

# Exact Speed and Transmission Cost in a Simple One-Dimensional Wireless Delay-Tolerant Network

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**Abstract**—We study a simple one-dimensional, discrete-time network model that consists of two nodes moving on a discrete circle, changing their direction of movement randomly, and a single packet travelling in the clockwise direction, using combinations of transmissions between the two nodes (when they are co-located) and physical transports on their buffers. In this setting, we provide exact, explicit expressions for the long-term averages of the packet speed and the wireless transmission cost. Our work is a first step towards providing simple and exact results for more realistic wireless delay-tolerant network models.

**Index Terms**—Delay, mobile network, mobility, one-dimensional network, wireless delay-tolerant network.

## I. INTRODUCTION

Numerous wireless mobile delay-tolerant network (DTN) models have recently been proposed, where packets travel to their destinations through combinations of both wireless transmissions and physical transports on the buffers of nodes, resulting in packet delivery delays that are much larger than typically expected. Examples include satellite, vehicular, pocket-switched, deep-space, and even sensor networks [1]. In some cases, physical transports (and the induced delay) are unavoidable; in other cases, they are tolerated in order to improve other performance figures, such as energy and spectral efficiency. Therefore, understanding the fundamental tradeoffs involving delay in such networks is of great current interest.

A variety of approaches have been taken towards this goal. In [2], cost/speed tradeoffs are studied for an infinite number of nodes moving in  $\mathbb{R}^2$  under a random way-point model; the authors use tools from stochastic geometry, resorting to approximations in order to make the analysis tractable. In [3], a related throughput/delay tradeoff is examined, in the well-known Gupta/Kumar setting: Asymptotic results are established as the number of nodes increases, and the relevant quantities are computed up to a multiplicative constant. In contrast, in [4] the authors study cost/delay tradeoffs in networks whose topology evolution is completely known in advance, using tools from the theory of dynamic networks and flows.

Due to the emergence of vehicular networks, where nodes are constrained to move on roads, numerous works have focused on one-dimensional networks. Notably, [5] considers such a bidirectional model with two lanes of node traffic moving with a fixed common speed in opposite directions. The packet propagation speed undergoes a phase transition as the density of nodes increases: For low densities it matches

the node speed, whereas for densities above a threshold it increases quasi-exponentially. In [6] the same authors consider a multi-lane setting with time-varying radio ranges. In [7] a multi-lane setting is also adopted and, in addition, node speeds change randomly, according to a lane-specific distribution; here, an accurate estimate for the packet speed is provided. Connectivity problems in a similar setting are studied in [8].

In this work, we study a simple discrete-time network consisting of two nodes and a single packet. The nodes move randomly on a discrete circle of  $N$  locations, changing their direction of travel with probability  $\epsilon$  in each time slot. The packet travels in the clockwise direction using a combination of physical transports (on the buffers of the two nodes) and wireless transmissions (which take place only when the two nodes are co-located, the current carrier of the packet is travelling in the counter-clockwise direction, and the other node is travelling in the clockwise direction). In this setting, we provide exact, explicit expressions for the long-term averages of (i) the packet speed (i.e., the rate of progress in the clockwise direction), and (ii) the wireless transmission cost (i.e., the wireless transmissions per time slot).

Although the model considered here is certainly too simple as a description of realistic applications, we expect that the resulting analysis could provide a first step towards identifying the appropriate technical tools and ideas that may facilitate the analysis of more complex scenarios arising in practice. Finally we remark that, in contrast with the earlier works mentioned above, our results are exact, and given in terms of simple, explicit expressions in terms of the problem parameters; furthermore, these results are not only on the packet speed, but also on the wireless transmission cost.

