DIFFUSION IN RANDOM ENVIRONMENT AND THE RENEWAL THEOREM

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ABSTRACT. According to a theorem of S. Schumacher and T. Brox, for a diffusion X in a Brownian environment it holds that $(X_t - b_{\log t})/\log^2 t \to 0$ in probability, as $t \to \infty$, where b is a stochastic process having an explicit description and depending only on the environment. We compute the distribution of the number of sign changes for b on an interval $[1, x]$ and study some of the consequences of the computation; in particular we get the probability of b keeping the same sign on that interval. These results have been announced in 1999 in a non-rigorous paper by P. Le Doussal, C. Monthus, and D. Fisher and were treated with a Renormalization Group analysis. We prove that this analysis can be made rigorous using a path decomposition for the Brownian environment and renewal theory. Finally, we comment on the information these results give about the behavior of the diffusion.

1. INTRODUCTION

On the space $\mathcal{W} := C(\mathbb{R})$ consider the topology of uniform convergence on compact sets, the corresponding σ-field of the Borel sets, and $\mathbb{P}$ the measure on $\mathcal{W}$ under which the coordinate the processes $\{w(t) : t \geq 0\}, \{w(-t) : t \geq 0\}$ are independent standard Brownian motions.

Also let $\Omega := C([0, +\infty))$, and equip it with the σ-field of Borel sets derived from the topology of uniform convergence on compact sets. For $w \in \mathcal{W}$, we denote by $\mathbf{P}_w$ the probability measure on Ω such that $\{\omega(t) : t \geq 0\}$ is a diffusion with $\omega(0) = 0$ and generator

$$\frac{1}{2} e^{w(x)} \frac{d}{dx} \left(e^{-w(x)} \frac{d}{dx} \right).$$

The construction of such a diffusion is done with scale and time transformation from a one-dimensional Brownian motion (see, e.g., Schumacher (1985), Shi (2001)). Using this construction, it is easy to see that for $\mathbb{P}$-almost all $w \in \mathcal{W}$ the diffusion does not explode in finite time; and on the same set of w's it satisfies the formal SDE

$$
(1) \qquad \begin{array}{rl}
d\omega(t) = & d\beta(t) - \frac{1}{2} w'(\omega(t))dt, \\
\omega(0) = & 0,
\end{array}
$$

where β is a one-dimensional standard Brownian motion.

Then consider the space $\mathcal{W} \times \Omega$, equip it with the product σ-field, and take the probability measure defined by

$$d\mathcal{P}(w, \omega) = d\mathbf{P}_w(\omega)d\mathbb{P}(w).$$

The marginal of $\mathcal{P}$ in Ω gives a process that is known as diffusion in a random environment; the environment being the function w.

S. Schumacher [Schumacher (1985, 1984)] proved the following result.

Fact 1. *There is a process* $b : [0, \infty) \times \mathcal{W} \to \mathbb{R}$ *such that for the formal solution* ω *of* (1) *it holds*

$$(2) \qquad \frac{\omega_t}{(\log t)^2} - b_1(w^{(\log t)}) \to 0 \ in \ \mathcal{P} \ as \ t \to +\infty,$$

where for $r > 0$ *we let* $w^{(r)}(s) = r^{-1}w(sr^2)$ *for all* $s \in \mathbb{R}$.

We will define the process b soon. This result shows the dominant effect of the environment, through the process b, on the asymptotic behavior of the diffusion. The results we prove in this paper concern the process b. In subsection 1.2 we commend on their implications for the behavior of the diffusion itself.

Besides this diffusion model, there is a discrete time and space analog, known as Sinai's walk, which was studied first. Sinai's pioneering paper Sinai (1982) identified the role of the process b in the analogous to (2) limit theorem for the walk. Then S. Schumacher proved in Schumacher (1984) (see also Schumacher (1985) for the results without the proofs) a more general statement than the above proposition where the environment w was not necessarily a two sided Brownian motion, while T. Brox [Brox (1986)] gave a different proof in the Brownian case. H. Kesten [Kesten (1986)] computed the density of b_1 in the case we consider, and Tanaka [Tanaka (1987b)] generalized the computation to the case that w is a two sided symmetric stable process. Localization results have been given for the Sinai walk by Golosov (Golosov (1984), actually, for the reflected walk) and for the diffusion model by Tanaka [Tanaka (1994)]. Also Tanaka [Tanaka (1987a), Tanaka (1987b)] studied the cases where the environment is non-positive reflecting Brownian motion, non-negative reflecting Brownian motion, or Brownian motion with drift. Finer results on the asymptotics of Sinai's have been obtained by Z. Shi and Y. Hu. A survey of some of them as well as a connection between Sinai's walk and diffusion in random environment is given in Shi (2001). Another connection is established in [Seignourel (2000)].

