On sequential maxima of exponential sample means, with an application to ruin probability

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Abstract: We obtain the distribution of the maximal average in a sequence of independent identically distributed exponential random variables. Surprisingly enough, it turns out that the inverse distribution admits a simple closed form. An application to ruin probability in a risk-theoretic model is also given.

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1. Introduction

Consider a sequence $(X_i)_{i\geq 1}$ of independent identically distributed (i.i.d.) random variables, each having exponential distribution with mean 1. For each $i \in \mathbb{N}^+$ define the sample mean of the first i variables as $\overline{X}_i := (X_1+X_2+\cdots+X_i)/i$. The supremum of this sequence

$$Z_\infty := \sup\{\bar{X}_i : i \in \mathbb{N}^+\}$$

is finite because the sequence converges to 1 with probability 1.

In this note we compute the distribution function, F_∞, of Z_∞. In fact, what has nice form is the inverse of this distribution function. Our main result is the following.

Theorem 1. (a) *Z_∞ has distribution function*

$$F_\infty(x) = 1 - \sum_{k=1}^{\infty} \frac{k^{k-1}}{k!} x^{k-1} e^{-kx}$$

for $x > 0$, and density which is continuous on $\mathbb{R}\backslash\{1\}$, positive on $(1, \infty)$, and zero on $(-\infty, 1)$.
(b) *The restriction of F_∞ on $(1, \infty)$ is one to one and onto $(0, 1)$ with inverse*

$$F_\infty^{-1}(u) = \frac{-\log(1-u)}{u} \quad \textit{for all } u \in (0, 1). \tag{1}$$

Remark 1. (a) For F_∞ we have the alternative expression

$$F_\infty(x) = 1 + \frac{1}{x} W_0(-xe^{-x})$$

where W_0 is the principal branch of the Lambert W function, that is, the inverse function of $x \mapsto xe^x, x \geq 1$; see [2]. Indeed, the power series $\sum_{k=1}^{\infty} \frac{k^{k-1}}{k!} y^k$ has interval of convergence $[-1/e, 1/e]$ and equals $-W_0(-y)$.

(b) Clearly, the results of the theorem extend immediately to the case that the X_i's are i.i.d. and $X_1 = aY + b$ with $a > 0$, $b \in \mathbb{R}$ and $Y \sim \mathrm{E}xp(1)$. However, we were not able to find an explicit formula for the distribution of Z_∞ for any other distribution of the X_i's.

(c) Although it is intuitively clear that $F_\infty(x) > 0$ for $x > 1$, it is not entirely obvious how to verify it by direct calculations. However, this fact is evident from Theorem 1.

(d) Formula (1) enables the explicit calculation of the percentiles of F_∞. Therefore, the result is useful for the following kind of problems: Suppose that a quality control machine calculates subsequent averages, and alarms if some average $\bar{X}_n$ is greater than c, where c is a predetermined constant such that the probability of false alarm is small, say α. For $\alpha \in (0,1)$, the upper percentage point of F_∞ (that is, the point c_α with $F_\infty(c_\alpha) = 1 - \alpha$) is given by $c_\alpha = \frac{-\log \alpha}{1-\alpha}$, and thus the proper value of c is $c = c_\alpha$.

If in the definition of Z_∞ we discard the first $n-1$ values of $\bar{X}_i$, we obtain the random variable

$$M_n := \sup\{\bar{X}_i : i \geq n\}$$

for which, however, (for $n \geq 2$) the distribution function is quite complicated even for the exponential case. For instance, the distribution of M_2 is given by (we omit the details)

$$F_{M_2}(x) = F_\infty(x) + e^{-2x} \frac{F_\infty(x)}{1 - F_\infty(x)}, \quad x \geq 0.$$

What we can compute is the asymptotic distribution of $\sqrt{n}(M_n - 1)$ as $n \to \infty$. This distribution is the same for a large class of distributions of the X_i's, as the following theorem shows.

Theorem 2. *Assume that the $(X_i)_{i \geq 1}$ are i.i.d. with mean 0, variance 1, and there is $p > 2$ with $\mathbb{E}|X_1|^p < \infty$. Let $M_n := \sup\{\bar{X}_i : i \geq n\}$ for all $n \in \mathbb{N}^+$. Then,*

$$\sqrt{n} M_n \Rightarrow |Z|$$

where $Z \sim N(0,1)$ is a standard normal random variable.

