Research Article

The Homological Kähler-De Rham Differential Mechanism part I: Application in General Theory of Relativity

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The mechanism of differential geometric calculus is based on the fundamental notion of a connection on a module over a commutative and unital algebra of scalars defined together with the associated de Rham complex. In this communication, we demonstrate that the dynamical mechanism of physical fields can be formulated by purely algebraic means, in terms of the homological Kähler-De Rham differential schema, constructed by connection inducing functors and their associated curvatures, independently of any background substratum. In this context, we show explicitly that the application of this mechanism in General Relativity, instantiating the case of gravitational dynamics, is related with the absolute representability of the theory in the field of real numbers, a byproduct of which is the fixed background manifold construct of this theory. Furthermore, the background independence of the homological differential mechanism is of particular importance for the formulation of dynamics in quantum theory, where the adherence to a fixed manifold substratum is problematic due to singularities or other topological defects.

1. Introduction

From a category-theoretic viewpoint, the generative mechanism of differential geometric calculus is a consequence of the existence of a pair of adjoint functors, expressing the conceptually inverse algebraic processes of infinitesimally extending and restricting the scalars. Classically, the algebraic differential mechanism is based on the fundamental notion of a connection on a module over a commutative and unital algebra of scalars defined together with the associated de Rham complex [1–3]. A connection on a module induces a process of infinitesimal extension of the scalars of the underlying algebra, which is interpreted geometrically as a process of first-order parallel transport along infinitesimally variable

paths in the geometric spectrum space of this algebra. The next stage of development of the differential mechanism involves the satisfaction of appropriate global requirements referring to the transition from the infinitesimal to the global level. These requirements are of a homological nature and characterize the integrability property of the variation process induced by a connection. Moreover, they are properly addressed by the construction of the De Rham complex associated to an integrable connection. The nonintegrability of a connection is characterized by the notion of curvature bearing the semantics of observable disturbances to the process of cohomologically unobstructed variation induced by the corresponding connection.

It is instructive to emphasize that the conceptualization of the classical differential mechanism along these lines does not presuppose the existence of an underlying differential manifold. This observation is particularly important because it provides the possibility of abstracting in functorial terms both the definition of a connection and the associated De Rham complex as well.

The above are of particular significance in relation to current research in theoretical physics devoted to the construction of a tenable quantum theory of gravity, conceived as an extensive unifying framework of both General Relativity and Quantum Theory [4–14]. It has been generally argued that these fundamental physical theories are based on incompatible conceptual and mathematical foundations. In this sense, the task of their reconciliation in a unifying framework, which respects the constraints posed by both theories, requires a radical revision, or at least a careful rethinking, of our current understanding of the basic notions, such as the conception of spacetime, physical fields, localization, observables, and dynamics.

In this communication, we would like to draw attention to these issues from the algebraic, categorical, and sheaf-theoretic perspective of modern mathematics [1, 2, 15–20]. An initial motivation regarding the relevance of the categorical viewpoint originates from the realization that both of our fundamental theories can be characterized in general terms as special instances of the replacement of the constant by the variable. The semantics of this transition, for both General Relativity and Quantum Theory, may be incorporated in an algebraic sheaf-theoretic homological framework, that hopefully provides the crucial pointers for the schematism of the essential concepts needed for the intelligibility of dynamics in the quantum regime, respecting the normative requirements of these theories. Epigrammatically, it is instructive to remark that, in the case of General Relativity, this process takes place through the rejection of the fixed kinematical structure of spacetime, by making the metric into a dynamical object determined solely by the solution of Einstein's field equations. In the case of Quantum Theory, the process of replacement of the constant by the variable, is signified by the imposition of Heisenberg's uncertainty relations, that determines the limits for simultaneous measurements of certain pairs of complementary physical properties, like position and momentum. From a mathematical point of view, the general process of semantic transition from constant to variable structures is being effectuated by passing to appropriate categories of presheaves or sheaves, conceived as categorical universes of variable sets, whose variation is being considered over generalized localization domains. Thus, there exists the possibility of comprehending uniformly the difference in the distinct instances of replacement of the constant by the variable, as they are explicated in the concrete cases of General Relativity and Quantum Theory, respectively, by employing different sheaf-theoretic universes, corresponding to the particular localization properties of observables in each theory. Of course, this strategy would be fruitful in a unifying perspective, if we managed to disassociate the dependence of dynamics in the regime of each theory from any fixed background spatiotemporal reference. Equivalently stated, the dynamical mechanism should