In Le Doussal et al. (1999), P. Le Dousal, C. Monthus, and D. Fisher proposed a new method for tackling questions related to asymptotic properties of Sinai's walk, and using it they gave a host of results. The method is a Renormalization Group analysis and it has consequences agreeing with rigorously proved results (e.g., Dembo et al. (2001), Kesten (1986)) . This is the starting point of the present paper. In the context of diffusion in random environment, we show how one can justify the Renormalization Group method using two tools. The first is a path decomposition for a two sided standard Brownian motion; the second is the renewal theorem. Our main results illustrate the use of the method and the way we justify it.

The structure of the paper is as follows. In the remaining of the introduction we state our results. In Section 2 we provide all the necessary machinery for the proofs, which are given in Section 3. Some technical lemmata that we use are proved in Section 4.

We begin by defining the process b.

For a function $w : \mathbb{R} \to \mathbb{R}$, $x > 0$ and $y_0 \in \mathbb{R}$ we say that w **admits an x-minimum at** y_0 if there are $\alpha, \beta \in \mathbb{R}$ with $\alpha < y_0 < \beta$, $w(y_0) = \inf\{w(y) : y \in [\alpha, \beta]\}$ and $w(\alpha) \geq w(y_0) + x$, $w(\beta) \geq w(y_0) + x$. We say that w **admits an x-maximum at** y_0 if $-w$ admits an x-minimum at y_0.
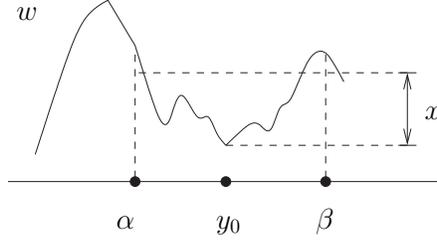

FIGURE 1. w admits an x-minimum at y_0

For convenience, we will call a point where w admits an x-maximum or x-minimum an x-maximum or an x-minimum respectively.

We denote by $R_x(w)$ the set of x-extrema of w and define

$$\mathcal{W}_1 := \left\{ w \in \mathcal{W} : \begin{array}{c} \text{For every } x > 0 \text{ the set } R_x(w) \text{ has no accumulation point in } \mathbb{R}, \\ \text{it is unbounded above and below,} \\ \text{and the points of } x\text{-maxima and } x\text{-minima alternate.} \end{array} \right\}.$$

Thus, for $w \in \mathcal{W}_1$ and $x > 0$ we can write $R_x(w) = \{x_k(w,x) : k \in \mathbb{Z}\}$ with $(x_k(w,x))_{k \in \mathbb{Z}}$ strictly increasing, $x_0(w,x) \leq 0 < x_1(w,x)$, $\lim_{k \to -\infty} x_k(w,x) = -\infty$, $\lim_{k \to \infty} x_k(w,x) = \infty$. It holds that $\mathbb{P}(\mathcal{W}_1) = 1$, and the easy proof of this fact is given in Lemma 8.

Definition 1. The process $b : [0, +\infty) \times \mathcal{W} \to \mathbb{R}$ is defined for $x > 0$ and $w \in \mathcal{W}_1$ as

$$b_x(w) := \begin{cases} x_0(w,x) & \text{if } x_0(w,x) \text{ is an } x\text{-minimum}, \\ x_1(w,x) & \text{else,} \end{cases}$$

and $b_x(w) = 0$ if $x = 0$ or $w \in \mathcal{W} \setminus \mathcal{W}_1$.

Remark 1. In the definition of $b_x(w)$ we do not make use of the entire sequence of x-extrema. The reason we introduce this sequence is that we plan to study the evolution of the process b as x increases. Since $R_{\tilde{x}}(w) \subset R_x(w)$ for $x < \tilde{x}$, the later values of $b.(w)$ are elements of $R_x(w)$. For $\tilde{x}$ large enough, the points $x_0(w,x)$, $x_1(w,x)$ will not be $\tilde{x}$-extrema.

Remark 2. We will decompose the process w at the endpoints of the intervals $\{[x_k(w,x), x_{k+1}(w,x)] : k \in \mathbb{Z}\}$ and study its restriction to each of them. Of course $[x_0(w,x), x_1(w,x)]$ has a particular importance for the process b; and it is in the study of $w|[x_0(w,x), x_1(w,x)]$ that the renewal theorem enters (see Lemma 1).