It is easy to see that under the assumptions of Theorem 2, by the law of the iterated logarithm, it holds

$$\limsup_{n \to \infty} \frac{\sqrt{n}}{\sqrt{2 \log \log n}} M_n = 1.$$

2. Proofs

Proof of Theorem 1. (a) For each $n \in \mathbb{N}^+$ consider the random variable

$$Z_n := \max\left\{\bar{X}_1, \bar{X}_2, \ldots, \bar{X}_n\right\}$$

and call F_n its distribution function. The sequence $(Z_n)_{n\geq 1}$ is increasing and $Z_\infty = \lim_{n\to\infty} Z_n$, $F_\infty(x) = \lim_{n\to\infty} F_n(x)$. We will compute F_n recursively.

For $n \in \mathbb{N}^+$ and $x \geq 0$ we have

$$\begin{aligned}
F_{n+1}(x) &= \Pr[X_1 \leq x, X_1 + X_2 \leq 2x, \ldots, X_1 + X_2 + \cdots + X_{n+1} \leq (n+1)x] \\
&= \int_0^x \int_0^{2x-y_1} \cdots \int_0^{(n+1)x-(y_1+y_2+\cdots+y_n)} e^{-(y_1+y_2+\cdots+y_{n+1})} d\mathbf{y}_{n+1} \\
&= \int_0^x \int_0^{2x-y_1} \cdots \int_0^{nx-(y_1+y_2+\cdots+y_{n-1})} \left\{e^{-(y_1+y_2+\cdots+y_n)} - e^{-(n+1)x}\right\} d\mathbf{y}_n \\
&= F_n(x) - e^{-(n+1)x}\,\mathrm{Vol}(K_n(x))
\end{aligned}$$

where $d\mathbf{y}_k = dy_k \cdots dy_2 dy_1$ and

$$K_n(x) := \{(y_1, y_2, \ldots, y_n) \in \mathbb{R}_+^n : \ 0 \leq y_1 + \cdots + y_i \leq ix, \quad i = 1, 2, \ldots, n\}.$$

Note that $F_1(x) = 1 - e^{-x}$ and introduce the convention $\mathrm{Vol}(K_0(x)) = 1$. It follows that $F_n(x) = 1 - \sum_{k=1}^n \mathrm{Vol}(K_{k-1}(x))e^{-kx}$ and from Lemma 1, below, we get the explicit form

$$F_n(x) = 1 - \sum_{k=1}^n \frac{k^{k-1}}{k!} x^{k-1} e^{-kx}, \text{ for all } \ x \geq 0,\ n \in \mathbb{N}^+.$$

This implies the first formula for F_∞. By the law of large numbers, we get that $F_\infty(x) = 0$ for all $x \in (-\infty, 1)$, and thus, the derivative of F_∞ in $\mathbb{R}\backslash\{1\}$ is

$$f_\infty(x) := \mathbf{1}_{x>1} \sum_{k=1}^\infty \frac{k^{k-1}}{k!} \left(k - \frac{k-1}{x}\right) x^{k-1} e^{-kx}.$$

(b) First we rewrite F_∞ in a more convenient form. The fact that $F_\infty(x) = 0$ for $x \in [0, 1)$ implies the remarkable identity (see Fig. 1)

$$\sum_{k=1}^\infty \frac{k^{k-1}}{k!} x^{k-1} e^{-kx} = 1 \quad \text{for all } x \in [0, 1). \tag{2}$$

Our aim is to compute the value of the series in the left hand side also for $x \geq 1$. The series converges uniformly for $x \in [0, \infty)$ because

$$\sup_{x\geq 0} \frac{k^{k-1}}{k!} x^{k-1} e^{-kx} = \frac{(k-1)^{k-1}}{k!} e^{-(k-1)} \sim \frac{1}{k^{3/2}\sqrt{2\pi}},$$

which is summable in k. Thus, by continuity, (2) holds also for $x = 1$. Now we rewrite (2) in the form

$$\sum_{k=1}^\infty \frac{k^{k-1}}{k!} (xe^{-x})^k = x \text{ for all } x \in [0, 1]. \tag{3}$$