be ideally formulated functorially and purely algebraically. The benefit of such a formulation has to do with the fact that, because of its functoriality, it can be algebraically forced uniformly inside the appropriate localization environments of the above theories. Thus, both of these theories can be treated homogenously regarding their dynamical mechanism, whereas, their difference can be traced to the distinctive localization properties of the observables employed in each case. In particular, the functorial representation of general relativistic gravitational dynamics induces a reformulation of the issue of quantization as a problem of selection of an appropriate sheaf-theoretic localization environment, in accordance with the behavior of observables in that regime, that effectuates the difference in the semantic interpretation of the dynamical machinery corresponding to the transition from the classical to the quantum case. In this work, we demonstrate that such an algebraic dynamical mechanism can be actually constructed using methods of homological algebra. In this context, we show explicitly that the application of this mechanism in General Relativity, instantiating the case of gravitational dynamics, is related with the absolute representability of the theory in the field of real numbers, a byproduct of which is the fixed background manifold construct of this theory. In a subsequent paper, we are going to explain the applicability of the homological Kähler-De Rham differential mechanism to the problem of formulating dynamics in the quantum regime, according to the preceding remarks, by using the methodology of sheaf-theoretic localization of quantum observable algebras.

A basic nucleus of ideas, technical methods and results related with this program has been already communicated in [21], see also [22, 23]. The general conceptual and technical aspects of the framework of sheaf-theoretic differential geometry have been presented in [3]. It is instructive to mention that the first explicit suggestion of approaching the problem of quantization of gravitational dynamics along sheaf-theoretic lines has appeared in the literature in [24, 25].

2. Algebraic Generation of Dynamics in General Relativity

The basic conceptual and technical issue pertaining to the current research attempts towards the construction of a tenable theory of gravitational interactions in the quantum regime, refers to the problem of independence of this theory from a fixed spacetime manifold substratum. In relation to this problem, we demonstrate the existence and functionality of an algebraic mechanism of modeling general relativistic dynamics functorially, constructed by means of connection inducing functors and their associated curvatures, the latter being, also, as a consequence, independent of any background substratum.

The basic defining feature of General Relativity, in contradistinction to Newtonian classical theory, as well as Special Relativity, is the abolishment of any fixed preexisting kinematical framework by means of dynamicalization of the metric tensor. This essentially means that the geometrical relations defined on a four dimensional smooth manifold, making it into a spacetime, become variable. Moreover, they are constituted dynamically by the gravitation field, as well as other fields from which matter can be derived, by means of Einstein's field equations, through the imposition of a compatibility requirement relating the metric tensor, which represents the spacetime geometry, with the affine connection, which represents the gravitational field. The dynamic variability of the geometrical structure on the spacetime manifold constitutes the means of dynamicalization of geometry in the descriptive terms of General Relativity, formulated in terms of the differential geometric framework on smooth manifolds. The intelligibility of the framework is enriched by the

imposition of the principle of general covariance of the field equations under arbitrary coordinate transformations of the points of the manifold preserving the differential structure, identified as the group of manifold diffeomorphisms. As an immediate consequence, the points of the manifold lose any intrinsic physical meaning, in the sense that, they are not dynamically localizable entities in the theory. Most importantly, manifold points assume an indirect reference as indicators of spacetime events only after the dynamical specification of geometrical relations among them, as particular solutions of the generally covariant field equations. From an algebraic viewpoint [1], a real differential manifold M can be recovered completely from the $\mathcal{R}$-algebra $\mathcal{C}^\infty(M)$ of smooth real-valued functions on it, and, in particular, the points of M may be recovered from the algebra $\mathcal{C}^\infty(M)$ as the algebra morphisms $\mathcal{C}^\infty(M) \to \mathcal{R}$.

In this sense, manifold points constitute the $\mathcal{R}$-spectrum of $\mathcal{C}^\infty(M)$, being isomorphic with the maximal ideals of this algebra. Notice that the $\mathcal{R}$-algebra $\mathcal{C}^\infty(M)$ is a commutative algebra that contains the field of real numbers $\mathcal{R}$ as a distinguished subalgebra. This particular specification incorporates the physical assumption that our form of observation is being represented globally by evaluations in the field of real numbers. In the setting of General Relativity, the form of observation is being coordinatized by means of a commutative unital algebra of scalar coefficients, called an algebra of observables, identified as the $\mathcal{R}$-algebra of smooth real-valued functions $\mathcal{C}^\infty(M)$. Hence, the background substratum of the theory remains fixed as the $\mathcal{R}$-spectrum of the coefficient algebra of scalars of that theory, and, consequently, the points of the manifold M, although not dynamically localizable degrees of freedom of General Relativity, are precisely the semantic information carriers of an absolute representability principle, formulated in terms of global evaluations of the algebra of observables in the field of real numbers. Of course, at the level of the $\mathcal{R}$-spectrum of $\mathcal{C}^\infty(M)$, the only observables are the smooth functions evaluated over the points of M. In physical terminology, the introduction of new observables is conceived as the result of interactions caused by the presence of a physical field, identified with the gravitational field in the context of General Relativity.