Remark 3. It is clear that b satisfies $b_x(\frac{1}{a}w(c\cdot)) = \frac{1}{c}b_{ax}(w)$ for all $a, c, x > 0$, and $w \in \mathcal{W}$. So that the quantity $b_1(w^{(\log t)})$ appearing in (2) equals also $b_{\log t}(w)/(\log t)^2$.

1.1. **Sign changes of** b**.** For $x \geq 1$ define on $\mathcal{W}_1$ the random variable

$$k(x) = \# \text{ times } b.(w) \text{ has changed sign in } [1, x].$$

The main result of the paper is the computation of the generating function of $k(x)$.

Theorem. *For $x \geq 1$ and $z \in \mathbb{C}$ with $|z| < 1$, it holds*

(3) $$\mathbb{E}(z^{k(x)}) = c_1(z)\ x^{\lambda_1(z)} + c_2(z)\ x^{\lambda_2(z)},$$

where

$$\lambda_1(z) = \frac{-3 + \sqrt{5 + 4z}}{2}, \quad \lambda_2(z) = \frac{-3 - \sqrt{5 + 4z}}{2},$$

and

$$c_1(z) = ((z-1)/3 - \lambda_2(z))/(\lambda_1(z) - \lambda_2(z)), \quad c_2(z) = (-(z-1)/3 + \lambda_1(z))/(\lambda_1(z) - \lambda_2(z)).$$

From this we extract several corollaries. Corollary 1 and Corollary 3 are immediate, while the rest require some work and are proved in section 3.

Corollary 1 follows by taking $z \to 0$ in (3).

Corollary 1.

(4) $$\mathbb{P}(b.(w) \text{ keeps the same sign in } [1, x])/x^{(-3+\sqrt{5})/2} \to 1/2 + 7\sqrt{5}/30$$

as $x \to +\infty$.

Corollary 2. *The increasing process of points $(X_k)_{k \geq 1}$ where b changes sign in $[1, +\infty)$ has the form $X_k = X_1 r_1 \cdots r_{k-1}$, $k \geq 2$, where X_1 is the smallest such point and the r_i's are i.i.d with density*

(5) $$f(r) = \frac{1}{\lambda_1 - \lambda_2}(r^{\lambda_1 - 1} - r^{\lambda_2 - 1}), \quad r \geq 1,$$

where $\lambda_1 = \lambda_1(0), \lambda_2 = \lambda_2(0)$.

Now observe that

$$k(t) = \sup\{n \in \mathbb{N} : X_n \leq t\} = \sup\{n \in \mathbb{N} : \log X_1 + \log r_1 + \cdots + \log r_{n-1} \leq \log t\}.$$

Since $\mathbb{E}(\log r_1) = 3$ and $\log X_1$ is finite a.s. (e.g., by Corollary 1, $\mathbb{E}(\log X_1) < +\infty$), the next statement follows from renewal theory.

Corollary 3. $k(t)/\log t \to 1/3$ *as $t \to +\infty$ $\mathbb{P}$-a.s.*

Corollary 2 allows the following strengthening of the above theorem.

Corollary 4. *Relation (3) holds for all $z \in \mathbb{C} \setminus (-\infty, -5/4]$.*

For $t \in (0, \infty)$ consider the random variable $k(e^t)/t$ and let μ_t be its distribution measure. Then the following holds.

Corollary 5. *The family of measures $(\mu_t)_{t>0}$ satisfies a Large Deviation Principle with speed t and good rate function*

$$I(x) = \begin{cases} x \ \log\left(2x(x + \sqrt{x^2 + 5/4})\right) + \frac{3}{2} - (x + \sqrt{x^2 + 5/4}) & \text{if } x \in [0, +\infty), \\ +\infty & \text{if } x \in (-\infty, 0). \end{cases}$$

In Le Doussal et al. (1999), Corollary 2 appears in paragraph IV.B with a different justification. We state it here because we need it for the proof of Corollary 5. The large deviation result of Corollary 5 is the precise mathematical interpretation of the discussion in paragraph IV.A of the same paper.

In Corollary 1, the condition $k(x) = 0$ means that the process b tends to keep the diffusion away form zero in the time interval $[e, e^x]$ (since the diffusion localizes around b, and b keeps sign on $[1, x]$). For the event that the diffusion *hits* zero there are two interesting relevant papers. The first one is by Y. Hu [Hu (2000)] who treats the annealed asymptotics of the first time of hitting zero after time t as $t \to +\infty$. The second is by F. Comets and S. Popov [Comets and Popov (2003)] and refers to a related model. That is, it considers a process $(X_t)_{t \geq 0}$ on $\mathbb{Z}$ that runs in continuous time in an environment ω satisfying the conditions of the Sinai model and studies, among other things, the asymptotics of the quenched probability $\mathbf{P}_w(X_t = 0 | X_0 = 0)$ as $t \to +\infty$.