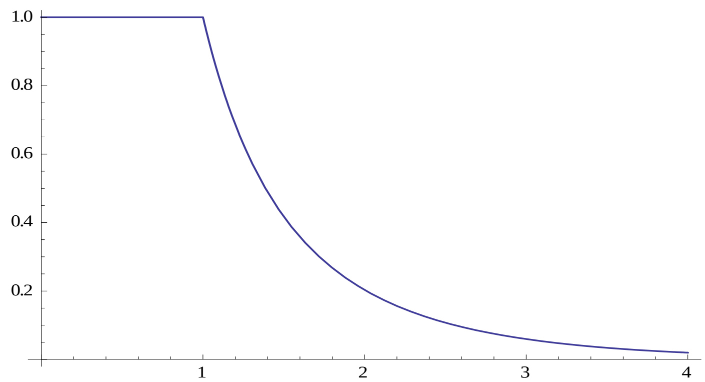

FIG 1. *The series* (2) *in the interval* $0 \le x \le 4$.

The power series $h(y) := \sum_{k=1}^{\infty} \frac{k^{k-1}}{k!} y^k$ is strictly increasing in $[0, e^{-1}]$ and thus (3) says that h is the inverse function of the restriction, g_r, on $[0,1]$ of the function $g : [0,\infty) \to [0, e^{-1}]$ with $g(x) = xe^{-x}$. The function g is continuous, strictly increasing in $[0,1]$, and strictly decreasing in $[1,\infty)$ with $g(0) = 0, g(1) = e^{-1}, g(\infty) = 0$. Thus, for each $x \in [1,\infty)$, there exists a unique $t = t(x) \in (0,1]$ such that $g_r(t) = xe^{-x}$, i.e., $te^{-t} = xe^{-x}$; hence, we define

$$t(x) := g_r^{-1}(xe^{-x}) = h(xe^{-x}), \quad x \ge 0. \tag{4}$$

Since $t(x) = x$ for $x \in [0,1]$, we have

$$F_\infty(x) = \begin{cases} 0, & \text{if} \quad x \le 1, \\ 1 - \frac{t(x)}{x}, & \text{if} \quad x \ge 1. \end{cases} \tag{5}$$

Now for any fixed $u \in (0,1)$, the relation $F_\infty(x) = u$ gives $x - t(x) = xu$ so that $t(x) = (1-u)x$. Consequently,

$$e^{xu} = \frac{e^{-t(x)}}{e^{-x}} = \frac{x}{t(x)} = \frac{1}{1-u}.$$

Thus, $x = -\log(1-u)/u$ and the proof is complete. □

Remark 2. From the well-known relation $\mathbb{E} Z_n^\alpha = \alpha \int_0^\infty x^{\alpha-1}(1 - F_n(x))dx$ for $\alpha > 0$, we obtain a simple expression for the moments:

$$\mathbb{E} Z_n^\alpha = \alpha \sum_{k=1}^{n} \frac{\Gamma(\alpha + k - 1)}{k^\alpha k!}.$$

In particular,

$$\mathbb{E} Z_n = \sum_{k=1}^n \frac{1}{k^2}, \quad \mathbb{E} Z_n^2 = 2\sum_{k=1}^n \frac{1}{k^2}, \quad \mathbb{E} Z_n^3 = 3\sum_{k=1}^n \frac{1}{k^2} + 3\sum_{k=1}^n \frac{1}{k^3}.$$

Since $Z_n \nearrow Z_\infty$ with probability one, the above relations combined with the monotone convergence theorem give the moments of Z_∞ and in particular that it has mean $\frac{\pi^2}{6}$ and variance $\frac{\pi^2}{6}(2-\frac{\pi^2}{6})$.

The next lemma is a special case of Theorem 1 in [5] (see relation (7) in that paper), however, to keep the exposition self-contained, we provide a proof.

Lemma 1. *For $x \ge 0$, $x+t \ge 0$, and $n \in \mathbb{N}^+$, define*

$$K_n(x,t) := \{(y_1, y_2, \dots, y_n) \in \mathbb{R}_+^n : y_1 + \cdots + y_i \le ix + t \ \ \text{for all } i = 1, 2, \dots, n\}.$$

Then,

$$V_n(x,t) := \mathrm{Vol}(K_n(x,t)) = \frac{1}{n!}(x+t)((n+1)x+t)^{n-1}, \quad n = 1, 2, \dots, \tag{6}$$

and, in particular, setting $t=0$, $\mathrm{Vol}(K_n(x)) = \frac{1}{n!}(n+1)^{n-1}x^n$.

