QUANTUM EVENT STRUCTURES FROM THE PERSPECTIVE OF GROTHENDIECK TOPOI

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Abstract

We develop a categorical scheme of interpretation of quantum event structures from the viewpoint of Grothendieck topoi. The construction is based on the existence of an adjunctive correspondence between Boolean presheaves of event algebras and Quantum event algebras, which we construct explicitly. We show that the established adjunction can be transformed to a categorical equivalence if the base category of Boolean event algebras, defining variation, is endowed with a suitable Grothendieck topology of covering systems. The scheme leads to a sheaf theoretical representation of Quantum structure in terms of variation taking place over epimorphic families of Boolean reference frames.

Key words: Quantum event structures, Boolean reference frames, Topos, Adjunction, Sheaves, Grothendieck Topology .

1 INTRODUCTION

The foundational issues implied by the structure of events displayed by the behavior of quantum mechanical systems [1-3] deserve special attention since they constitute a conceptual shift in the globally Boolean descriptive rules characterizing classical systems. Most importantly, they are amenable to an analysis which is based on a simple, but rich in consequences, methodological principle. According to this, we are guided in studying a globally non-Boolean object, like a quantum algebra of events (or quantum logic), in terms of structured families of simpler, adequately comprehended local objects (in our case Boolean event algebras), which have to satisfy certain compatibility relations, and also, a closure constraint. Hence Boolean descriptive contexts are not abandoned once and for all, but instead are used locally, accomplishing the task of providing partial congruent relations with globally non-Boolean objects, the internal structure and functioning of which, is being hopefully recovered by the interconnecting machinery governing the local objects. This point of view inevitably leads to a relativistic conception of quantum theory as a whole, and stresses the contextual character of the theory. In order to reveal these aspects of quantum theory, which pertain the nature of quantum events, a suitable mathematical language has to be used. The criterion for choosing an appropriate language is rather emphasis in the form of the structures and the universality of the constructions involved. The ideal candidate for this purpose is provided by category theory [4-10]. Subsequently, we will see that sheaf theory [11-13] is the appropriate mathematical vehicle to carry out the program implied by the aforementioned methodological principle of enquiry.

In a previous work [14,15], we have constructed a representation of quantum event algebras in terms of compatible families of Boolean localization systems. This representation has been motivated by the physical significance of Kochen-Specker theorem, understood as expressing the impossibility of probing the entire manifestation of a quantum system, in terms of observable attributes, with the use of a single universal Boolean device. In this paper, we will focus on an equivalent sheaf-theoretical conception of quantum event algebras (and subsequently quantum logics), based on the notion of a topology on a category, and the construction of sheaves for this topology. This conception sheds more light on the connection between a quantum algebra of events and its underlying building blocks of Boolean algebras, and clarifies the intrinsic contextuality of quantum events.

The concept of sheaf expresses essentially gluing conditions, namely the way by which local data can be collated in global ones. It is the appropriate mathematical tool for the formalization of the relations between covering systems and properties, and, furthermore, provides the means for studying the global consequences of locally defined properties. The notion of local is characterized using a topology (in the general case a Grothendieck topology on a category), the axioms of which express closure conditions on the collection of covers. Essentially a map which assigns a set to each object of a topology is called a sheaf if the map is defined locally, or else the value of the map on an object can be uniquely obtained from its values on any cover of that object. Categorically speaking, besides mapping each object to a set, a sheaf maps each arrow in the topology to a restriction function in the opposite direction. We stress the point that the transition from locally defined properties to global consequences happens via a compatible family of elements over a cover of the complex object. A cover of the global, complex object can be viewed as providing a decomposition of that object into simpler objects. The sheaf assigns a set to each element of the cover, or else each piece of the original object. A choice of elements from these sets, one for each piece, forms a compatible family if the choice respects the mappings by the restriction functions and if the elements chosen agree whenever two pieces of the cover overlap. If such a locally compatible choice induces a unique choice for the object being covered, a global choice, then the condition for being a sheaf is satisfied. We note that in general, there will be more locally defined or partial choices than globally defined ones, since not all partial choices need be extendible to global ones, but a compatible family of partial choices uniquely extends to a global one.

In the following sections we shall see that a quantum event algebra can be understood as a sheaf for a suitable Grothendieck topology on the category of Boolean subalgebras of it. We mention that, contextual topos theoretical approaches to quantum structures have been also considered, from a different viewpoint in [16,17], and discussed in [18-20].

2 THE ADJOINT FUNCTORIAL BOOLEAN-QUANTUM RELATION

2.1 CATEGORIES OF BOOLEAN AND QUANTUM STRUCTURES

Category theory provides a general apparatus for dealing with mathematical structures and their mutual relations and transformations. The basic categorical principles that we adopt in the subsequent analysis are summarized as follows:

[i] To each species of mathematical structure, there corresponds a **category** whose objects have that structure, and whose morphisms preserve it.

[ii] To any natural construction on structures of one species, yielding structures of another species, there corresponds a **functor** from the category of first species to the category of the second.

Categories: A category $\mathcal{C}$ is a class of objects and morphisms of objects such that the following properties are satisfied:

[1]. For any objects X, Y all morphisms $f : X \to Y$ form a set denoted $Hom_{\mathcal{C}}(X, Y)$;

[2]. For any object X an element $id_X \in Hom_{\mathcal{C}}(X, X)$ is distinguished; it is called the identity morphism;

[3]. For arbitrary objects X, Y, Z the set mapping is defined

$$Hom_{\mathcal{C}}(X, Y) \times Hom_{\mathcal{C}}(Y, Z) \to Hom_{\mathcal{C}}(X, Z)$$

For morphisms $g \in Hom_{\mathcal{C}}(X, Y)$, $h \in Hom_{\mathcal{C}}(Y, Z)$ the image of the pair (g, h) is called the composition; it is denoted $h \circ g$. The composition operation is associative.

[4]. For any $f \in Hom_{\mathcal{C}}(X, Y)$ we have $id_Y \circ f = f \circ id_X = f$.

For an arbitrary category $\mathcal{C}$ the opposite category $\mathcal{C}^{op}$ is defined in the following way: the objects are the same, but $Hom_{\mathcal{C}^{op}}(X, Y) = Hom_{\mathcal{C}}(Y, X)$, namely all arrows are inverted.

Functors: Let $\mathcal{C}$, $\mathcal{D}$ be categories; a covariant functor $\mathbf{F} : \mathcal{C} \to \mathcal{D}$ is a class mapping that transforms objects to objects and morphisms to morphisms preserving compositions and identity morphisms:

$$\mathbf{F}(id_X) = id_{\mathbf{F}(X)}; \mathbf{F}(g \circ f) = \mathbf{F}(g) \circ \mathbf{F}(f)$$

A contravariant functor $\mathbf{F} : \mathcal{C} \to \mathcal{D}$ is, by definition, a covariant functor $\mathbf{F} : \mathcal{C} \to \mathcal{D}^{op}$.

A **Classical event structure** is a small category, denoted by $\mathcal{B}$, and called the category of Boolean event algebras. Its objects are Boolean algebras of events, and its arrows are Boolean algebraic homomorphisms.

A **Quantum event structure** is a small category, denoted by $\mathcal{L}$, and called the category of Quantum event algebras.

Its objects are Quantum algebras of events, that is, partially ordered sets of Quantum events, endowed with a maximal element 1, and with an operation of orthocomplementation $[-]^* : L \longrightarrow L$, which satisfy, for all $l \in L$ the following conditions: [a] $l \leq 1$, [b] $l^{**} = l$, [c] $l \vee l^* = 1$, [d] $l \leq \acute{l} \Rightarrow \acute{l}^* \leq l^*$, [e] $l \bot \acute{l} \Rightarrow l \vee \acute{l} \in L$, [f] $l \vee \acute{l} = 1, l \wedge \acute{l} = 0 \Rightarrow l = \acute{l}^*$, where $0 := 1^*$, $l \bot \acute{l} := l \leq \acute{l}^*$, and the operations of meet $\wedge$ and join $\vee$ are defined as usually.