Algebraically, the process of extending the form of observation with respect to the algebra of scalars we have started with, that is $\mathcal{A} = \mathcal{C}^\infty(M)$, due to field interactions, is described by means of a fibering, defined as an injective morphism of $\mathcal{R}$-algebras $\iota : \mathcal{A} \hookrightarrow \mathcal{B}$. Thus, the $\mathcal{R}$-algebra $\mathcal{B}$ is considered as a module over the algebra $\mathcal{A}$. A section of the fibering $\iota : \mathcal{A} \hookrightarrow \mathcal{B}$, is represented by a morphism of $\mathcal{R}$-algebras $s : \mathcal{B} \to \mathcal{A}$, left inverse to ι, that is $\iota \circ s = id_B$. The fundamental extension of scalars of the $\mathcal{R}$-algebra $\mathcal{A}$ is obtained by tensoring $\mathcal{A}$ with itself over the distinguished subalgebra of the reals, that is $\iota : \mathcal{A} \hookrightarrow \mathcal{A} \otimes_\mathcal{R} \mathcal{A}$. Trivial cases of scalars extensions, in fact isomorphic to $\mathcal{A}$, induced by the fundamental one, are obtained by tensoring $\mathcal{A}$ with $\mathcal{R}$ from both sides, that is $\iota_1 : \mathcal{A} \hookrightarrow \mathcal{A} \otimes_\mathcal{R} \mathcal{R}$, $\iota_2 : \mathcal{A} \hookrightarrow \mathcal{R} \otimes_\mathcal{R} \mathcal{A}$.

The basic idea of Riemann that has been incorporated in the context of General Relativity is that geometry should be built from the infinitesimal to the global. Geometry in this context is understood in terms of metric structures that can be defined on differential manifolds. If we adopt the algebraic/physical viewpoint, geometry as a result of interactions, requires the extension of scalars of the algebra $\mathcal{A}$ by infinitesimal quantities, defined as a fibration

$$d_* : \mathcal{A} \hookrightarrow \mathcal{A} \oplus \mathbf{M} \cdot \epsilon,$$

$$f \longmapsto f + d_*(f) \cdot \epsilon,$$

$$(2.1)$$

where $d_*(f) =: df$ is considered as the infinitesimal part of the extended scalar, and ϵ the infinitesimal unit obeying $\epsilon^2 = 0$ [21]. The algebra of infinitesimally extended scalars $[\mathscr{A} \oplus \mathbf{M} \cdot \epsilon]$ is called the algebra of dual numbers over $\mathscr{A}$ with coefficients in the $\mathscr{A}$-module $\mathbf{M}$. It is immediate to see that the algebra $[\mathscr{A} \oplus \mathbf{M} \cdot \epsilon]$, as an abelian group is just the direct sum $\mathscr{A} \oplus \mathbf{M}$, whereas, the multiplication is defined by

$$(f + df \cdot \epsilon) \bullet (f' + df' \cdot \epsilon) = f \cdot f' + (f \cdot df' + f' \cdot df) \cdot \epsilon. \tag{2.2}$$

Note that, we further require that the composition of the augmentation $[\mathscr{A} \oplus \mathbf{M} \cdot \epsilon] \rightarrow \mathscr{A}$, with d_* is the identity. Equivalently, the above fibration can be formulated as a derivation [26], that is as an additive morphism:

$$d : \mathscr{A} \longrightarrow \mathbf{M} \tag{2.3}$$

which satisfies the Leibniz rule, that is,

$$d(f \cdot g) = f \cdot dg + g \cdot df. \tag{2.4}$$

Since the formal symbols of differentials $\{df, f \in \mathscr{A}\}$, are reserved for the universal derivation, the $\mathscr{A}$-module $\mathbf{M}$ is identified as the free $\mathscr{A}$-module $\mathbf{\Omega}^1(\mathscr{A}) \equiv \mathbf{\Omega}(\mathscr{A}) \equiv \mathbf{\Omega}$ of 1-forms generated by these formal symbols, modulo the Leibniz constraint, where the scalars of the distinguished subalgebra $\mathcal{R}$, that is the real numbers, are treated as constants.

In this purely algebraic context, the fundamental insight of Kähler has been that the free $\mathscr{A}$-module $\mathbf{\Omega}$ can be constructed explicitly from the fundamental form of scalars extension of $\mathscr{A}$, that is $\iota : \mathscr{A} \hookrightarrow \mathscr{A} \otimes_{\mathcal{R}} \mathscr{A}$ by considering the morphism:

$$\mu : \mathscr{A} \otimes_{\mathcal{R}} \mathscr{A} \longrightarrow \mathscr{A},$$
$$\sum_i f_i \otimes g_i \longmapsto \sum_i f_i \cdot g_i. \tag{2.5}$$

Then, by taking the kernel of this morphism of algebras, that is, the ideal:

$$\mathbf{I} = \mathbf{Ker}\mu = \{\theta \in \mathscr{A} \otimes_{\mathcal{R}} \mathscr{A} : \mu(\theta) = 0\} \subset \mathscr{A} \otimes_{\mathcal{R}} \mathscr{A} \tag{2.6}$$

it can be proved that the morphism of $\mathscr{A}$-modules:

$$\Sigma : \mathbf{\Omega} \longrightarrow \frac{\mathbf{I}}{\mathbf{I}^2},$$
$$df \longmapsto 1 \otimes f - f \otimes 1 \tag{2.7}$$

is an isomorphism.