Proof. Clearly $V_1(x,t) = x+t$ and for $n \ge 1$

$$\begin{aligned} V_{n+1}(x,t) &= \int_0^{x+t} \int_0^{2x+t-y_1} \cdots \int_0^{(n+1)x+t-(y_1+y_2+\cdots+y_n)} d\mathbf{y}_{n+1} \\ &= \int_0^{x+t} \int_0^{x+(x+t-y_1)} \cdots \int_0^{nx+(x+t-y_1)-(y_2+\cdots+y_n)} d\mathbf{y}_{n+1} \\ &= \int_0^{x+t} V_n(x, x+t-y_1) dy_1. \end{aligned} \tag{7}$$

The claim follows by induction on n. □

It is consistent with the recursion (7) for V_n and (6) to define $V_0(x,t) := 1$ so that (6) holds for all $n \in \mathbb{N}^+ \cup \{0\}$. This agrees with the convention $\mathrm{Vol}(K_0(x)) = 1$ we made in the proof of Theorem 1(a).

Proof of Theorem 2. By Theorem 2.2.4 in [3] we may assume that we can place $(X_i)_{i\ge1}$ in the same probability space with a standard Brownian motion $(W_s)_{s\ge0}$, so that, with probability 1, we have $|n\bar{X}_n - W_n|/n^{1/p}(\log n)^{1/2} \to 0$ as $n \to \infty$. This implies that

$$\lim_{n\to\infty} \sqrt{n}\left(M_n - \sup_{k\in\mathbb{N}, k\ge n} \frac{W_k}{k}\right) = 0$$

with probability 1. On the other hand, with probability one, we have for all large n the bound $\sup_{s\in[n,n+1]} |W_s - W_n| \le 2\sqrt{\log n}$, thus

$$\lim_{n\to\infty} \sqrt{n}\left(\sup_{k\in\mathbb{N}, k\ge n} \frac{W_k}{k} - \sup_{s\ge n} \frac{W_s}{s}\right) = 0.$$

Finally, by scaling and time inversion, we conclude that

$$\sqrt{n} \sup_{s\ge n} \frac{W_s}{s} \stackrel{d}{=} \sup_{s\ge1} \frac{W_s}{s} \stackrel{d}{=} \sup_{s\in[0,1]} W_s \stackrel{d}{=} |W_1|,$$

and the proof is complete. □

3. An application to ruin probability

Following the same steps as in the proof of Theorem 1(b), one can evaluate the distribution function, $F_{n;\lambda}$, of the random variable

$$Z_{n;\lambda} := \max\left\{\frac{X_1}{1+\lambda}, \frac{X_1+X_2}{2+\lambda}, \ldots, \frac{X_1+X_2+\cdots+X_n}{n+\lambda}\right\}$$

for all $\lambda > -1$ and $n \in \mathbb{N}^+$. Indeed, using (6) and induction on n it is easily verified that for all $x \geq 0$ we have

$$F_{n;\lambda}(x) = 1 - (1+\lambda)e^{-\lambda x}\sum_{k=1}^{n}\frac{k(k+\lambda)^{k-2}}{k!}x^{k-1}e^{-kx}.$$

Thus, the distribution function of $Z_{\infty,\lambda} := \lim_{n\to\infty} Z_{n;\lambda}$ equals

$$F_{\infty;\lambda}(x) = 1 - (1+\lambda)e^{-\lambda x}\sum_{k=1}^{\infty}\frac{k(k+\lambda)^{k-2}}{k!}x^{k-1}e^{-kx} \tag{8}$$

$$= 1 - \frac{t(x)}{x}e^{\lambda(t(x)-x)}, \tag{9}$$

where the function t is defined by (4). To justify the equality (9), we use the same arguments that lead from (2) to (5). Similarly as in Theorem 1(b), we find that $F_{\infty;\lambda}$ is zero in $(-\infty, 1]$, strictly increasing in $[1, \infty)$ with range $[0, 1)$, and its distribution inverse is given by

$$F^{-1}_{\infty;\lambda}(u) = \frac{-\log(1-u)}{1-(1-u)^{\frac{1}{1+\lambda}}} \times \frac{1}{\lambda+1}, \quad 0 < u < 1. \tag{10}$$

Remark 3. By the law of large numbers, the series in the right hand side of (8) equals to one for all $x \in [0, 1]$. Therefore, setting $x = \alpha$, $1 + \lambda = \theta$ and $k \to k + 1$, the function

$$p(k; \alpha, \theta) = \theta e^{-\alpha(\theta+k)}\frac{\alpha^k(k+\theta)^{k-1}}{k!}$$

defines a probability mass function supported on $\mathbb{N}^+ \cup \{0\}$, known (after a suitable re-parametrization) as *generalized Poisson distribution* with parameter $(\alpha, \theta) \in [0, 1] \times (0, \infty)$; see [1] and references therein.