1.2. **The process b and the diffusion.** The results of the previous subsection concern the process b, which is a functional of the environment. And our motivation for studying b was the localization results involving this process (the simplest being Fact 1, and keep in mind Remark 3). An obvious question is what we can infer about the behavior of the diffusion from our results.

Using the representation of the diffusion as a time and scale change of Brownian motion, one can show easily that the diffusion is recurrent and 0 is a regular point for $(0, +\infty)$ and $(-\infty, 0)$. Consequently, for all $c > 0$, the diffusion visits 0 in the time interval $[c, +\infty)$ and in its first visit there it scores an infinite number of sign changes. So there can be no direct connection with the corresponding number for the process b. One can consider, say, the number of sign changes of the diffusion between times where the diffusion achieves a positive or negative record value. This number is finite on compact intervals of $(0, +\infty)$ but still not related with the sign changes of b (and it is not hard to see this).

When $b_{\log}$. jumps to a new value, what happens is not that just the diffusion goes through that value shortly before or after that. As it is known (see, e.g., Brox (1986), Tanaka (1988)), the impact of the jump is that the diffusion goes to that value and it is trapped in its neighborhood for a large amount of time. The way to detect the approximate location of such values when we observe the diffusion in real time (i.e., at time t we know $\omega|[0, t]$) is to find the site the diffusion has spent the most time thus far.

To make the last statement precise, we use the local time process $\{L_\omega(t, x) : t \geq 0, x \in \mathbb{R}\}$ that corresponds to the diffusion ω. This process is jointly continuous, and with probability one satisfies

$$\int_0^t f(\omega(s)) ds = \int_{\mathbb{R}} f(x) L_\omega(t, x) dx$$

for all $t \geq 0$ and any bounded Borel function $f \in \mathbb{R}^{\mathbb{R}}$.

For a fixed $t > 0$, the set $\mathfrak{F}(t) := \{x \in \mathbb{R} : L_\omega(t, x) = \sup_{y \in \mathbb{R}} L_\omega(t, y)\}$ of the points with the most local time at time t is nonempty and compact. Any point there is called a **favorite point** of the diffusion at time t. Define $F : (0, +\infty) \to \mathbb{R}$ with $F(t) = \inf \mathfrak{F}(t)$, the smallest favorite point at time t. Also, for $x > 1$ define the interval $I(x) := (x - (\log x)^5, x + (\log x)^5)$.

In a work in progress we expect to prove the following.

Claim. *With $\mathcal{P}$ probability one, there is a strictly increasing sequence $(t_n(\omega, w))_{n \geq 1}$ converging to infinity so that if we denote by $(x_n(w, t_1))_{n \geq 1}$ the sequence of consecutive values that $b_{\log}$. takes on the interval $(t_1, +\infty)$ (remember that b is a step function in any interval $[x, +\infty)$ with $x > 0$), then*

$$F((t_n, t_{n+1})) \subset I(x_n) \text{ for all } n \geq 1,$$

and $x_n = b_{\log t}$ for some $t \in (t_n, t_{n+1})$. We abbreviated $t_n(\omega, w), x_n(w, t_1)$ to t_n, x_n.

Observe that for big x, the interval $I(x)$ is a relatively small neighborhood of x. Thus, the claim says that, after some point, the function F "almost tracks" the values of the process $b_{\log}$. with the same order and at about the same time. Consequently, the number of sign changes of $b_{\log}$. on an interval (s, t) (with s, t large) would correspond to the number of sign changes of F on approximately the same interval. Or, more precisely, the number of sign changes of F on (s, t) and the corresponding number for $b_{\log}$. differ by at most two. It is easy to see that the last statement follows from the Claim above.

2. Preliminaries

As a first step towards the study of the process b, we look at the law of the Brownian path between two consecutive x-extrema as well as the way these pieces are put together to constitute the entire path.

The first piece of information is provided by Proposition of §1 in Neveu and Pitman (1989).