Its arrows are Quantum algebraic homomorphisms, that is maps $L \xrightarrow{H} K$, which satisfy, for all $k \in K$ the following conditions: [a] $H(1) = 1$, [b] $H(k^*) = [H(k)]^*$, [c] $k \leq \acute{k} \Rightarrow H(k) \leq H(\acute{k})$, [d] $k \bot \acute{k} \Rightarrow H(k \vee \acute{k}) \leq H(k) \vee H(\acute{k})$.

It is instructive to emphasize that the definition of a Quantum algebra of events is general enough to accommodate well known special cases, like the lattice of closed subspaces of a complex Hilbert space used in the majority of discussions on the subject of Quantum logic. Moreover, the above definition is not restrictive, since we could legitimately use as a definition of a Quantum event algebra the well-known case of an orthomodular orthoposet, without any change in the arguments of the paper. A detailed discussion of these issues is contained in the Appendix (A.1).

We note parenthetically, that both the categories $\mathcal{B}$ and $\mathcal{L}$ are algebraic categories, and have arbitrary colimits [5].

2.2 BOOLEAN COEFFICIENT FUNCTORS AND FIBRATIONS

We define a modeling or coordinatization functor, $\mathbf{A} : \mathcal{B} \longrightarrow \mathcal{L}$, which assigns to Boolean event algebras in $\mathcal{B}$, that instantiates a model category, the underlying quantum event algebras from $\mathcal{L}$, and to Boolean homomorphisms the underlying quantum algebraic homomorphisms. Hence $\mathbf{A}$ acts as a forgetful functor, forgetting the extra Boolean structure of $\mathcal{B}$.

If $\mathcal{B}^{op}$ is the opposite category of $\mathcal{B}$, then $\mathbf{Sets}^{\mathcal{B}^{op}}$ denotes the functor category of presheaves on Boolean event algebras, with objects all functors $\mathbf{P} : \mathcal{B}^{op} \longrightarrow \mathbf{Sets}$, and morphisms all natural transformations between such functors. Each object $\mathbf{P}$ in this category is a contravariant set-valued functor on $\mathcal{B}$, called a presheaf on $\mathcal{B}$. The functor category of presheaves on Boolean event algebras $\mathbf{Sets}^{\mathcal{B}^{op}}$, provides an instantiation of a structure known as topos. A topos exemplifies a well defined notion of variable set. It can be conceived as a local mathematical framework corresponding to a generalized model of set theory or as a generalized space.

For each Boolean algebra B of $\mathcal{B}$, $\mathbf{P}$(B) is a set, and for each arrow $f : C \longrightarrow B$, $\mathbf{P}(f) : \mathbf{P}(B) \longrightarrow \mathbf{P}$(C) is a set function. If $\mathbf{P}$ is a presheaf on $\mathcal{B}$ and $p \in \mathbf{P}$(B), the value $\mathbf{P}(f)(p)$ for an arrow $f : C \longrightarrow B$ in $\mathcal{B}$ is called the restriction of x along f and is denoted by $\mathbf{P}(f)(p) = p \circ f$.

Each object B of $\mathcal{B}$ gives rise to a contravariant Hom-functor $\mathbf{y}[B] := Hom_{\mathcal{B}}(-, B)$. This functor defines a presheaf on $\mathcal{B}$. Its action on an object C of $\mathcal{B}$ is given by

$$\mathbf{y}[B](C) := Hom_{\mathcal{B}}(C, B)$$

whereas its action on a morphism $D \stackrel{x}{\longrightarrow} C$, for $v : C \longrightarrow B$ is given by

$$\mathbf{y}[B](x) : Hom_{\mathcal{B}}(C, B) \longrightarrow Hom_{\mathcal{B}}(D, B)$$

$$\mathbf{y}[B](x)(v) = v \circ x$$

Furthermore $\mathbf{y}$ can be made into a functor from $\mathcal{B}$ to the contravariant functors on $\mathcal{B}$

$$\mathbf{y} : \mathcal{B} \longrightarrow \mathbf{Sets}^{\mathcal{B}^{op}}$$

such that $B \mapsto Hom_{\mathcal{B}}(-, B)$. This is called the Yoneda embedding and it is a full and faithful functor.

Next we construct the category of elements of $\mathbf{P}$, denoted by $\mathbf{G}(\mathbf{P}, \mathcal{B})$. Its objects are all pairs (B, p), and its morphisms $(\acute{B}, \acute{p}) \rightarrow (B, p)$ are those morphisms $u : \acute{B} \rightarrow B$ of $\mathcal{B}$ for which $pu = \acute{p}$. Projection on the second coordinate of $\mathbf{G}(\mathbf{P}, \mathcal{B})$, defines a functor $\mathbf{G}(\mathbf{P}) : \mathbf{G}(\mathbf{P}, \mathcal{B}) \rightarrow \mathcal{B}$. $\mathbf{G}(\mathbf{P}, \mathcal{B})$ together with the projection functor $\mathbf{G}(\mathbf{P})$ is called the split discrete fibration induced by $\mathbf{P}$, and $\mathcal{B}$ is the base category of the fibration as in the Diagram below. We note that the fibers are categories in which the only arrows are identity arrows. If B is an object of $\mathcal{B}$, the inverse image under $\mathbf{G}(\mathbf{P})$ of B is simply the set $\mathbf{P}(\mathrm{B})$, although its elements are written as pairs so as to form a disjoint union. The construction of the fibration induced by $\mathbf{P}$, is called the Grothendieck construction [13].

$$\begin{array}{ccc} \mathbf{G}(\mathbf{P}, \mathcal{B}) & & \\ {\scriptstyle \mathbf{G}(\mathbf{P})}\downarrow & & \\ \mathcal{B} & \xrightarrow{\mathbf{P}} & \mathbf{Sets} \end{array}$$

2.3 ADJUNCTIVE CORRESPONDENCE OF BOOLEAN PRESHEAVES WITH QUANTUM ALGEBRAS

The adjunctive correspondence, which will be proved in what follows, provides the conceptual ground, concerning the representation of quantum event algebras in terms of sheaves of structured families of Boolean event algebras, and is based on the categorical construction of colimits over the category of elements of a presheaf of Boolean algebras $\mathbf{P}$.

If we consider the category of quantum event algebras $\mathcal{L}$ and the coefficient functor $\mathbf{A}$, we can define the functor $\mathbf{R}$ from $\mathcal{L}$ to the category of presheaves of Boolean event algebras given by:

$$\mathbf{R}(L) : B \mapsto Hom_{\mathcal{L}}(\mathbf{A}(B), L)$$

A natural transformation τ between the presheaves on the category of Boolean algebras $\mathbf{P}$ and $\mathbf{R}(L)$, $\tau : \mathbf{P} \longrightarrow \mathbf{R}(L)$ is a family τ_B indexed by Boolean algebras B of $\mathcal{B}$ for which each τ_B is a map of sets,

$$\tau_B : \mathbf{P}(B) \rightarrow Hom_{\mathcal{L}}(\mathbf{A}(B), L)$$

such that the diagram of sets below commutes for each Boolean homomorphism $u : \acute{B} \to B$ of $\mathcal{B}$.

$$\begin{array}{ccc}
\mathbf{P}(B) & \xrightarrow{\tau_B} & Hom_{\mathcal{L}}(\mathbf{A}(B), L) \\
\mathbf{P}(u) \downarrow & & \downarrow {}^{*}\mathbf{A}(u) \\
\mathbf{P}(\acute{B}) & \xrightarrow{\tau_B} & Hom_{\mathcal{L}}(\mathbf{A}(\acute{B}), L)
\end{array}$$

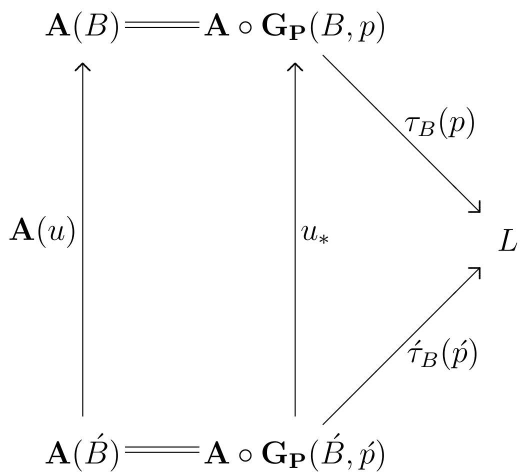
From the perspective of the category of elements of the Boolean algebras-variable set P the map τ_B, defined above, is identical with the map:

$$\tau_B : (B, p) \rightarrow Hom_{\mathcal{L}}(\mathbf{A} \circ G_{\mathbf{P}}(B, p), L)$$

Subsequently such a τ may be represented as a family of arrows of $\mathcal{L}$ which is being indexed by objects (B, p) of the category of elements of the presheaf of Boolean algebras $\mathbf{P}$, namely

$$\{\tau_B(p) : \mathbf{A}(B) \to L\}_{(B,p)}$$

Thus, according to the point of view provided by the category of elements of $\mathbf{P}$, the condition of the commutativity of the diagram on the top, is equivalent to the condition that for each arrow u the following diagram commutes.