Consider now the following risk model. Assume that the aggregate claim at time n is described by $S_n := X_1 + \cdots + X_n$, where the $(X_i)_{i\geq 1}$ are i.i.d. with $\mathbb{E}X_1 = 1$, the premium rate (per time unit) is $c = 1 + \theta > 0$ (θ is the safety loading of the insurance), and the initial capital is $u > -(1+\theta)$, where negative initial capital is allowed for technical reasons. The risk process is defined by

$$U_n = u + cn - S_n, \quad n \in \mathbb{N}^+.$$

Clearly, the ruin probability

$$\psi(u) := \Pr(U_n < 0 \quad \text{for some} \quad n \in \mathbb{N}^+) \tag{11}$$

is of fundamental importance. Our explicit formulae are useful in computing the minimum initial capital needed to ensure that $\psi(u)$ is small. The particular problem (for general claims) has been studied in [4], under the name *discrete-time surplus-process model*. It is well-known that $\psi(u) = 1$ when $c \le 1$, no matter how large u is, because $\mathbb{E}X_i = 1$. Hence, the problem is meaningful only for $c > 1$, i.e., $\theta > 0$.

Theorem 3. *Assume that the i.i.d. individual claims $(X_i)_{i\ge1}$ are exponential random variables with mean 1, fix $\alpha \in (0,1)$ and $\theta > 0$, and set $c = 1+\theta$. Then,*
(a) *the ruin probability (11) is given by*

$$\psi(u) = \begin{cases} \frac{t(c)}{c} \exp\left(-u\left(1 - \frac{t(c)}{c}\right)\right), & \text{if } u > -c, \\ 1 & \text{if } u \le -c, \end{cases} \tag{12}$$

where the function t is given by (4);
(b) *the minimum initial capital $u = u(\alpha,\theta)$ needed to ensure that $\psi(u) \le \alpha$ is given by the unique root of the equation*

$$(1+\theta+u)\left(1 - \alpha^{\frac{1+\theta}{1+\theta+u}}\right) = -\log\alpha, \quad u > -(1+\theta). \tag{13}$$

Proof. (a) For $u > -c$, we can use (9) to get

$$\psi(u) = 1 - F_{\infty;u/c}(c) = \frac{t(c)}{c} e^{(u/c)(t(c)-c)},$$

which is (12). Then, the definition of t shows that $\lim_{u\to -c^+}\psi(u) = \frac{t(c)e^{-t(c)}}{ce^{-c}} = 1$, and the monotonicity of ψ implies that $\psi(u) = 1$ for $u \le -c$.

(b) By the formula of part (a), the function ψ is strictly decreasing in the interval $(-c,\infty)$ and maps that interval to $(0,1)$. Therefore, there is a unique $u = u(\alpha,\theta) > -c$ such that $\psi(u) = \alpha$. Let $\lambda := u/c$, which is greater than -1. Then, using (10), we see that

$$\psi(u) = \alpha \Leftrightarrow F_{\infty;\lambda}(c) = 1-\alpha \Leftrightarrow c = F^{-1}_{\infty;\lambda}(1-\alpha) = \frac{-\log\alpha}{(1+\lambda)\left(1-\alpha^{\frac{1}{1+\lambda}}\right)}.$$

We substitute $c = 1+\theta, \lambda = u/(1+\theta)$, and the above equivalences show that u is the unique solution of

$$\left(1 + \frac{u}{1+\theta}\right)\left(1 - \alpha^{\frac{1+\theta}{1+\theta+u}}\right) = \frac{-\log\alpha}{1+\theta}.$$

$\square$

The exact values of u in (13) are in perfect agreement with the numerical approximations given in the last line of Table 1 in [4]. Notice that the initial capital u can be negative sometimes, e.g., $u(.5,.5) \simeq -.3107$.

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