## II. MODEL

Let  $S = \{0, 1, \dots, N-1\} = \mathbb{Z}/N\mathbb{Z}$  denote the discrete  $N$ -circle, for a fixed odd  $N \geq 3$  ( $N$  is assumed to be odd in order to avoid uninteresting technicalities stemming from the fact that the resulting chain  $\Phi$  below is periodic when  $N$  is even). We place 2 independent random walkers  $X_t = (X_t(1), X_t(2))$  on  $S$ , and with each walker  $i = 1, 2$  we associate a random direction  $D_t(i)$  at time  $t$ , where  $D_t(i)$  is either  $+1$  (clockwise motion) or  $-1$  (counter-clockwise motion). The initial positions  $X_0$  and directions  $D_0 = (D_0(1), D_0(2))$  are arbitrary. The Markov chain  $\{(X_t, D_t) ; t \geq 0\}$  evolves on the state space  $S^2 \times \{-1, +1\}^2$  as follows.

Let  $\{Z_t = (Z_t(1), Z_t(2))\}$  be a sequence of independent Bernoulli random variables with parameter  $\epsilon \in (0, 1)$ , and  $\{U_t = (U_t(1), U_t(2))\}$  be independent  $\pm 1$  uniform random variables. Given the current state  $(X_t, D_t)$ , each walker  $i$  takes a step in the direction given by  $D_t(i)$ ,

$$X_{t+1}(i) = X_t(i) + D_t(i) \pmod{N},$$

for  $t \geq 0, i = 1, 2$  and then decides to either continue moving in the same direction with probability  $(1 - \epsilon)$ , or to choose a new, random direction, with probability  $\epsilon$ :

$$D_{t+1}(i) = (1 - Z_t(i))D_t(i) + Z_t(i)U_t(i),$$

for all  $t \geq 0, i = 1, 2$ .

A single packet is travelling on  $S$ . To track its movement, we define an index process  $\{I_t\}$  evolving on  $\{1, 2\}$ , with  $I_0$  chosen arbitrarily and  $I_t$  trying to track walkers that move clockwise: Given  $(X_t, D_t, I_t = i)$ , let  $(X_{t+1}, D_{t+1})$  be defined as above. If  $D_{t+1}(i) = -1$  and the other walker  $i'$  is in the same location at time  $t$ , but its direction is  $+1$ , then  $I_{t+1} = i'$ , i.e., a transmission takes place. In all other cases,  $I_{t+1} = I_t = i$ .

It is easy to see from the above construction that  $\Phi = \{\Phi(t) = (X_t, D_t, I_t); t \geq 0\}$  is an irreducible and aperiodic chain on the state space  $\Sigma$  consisting of all configurations of the form,

$$(x(1), x(2), d(1), d(2), i) \in S^2 \times \{+1, -1\}^2 \times \{1, 2\},$$

except those where  $d(i) = -1, x(1) = x(2)$  and  $d(i') = +1$ . Moreover, the unique invariant distribution of the chain  $\{(X_t, D_t)\}$  is uniform: The positions  $X_t(i)$  are independent of each other and uniformly distributed on  $S$ , and the directions  $D_t(i)$  are independent of the positions  $X_t$  and each  $D_0(i) = \pm 1$  with probability  $1/2$ , independently of the others.

The main goal of this work is to answer the following questions: (i) What is the limiting distribution of the direction  $D_t(I_t)$  of the message at time  $t$ ? (ii) What is the long-term average speed of the packet? (iii) What is the long-term average number of transmissions per unit time?

### III. RESULTS

Let  $P_\pi$  and  $P_\phi$  denote the distribution of the Markov chain  $\Phi$  when the initial state is distributed according to  $\pi$  or when it is equal to a state  $\phi \in \Sigma$ , respectively.

**Theorem 1 (Packet speed).** *For any initial state, the long-term average packet speed is*

$$\begin{aligned} s &:= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{t=0}^{n-1} D_t(I_t) = E_\pi[D_1(I_1)] \\ &= \frac{2 - \epsilon}{2(2 + \epsilon(N - 2))}, \quad a.s., \end{aligned}$$

where  $\pi$  denotes the unique stationary distribution of the chain  $\Phi = \{\Phi(t) = (X_t, D_t, I_t); t \geq 0\}$ .