Consequently, according to the diagram above, the arrows $\tau_B(p)$ form a cocone from the functor $\mathbf{A} \circ G_{\mathbf{P}}$ to the quantum event algebra L. The categorical definition of colimit, points to the conclusion that each such cocone emerges by the composition of the colimiting cocone with a unique arrow from the colimit $\mathbf{LP}$ to the quantum event algebra object L. Equivalently, we conclude that there is a bijection, natural in $\mathbf{P}$ and L as follows:

$$Nat(\mathbf{P}, \mathbf{R}(L)) \cong Hom_{\mathcal{L}}(\mathbf{LP}, L)$$

The established bijective correspondence, interpreted functorially, says that the functor $\mathbf{R}$ from $\mathcal{L}$ to presheaves given by

$$\mathbf{R}(L) : B \mapsto Hom_{\mathcal{L}}(\mathbf{A}(B), L)$$

has a left adjoint $\mathbf{L} : \mathbf{Sets}^{\mathcal{B}^{op}} \to \mathcal{L}$, which is defined for each presheaf of Boolean algebras $\mathbf{P}$ in $\mathbf{Sets}^{\mathcal{B}^{op}}$ as the colimit

$$\mathbf{L}(\mathbf{P}) = Colim\{\mathbf{G}(\mathbf{P}, \mathcal{B}) \xrightarrow{\mathbf{G_P}} \mathcal{B} \xrightarrow{\mathbf{A}} \mathcal{L}\}$$

An explicit construction of colimits of the above form is presented in detail in the Appendix (A.2).

Consequently there is a pair of adjoint functors $\mathbf{L} \dashv \mathbf{R}$ as follows:

$$\mathbf{L} : \mathbf{Sets}^{\mathcal{B}^{op}} \leftrightarrows \mathcal{L} : \mathbf{R}$$

Thus we have constructed an adjunction which consists of the functors $\mathbf{L}$ and $\mathbf{R}$, called left and right adjoints with respect to each other respectively, as well as the natural bijection

$$\begin{array}{ccc} Nat(\mathbf{P}, \mathbf{R}(L)) & \xrightarrow{\mathbf{r}} & Hom_{\mathcal{L}}(\mathbf{LP}, L) \\ \| & & \| \\ Nat(\mathbf{P}, \mathbf{R}(L)) & \xleftarrow{\mathbf{l}} & Hom_{\mathcal{L}}(\mathbf{LP}, L) \end{array}$$

$$Nat(\mathbf{P}, \mathbf{R}(L)) \cong Hom_{\mathcal{L}}(\mathbf{LP}, L)$$

Furthermore it has been shown [14] that the categorical construction of this colimit as a coequalizer of a coproduct reveals the fact that this left adjoint is like the tensor product $-\otimes_{\mathcal{B}}\mathbf{A}$.

$$\coprod_{v:\acute{B}\to B}\mathbf{A}(\acute{B}) \;\underset{\eta}{\overset{\zeta}{\rightrightarrows}}\; \coprod_{(B,p)}\mathbf{A}(B)\xrightarrow{\chi}\mathbf{P}\otimes_{\mathcal{B}}\mathbf{A}$$

In the diagram above the second coproduct is over all the objects (B,p) with $p \in \mathbf{P}(B)$ of the category of elements, while the first coproduct is over all the maps $v : (\acute{B},\acute{p})\longrightarrow(B,p)$ of that category, so that $v : \acute{B}\longrightarrow B$ and the condition $pv = \acute{p}$ is satisfied. We conclude that the colimit $\mathbf{L}_A(P)$ can be equivalently presented as the coequalizer of the diagram above.

The physical meaning of the adjunction between presheaves of Boolean event algebras and Quantum event algebras is crystallized if we consider that the adjointly related functors are associated with the process of encoding information relevant to the structural form of their domain and codomain categories. Let us consider that $\mathbf{Sets}^{\mathcal{B}^{op}}$ is the universe of Boolean event structures modelled in **Sets** by observers, and $\mathcal{L}$ that of Quantum event structures. In the proposed interpretation the functor $\mathbf{L} : \mathbf{Sets}^{\mathcal{B}^{op}}\longrightarrow\mathcal{L}$ can be comprehended as a translational code from Boolean windows to the Quantum species of event structure, whereas the functor $\mathbf{R} : \mathcal{L}\longrightarrow\mathbf{Sets}^{\mathcal{B}^{op}}$ as a translational code in the inverse direction. In general, the content of the information is not possible to remain completely invariant translating from one language to another and back. However, there remain two ways for a Boolean-event algebra variable set $\mathbf{P}$, or else Boolean window to communicate a message to a quantum event algebra L. Either the information is given in Quantum terms with $\mathbf{P}$ translating, which we can be represented as the quantum homomorphism $\mathbf{LP} \longrightarrow L$, or the information is given in Boolean terms with L translating, that, in turn, can be represented as the natural transformation $\mathbf{P} \longrightarrow \mathbf{R}(L)$. In the first case, from the perspective of L information is being received in quantum terms, while in the second, from the perspective of $\mathbf{P}$ information is being sent in Boolean terms. The natural bijection then corresponds to the assertion that these two distinct ways of communicating are equivalent. Thus, the fact that these two functors are adjoint, expresses a relation of variation regulated by two poles, with respect to the meaning of the information related to observation. In this sense, the adjunction provides the tool for relating relations, by specifying the conditions for a consistent notion of mutually dependent variation, in association to the interpretation of the information content shared by the Boolean and Quantum species of structure. We argue that the totality of the content of information included in the quantum species of event structure remains invariant under Boolean encodings, corresponding to local Boolean modeling algebras, if and only if the adjunctive correspondence can be appropriately restricted to an equivalence of the functorially correlated categories. In the following sections we will show that this task can be accomplished by defining a suitable Grothendieck topology on the category of Boolean event algebras, that, essentially permits the comprehension of a quantum event structure as a sheaf for the specified covering system of the base Boolean category introducing variation, or equivalently as a Grothendieck topos. Subsequently the categorical equivalence will signify an invariance in the translational code of communication between Boolean windows and Quantum systems.

3 MOTIVATING TOPOLOGIES ON CATEGORIES

Our purpose is to show that the functor $\mathbf{R}$ from $\mathcal{L}$ to presheaves given by

$$\mathbf{R}(L) : B \mapsto Hom_{\mathcal{L}}(\mathbf{A}(B), L)$$

sends quantum event algebras L in $\mathcal{L}$ not just into presheaves, but into sheaves for a suitable Grothendieck topology $\mathbf{J}$ on the category of Boolean event algebras $\mathcal{B}$, so that the fundamental adjunction will restrict to an equivalence of categories $\mathbf{Sh}(\mathcal{B}, \mathbf{J}) \cong \mathcal{L}$.

We note at this point that the usual notion of sheaf, in terms of coverings, restrictions, and collation, can be defined and used not just in the spatial sense, namely on the usual topological spaces, but in a generalized spatial sense, on more general topologies (Grothendieck topologies). In the usual definition of a sheaf on a topological space we use the open neighborhoods U of a point in a space X; such neighborhoods are actually monic topological maps $U \to X$. The neighborhoods U in topological spaces can be replaced by maps $V \to X$ not necessarily monic, and this can be done in any category with pullbacks. In effect a covering by open sets can be replaced by a new notion of covering provided by a family of maps satisfying certain conditions.