In order to show this, we notice that the fractional object $\mathbf{I}/\mathbf{I}^2$ has an $\mathscr{A}$-module structure defined by:

$$f \cdot (f_1 \otimes f_2) = (f \cdot f_1) \otimes f_2 = f_1 \otimes (f \cdot f_2). \tag{2.8}$$

for $f_1 \otimes f_2 \in \mathbf{I}$, $f \in \mathcal{A}$. We can check that the second equality is true by proving that the difference of $(f \cdot f_1) \otimes f_2$ and $f_1 \otimes (f \cdot f_2)$ belonging to $\mathbf{I}$ is actually an element of $\mathbf{I}^2$, namely, the equality is true modulo $\mathbf{I}^2$. So we have

$$(f \cdot f_1) \otimes f_2 - f_1 \otimes (f \cdot f_2) = (f_1 \otimes f_2) \cdot (f \otimes 1 - 1 \otimes f). \tag{2.9}$$

The first factor of the above product of elements belongs to $\mathbf{I}$ by assumption, whereas the second factor also belongs to $\mathbf{I}$, since we have that

$$\mu(f \otimes 1 - 1 \otimes f) = 0. \tag{2.10}$$

Hence, the product of elements above belongs to $\mathbf{I} \cdot \mathbf{I} = \mathbf{I}^2$. Consequently, we can define a morphism of $\mathcal{A}$-modules:

$$\Sigma : \mathbf{\Omega} \longrightarrow \frac{\mathbf{I}}{\mathbf{I}^2},$$
$$df \longmapsto 1 \otimes f - f \otimes 1. \tag{2.11}$$

Now, we construct the inverse of that morphism as follows: the $\mathcal{A}$-module $\mathbf{\Omega}$ can be made an ideal in the algebra of dual numbers over $\mathcal{A}$, namely, $\mathcal{A} \oplus \mathbf{\Omega} \cdot \epsilon$. Moreover, we can define the morphism of algebras:

$$\mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A} \oplus \mathbf{\Omega} \cdot \epsilon,$$
$$(f_1, f_2) \longmapsto f_1 \cdot f_2 + f_1 \cdot df_2 \epsilon. \tag{2.12}$$

This is an $\mathcal{R}$-bilinear morphism of algebras, and thus, it gives rise to a morphism of algebras:

$$\Theta : \mathcal{A} \otimes_{\mathcal{R}} \mathcal{A} \longrightarrow \mathcal{A} \oplus \mathbf{\Omega} \cdot \epsilon. \tag{2.13}$$

Then, by definition, we have that $\Theta(\mathbf{I}) \subset \mathbf{\Omega}$, and also, $\Theta(\mathbf{I}^2) = 0$. Hence, there is obviously induced morphism of $\mathcal{A}$-modules:

$$\mathbf{\Omega} \longleftarrow \frac{\mathbf{I}}{\mathbf{I}^2} \tag{2.14}$$

which is the inverse of Σ. Consequently, we conclude that

$$\mathbf{\Omega} \cong \frac{\mathbf{I}}{\mathbf{I}^2}. \tag{2.15}$$

Thus, the free $\mathcal{A}$-module $\mathbf{\Omega}$ of 1-forms is isomorphic with the free $\mathcal{A}$-module $\mathbf{I}/\mathbf{I}^2$ of Kähler differentials of the algebra of scalars $\mathcal{A}$ over $\mathcal{R}$, conceived as distinguished ideals within the

algebra of infinitesimally extended scalars $[\mathcal{A} \oplus \Omega \cdot \epsilon]$ due to interaction, according to the following split short exact sequence:

$$\Omega \hookrightarrow \mathcal{A} \oplus \Omega \cdot \epsilon \twoheadrightarrow \mathcal{A} \qquad (2.16)$$

or equivalently formulated as:

$$0 \longrightarrow \Omega \longrightarrow \mathcal{A} \otimes_{\mathcal{R}} \mathcal{A} \longrightarrow \mathcal{A}. \qquad (2.17)$$

By dualizing, we obtain the dual $\mathcal{A}$-module of Ω, that is, $\Xi := \mathrm{Hom}(\Omega, \mathcal{A})$. Consequently, we have at our disposal, expressed in terms of infinitesimal scalars extension of the algebra $\mathcal{A}$, semantically intertwined with the generation of geometry as a result of interaction, new types of observables related with the incorporation of differentials and their duals, called vectors.