Theorem 1 answers Question (ii) of Section II. The answer to Question (i) is a simple consequence of Theorem 1:

**Corollary 2 (Packet direction).** *For any initial state  $\Phi(0) = \phi \in \Sigma$ , the steady state direction of the packet is:*

$$\begin{aligned} d &:= P_\pi(D_1(I_1) = +1) = \lim_{t \rightarrow \infty} P_\phi(D_t(I_t) = +1) \\ &= \frac{s + 1}{2} = \frac{6 + \epsilon(2N - 5)}{4(2 + \epsilon(N - 2))}. \end{aligned}$$

We note that the speed  $s = s(N, \epsilon)$  is decreasing in both  $N$  and  $\epsilon$ ; see Fig. 1. In the boundary case  $\epsilon = 0$ , the speed  $s(N, \epsilon)$  is either  $+1$  or  $-1$ , depending on the initial directions of the two walkers. Therefore,  $s(N, \epsilon)$  is discontinuous at  $\epsilon = 0$ , since  $s(N, \epsilon) \uparrow 1/2$  as  $\epsilon \downarrow 0$ , for any  $N$ .

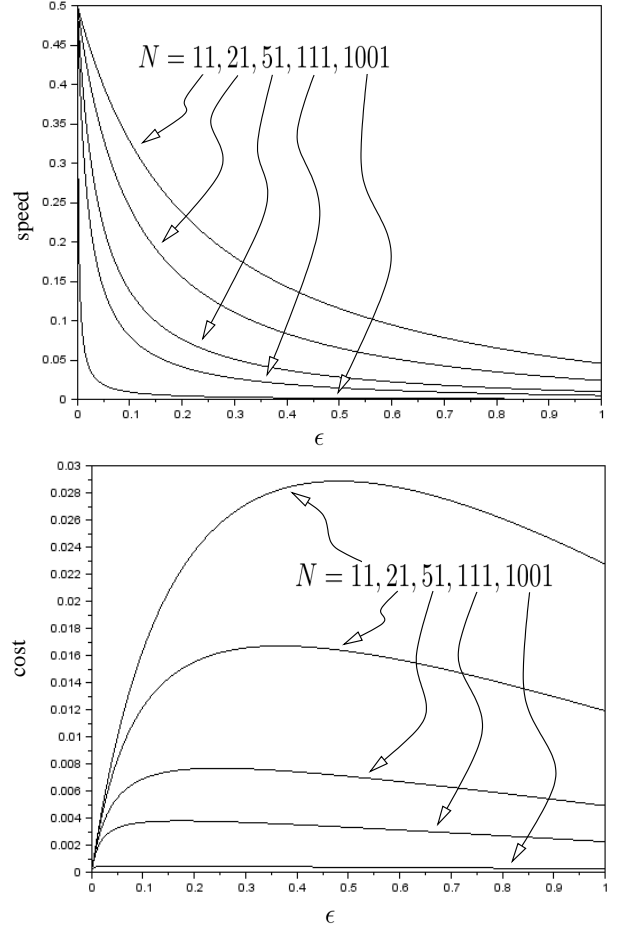


Fig. 1. Plots of the asymptotic speed  $s = s(N, \epsilon)$  and cost  $c = c(N, \epsilon)$  of the packet, as a function of  $\epsilon$ , for  $N = 11, 21, 51, 111$  and  $1001$ .

Also we observe that, for a large circle, i.e.,  $N \gg 1$ , and a small rate of direction updates  $\epsilon \approx 0$ , we have the following interesting scaling limit: If  $\epsilon = \gamma/N$  for some  $\gamma > 0$ , then the speed  $s(N, \epsilon) \rightarrow (2 + \gamma)^{-1}$  as  $N \rightarrow \infty$ . This suggests the following question: Is there a space-time scaling of this family of chains that converges to the obvious continuous-time/continuous space version of this problem?

Our second main result determines the long-term cost of the packet transmissions, i.e., the steady-state average number of transmissions  $c = c(N, \epsilon)$  per unit time. This answers the third question stated in Section II.

