatial connotation reference context. We have argued that this is an essential and necessary reconceptualization of the meaning of locality in relational information terms forced by the quantum description of physical systems.

2. A suitable localization process of global quantum event and probabilistic structures that respects the premises of the quantum theory of measurement is being formulated in terms of Boolean localization systems, described categorically in terms of an appropriate Grothendieck topology, that incorporate the constitutive requirements of the notion of Boolean localization in functorial relational terms.

3. Global quantum event and probabilistic structures are being functionally and functorially dependent on generalized topological localization measures induced by the preparation of Boolean structured domains of measurement, not necessarily based on the existence of an underlying set-theoretic structure of points on the real line.

4. The sheaf-theoretic semantic transition of quantum measure algebras has been forced by means of gluing cocycle conditions over an explicitly constructed uniform and homologous fibered representation of quantum states with respect to local Boolean reference frames for the Grothendieck topology of epimorphic families. According to this representation, quantum states have been conceptualized as equivalence classes of local Boolean coordinates

with respect to those reference frames. Subsequently, it has been constructed an isomorphic representation of quantum measure algebras with colimits taken in the categories of elements of sheaves of Boolean reference frames.

5. The physical significance of the sheaf-theoretic representation of quantum measure algebras is encapsulated in the realization that the whole information content of a quantum measure algebra is preserved by the action of some covering system, if and only if that system forms a Boolean localization system. Hence, the significance of a quantum measure algebra is shifted from the orthoposet axiomatization at the level of events, to the sheaf-theoretic gluing conditions at the level of Boolean localization systems.

6. The preservation of quantum information property according to the above is being formally established by the counit of the related adjunction isomorphism. More specifically, the surjective property of the counit guarantees that the Boolean localization measures, representing objects in the category of elements of the sheaf $\int(\mathbf{R}({}^pL), \mathcal{C})$, cover entirely a quantum measure algebra ${}^pL$, whereas its injective property guarantees that any two covers are compatible in a system of localizations. Moreover, since the counit is also a homomorphism, it preserves the algebraic structure.

7. The physical content of the sheaf-theoretic representation of quantum events algebras can be formulated in terms of a functoriality property. According to this, the information content of a quantum measure algebra is covariant under the groupoid of gluing isomorphisms between overlapping local Boolean reference frames, along their intersections, in a Boolean localization system.

8. In the physical state of affairs, each cover corresponds to a Boolean measure algebra of events realized locally (with respect to the Grothendieck topology of epimorphic families) in a measurement situation. The equivalence classes of local Boolean measure coefficients represent quantum states in ${}^{p}L$, via the sheaf-theoretic pullback compatibility conditions. In this sense, the notion of quantum probability is basically classical when interpreted locally *à la Grothendieck*. Moreover, the probabilities of actualization of events in equivalent local measurement environments are equal.

9. Conclusively, the structure of a quantum measure algebra is being generated by the information that its structure preserving morphims, encoded as Boolean covers in localization systems carry, as well as their compatibility relations. Most significantly, the same compatibility conditions provide the necessary relations for understanding a system of localizations for a quantum probabilistic structure, as a structure sheaf of Boolean measure coefficients associated with local contexts of measurement.

Finally, it would be instructive to comment briefly on the possible implications of the proposed topos-theoretic interpretation schema of quantum structures, based on a reconceptualization of the notion of physical localization, in relation to the ongoing research on quantum relativity and quantum gravity. A preliminary account of the attempt to establish a connective link with the conception of a categorical theory of covariant quantum gravitational dynamics based on the utilization of topological localization systems in the physical "continuum" is in the phase of intense development, while some basic ideas and results related with this program have been already communicated [28]. In the context of that work we initiate a sheaf-theoretic dynamical analysis of quantum observable structures by synthesizing the flexible categorical machinery of Grothendieck topoi, together with, the powerful sheaf-theoretic methodology of Mallios's Abstract Differential Geometry [18].

The crucial physical issue incorporated in the idea of generalized topological localization processes, conceived in the sense of Grothendieck topologies on a base category of structured reference contexts, is related to a novel topos-theoretic conception of the physical "continuum". According to this conception the quantum regime of observable dynamical phenomena should be understood in functorial terms of categorically localized information, and not in the restricted classical localization terms conceived by means of metrical properties on a pre-existing smooth set-theoretic spacetime manifold. Subsequently, that semantic transition can be implemented conceptually and technically by the replacement of the classical variable metrical ruler of localization on a smooth background spacetime manifold, with a variable sheaf-cohomological ruler of categorical localization in a Grothendieck topos, that captures the relational information of observables in the quantum regime, filtered through local reference frames in that topos. Then, the dynamical properies of quantum structures can be addressed to the global topos-theoretic dynamics generated by interlocking diagrams of local frames in that topos, giving rise to generalized De Rham complexes of sheaves encapsulating cohomologically the corresponding dynamical behavior.

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