Let us now explain the function of geometry, as related with the infinitesimally extended algebras of scalars defined above, in the context of General Relativity. As we have argued before, the absolute representability principle of this theory, necessitates that our form of observation is semantically equivalent with real numbers representability. This means that all types of observables should possess uniquely defined dual types of observables, such that their representability can be made possible by means of real numbers. This is exactly the role of a geometry induced by a metric g. Concretely, a metric structure assigns a unique dual to each observable, by effectuating an isomorphism $\tilde{g}$ between the $\mathcal{A}$-module Ω and its dual $\mathcal{A}$-module $\Xi = \mathrm{Hom}(\Omega, \mathcal{A})$, that is,

$$\tilde{g} : \Omega \longrightarrow \Xi \qquad (2.18)$$

with

$$\tilde{g} : \Omega \cong \Xi,$$
$$df \longmapsto v_f := \tilde{g}(df). \qquad (2.19)$$

Equivalently, a metric g stands for an $\mathcal{R}$-valued symmetric bilinear form on Ω, that is, g $\Omega \times \Omega \to \mathcal{R}$, yielding an invertible $\mathcal{R}$-linear morphism $\tilde{g} : \Omega \to \Xi$. Notice that for $df, dh \in \Omega$, a symmetric bilinear form g acts, via $\tilde{g}$, on df to give an element of the dual $\tilde{g}(df) \in \Xi$, which then acts on dh to give $(\tilde{g}(df))(dh) = (\tilde{g}(dh))(df)$, or equivalently, $v_f(dh) = v_h(df) \in \mathcal{R}$. Also note that the invertibility of $\tilde{g}$ amounts to the property of nondegeneracy of g, meaning that for each $df \in \Omega$, there exists $dh \in \Omega$, such that $(\tilde{g}(df))(dh) = v_f(dh) \neq 0$.

Thus, the functional role of a metric geometry forces the observation of extended scalars, by means of representability in the field of real numbers, and is reciprocally conceived as the result of interactions causing infinitesimal variations in the scalars of the $\mathcal{R}$-algebra $\mathcal{A}$.

Before proceeding further, it is instructive at this point to clarify the meaning of a universal derivation, playing a paradigmatic role in the construction of extended algebras of

scalars as above, in appropriate category-theoretic terms as follows [21]: the covariant functor of left $\mathscr{A}$-modules valued derivations of $\mathscr{A}$:

$$\overleftarrow{\nabla}_{\mathscr{A}}(-) : \mathscr{M}^{(\mathscr{A})} \longrightarrow \mathscr{M}^{(\mathscr{A})},$$
$$\mathbf{N} \longmapsto \overleftarrow{\nabla}_{\mathscr{A}}(N) \tag{2.20}$$

is being representable by the left $\mathscr{A}$-module of 1-forms $\mathbf{\Omega}^1(\mathscr{A})$ in the category of left $\mathscr{A}$-modules $\mathscr{M}^{(\mathscr{A})}$, according to the isomorphism:

$$\overleftarrow{\nabla}_{\mathscr{A}}(N) \cong \mathrm{Hom}_{\mathscr{A}}\left(\mathbf{\Omega}^1(\mathscr{A}), N\right). \tag{2.21}$$

This isomorphism is induced by the derivation $d : \mathscr{A} \rightarrow \mathbf{\Omega}^1(\mathscr{A})$. Thus, $\mathbf{\Omega}^1(\mathscr{A})$ is characterized categorically as a universal object in $\mathscr{M}^{(\mathscr{A})}$, and the derivation $d : \mathscr{A} \rightarrow \mathbf{\Omega}^1(\mathscr{A})$ as the universal derivation [21]. Furthermore, we can define algebraically, for each $n \in N$, $n \geq 2$, the n-fold exterior product

$$\mathbf{\Omega}^n(\mathscr{A}) = \bigwedge^n \mathbf{\Omega}^1(\mathscr{A}), \tag{2.22}$$

where $\mathbf{\Omega}(\mathscr{A}) := \mathbf{\Omega}^1(\mathscr{A})$, $\mathscr{A} := \mathbf{\Omega}^0(\mathscr{A})$, and finally show analogously that the left $\mathscr{A}$-modules of n-forms $\mathbf{\Omega}^n(\mathscr{A})$ in $\mathscr{M}^{(\mathscr{A})}$ are representable objects in $\mathscr{M}^{(\mathscr{A})}$ of the covariant functor of left $\mathscr{A}$-modules valued n-derivations of $\mathscr{A}$, denoted by $\overleftarrow{\nabla}^n_{\mathscr{A}}(-) : \mathscr{M}^{(\mathscr{A})} \rightarrow \mathscr{M}^{(\mathscr{A})}$. We conclude that, all infinitesimally extended algebras of scalars that have been constructed from $\mathscr{A}$ by fibrations, presented equivalently as derivations, are representable as left $\mathscr{A}$-modules of n-forms $\mathbf{\Omega}^n(\mathscr{A})$ in the category of left $\mathscr{A}$-modules $\mathscr{M}^{(\mathscr{A})}$.