**Theorem 3 (Transmission cost).** *For any initial state, the long-term average cost of packet transmissions is:*

$$\begin{aligned} c &:= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{t=0}^{n-1} \mathbb{I}_{\{I_{t+1} \neq I_t\}} = P_\pi(I_2 \neq I_1) \\ &= \frac{\epsilon(2 - \epsilon)}{4[2 + \epsilon(N - 2)]}, \quad a.s. \end{aligned}$$

Note that the cost  $c = c(N, \epsilon)$  is decreasing in  $N$ , and for each fixed  $N$  it is a concave function of  $\epsilon$ ; see Fig. 1. Also, unlike the speed  $s = s(N, \epsilon)$ ,  $c = c(N, \epsilon)$  is continuous and equal to zero at  $\epsilon = 0$ . It is also interesting to observe the following simple, scale-free relationship between the speed and the cost:  $c(N, \epsilon) = (\epsilon/2)s(N, \epsilon)$ , for all  $N$  and  $\epsilon$ . Therefore, on the average, the packet travels a (clockwise) distance of  $2/\epsilon$  units between successive jumps, regardless of the value of  $N$ .

In terms of the speed/cost tradeoff, note that for each  $N$  there is an  $\epsilon^*$  below which the speed increases *and* the cost decreases as  $\epsilon \downarrow 0$ . Furthermore, in the limit  $N \rightarrow \infty$  with  $\epsilon = \gamma/N$  discussed earlier, although the speed  $c(N, \epsilon)$  converges to a finite, nonzero constant, the cost  $c(N, \epsilon)$  decays to zero as  $O(1/N)$ . This suggests that, if such a protocol were to be implemented in practice, it is the relatively smaller values of  $\epsilon$  that would be most effective in the long run.

#### IV. PROOF OF THEOREM 1

Consider the reduced chain,

$$\Psi = \{\Psi(t) = (Y_t = X_t(1) - X_t(2), D_t(1), D_t(2), I_t); t \geq 0\}$$

where the differences  $Y_t = X_t(1) - X_t(2)$  are taken modulo  $N$ . Clearly  $\Psi$  is irreducible and aperiodic on the corresponding reduced state space  $\Sigma_\psi$  consisting of all configurations of the form,

$$(y, d, d', i) \in S \times \{+1, -1\}^2 \times \{1, 2\},$$

except  $(0, +1, -1, 2)$  and  $(0, -1, +1, 1)$ . Let  $\pi_\psi$  denote the unique invariant measure of  $\Psi$ . The limit in the theorem exists a.s. by ergodicity; in order to compute its actual value, we define the regeneration time

$$T = \inf\{t \geq 1; Y_t = 0 \text{ and } D_t(1) \neq D_t(2)\}, \quad (1)$$

and we consider two special states of  $\Psi$ :  $\psi_1 = (0, +1, -1, 1)$  and  $\psi_2 = (0, -1, +1, 2)$ . Let  $\nu$  denote the probability measure on  $\Sigma_\psi$  given by,

$$\nu = \frac{1}{2}\delta_{\psi_1} + \frac{1}{2}\delta_{\psi_2}, \quad (2)$$

where, as usual,  $\delta_x$  denotes the unit mass at  $x$ . Then  $T$  is indeed a regeneration time for  $\nu$  in the sense that, with  $\Psi(0) \sim \nu$ , we also have  $\Psi(T) \sim \nu$ . We will use the following general version of Kac's formula.

**Lemma 4.** *For any function  $f : \Sigma_\psi \rightarrow \mathbb{R}$ :*

$$E_\nu \left[ \sum_{t=0}^{T-1} f(\Psi(t)) \right] = E_\nu(T) \pi_\psi(f).$$

To apply Lemma 4, we first compute  $E_\nu(T)$ :