For an object B of $\mathcal{B}$, we consider indexed families

$$\mathbf{S} = \{\psi_i : B_i \to B, i \in I\}$$

of maps to B, and we assume that for each object B of $\mathcal{B}$ we have a set $\mathbf{\Lambda}(B)$ of certain such families satisfying conditions to be specified later. These families play the role of coverings of B under those conditions. For the coverings provided, it is possible to construct the analogue of the topological definition of a sheaf, where as presheaves on $\mathcal{B}$ we consider the functors $\mathbf{P} : \mathcal{B}^{op} \to \mathbf{Sets}$. According to the topological definition of a sheaf on a space we demand that for each open cover $\{U_i, i \in I\}$ of some U, every family of elements $\{p_i \in \mathbf{P}(U_i)\}$ that satisfy the compatibility condition on the intersections $U_i \cap U_j, \forall i, j$, are pasted together as a unique element $p \in \mathbf{P}(U)$. Imitating the above procedure for any covering $\mathbf{S}$ of an object B, and replacing the intersection $U_i \cap U_j$ by the pullback $B_i \times_B B_j$ in the general case, according to the diagram

$$
\begin{array}{ccc}
B_i \times_B B_j & \xrightarrow{g_{ij}} & B_j \\
\Big\downarrow{\scriptstyle h_{ij}} & & \Big\downarrow{\scriptstyle \psi_j} \\
B_i & \xrightarrow{\psi_i} & B
\end{array}
$$

we effectively obtain for a given presheaf $\mathbf{P} : \mathcal{B}^{op} \rightarrow \mathbf{Sets}$ a diagram of sets as follows

$$
\begin{array}{ccc}
\mathbf{P}(B_i \times_B B_j) & \xrightarrow{\mathbf{P}(g_{ij})} & \mathbf{P}(B_j) \\
\Big\downarrow{\scriptstyle \mathbf{P}(h_{ij})} & & \Big\downarrow{\scriptstyle \mathbf{P}(\psi_j)} \\
\mathbf{P}(\mathbf{B_i}) & \xrightarrow{\mathbf{P}(\psi_i)} & \mathbf{P}(B)
\end{array}
$$

In this case the compatibility condition for a sheaf takes the form: if $\{p_i \in \mathbf{P}_i, i \in I\}$ is a family of compatible elements, namely satisfy $p_i h_{ij} = p_j g_{ij}, \forall i, j$, then a unique element $p \in \mathbf{P}(B)$ is being determined by the family such that $p \cdot \psi_i = p_i, \forall i \in I$, where the notational convention $p \cdot \psi_i = \mathbf{P}(\psi_i)(p)$ has been used . Equivalently this condition can be expressed in the categorical terminology by the requirement that in the diagram

$$
\prod_{i,j} \mathbf{P}(B_i \times_B B_j) \;\leftleftarrows\; \prod_i \mathbf{P}(B_i) \;\xleftarrow{\;e\;}\; \mathbf{P}(B)
$$

the arrow e, where $e(p) = (p \cdot \psi_i, i \in I)$ is an equalizer of the maps $(p_i, i \in I) \rightarrow (p_i h_{ij}; i, j \in I \times I)$ and $(p_i, i \in I) \rightarrow (p_i g_{ij}; i, j \in I \times I)$ correspondingly.

The specific conditions that the elements of the set $\Lambda(B)$, or else the coverings of B, have to satisfy naturally lead to the notion of a Grothendieck topology on the category $\mathcal{B}$.

4 GROTHENDIECK TOPOLOGIES

We start our discussion by explicating the notion of a pretopology on the category of Boolean event algebras $\mathcal{B}$. A pretopology on B is a system $\mathbf{\Lambda}$ where for each onject B there is a set $\mathbf{\Lambda}(B)$. Each $\mathbf{\Lambda}(B)$ contains indexed families of $\mathcal{B}$-morphisms

$$\mathbf{S} = \{\psi_i : B_i \to B, i \in I\}$$

of maps to B such that the following conditions are satisfied:

(1) For each B in $\mathcal{B}$, $\{id_B\} \in \mathbf{\Lambda}(\mathbf{B})$;

(2) If $C \to B$ in $\mathcal{B}$ and $\mathbf{S} = \{\psi_i : B_i \to B, i \in I\} \in \mathbf{\Lambda}(B)$ then $\{\psi_1 : C \times_B B_i \to B, i \in I\} \in \mathbf{\Lambda}(C)$. Note that ψ_1 is the pullback in $\mathcal{B}$ of ψ_i along $C \to B$;

(3) If $\mathbf{S} = \{\psi_i : B_i \to B, i \in I\} \in \mathbf{\Lambda}(B)$, and for each $i \in I$, $\{{\psi_{ik}}^i : C_{ik} \to B_i, k \in K_i\} \in \mathbf{\Lambda}(B_i)$, then $\{{\psi_{ik}}^i \circ \psi_i : C_{ik} \to B_i \to B, i \in I; k \in K_i\} \in \mathbf{\Lambda}(B)$. Note that C_{ik} is an example of a double indexed object rather than the intersection of C_i and C_k.

The notion of a pretopology on the category of Boolean algebras $\mathcal{B}$ is a categorical generalization of a system of set-theoretical covers on a topology $\mathbf{T}$, where a cover for $U \in \mathbf{T}$ is a set $\{U_i : U_i \in \mathbf{T}, \mathbf{i} \in \mathbf{I}\}$ such that $\cup U_i = U$. The generalization is achieved by noting that the topology ordered by inclusion is a poset category and that any cover corresponds to a collection of inclusion arrows $U_i \to U$. Given this fact, any family of arrows contained in $\mathbf{\Lambda}(B)$ of a pretopology is a cover as well.

The passage from a pretopology to a categorial or Grothendieck topology on the category of Boolean algebras takes place through the introduction of appropriate covering devices, called covering sieves. For an object B in $\mathcal{B}$, a B-sieve is a family R of $\mathcal{B}$-morphisms with codomain B, such that if $C \to B$ belongs to R and $D \to C$ is any $\mathcal{B}$-morphism , then the composite $D \to C \to B$ belongs to R.

A Grothendieck topology on the category of Boolean algebras $\mathcal{B}$, is a system J of sets, $J(B)$ for each B in $\mathcal{B}$, where each $J(B)$ consists of a set of B-sieves, (called the covering sieves), that satisfy the following conditions:

1. For any B in $\mathcal{B}$ the maximal sieve $\{g : cod(g) = B\}$ belongs to $J(B)$ (maximality condition).

2. If R belongs to $J(B)$ and $f : C \to B$ is a $\mathcal{B}$-morphism, then $f^*(R) = \{h : C \to B, f \cdot h \in R\}$ belongs to $J(C)$ (stability condition).

3. If R belongs to $J(B)$ and S is a sieve on C, where for each $f : C \to B$

belonging to R, we have $f^*(S)$ in $J(C)$, then S belongs to $J(B)$ (transitivity condition).

The small category $\mathcal{B}$ together with a Grothendieck topology $\mathbf{J}$, is called a site. A sheaf on a site $(\mathcal{B}, \mathbf{J})$ is defined to be any contravariant functor $\mathbf{P} : \mathcal{B}^{op} \to \mathbf{Sets}$, satisfying the equalizer condition expressed in terms of covering sieves S for B, as in the following diagram in $\mathbf{Sets}$:

$$\prod_{f \cdot g \in S} \mathbf{P}(domg) \quad \overleftarrow{\overleftarrow{\qquad\qquad}} \quad \prod_{f \in S} \mathbf{P}(domf) \xleftarrow{\quad e \quad} \mathbf{P}(B)$$

If the above diagram is an equalizer for a particular covering sieve S, we obtain that $\mathbf{P}$ satisfies the sheaf condition with respect to the covering sieve S.

A Grothendieck topos over the small category $\mathcal{B}$ is a category which is equivalent to the category of sheaves $\mathbf{Sh}(\mathcal{B}, \mathbf{J})$ on a site $(\mathcal{B}, \mathbf{J})$. We note that a category of sheaves $\mathbf{Sh}(\mathcal{B}, \mathbf{J})$ on a site $(\mathcal{B}, \mathbf{J})$ is a full sucategory of the functor category of presheaves $\mathbf{Sets}^{\mathcal{B}^{op}}$.