We emphasize that the physical intelligibility of the algebraic differential mechanism is based on the conception that infinitesimal variations in the scalars of $\mathscr{A}$ are caused by interactions, meaning that they are being effectuated by the presence of a physical field, identified as the gravitational field in the context of General Relativity. Thus, it is necessary to establish a purely algebraic representation of the notion of physical field and explain the functional role it assumes for the interpretation of the theory.

3. Functorial Representation of the Gravitational Field

The key idea serving the purpose of modeling the notion of the gravitational field in General Relativity algebraically amounts to expressing the process of infinitesimal scalars extension in functorial terms, and by anticipation, identifies the functor of infinitesimal scalars extension due to gravitational interaction with the physical field that causes it.

Regarding the first step of this strategy, we clarify that the general process of scalars extension from an algebra $\mathscr{S}$ to an algebra $\mathscr{T}$ is represented functorially by means of the functor of scalars extension, from $\mathscr{S}$ to $\mathscr{T}$ as follows:

$$\mathbf{Ext}^{\mathscr{S}}_{\mathscr{T}} := [-] \otimes_{\mathscr{S}} \mathscr{T} : \mathscr{M}^{(\mathscr{S})} \longrightarrow \mathscr{M}^{(\mathscr{T})},$$
$$\mathbf{E} \longmapsto \mathscr{T} \otimes_{\mathscr{S}} \mathbf{E}. \tag{3.1}$$

The functor $\mathbf{Ext}^{\mathcal{S}}{}_{\mathcal{T}}$ is called the extension of scalars functor from the category of $\mathcal{S}$-modules to the category of $\mathcal{T}$-modules. Furthermore, the algebraic functorial processes of restriction and extension of scalars are conceptually inverse, namely, given a morphism of commutative and unital algebras of scalars $\mathcal{S} \rightarrow \mathcal{T}$, and there exists a categorical adjunction

$$\mathbf{L} : \mathcal{M}^{(\mathcal{S})} \rightleftarrows \mathcal{M}^{(\mathcal{T})} : \mathbf{R}, \tag{3.2}$$

where $\mathbf{Ext}^{\mathcal{S}}{}_{\mathcal{T}} := \mathbf{L}$ is left adjoint to $\mathbf{Res}^{\mathcal{T}}{}_{\mathcal{S}} := \mathbf{R}$.

The second step involves the application of the functorial algebraic procedure for the case admitting the identifications $\mathcal{S} = \mathcal{A}$, $\mathcal{T} = [\mathcal{A} \oplus \Omega^1(\mathcal{A}) \cdot \epsilon]$ corresponding to infinitesimal scalars extension. Consequently, the physical field as the causal agent of interactions admits a purely algebraic description as the functor of infinitesimal scalars extension, called a connection-inducing functor:

$$\widehat{\nabla} : \mathcal{M}^{(\mathcal{A})} \longrightarrow \mathcal{M}^{(\mathcal{A} \oplus \Omega^1(\mathcal{A}) \cdot \epsilon)},$$
$$\mathbf{E} \longmapsto \left[\mathcal{A} \oplus \Omega^1(\mathcal{A}) \cdot \epsilon\right] \otimes_{\mathcal{A}} \mathbf{E}. \tag{3.3}$$

Thus, the effect of the action of the physical field on the vectors of the left $\mathcal{A}$-module $\mathbf{E}$ can be expressed by means of the following morphism of left $\mathcal{A}$-modules:

$$\nabla^{\star}{}_{\mathbf{E}} : \mathbf{E} \longrightarrow \mathbf{E} \oplus \left[\Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathbf{E}\right] \cdot \epsilon. \tag{3.4}$$

The irreducible amount of information incorporated in the above morphism can be, equivalently, expressed as a connection on $\mathbf{E}$, namely, as an $\mathcal{R}$-linear morphism of $\mathcal{A}$-modules [3]:

$$\nabla_{\mathbf{E}} : \mathbf{E} \longrightarrow \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathbf{E} = \mathbf{E} \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) = \Omega^1(\mathbf{E}) \tag{3.5}$$

such that, the following Leibniz type constraint is satisfied:

$$\nabla_{\mathbf{E}}(f \cdot v) = f \cdot \nabla_{\mathbf{E}}(v) + df \otimes v \tag{3.6}$$

for all $f \in \mathcal{A}$, $v \in \mathbf{E}$.