**Lemma 5.**  $E_\nu(T) = 2N$ .

**PROOF.** Consider the (further restricted) chain  $\Upsilon = \{\Upsilon(t) = (Y_t, D_1(t), D_2(t)); t \geq 0\}$  on the state space  $S \times \{+1, -1\}^2$ , and note that its unique invariant measure  $\rho$  is uniform. Write,  $D = \{(0, +1, -1), (0, -1, +1)\}$ , let  $\rho_D$  denote the measure  $\rho$  conditioned on  $D$ , and let,

$$T_D^+ = \inf\{t \geq 1; \Upsilon(t) \in D\},$$

so that, in fact,  $T_D^+ = T$ . Then, by Kac's formula [9], we have,

$$E_\nu(T) = E_{\rho_D}(T_D^+) = \frac{1}{\rho(D)} = \frac{4N}{|D|} = 2N,$$

as claimed.  $\square$

The central step in the proof of the theorem is an application of Lemma 4 with  $f(\Psi(t)) = D_t(I_t)$ , which, combined with Lemma 5 gives us that  $s = \pi(D_1(I_1)) = \pi_\psi(D_1(I_1))$  equals

$$\begin{aligned} s &= \frac{1}{2N} E_\nu \left[ \sum_{t=0}^{T-1} D_t(I_t) \right] = \frac{1}{4N} E_{\psi_1} \left[ \sum_{t=0}^{T-1} D_t(1) \right] \\ &\quad + \frac{1}{4N} E_{\psi_2} \left[ \sum_{t=0}^{T-1} D_t(2) \right] = \frac{1}{2N} E_{\psi_1} \left[ \sum_{t=0}^{T-1} D_t(1) \right], \end{aligned}$$

where the sums above (and in what follows) correspond to addition over  $\mathbb{Z}$  (as opposed to modulo  $N$  addition over  $S$ ). Therefore, writing  $X_{t+1}^*(i) = X_t^*(i) + D_t(i)$ , for  $i = 1, 2$ ,  $t \geq 1$ , we have,

$$\begin{aligned} s &= \frac{1}{2N} E_{\psi_1}(X_T^*(1)) \\ &= \frac{1}{2N} E_{\psi_1} \left( \frac{X_T^*(1) - X_T^*(2)}{2} \right) \\ &\quad + \frac{1}{2N} E_{\psi_1} \left( \frac{X_T^*(1) + X_T^*(2)}{2} \right) \\ &= \frac{1}{2N} E_{\psi_1} \left( \frac{X_T^*(1) - X_T^*(2)}{2} \right), \end{aligned}$$

where we noted that  $E_{\psi_1}(X_T^*(1) + X_T^*(2))$  is zero by the symmetry of the distribution of the independent increments  $\{D_t\}$ , which implies that the law (conditional on  $\psi_1$ ) of  $X_T^*(1)$  is the same as that of  $-X_T^*(2)$ .

Now write

$$A = \{(0, +1, +1), (0, +1, -1), (0, -1, +1), (0, -1, -1)\},$$

and let  $T_A^+$  denote the first time when the two walkers meet,

$$T_A^+ = \inf\{t \geq 1; \Upsilon(t) \in A\} = \inf\{t \geq 1; Y_t = 0\},$$

so that  $T_A^+$  can be expressed in terms of either  $\Psi$  or  $\Upsilon$ . We observe that, at time  $T_A^+$ , either the two walkers decide to go in opposite directions, in which case  $T_A^+ = T$ , or they continue moving together until they choose opposite directions, in which case the difference of their locations  $X_t^*(i)$  stays constant; therefore,

$$s = \frac{1}{2N} E_{\psi_1} \left( \frac{X_{T_A^+}^*(1) - X_{T_A^+}^*(2)}{2} \right).$$

Since the last expectation above is conditioned on the two walkers starting from the same position, in opposite directions, and with the first one moving in the positive (clockwise) direction, there are exactly two possible scenarios for their first meeting time  $T_A^+$ : In the first scenario, at time  $t = T_A^+ - 1$  walker 1 is two steps “ahead” in the clockwise direction of walker 2 (as they are, e.g., at time  $t = 1$ ). In this case, we will necessarily have  $X_{T_A^+}^*(1) - X_{T_A^+}^*(2) = 0$ . We call this event  $C$ . In the second scenario, the relative positions of the two walkers at time  $t = T_A^+ - 1$  will be reversed, which necessarily means that the first walker travelled a whole circle “around” the second one before they met, so that (since  $N$  is odd) on  $C^c$ , we must have  $X_{T_A^+}^*(1) - X_{T_A^+}^*(2) = 2N$ . Therefore,

$$s = \frac{1}{2N} \left[ \frac{0}{2} \cdot P_{\psi_1}(C) + \frac{2N}{2} \cdot P_{\psi_1}(C^c) \right] = \frac{1}{2} P_{\psi_1}(C^c). \quad (3)$$