5 CONSTRUCTION OF A GROTHENDIECK TOPOLOGY ON $\mathcal{B}$

We remind that our purpose is to show that the functor $\mathbf{R}$ from $\mathcal{L}$ to presheaves, $\mathbf{R}(L) : B \mapsto Hom_{\mathcal{L}}(\mathbf{A}(B), L)$, transforms quantum event algebras

L in $\mathcal{L}$ not just into presheaves, but into sheaves for a suitable Grothendieck topology $\mathbf{J}$ on the category of Boolean event algebras $\mathcal{B}$. Under these circumstances, the fundamental adjunction will restrict to an equivalence of categories $\mathbf{Sh}(\mathcal{B}, \mathbf{J}) \cong \mathcal{L}$.

5.1 $\mathcal{B}$ AS A GENERATING SUBCATEGORY OF $\mathcal{L}$

We consider $\mathcal{B}$ as a full subcategory of $\mathcal{L}$, whose set of objects $\{B_i / i \in I\}$, I: index set, generate $\mathcal{L}$, in the sense that,

$$B_i \xrightarrow{w} L \overset{v}{\underset{u}{\rightrightarrows}} K$$

the identity $w \cdot v = w \cdot u$, for every arrow $w : B_i \to L$, and every B_i, implies that $v = u$. Equivalently we can say that the set of all arrows $w : B_i \to L$, constitute an epimorphic family. We may verify this claim if we take into account the adjunction and observe that objects of $\mathcal{L}$ are being constructed as colimits over the category of elements of presheaves over $\mathcal{B}$. Since objects of $\mathcal{L}$ are constructed as colimits of this form, whenever two parallel arrows

$$L \overset{v}{\underset{u}{\rightrightarrows}} K$$

are different, there is an arrow $w : B_i \to L$ from some B_i in $\mathcal{B}$, such that $vw \neq uw$.

Since we assume that $\mathcal{B}$ is a full subcategory of $\mathcal{L}$ we omit the explicit presence of the coordinatization functor $\mathbf{A}$ in the subsequent discussion.

The consideration that $\mathcal{B}$ is a generating subcategory of $\mathcal{L}$ points exactly to the depiction of the appropriate Grothendieck topology on $\mathcal{B}$, that accomplishes our purpose of comprehending quantum event algebras as sheaves on $\mathcal{B}$.

We assert that a sieve S on a Boolean algebra B in $\mathcal{B}$ is to be a covering sieve of B, when the arrows $s : C \to B$ belonging to the sieve S together form an epimorphic family in $\mathcal{L}$. This requirement may be equivalently expressed in terms of a map

$$G_S : \coprod_{(s:C \to B) \in S} C \to B$$

being an epi in $\mathcal{L}$.

5.2 THE GROTHENDIECK TOPOLOGY OF EPIMORPHIC FAMILIES

We will show that the choice of covering sieves on Boolean algebras B in $\mathcal{B}$, as being epimorphic families in $\mathcal{L}$, does indeed define a Grothendieck topology on $\mathcal{B}$.

First of all we notice that the maximal sieve on each Boolean algebra B, includes the identity $B \to B$, thus it is a covering sieve. Next, the transitivity property of the depicted covering sieves is obvious. It remains to demonstrate that the covering sieves remain stable under pullback. For this purpose we consider the pullback of such a covering sieve S on B along any arrow $h : B' \rightarrow B$ in $\mathcal{B}$

$$\begin{array}{ccc} \coprod_{s \in S} C \times_B \acute{B} & \longrightarrow & \acute{B} \\ \downarrow & & \downarrow h \\ \coprod_{s \in S} C & \xrightarrow{G} & B \end{array}$$

The Boolean algebras B in $\mathcal{B}$ generate the category of quantum event algebras $\mathcal{L}$, hence, there exists for each arrow $s : D \rightarrow B$ in S, an epimorphic family of arrows $\coprod [B]^s \rightarrow D \times_B \acute{B}$, or equivalently $\{[B]^s{}_j \rightarrow D \times_B \acute{B}\}_j$, with each domain $[B]^s$ a Boolean algebra.

Consequently the collection of all the composites:

$$[B]^s{}_j \rightarrow D \times_B \acute{B} \rightarrow \acute{B}$$

for all $s : D \rightarrow B$ in S, and all indices j together form an epimorphic family in $\mathcal{L}$, that is contained in the sieve $h^*(S)$, being the pullback of S along $h : B \rightarrow \acute{B}$. Therefore the sieve $h^*(S)$ is a covering sieve.

It is important to construct the representation of covering sieves within the category of Boolean event algebras $\mathcal{B}$. This is possible, if we first observe that for an object C of $\mathcal{B}$, and a covering sieve for the defined Grothendieck topology on $\mathcal{B}$, the map

$$G_S : \coprod_{(s:C\to B)\in S} C \to B$$

being an epi in $\mathcal{L}$, can be equivalently presented as the coequalizer of its kernel pair, or else the pullback of G_S along itself

$$\begin{array}{ccc} \coprod_{\acute{s}} \acute{D} \times_C \coprod_s D & \longrightarrow & \coprod_s D \\ \Big\downarrow & & \Big\downarrow G_S \\ \coprod_{\acute{s}} \acute{D} & \xrightarrow{\;G_S\;} & C \end{array}$$

Furthermore, using the fact that pullbacks in $\mathcal{L}$ preserve coproducts, the epimorhic family associated with a covering sieve on C, admits the following coequalizer presentation

$$\coprod_{\acute{s},s} \acute{D} \times_C D \;\; \overset{q_1}{\underset{q_2}{\rightrightarrows}} \;\; \coprod_s D \xrightarrow{\;G\;} C$$

Moreover, since the the category $\mathcal{B}$ is a generating subcategory of $\mathcal{L}$, for each pair of arrows $s : D \to C$ and $\acute{s} : \acute{D} \to C$ in the covering sieve S on the Boolean algebra C, there exists an epimorphic family $\{B \to \acute{D} \times_C D\}$, such that each domain B is a Boolean algebra object in $\mathcal{B}$.

Consequently, each element of the epimorphic family associated with a covering sieve S on a Boolean algebra C is represented by a commutative diagram in $\mathcal{B}$ of the following form

$$\begin{array}{ccc} B & \xrightarrow{\;l\;} & D \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle s} \\ \acute{D} & \xrightarrow{\;\acute{s}\;} & C \end{array}$$

At a further step we may compose the representation of epimorphic families by commutative squares in $\mathcal{B}$, obtained previously, with the coequalizer presentation of the same epimorphic families. The composition results in a new coequalizer diagram in $\mathcal{B}$ of the following form:

$$\coprod_B B \quad \underset{y_2}{\overset{y_1}{\rightrightarrows}} \quad \coprod_s D \xrightarrow{\;G\;} C$$

where the first coproduct is indexed by all B in the commutative diagrams in $\mathcal{B}$, representing elements of epimorphic families.

5.3 $\mathbf{R}(L)$ AS A J-Sheaf

For each quantum event algebra L in $\mathcal{L}$, we consider the contravariant Hom-functor $\mathbf{R}(L) = Hom_{\mathcal{L}}(-, L)$ in $\mathbf{Sets}^{\mathcal{B}^{op}}$. If we apply this representable functor to the latter coequalizer diagram we obtain an equalizer diagram in **Sets** as follows:

$$\prod_B Hom_{\mathcal{L}}(B,L) \; \overleftarrow{\overleftarrow{\qquad\qquad}} \; \prod_{s\in S} Hom_{\mathcal{L}}(D,L) \longleftarrow \qquad Hom_{\mathcal{L}}(C,L)$$

where the first product is indexed by all B in the commutative diagrams in $\mathcal{B}$, representing elements of epimorphic families.

The equalizer in **Sets**, thus obtained, says explicitly that the contravariant Hom-functor $\mathbf{R}(L) = Hom_{\mathcal{L}}(-,L)$ in $\mathbf{Sets}^{\mathcal{B}^{op}}$, satisfies the sheaf condition for the covering sieve S. Moreover, the equalizer condition holds for every covering sieve in the Grothendieck topology of epimorphic families. By rephrasing the above, we conclude that the representable Hom-functor $\mathbf{R}(L)$ is a sheaf for the Grothendieck topology of epimorphic families defined on the category of Boolean event algebras.

