Preface

The aim of the authors is to lay down the foundations of the projective systems of various geometrical structures modelled on Banach spaces, eventually leading to homologous structures in the framework of Fréchet differential geometry, by overcoming some of the inherent deficiencies of Fréchet spaces. We elaborate this brief description in the sequel.

Banach spaces, combining a metric topology (subordinate to a norm), and a linear space structure (for representing derivatives as linear approximations to functions in order to do calculus), provide a very convenient setting for many problems in functional analysis, which we need for handling calculus on function spaces, usually infinite dimensional. They are a relatively gentle extension from experience on finite dimensional spaces, since many topological properties of spaces and groups of linear maps, as well as many of the existence and uniqueness theorems for solutions of differential equations carry over to the infinite dimensional case.

Manifolds and fibre bundles modelled on Banach spaces arise from the synthesis of differential geometry and functional analysis, thus leading to important examples of global analysis. Indeed, many spaces of (differentiable) maps between appropriate manifolds admit the structure of Banach manifolds (see, for instance, J. Eells [Eel66, § 6]).

On the other hand, as mentioned also in [Eel66], Riemannian manifolds, represented as rigid maps on infinite dimensional function spaces, arise as configuration spaces of dynamical systems, with metrics interpreted as kinetic energy. Much of the calculus of variations and Morse theory is concerned with a function space in differential geometry—the Euler-Lagrange operator of a variational problem is interpreted as a gradient vector field, with integral curves the paths of steepest ascent. Some eigenvalue problems in integral and differential equations are interpretable via Lagrangian multipliers, involving infinite dimensional function spaces from differential geometry—such as focal point theory and geometric consequences of the inverse function theorem in infinite dimensions.

However, in a number of situations that have significance in global analysis and physics, for example, physical field theory, Banach space representations break down. A first step forward is achieved by weakening the topological requirements: Instead of a norm, a family of seminorms is considered. This leads to Fréchet spaces, which do have a linear structure and their topology is defined through a sequence of seminorms.

Although Fréchet spaces seem to be very close to Banach spaces, a number of critical deficiencies emerge in their framework. For instance, despite the progress in particular cases, they lack a general solvability theory of differential equations, even the linear ones; also, the space of continuous linear morphisms between Fréchet spaces does not remain in the category, and the space of linear isomorphisms does not admit a reasonable Lie group structure.

The situation becomes much more complicated when we consider manifolds modelled on Fréchet spaces. Fundamental tools such as the exponential map of a Fréchet-Lie group may not exist. Additional complications become particularly noticeable when we try to collect Fréchet spaces together to form bundles (over manifolds modelled on atlases of Fréchet spaces), in order to develop geometrical operators like covariant derivatives and curvature to act on sections of bundles. The structure group of such bundles, being the general linear group of a Fréchet space, is not a Lie group—even worse, it does not have a natural topological structure. Parallel translations do not necessarily exist because of the inherent difficulties in solving differential equations within this framework, and so on.

This has relevance to real problems. The space of smooth functions $C^\infty(I, \mathbb{R})$, where I is a compact interval of $\mathbb{R}$, is a Fréchet space. The space $C^\infty(M, V)$, of smooth sections of a vector bundle V over a compact smooth Riemannian manifold M with covariant derivative ∇, is a Fréchet space. The C^∞ Riemannian metrics on a fixed closed finite-dimensional orientable manifold has a Fréchet model space. Fréchet spaces of sections arise naturally as configurations of a physical field. Then the moduli space, consisting of inequivalent configurations of the physical field, is the quotient of the infinite-dimensional configuration space $\mathcal{X}$ by the appropriate symmetry gauge group. Typically, $\mathcal{X}$ is modelled on a Fréchet space of smooth sections of a vector bundle over a closed manifold.

Despite their apparent differences, the categories of Banach and Fréchet spaces are connected through projective limits. Indeed, the limiting real product space $\mathbb{R}^\infty = \lim_{n\to\infty} \mathbb{R}^n$ is the simplest example of this situation. Taking notice of how $\mathbb{R}^\infty$ arises from $\mathbb{R}^n$, this approach extends to arbitrary Fréchet spaces, since always they can be represented by a countable sequence of Banach spaces in a somewhat similar manner. Although careful concentration to the above example is salutary, (bringing to mind the story of the mathematician drafted to work on a strategic radar project some 70 years ago, who when told of the context said "but I only know Ohms Law!" and the response came, "you only need to know Ohms Law, but you must know it very, very well"), it should be emphasized that the mere properties of $\mathbb{R}^\infty$ do not answer all the questions and problems referring to the more complicated geometrical structures mentioned above.

The approach adopted is designed to investigate, in a systematic way, the extent to which the shortcomings of the Fréchet context can be worked round by viewing, under sufficient conditions, geometrical objects and properties in this context as limits of sequences of their Banach counterparts, thus exploiting the well developed geometrical tools of the latter. In this respect, we propose, among other generalizations, the replacement of certain pathological structures and spaces such as the structural group of a Fréchet bundle, various spaces of linear maps, frame bundles, connections on principal and vector bundles etc., by appropriate entities, susceptible to the limit process. This extends many classical results to our framework and, to a certain degree, bypasses its drawbacks.

Apart from the problem of solving differential equations, much of our work is motivated also by the need to endow infinite-dimensional Lie groups with an exponential map [a fact characterizing–axiomatically–the category of (infinite-dimensional) *regular* Lie groups]; the differential and vector bundle structure of the set of infinite jets of sections of a Banach vector bundle (compare with the differential structure described in [Tak79]); the need to put in a wider perspective particular cases of projective limits of manifolds and Lie groups appearing in physics (see e.g. [AM99], [AI92], [AL94], [Bae93]) or in various groups of diffeomorphisms (e.g. [Les67], [Omo70]).