Finally we compute the probability of the event  $C$ :

**Lemma 6.**  $P_{\psi_1}(C^c) = \frac{2-\epsilon}{2+\epsilon(N-2)}$ .

PROOF. Here we consider the chain  $\Upsilon^* = \{\Upsilon^*(t) = (Y_t^*, D_1(t), D_2(t)) ; t \geq 0\}$  on  $\Sigma^* = \mathbb{Z} \times \{+1, -1\}^2$ , where  $Y_t^* = X_t^*(1) - X_t^*(2)$ . Note that, for the state  $u_1 := (0, +1, -1)$ , the initial condition  $\Upsilon^*(0) = u_1$  corresponds to  $\Psi(0) = \psi_1$ .

We will only need to examine the evolution of  $\Upsilon^*$  until time  $t = T_A^+$ , which, since  $N$  is odd, can equivalently be expressed as,

$$T_A^+ = \inf\{t \geq 1 ; Y_t^* = 0 \pmod{2N}\},$$

and the same argument as in the last paragraph before the statement of the lemma shows that, given  $\Upsilon^*(0) = u_1$ , the only two possible values of  $Y_{T_A^+}^*$  are 0 and  $2N$ , on  $C$  and on  $C^c$ , respectively. Therefore, letting,

$$\begin{aligned} T_R &= \min\{t \geq 1 ; Y_t^* = 0\}, \\ \text{and } T_L &= \min\{t \geq 1 ; Y_t^* = 2N\}, \end{aligned}$$

we have that  $T_A^+ = \min\{T_L, T_R\}$  and that  $P_{\psi_1}(C^c) = P_{\psi_1}(T_L < T_R)$ ; cf. Fig. 2.

Next, for the computation of  $P_{\psi_1}(T_L < T_R)$ ; it will suffice to consider the *trace* of  $\Upsilon^*$  on the set,

$$\Sigma^t := \{0, 2, 4, \dots, 2N\} \times \{(+1, -1), (-1, +1)\} \subset \Sigma^*;$$

cf. [10]. The evolution of this Markov chain is fairly simple and its transition probabilities are easy to compute; e.g., the probability of the transition from  $(0, +1, -1)$  to  $(2, +1, -1)$  is equal to,

$$\left(1 - \frac{\epsilon}{2}\right)^2 + \left(\frac{\epsilon}{2}\right)\left(1 - \frac{\epsilon}{2}\right)\frac{1}{2} + \left(\frac{\epsilon}{2}\right)\left(1 - \frac{\epsilon}{2}\right)\frac{1}{2} = 1 - \frac{\epsilon}{2}.$$

The first term above corresponds to the case when the two walkers both maintain their original directions after their first step; the second term corresponds to the case when only the first walker changes direction, after which they keep moving at a distance two apart, until one of them changes direction again and they either reach the state  $(2, +1, -1)$  or the state  $(2, -1, +1)$ , each having probability  $1/2$  by symmetry; and

the third term corresponds to the case when only the second walker changes direction after their first step, and its value is the same as the second term again by symmetry. The remaining transition probabilities can be similarly computed; see Fig. 2.

Finally, for every state

$$u \in \{0, 2, 4, \dots, 2N\} \times \{(+1, -1), (-1, +1)\}$$

we define  $h(u) = P_u(T_L < T_R)$ , so that  $h(u_1) = P_{\psi_1}(C^c)$ . Writing  $L$  and  $R$  for the states  $(2N - 2, +1, -1)$  and  $(2, -1, +1)$ , respectively, we have  $h(L) = 1$ ,  $h(R) = 0$ , and in fact it is easy to see that the one-step conditional expectation of  $h$  given any state  $u \neq (2N, +1, -1)$  or  $(0, -1, +1)$ , is equal  $h(u)$ . This relationship can be expressed as a simple recursion: Letting  $f(k) = h(2k, +1, -1)$  and  $g(k) = h(2k + 2, -1, +1)$ :

$$\begin{aligned} f(k) &= (1 - \epsilon/2)f(k+1) + (\epsilon/2)g(k), \quad 0 \leq k \leq N-1 \\ g(k+1) &= (1 - \epsilon/2)g(k) + (\epsilon/2)f(k+1), \quad 0 \leq k \leq N-1 \\ g(0) &= 0 \text{ and } f(N-1) = 1. \end{aligned}$$