6 EQUIVALENCE OF GROTHENDIECK TOPOS $\mathbf{Sh}(\mathcal{B},\mathbf{J})$ WITH $\mathcal{L}$

We claim, that if the functor $\mathbf{R}$ from $\mathcal{L}$ to presheaves $\mathbf{R}(L) : B \mapsto Hom_{\mathcal{L}}(\mathbf{A}(B),L)$ sends quantum event algebras L in $\mathcal{L}$ not just into presheaves, but into sheaves for the Grothendieck topology of epimorphic families, $\mathbf{J}$, on the category of Boolean event algebras $\mathcal{B}$, the fundamental adjunction restricts to an equivalence of categories $\mathbf{Sh}(\mathcal{B},\mathbf{J}) \cong \mathcal{L}$. The proof of this claim is presented in detail below and occupies the remainder of Section 6.

6.1 COVERING SIEVES ON QUANTUM EVENT ALGEBRAS

If we consider a quantum event algebra L, and all quantum algebraic homomorphisms of the form $\psi : E \to L$, with domains E, in the generating subcategory of Boolean algebras $\mathcal{B}$, then the family of all these maps ψ, constitute an epimorphism:

$$T : \coprod_{(E \in \mathcal{B}, \psi : E \to L)} E \to L$$

We notice that the quantum algebraic epimorphism t is actually the same as the map

$$T : \coprod_{(E \in \mathcal{B}, \psi : \mathbf{A}(E) \to L)} \mathbf{A}(E) \to L$$

since the coordinatization functor $\mathbf{A}$, is, by the fact that $\mathcal{B}$ is a full subcategory of $\mathcal{L}$, just the inclusion functor $\mathbf{A} : \mathcal{B} \hookrightarrow \mathcal{L}$.

Subsequently, we may use the same arguments as in the discussion of the Grothendieck topology of epimorphic families of the previous section, in order to assert that the epimorphism T can be presented in the form of a coequalizer diagram in $\mathcal{L}$ as follows:

$$\coprod_{\nu} B \quad \underset{y_2}{\overset{y_1}{\rightrightarrows}} \quad \coprod_{(E \in \mathcal{B}, \psi : E \to L)} E \xrightarrow{T} L$$

where the first coproduct is indexed by all ν, representing commutative diagrams in $\mathcal{L}$, of the form [DI]:

$$
\begin{array}{ccc}
B & \xrightarrow{\ l\ } & E \\
\downarrow{\scriptstyle k} & & \downarrow{\scriptstyle \psi} \\
\acute{E} & \xrightarrow{\ \acute{\psi}\ } & L
\end{array}
$$

where B, E, $\acute{E}$ are objects in the generating subcategory $\mathcal{B}$ of $\mathcal{L}$.

We say that a sieve on a quantum event algebra defines a covering sieve by objects of its generating subcategory $\mathcal{B}$, when the quantum algebraic homomorphisms belonging to the sieve define an epimorphism

$$T : \coprod_{(E \in \mathcal{B}, \psi : \mathbf{A}(E) \to L)} \mathbf{A}(E) \to L$$

In this case the epimorphic families of quantum algebraic homomorphisms constituting covering sieves of quantum event algebras fit into coequalizer diagrams of the latter form [DI].

From the physical point of view covering sieves by Boolean algebras, are equivalent with Boolean localization systems of quantum event algebras [14, 15]. These localization systems filter the information of the quantum species of structure through Boolean domains, associated with procedures of measurement of observables. The key idea behind the notion of a system of localizations for a quantum event algebra amounts to coordinatizing the information contained in a quantum event algebra L in $\mathcal{L}$ by means of structure preserving morphisms $B \longrightarrow L$ having as their domains, locally defined

Boolean event algebras B in $\mathcal{B}$ for measurement of observables. Any single map from a Boolean domain to a quantum event algebra is not enough for a complete determination of its information content, and hence, it contains only a limited amount of information about it. This problem is tackled by employing a sufficient amount of maps, organized in terms of covering sieves, from the coordinatizing Boolean domains to a quantum event algebra simultaneously, so as to cover it completely. These maps play exactly the role of covers for the filtration of the information associated with a quantum event structure, in terms of Boolean coefficients, associated with measurement situations. The introduction of covering sieves, is furthermore, motivated by the consequences of Kochen-Specker theorem, according to which, it is not possible to characterize completely a quantum system in terms of observable attributes by employing a single Boolean experimental device globally. On the other side, in every concrete experimental context, the set of events that have been actualized in this context forms a Boolean algebra. Consequently, any Boolean domain cover in a covering sieve for quantum event algebra, corresponds to a set of Boolean events that become actualized in the experimental context of B. Moreover, the organization of covering sieves in terms of the requirements characterizing a categorical topology, physically correspond to conditions for the compatibility of the information content gathered in different Boolean filtering mechanisms associated with measurement of observables. In this manner, covering sieves of quantum event algebras incorporate all the necessary conditions for the analysis of the information content of a quantum event structure in terms of a sheaf of Boolean coefficients, for the Grothendieck topology specified, associated with measurement localization contexts. The sheaf concept is introduced to express precisely the pasting conditions that the locally defined Boolean covers have to satisfy on their overlapping regions, or else, the specification by which local data, providing Boolean coefficients obtained in measurement situations, can be collated.

6.2 UNIT AND COUNIT OF THE ADJUNCTIVE CORRESPONDENCE

We focus again our attention in the fundamental adjunction and investigate the unit and the counit of it. For any presheaf $\mathbf{P} \in \mathbf{Sets}^{B^{op}}$, we deduce that the unit $\delta_{\mathbf{P}} : \mathbf{P} \longrightarrow Hom_{\mathcal{L}}(\mathbf{A}(\_), \mathbf{P} \otimes_{\mathcal{B}} \mathbf{A})$ has components:

$$\delta_{\mathbf{P}}(B) : \mathbf{P}(B) \longrightarrow Hom_{\mathcal{L}}(\mathbf{A}(B), \mathbf{P} \otimes_{\mathcal{B}} \mathbf{A})$$

for each Boolean algebra object B of $\mathcal{B}$.

If we make use of the representable presheaf $y[B]$ we obtain

$$\delta_{\mathbf{y}[B]} : \mathbf{y}[B] \rightarrow Hom_{\mathcal{L}}(\mathbf{A}(\_), \mathbf{y}[B] \otimes_{\mathcal{B}} \mathbf{A})$$

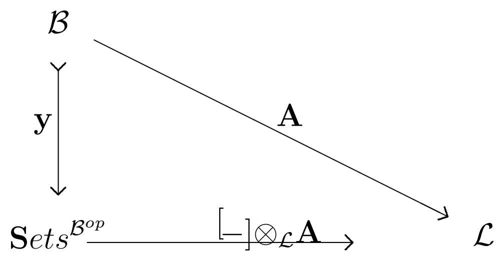
Hence for each object B of $\mathcal{B}$ the unit, in the case considered, corresponds to a map

$$\mathbf{A}(B) \to \mathbf{y}[B] \otimes_{\mathcal{B}} \mathbf{A}$$

But, since

$$\mathbf{y}[B] \otimes_{\mathcal{B}} \mathbf{A} = \mathbf{L}_A \mathbf{y}[B](B) \cong \mathbf{A} \circ \mathbf{G}_{\mathbf{y}[B]}(B, 1_B) = \mathbf{A}(B)$$

the unit for the representable presheaf of Boolean algebras, which is a sheaf for the Grothendieck topology of epimorphic families, is clearly an isomorphism. By the preceding discussion we can see that the diagram commutes

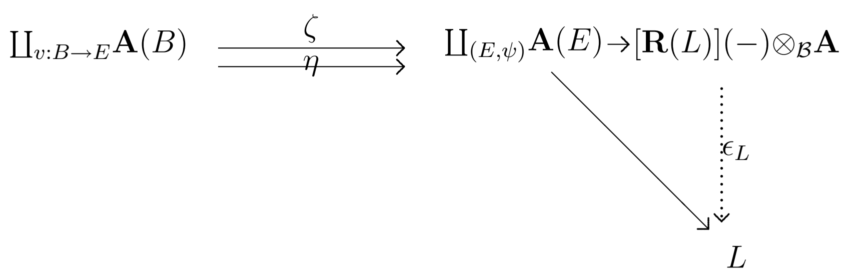


Thus the unit of the fundamental adjunction referring to the representable sheaf $\mathbf{y}[B]$ of the category of Boolean event algebras provides a map (quantum algebraic homomorphism) $\mathbf{A}(B) \longrightarrow \mathbf{y}[B] \otimes_{\mathcal{B}} \mathbf{A}$ which is an isomorphism.