In the context of General Relativity, the absolute representability principle over the field of real numbers necessitates as we have explained above the existence of uniquely defined duals of observables. Thus, the gravitational field is identified with a linear connection on the $\mathcal{A}$-module $\Xi = \mathrm{Hom}(\Omega, \mathcal{A})$, being isomorphic with Ω^1, by means of a metric

$$\widetilde{g} : \Omega^1 \simeq \Xi = \Omega^{1*} \tag{3.7}$$

and consequently, it may be represented by the pair (Ξ, ∇_{Ξ}). The metric compatibility of the connection required by the theory is simply expressed as:

$$\nabla_{\mathrm{Hom}_{\mathscr{A}}(\Xi, \Xi^*)}(\widetilde{g}) = 0. \tag{3.8}$$

It is instructive to emphasize that the functorial conception of physical fields in general, according to the proposed schema, based on the notion of causal agents of infinitesimal scalars extension, does not depend on any restrictive representability principle, like the absolute representability principle over the real numbers, imposed by General Relativity. Consequently, the meaning of functoriality implies covariance with respect to representability, and thus, covariance with respect to generalized geometric realizations. In the same vein of ideas, the reader has already noticed that all the algebraic arguments refer, on purpose, to a general observable algebra $\mathscr{A}$, that has been identified with the $\mathcal{R}$-algebra $\mathcal{C}^\infty(M)$ in the model case of General Relativity. Of course, the functorial mechanism of understanding the notion of interaction should not depend on the observable algebras used for the particular manifestations of it, thus, the only actual requirement for the intelligibility of functoriality of interactions by means of physical fields rests on the algebra-theoretic specification of what we characterize structures of observables. Put differently, the functorial coordinatization of the universal mechanism of encoding physical interactions in terms of observables, by means of causal agents, namely, physical fields effectuating infinitesimal scalars extension, should respect the algebra-theoretic structure.

4. The Algebraic De Rham Complex and the Role of Curvature

The next stage of development of the algebraic schema of comprehending the mechanism of dynamics involves the satisfaction of appropriate global constraints that impose consistency requirements referring to the extension from the infinitesimal to the global. These requirements are homological and characterize the integrability property of the variation process induced by a connection. Moreover, they are properly addressed by the construction of the De Rham complex associated to an integrable connection. For this purpose, it is necessary to review briefly the mathematical construction of algebraic de Rham complexes [1–3, 17]. In parallel, we provide the corresponding dynamical physical interpretation.

We start by reminding the algebraic construction, for each $n \in N$, $n \geq 2$, of the n-fold exterior product as follows: $\mathbf{\Omega}^n(\mathscr{A}) = \bigwedge^n \mathbf{\Omega}^1(\mathscr{A})$, where $\mathbf{\Omega}(\mathscr{A}) := \mathbf{\Omega}^1(\mathscr{A})$, $\mathscr{A} := \mathbf{\Omega}^0(\mathscr{A})$. We notice that there exists an $\mathcal{R}$-linear morphism:

$$d^n : \mathbf{\Omega}^n(\mathscr{A}) \longrightarrow \mathbf{\Omega}^{n+1}(\mathscr{A}) \tag{4.1}$$

for all $n \geq 0$, such that $d^0 = d$. Let $\omega \in \mathbf{\Omega}^n(\mathscr{A})$, then ω has the form:

$$\omega = \sum f_i \left(dl_{i1} \bigwedge \cdots \bigwedge dl_{in} \right) \tag{4.2}$$

with $f_i, l_{ij}, \in \mathscr{A}$ for all integers i, j. Further, we define

$$d^n(\omega) = \sum df_i \bigwedge dl_{i1} \bigwedge \cdots \bigwedge dl_{in}. \tag{4.3}$$

Then, we can easily see that the resulting sequence of $\mathcal{R}$-linear morphisms,

$$\mathcal{A} \longrightarrow \Omega^1(\mathcal{A}) \longrightarrow \cdots \longrightarrow \Omega^n(\mathcal{A}) \longrightarrow \cdots, \tag{4.4}$$

is a complex of $\mathcal{R}$-vector spaces, called the algebraic de Rham complex of $\mathcal{A}$. The notion of complex means that the composition of two consecutive $\mathcal{R}$-linear morphisms vanishes, that is $d^{n+1} \circ d^n = 0$, simplified symbolically as:

$$d^2 = 0. \tag{4.5}$$

If we assume that $(\mathbf{E}, \nabla_{\mathbf{E}})$ is an interaction field, defined by a connection $\nabla_{\mathbf{E}}$ on the $\mathcal{A}$-module $\mathbf{E}$, then $\nabla_{\mathbf{E}}$ induces a sequence of $\mathcal{R}$-linear morphisms:

$$\mathbf{E} \longrightarrow \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathbf{E} \longrightarrow \cdots \longrightarrow \Omega^n(\mathcal{A}) \otimes_{\mathcal{A}} \mathbf{E} \longrightarrow \cdots \tag{4.6}$$

or equivalently:

$$\mathbf{E} \longrightarrow \Omega^1(\mathbf{E}) \longrightarrow \cdots \longrightarrow \Omega^n(\mathbf{E}) \longrightarrow \cdots, \tag{4.7}$$