Adding the first two equations above shows that  $f(k) - g(k)$  is a constant, say  $A$ , independent of  $k$ , and substituting this in the recursion for  $g$  gives  $g(k) = Aek/(2 - \epsilon)$ . Similarly solving for  $f$  we obtain,  $f(k) = A + Aek/(2 - \epsilon)$ , and from the boundary values we can obtain that  $A = (2 - \epsilon)/(2 + \epsilon(N - 2))$ . Therefore, as claimed,

$$P_{\psi_1}(C^c) = h(u_1) = f(0) = A = (2 - \epsilon)/(2 + \epsilon(N - 2)). \quad \square$$

Combining (3) with Lemma 6 proves of Theorem 1.

## V. PROOF OF THEOREM 3

In the interest of space, some details will be omitted below. Recall the ergodic chain  $\Psi$  defined earlier. Write  $\Sigma_\psi$  for its state space,  $\pi_\psi$  for its unique invariant measure, and let  $\mathbf{P}(\psi, \psi') = \Pr(\Psi(t+1) = \psi' | \Psi(t) = \psi)$  denote its transition kernel. Consider the bivariate chain  $\tilde{\Psi} = \{\tilde{\Psi}(t) = (\Psi(t), \Psi(t+1)) ; t \geq 0\}$ . Then  $\tilde{\Psi}$  is also ergodic, with unique invariant measure,

$$\tilde{\pi}(\psi, \psi') = \pi_\psi(\psi)\mathbf{P}(\psi, \psi').$$

Therefore, the limit in the statement exists a.s. and it equals,

$$c := P_\pi(I_2 \neq I_1) = \tilde{\pi}\left(\mathbb{I}_{\{I_2 \neq I_1\}}\right) = \tilde{\pi}(B),$$

where  $B$  consists of the following 8 states,

$$\begin{aligned} &\left((0, 1, 1, 1), (0, -1, 1, 2)\right), \left((0, -1, -1, 1), (0, -1, 1, 2)\right), \\ &\left((0, 1, 1, 2), (0, 1, -1, 1)\right), \left((0, -1, -1, 2), (0, 1, -1, 1)\right), \\ &\left((2, -1, 1, 1), (0, -1, 1, 2)\right), \left((2, -1, 1, 2), (0, 1, -1, 1)\right), \\ &\left((-2, 1, -1, 1), (0, -1, 1, 2)\right), \left((-2, 1, -1, 2), (0, 1, -1, 1)\right). \end{aligned}$$

and where, with a slight abuse of notation, the negative values of the  $Y_t$  variables above are again interpreted modulo  $N$ .

Now, observe that  $\tilde{\pi}(B)$  can easily be shown to be,

$$\begin{aligned} &\frac{1}{2N} \frac{\epsilon}{2} \left(1 - \frac{\epsilon}{2}\right) \\ &+ \left(\pi_\psi(-2, +1, -1, 1) - \pi_\psi(2, -1, +1, 1) + \frac{1}{4N}\right) \left(\frac{\epsilon}{2}\right)^2 \\ &+ \left(\pi_\psi(2, -1, +1, 1) - \pi_\psi(-2, +1, -1, 1) + \frac{1}{4N}\right) \left(1 - \frac{\epsilon}{2}\right)^2 \end{aligned}$$