On the other side, for each quantum event algebra object L of $\mathcal{L}$ the counit is

$$\epsilon_L : Hom_{\mathcal{L}}(\mathbf{A}(\_), L) \otimes_{\mathcal{B}} \mathbf{A} \longrightarrow L$$

The counit corresponds to the vertical map in the following coequalizer diagram [DII]:



where the first coproduct is indexed by all arrows $v : B \to E$, with B, E objects of $\mathcal{B}$, whereas the second coproduct is indexed by all objects B in $\mathcal{B}$ and arrows $\psi : \mathbf{A}(E) \to L$, belonging to a covering sieve of L by objects of its generating subcategory.

It is important to notice the similarity in form of diagrams [DI] and [DII]. Based on this observation it is possible to prove that if the domain of the counit of the adjunction is restricted to sheaves for the Grothendieck topology of epimorphic families on $\mathcal{B}$, then the counit defines a quantum algebraic isomorphism

$$\epsilon_L : Hom_{\mathcal{L}}(\mathbf{A}(\_), L)\otimes_{\mathcal{B}}\mathbf{A} \simeq L$$

In order to substantiate our thesis we inspect diagrams [DI], and [DII], observing that it is enough to prove that the pairs of arrows (ζ, η) and (y_1, y_2) have isomorphic coequalizers, since, then, the counit is obviously an isomorphism. Thus, we wish to show that a covering sieve of a quantum event algebra

$$T : \coprod_{(E\in\mathcal{B},\psi:\mathbf{A}(E)\to L)}\mathbf{A}(E) \to L$$

is the coequalizer of (y_1, y_2) iff it is the coequalizer of (ζ, η). In the following discussion, we may omit the explicit presence of the inclusion functor $\mathbf{A}$, for the same reasons stated previously.

We consider a covering sieve of quantum event algebra L, consisting of quantum algebraic homomorhisms $T_{(E,\psi)}$, that together constitute an epimorphic family in $\mathcal{L}$. We observe that the condition $T \cdot y_1 = T \cdot y_2$ is equivalent to the condition [CI]

$$T_{(E,\psi)} \cdot l = T_{(\acute{E},\acute{\psi})} \cdot k$$

for each commutative square ν. Furthermore, the condition $T \cdot \zeta = T \cdot \eta$ is equivalent to the condition [CII]

$$T_{(E,\psi)} \cdot u = T_{(\acute{E},\psi \cdot u)}$$

for every Boolean homomorphism $u : \acute{E} \to E$, with B, E objects of $\mathcal{B}$ and $\psi : E \to L$, belonging to a covering sieve of L by objects of its generating subcategory. Therefore our thesis is proved if we show that $[CI] \Leftrightarrow [CII]$.

On the one hand, $T \cdot \zeta = T \cdot \eta$, implies for every commutative diagram of the form ν:

$$\begin{array}{ccc} B & \xrightarrow{\ l\ } & E \\ \Big\downarrow{\scriptstyle k} & & \Big\downarrow{\scriptstyle \psi} \\ \acute{E} & \xrightarrow{\ \acute{\psi}\ } & L \end{array}$$

the following relations:

$$T_{(E,\psi)} \cdot l = T_{(B,\psi \cdot l)} = T_{(B,\acute{\psi} \cdot k)} = T_{(\acute{E},\acute{\psi})} \cdot k$$

Thus $[CI] \Rightarrow [CII]$

On the other hand, $T \cdot y_1 = T \cdot y_2$, implies that for every Boolean homomorphism $u : \acute{E} \to E$, with B, E objects of $\mathcal{B}$ and $\psi : E \to L$, the diagram of the form ν

$$\begin{array}{ccc} \acute{E} & \xrightarrow{\ u\ } & E \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle \psi} \\ \acute{E} & \xrightarrow{\ \psi \cdot u\ } & L \end{array}$$

commutes and provides the condition

$$T_{(E,\psi)} \cdot u = T_{(\acute{E},\psi \cdot u)}$$

Thus $[CI] \Leftarrow [CII]$.

Consequently, the pairs of arrows (ζ, η) and (y_1, y_2) have isomorphic coequalizers, proving that the counit of the fundamental adjunction restricted to sheaves for the Grothendieck topology of epimorphic families on $\mathcal{B}$ is an isomorphism.

$$\epsilon_L : Hom_{\mathcal{L}}(\mathbf{A}(\_), L) \otimes_{\mathcal{B}} \mathbf{A} \simeq L$$

7 CONCLUSIONS

The representation of quantum event algebras as sheaves for the Grothendieck topology of epimorphic families on $\mathcal{B}$, through the counit isomorphism, and subsequently the comprehension of the category of quantum event algebras as a Grothendieck topos is of remarkable physical significance. If we remind the discussion of the physical meaning of the adjunction, expressed in terms of the information content, communicated between Boolean windows and quantum event algebras, we arrive to the following conclusion: the totality of the content of information included in the quantum species of event structure remains invariant under Boolean encodings, corresponding to local Boolean modeling algebras for measurement of observables, in covering sieves of quantum event algebras, if and only if the counit of the fundamental adjunction is a quantum algebraic isomorphism. Phrased differently, in this case, the category of quantum event algebras is equivalent to a Grothendieck topos for the covering sieves of epimorphic families from the the base Boolean localization category, or else, the category of sheaves for the Grothendieck topology of epimorphic families on the modeling generating subcategory of Boolean algebras. We may, furthermore, argue that the sheaf theoretical representation of a quantum event algebra reveals that its deep conceptual significance is related not to its poset axiomatization (which has been the starting point of almost all subsequent discussions of quantum logics), but, on the precise manner that distinct Boolean local contexts of observation are interconnected, so as its informational content is preserved in the totality of its operational encodings. By the latter, we precisely mean contextual operational procedures for probing the quantum regime of structure, which categorically give rise to covering sieves, substantiated as interconnected epimorphic families of the objects of the category of elements of the sheafified *Hom*-functor $\mathbf{R}(L)$. From a logical point of view these objects are comprehended as unsharp Boolean algebras of events, in agreement with the interpretational framework put forward in [15], and also, introduced from a non-categorical viewpoint in [21,22]. The sheaf theoretical representation expresses exactly the compatibility of these unsharp Boolean algebras of events on their overlaps in such a way as to leave invariant the amount of information contained in a quantum system. We may adopt the term Boolean reference frames to refer to these local contexts of encoding the information related to a quantum system, emphasizing their prominent role in the organization of meaning associated with a quantum algebra of events, through the establishment of covering sieves, that, precisely, mediate in the subsequent equivalence of quantum event algebras with Boolean localization systems. Moreover this rerminology signifies the intrinsic contextuality of quantum events, as filtered through the base localizing category, and is suggestive of the introduction of a relativity principle in the quantum level of observable structure related with the invariance of the informational content with respect to Boolean reference frames contained in covering sieves of quantum event algebras. This is a crucial observation concerning the interpretation of quantum event algebras as quantum logics. It underlines the fact that the conceptual significance of a logic of propositions referring to the description of a quantum system is to be sought, not at the level of non-contextual propositions forming the original axiomatized poset structure, but on the level of propositions holding in distinct Boolean reference frames. The latter are endowed with different unsharp Boolean propositional languages, not always compatible with each other. The sheaf theoretical representation contains the necessary and sufficient conditions for the compatibility of these languages associated with Boolean reference frames in covering sieves, such that the content of information associated with a quantum system is preserved under its operational unfoldments, in Boolean localization systems. The above remarks constitute a basis for a consistent interpretation of the category of logics of quantum propositions from the sheaf theoretical perspective of the present paper. Most significantly, this task, which will be presented in detail in a future work, is facilitated by the fact that the category of sheaves on the Boolean localizing category, is a Grothendieck topos, and consequently comes naturally equipped with an object of generalized truth values, called subobject classifier. This object of truth values, being remarkably a sheaf itself, namely an object of the Grothendieck topos, is the appropriate conceptual tool for the organization of the logical dimension of the information included in the category of quantum event algebras, as it is encoded in Boolean localization systems.