where the morphism

$$\nabla^n : \Omega^n(\mathcal{A}) \otimes_{\mathcal{A}} \mathbf{E} \longrightarrow \Omega^{n+1}(\mathcal{A}) \otimes_{\mathcal{A}} \mathbf{E} \tag{4.8}$$

is given by the formula

$$\nabla^n(\omega \otimes v) = d^n(\omega) \otimes v + (-1)^n \omega \wedge \nabla(v) \tag{4.9}$$

for all $\omega \in \Omega^n(\mathcal{A})$, $v \in \mathbf{E}$. It is immediate to see that $\nabla^0 = \nabla_{\mathbf{E}}$. Let us denote by

$$\mathbf{R}_\nabla : \mathbf{E} \longrightarrow \Omega^2(\mathcal{A}) \otimes_{\mathcal{A}} \mathbf{E} = \Omega^2(\mathbf{E}) \tag{4.10}$$

the composition $\nabla^1 \circ \nabla^0$. We see that $\mathbf{R}_\nabla$ is actually an $\mathcal{A}$-linear morphism, that is $\mathcal{A}$-covariant, and is called the curvature of the connection $\nabla_{\mathbf{E}}$. We note that for the case of the gravitational field (Ξ, ∇_Ξ), in the context of General Relativity, $\mathbf{R}_\nabla$ is equivalent to the Riemannian curvature of the spacetime manifold. We notice that, the latter sequence of $\mathcal{R}$-linear morphisms is actually a complex of $\mathcal{R}$-vector spaces if and only if

$$\mathbf{R}_\nabla = 0. \tag{4.11}$$

We say that the connection $\nabla_{\mathbf{E}}$ is integrable if $\mathbf{R}_\nabla = 0$, and we refer to the above complex as the de Rham complex of the integrable connection $\nabla_{\mathbf{E}}$ on $\mathbf{E}$ in that case. It is also usual to call a connection $\nabla_{\mathbf{E}}$ flat if $\mathbf{R}_\nabla = 0$. A flat connection defines a maximally undisturbed process of dynamical variation caused by the corresponding physical field. In this sense, a

nonvanishing curvature signifies the existence of disturbances from the maximally symmetric state of that variation. These disturbances can be cohomologically identified as obstructions to deformation caused by physical sources. In that case, the algebraic de Rham complex of the algebra of scalars $\mathcal{A}$ is not acyclic, namely, it has nontrivial cohomology groups. These groups measure the obstructions caused by sources and are responsible for a nonvanishing curvature of the connection. In the case of General Relativity, these disturbances are associated with the presence of matter distributions, being incorporated in the specification of the energy-momentum tensor. Taking into account the requirement of absolute representability over the real numbers, and thus considering the relevant evaluation trace operator by means of the metric, we arrive at Einstein's field equations, which in the absence of matter sources read

$$R(\nabla_\Xi) = 0, \tag{4.12}$$

where $R(\nabla_\Xi)$ denotes the relevant Ricci scalar curvature. In more detail, we first define the curvature endomorphism $\mathfrak{R}_\nabla \in \mathbf{E}nd(\Xi)$ [3], called Ricci curvature operator. Since the Ricci curvature $\mathfrak{R}_\nabla$ is matrix valued, by taking its trace using the metric, that is by considering its evaluation or contraction by means of the metric, we arrive at the definition of the Ricci scalar curvature $R(\nabla_\Xi)$ obeying the above equation.

5. Conclusions

The central focus of the studies pertaining to the current research attempts towards the construction of a tenable theory of gravitational interactions in the quantum regime revolves around the issue of background spacetime manifold independence, as it is evident from the literature [4–14], and especially, the criticism and suggestions offered in [22, 23, 27, 28]. In this communication, we have constructed a general functorial framework of modeling field dynamics, modeled on the conceptual basis of the Kähler-De Rham differential mechanism, using homological algebraic concepts and techniques. In particular, we have applied this framework in the case of General Relativity recovering the classical gravitational dynamics. The significance of the proposed functorial schema of dynamics lies on the fact that the coordinatization of the universal mechanism of encoding physical interactions in terms of observables, by means of causal agents, namely, physical fields effectuating infinitesimal scalars extension should respect only the algebra-theoretic structure of observables. Consequently, it is not constrained at all by the absolute representability principle over the field of real numbers, imposed by classical General Relativity, a byproduct of which is the fixed background manifold construct of that theory. In this vein of ideas, the requirement of background manifold independence can be attained, by rejecting the absolute representability of the classical theory over the real numbers, and thus, the fixed spacetime manifold substratum, while keeping at the same time, the homological machinery of functorial dynamics. In a following paper, we will explain in detail that the abolishment of the above absolute representability requirement of the classical theory, paving the way towards Quantum Relativity, can be achieved by effectuating a process of sheaf-theoretic localization suitable for the modeling of quantum phenomena [21, 29], based on the technique of (pre)-sheafification of observable algebras.

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