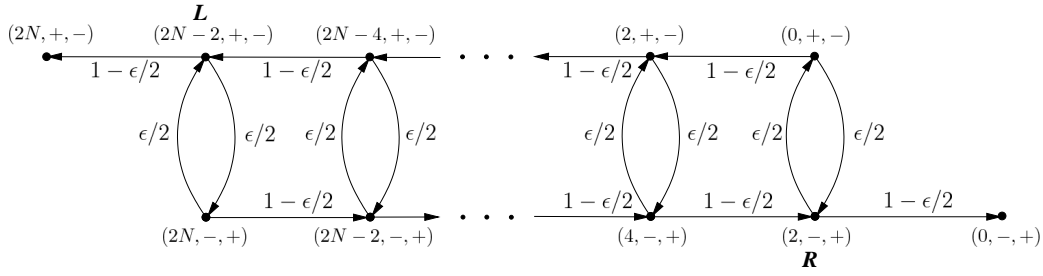


Fig. 2. Evolution of the trace of the chain  $\Upsilon^*$  on  $\{0, 2, 4, \dots, 2N\} \times \{(+1, -1), (-1, +1)\}$ .

where we used the fact that the invariant distribution of  $(X_t(1), X_t(2), D_t(1), D_t(2))$  is uniform, which implies that  $\pi_\psi(y, d, d', 1) + \pi_\psi(y, d, d', 2) = 1/(4N)$ , for any  $y \in S$  and  $d, d' \in \{+1, -1\}$ . Simplifying,  $\tilde{\pi}(B)$  equals,

$$\frac{1}{4N} + (1 - \epsilon)[\pi_\psi(2, -1, +1, 1) - \pi_\psi(-2, +1, -1, 1)], \quad (4)$$

and, by Lemma 4,

$$\begin{aligned} E_\nu(T)\pi_\psi(\psi) &= E_\nu \left[ \sum_{t=0}^{T-1} \mathbb{I}_{\{\Psi(t)=\psi\}} \right] \\ &= \frac{1}{2} E_{\psi_1} \left[ \sum_{t=0}^{T-1} \mathbb{I}_{\{\Psi(t)=\psi\}} \right] = \frac{1}{2} E_{u_1} \left[ \sum_{t=0}^{T-1} \mathbb{I}_{\{\Upsilon^*(t)=(y,d,d')\}} \right], \end{aligned}$$

so that, substituting this twice in (4), we have,

$$c = E_{u_1} \left[ \sum_{t=0}^{T-1} \left( \mathbb{I}_{\{\Upsilon^*(t)=(2,-1,+1)\}} - \mathbb{I}_{\{\Upsilon^*(t)=(2N-2,+1,-1)\}} \right) \right]$$

where we also used Lemma 5. By the definition of  $T$ , the value of the sum inside the expectation above is either  $0 - 1$  or  $1 - 0$ , and the corresponding probabilities can be found by looking at the trace of  $\Upsilon^*$  on  $\Sigma^t$ ; indeed referring to Fig. 2, and in the notation of the proof of Lemma 6, the case  $1 - 0$  has probability  $1 - P_{u_1}(T_L < T_R)$ , whereas the case  $0 - 1$  has probability  $P_{u_1}(T_L < T_R)$ . So,

$$c = \tilde{\pi}(B) = \frac{1}{4N} + \frac{(1 - \epsilon)}{4N} [1 - 2P_{u_1}(T_L < T_R)],$$

and now substituting the result of Lemma 6,

$$P_{u_1}(T_L < T_R) = \frac{2 - \epsilon}{2 + \epsilon(N - 2)},$$

and simplifying yields the claimed result.

## VI. CONCLUSIONS

The performance analysis of wireless delay-tolerant networks with nodes moving in a random fashion is typically quite difficult. In part for this reason, past works have focused on using relatively accurate network models together with approximations and/or asymptotic analysis. In this work, instead, we consider a very simple network model (perhaps the simplest non-trivial one) but arrive at exact, simple expressions for the quantities of interest. The intuition thus gained suggests the following scenarios for future work that could also lead

to exact, closed-form results: The same setting, but with more than two nodes; for *continuous* one-dimensional topologies as those considered in [5], [6], [7]; and in two-dimensional settings, e.g., the discrete or continuous torus.

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