A APPENDIX

A.1 On the definition of a quantum event algebra

The definition used in the paper is the following: A quantum event algebra is a partially ordered set of Quantum events, endowed with a maximal element 1, and with an operation of orthocomplementation $[-]^* : L \longrightarrow L$, which satisfy, for all $l \in L$ the following conditions: [a] $l \leq 1$, [b] $l^{**} = l$, [c] $l \vee l^* = 1$, [d] $l \leq \acute{l} \Rightarrow \acute{l}^* \leq l^*$, [e] $l \perp \acute{l} \Rightarrow l \vee \acute{l} \in L$, [f] $l \vee \acute{l} = 1, l \wedge \acute{l} = 0 \Rightarrow l = \acute{l}^*$, where $0 := 1^*$, $l \perp \acute{l} := l \leq \acute{l}^*$, and the operations of meet $\wedge$ and join $\vee$ are defined as usually.

We can check the following:

[1]. In the Hilbert space formalism of Quantum theory events are considered as closed subspaces of a seperable, complex Hilbert space corresponding to a physical system. Then the quantum event structure is identified with the lattice of closed subspaces of the Hilbert space, ordered by inclusion and carrying an orthocomplementation operation which is given by the orthogonal complement of the closed subspaces. For a seperable complex Hilbert space of dimension at least three, the lattice is also a quantum event algebra (the Hilbert space quantum event algebra).

[2]. Obviously every Boolean event algebra is also a quantum event algebra.

[3]. The Lindenbaum algebra corresponding to propositions describing the behavior of a quantum system is also a quantum event algebra.

If the reader does not feel comfortable with the definition of a quantum event algebra as above, it is possible to modify the definition slightly, without any change in the arguments of the paper, as follows:

A quantum event algebra is an orthomodular orthoposet. More concretely, each object L in the category $\mathcal{L}$, is considered as a partially ordered set of Quantum events, endowed with a maximal element 1, and with an operation of orthocomplementation $[-]^* : L \longrightarrow L$, which satisfy, for all $l \in L$, the following conditions: [a] $l \leq 1$, [b] $l^{**} = l$, [c] $l \vee l^* = 1$, [d] $l \leq \acute{l} \Rightarrow \acute{l}^* \leq l^*$, [e] $l \bot \acute{l} \Rightarrow l \vee \acute{l} \in L$, [f] for $l, \acute{l} \in L, l \leq \acute{l}$ implies that l and $\acute{l}$ are compatible, where $0 := 1^*$, $l \bot \acute{l} := l \leq \acute{l}^*$, and the operations of meet $\wedge$ and join $\vee$ are defined as usually. We also recall that $l, \acute{l} \in L$ are compatible if the sublattice generated by $\{l, l^*, \acute{l}, \acute{l}^*\}$ is a Boolean algebra, namely if it is a Boolean sublattice.

Furthermore it is obvious that if someone wishes may also impose a σ-completeness condition, namely that the join of countable families of pairwise orthogonal events must exist, in order to have a well defined theory of observables over L.

A.2 On the explicit construction of colimits

It is important to notice that the key colimit is defined over the category of elements of the functor

$$\mathbf{R}(L) : B \mapsto Hom_{\mathcal{L}}(\mathbf{A}(B), L)$$

by the relation

$$\mathbf{L}(\mathbf{P}) = Colim\{\mathbf{G}(\mathbf{P}, \mathcal{B}) \xrightarrow{\mathbf{G_P}} \mathcal{B} \xrightarrow{\mathbf{A}} \mathcal{L}\}$$

for the presheaf $\mathbf{P} = \mathbf{R}(L)$.

In order to cope with relevant ambiguities in the exposition of the arguments it is worthwhile to construct the colimit expicitly, and show that it is actually a quantum event algebra. For this purpose we consider the set:

$$\mathbf{L}(\mathbf{R}(L)) = \{(\psi_B, q)/(\psi_B : \mathbf{A}(B) \longrightarrow L) \in [\mathbf{G}(\mathbf{R}(\mathbf{L}), \mathcal{B})]_0, q \in \mathbf{A}(B)\}$$

We notice that if there exists $u : \psi_{\acute{B}} \rightarrow \psi_B$ such that: $u(\acute{q}) = q$ and $\psi_B \circ u = \psi_{\acute{B}}$, where $[\mathbf{R}(L)u](\psi_B) := \psi_B \circ u$ as usual, then we may define a transitive and reflexive relation $\Re$ on the set $\mathbf{L}(\mathbf{R}(L))$. Of course the inverse also holds true. We notice then that

$$(\psi_B \circ u, q)\Re(\psi_B, u(\acute{q}))$$

for any $u : \mathbf{A}(\acute{B}) \rightarrow \mathbf{A}(B)$ in the category $\mathcal{B}$. The next step is to make this relation also symmetric by postulating that for ζ, η in $\mathbf{L}(\mathbf{R}(L))$, where ζ, η

denote pairs in the above set, we have:

$$\zeta \sim \eta$$

if and only if $\zeta \Re \eta$ or $\eta \Re \zeta$. Finally by considering a sequence ξ_1, ξ_2, $\ldots$,ξ_k of elements of the set $\mathbf{L}(\mathbf{R}(L))$ and also ζ, η such that:

$$\zeta \sim \xi_1 \sim \xi_2 \sim \ldots \sim \xi_{k-1} \sim \xi_k \sim \eta$$

we may define an equivalence relation on the set $\mathbf{L}(\mathbf{R}(L))$ as follows:

$$\zeta \bowtie \eta := \zeta \sim \xi_1 \sim \xi_2 \sim \ldots \sim \xi_{k-1} \sim \xi_k \sim \eta$$

Then for each $\zeta \in \mathbf{L}(\mathbf{R}(L))$ we define the quantum at ζ as follows:

$$Q_\zeta = \{\iota \in \mathbf{L}(\mathbf{R}(L)) : \zeta \bowtie \iota\}$$

We finally put

$$\mathbf{L}(\mathbf{R}(L))/\bowtie := \{Q_\zeta : \zeta = (\psi_B, q) \in \mathbf{L}(\mathbf{R}(L))\}$$

and use the notation $Q_\zeta = \|(\psi_B, q)\|$. The set $\mathbf{L}(\mathbf{R}(L))/\bowtie$ is naturally endowed with a quantum algebra structure if we are careful to notice that:

[1]. The orthocomplementation is defined as: $Q_\zeta^* = \|(\psi_B, q)\|^* = \|(\psi_B, q^*)\|$.

[2]. The unit element is defined as: $\mathbf{1} = \|(\psi_B, 1)\|$.

[3]. The partial order structure on the set $\mathbf{L}(\mathbf{R}(L))/\bowtie$ is defined as:

$\|(\psi_B, q)\| \preceq \|(\psi_C, r)\|$ if and only if $d_1 \preceq d_2$ where we have made the following identifications: $\|(\psi_B, q)\| = \|(\psi_D, d_1)\|$ and $\|(\psi_C, r)\| = \|(\psi_D, d_2)\|$, with d_1, $d_2 \in \mathbf{A}(D)$ according to the pullback diagram of event algebras:

$$
\begin{array}{ccc}
\mathbf{A}(D) & \xrightarrow{\beta} & \mathbf{A}(B) \\
\Big\downarrow{\scriptstyle \gamma} & & \Big\downarrow \\
\mathbf{A}(C) & \longrightarrow & L
\end{array}
$$

such that $\beta(d_1) = q$, $\gamma(d_2) = r$. The rest of the requirements such that $\mathbf{L}(\mathbf{R}(L))/\bowtie$, namely the colimit in question, actually carries the structure of a quantum event algebra are obvious.

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