For the convenience of the reader, we give an outline of the presentation, referring for more details to the table of contents and the introduction to each chapter.

Chapter 1 introduces the basic notions and results on Banach manifolds and bundles, with special emphasis on their geometry. Since there is not a systematic treatment of the general theory of connections on Banach principal and vector bundles (apart from numerous papers, with some very fundamental ones among them), occasionally we include extra details on specific topics, according to the needs of subsequent chapters. With a few exceptions, there are not proofs in this chapter and the reader is guided to the literature for details. This is to keep the notes within a reasonable size; however, the subsequent chapters are essentially self-contained.

Chapter 2 contains a brief account of the structure of Fréchet spaces and the differentiability method applied therein. From various possible differentiability methods we have chosen to apply that of J.A. Leslie [Les67], [Les68], a particular case of Gâteaux differentiation which fits well to the structure of locally convex spaces, without recourse to other topologies. Among the main features of this chapter we mention the representation of a Fréchet space by a projective limit of Banach spaces, and that of some particular spaces of continuous linear maps by projective limits of Banach functional spaces, a fact not true for arbitrary spaces of linear maps. An application of the same representation is proposed for studying differential equations in Fréchet spaces, including also comments on other approaches to the same subject. Projective limit representations of various geometrical structures constitute one of the main tools of our approach.

Chapter 3 is dealing with the smooth structure, under appropriate conditions, of Fréchet manifolds arising as projective limits of Banach manifolds, as well as with topics related to their tangent bundles. The case of Fréchet-Lie groups represented by projective limits of Banach-Lie groups is also studied in detail, because of their fundamental role in the structure of Fréchet principal bundles. Such groups admit an exponential map, an important property not yet established for arbitrary Fréchet-Lie groups.

Chapter 4 is devoted to the study of projective systems of Banach principal bundles and their connections. The latter are handled by their connection forms, global and local ones. It is worthy of note that any Fréchet principal bundle, with structure group one of those alluded to in Chapter 3, is always representable as a projective limit of Banach principal bundles, while any connection on the former bundle is an appropriate projective limit of connections in the factor bundles of the limit. Here, related (or conjugate) connections, already treated in Chapter 1, provide an indispensable tool in the approach to connections in the Fréchet framework. We further note that the holonomy groups of the limit bundle do not necessarily coincide with the projective limits of the holonomy groups of the factor bundles. This is supported by an example after the study of flat bundles.

Chapter 5 is concerned with projective limits of Banach vector bundles. If the fibre type of a limit bundle is the Fréchet space $\mathbb{F}$, the structure of the vector bundle is fully determined by a particular group (denoted by $\mathcal{H}_0(\mathbb{F})$ and described in § 5.1), which replaces the pathological general linear group $\mathrm{GL}(\mathbb{F})$ of $\mathbb{F}$, thus providing the limit with the structure of a Fréchet vector bundle. The study of connections on vector bundles of the present type is deferred until Chapter 7.

Chapter 6 contains a collection of examples of Fréchet bundles realized as projective limits of Banach ones. Among them, we cite in particular the bundle $J^\infty(E)$ of infinite jets of sections of a Banach vector bundle E. This is a non trivial example of a Fréchet vector bundle, essentially motivating the conditions required to define the structure of an arbitrary vector bundle in the setting of Chapter 5. On the other hand, the generalized bundle of frames of a Fréchet vector bundle is an important example of a principal bundle with structure group the aforementioned group $\mathcal{H}_0(\mathbb{F})$.

Chapter 7 aims at the study of connections on Fréchet vector bundles the latter being in the sense of Chapter 5. The relevant notions of parallel displacement along a curve and the holonomy group are also examined. Both can be defined, despite the inherent difficulties of solving equations in Fréchet spaces, by reducing the equations involved to their counterparts in the factor Banach bundles.

Chapter 8 is mainly focused on the vector bundle structure of the second order tangent bundle of a Banach manifold. Such a structure is always defined once we choose a linear connection on the base manifold, thus a natural question is to investigate the dependence of the vector bundle structure on the choice of the connection. The answer relies on the possibility to characterize the second order differentials as vector bundle morphisms, which is affirmative if the connections involved are properly related (conjugate). The remaining part of the chapter is essentially an application of our methods to the second order Fréchet tangent bundle and the corresponding (generalized) frame bundle.

We conclude with a series of open problems or suggestions for further applications, within the general framework of our approach to Fréchet geometry, eventually leading to certain topics not covered here.

These notes are addressed to researchers and graduate students of mathematics and physics with an interest in infinite-dimensional geometry, especially that of Banach and Fréchet manifolds and bundles. Since we have in mind a wide audience, with possibly different backgrounds and interests, we have paid particular attention to the details of the exposition so that it is as far as possible self-contained. However, a familiarity with the rudiments of the geometry of manifolds and bundles (at least of finite dimensions) is desirable if not necessary.

It is a pleasure to acknowledge our happy collaboration, started over ten years ago by discussing some questions of common research interest and resulting in a number of joint papers. The writing of these notes is the outcome of this enjoyable activity. Finally, we are very grateful to an extremely diligent reviewer who provided many valuable comments and suggestions on an earlier draft, we have benefited much from this in the final form of the monograph.

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