Fact 2. *For every $x > 0$, the times of x-extrema of a Brownian motion $(w_t : t \in \mathbb{R})$, $w_0 = 0$ build a stationary renewal process $R_x(w) = \{x_k : k \in \mathbb{Z}\}$ with $(x_k)_{k \in \mathbb{Z}}$ strictly increasing and $x_0 \leq 0 < x_1$. The trajectories between consecutive x-extrema $(w_{x_k + t} - w_{x_k} : t \in [0, x_{k+1} - x_k]), k \in \mathbb{Z}$ are independent and for $k \neq 0$ identically distributed (up to changes of sign).*

In Lemma of §1 of the same paper [Neveu and Pitman (1989)] a description of each such trajectory is given which we quote (see Figure 2).
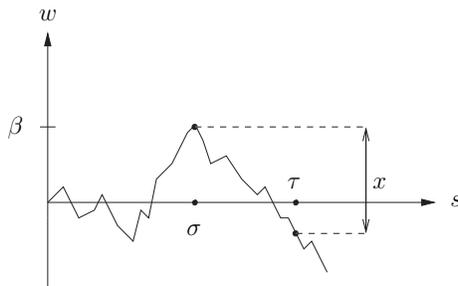

FIGURE 2. The graph of w until $M - w$ hits x

For $x, t \geq 0$ let

$$M_t := \sup\{w_s : s \in [0, t]\},$$
$$\tau := \min\{t : M_t = w_t + x\},$$
$$\beta := M_\tau,$$
$$\sigma := \max\{s \in [0, \tau] : w_s = \beta\}.$$

Fact 3. *The following hold:*

(i) *The two trajectories $(w_t : t \in [0, \sigma])$ and $(\beta - w_{\sigma+t} : t \in [0, \tau - \sigma])$ are independent.*

(ii) *β has exponential distribution with mean x.*

(iii) *The Laplace transform of the law of σ given β is*

$$\mathbb{E}[e^{-s\sigma}|\beta = y] = \exp\{-(y/x)\phi(sx^2)\} \qquad , s > 0.$$

(iv) *The Laplace transform of the law of $\tau - \sigma$ is*

$$\mathbb{E}[e^{-s(\tau-\sigma)}] = \psi(sx^2) \qquad , s > 0,$$

where $\phi(s) = \sqrt{2s} \coth \sqrt{2s} - 1$ and $\psi(s) = \sqrt{2s}/\sinh \sqrt{2s}$.

We call the translation $(w - w(x_k))|[x_k, x_{k+1}]$ of the trajectory of w between two consecutive x-extrema an x**-slope** (or a slope, when the value of x is clear or irrelevant), a slope that takes only non-negative values an **upward slope**, and a slope taking only non-positive values a **downward slope**. We call $(w - w(x_0))|[x_0, x_1]$ the **central** x**-slope**.

In the following we will use the operation of "gluing together" functions defined on compact intervals. For two functions $f : [\alpha, \beta] \to \mathbb{R}$, $g : [\gamma, \delta] \to \mathbb{R}$, by gluing g to the right of f we mean that we define a new function $j : [\alpha, \beta + \delta - \gamma] \to \mathbb{R}$ with

$$j(x) = \begin{cases} f(x) & \text{for } x \in [\alpha, \beta], \\ f(\beta) + g(x - \beta + \gamma) - g(\gamma) & \text{for } x \in [\beta, \beta + \delta - \gamma]. \end{cases}$$

It is clear that for all $k \neq 0$, if $(w - w(x_k))|[x_k, x_{k+1}]$ is an upward x-slope, then it is obtained by gluing together a trajectory of type $(w_t : t \in [0, \sigma])$ to the right of a trajectory of type $(\beta - w_{\sigma+t} : t \in [0, \tau - \sigma])$ and then translating the resulting path so that it starts at $(x_k, 0)$. Similarly for a downward slope.

For any x-slope $T : [\alpha, \beta] \to \mathbb{R}$ we call $l(T) := \beta - \alpha$, $h(T) := |T(\beta) - T(\alpha)|$ the **length** and the **height** of the slope respectively, and we denote by $\eta(T) := h(T) - x$ the "**excess height**" of T. Also we denote by $|T|$, $\theta(T)$, the slopes with domains $[\alpha, \beta]$, $[0, \beta - \alpha]$ and values $|T|(\cdot) = |T(\cdot)|$, $\theta(T)(\cdot) = T(\alpha + \cdot)$ respectively.

For any slope T, the slope $|\theta(T)|$ is in the set $\mathcal{S}$ defined by

$$\mathcal{S} := \left\{ f \subset [0, +\infty)^2 : \begin{array}{c} f \text{ is a function such that there is a } l(f) \geq 0 \text{ with} \\ \text{Domain}(f) = [0, l(f)], f \text{ continuous,} \\ \text{and } 0 = f(0) \leq f(x) \leq f(l(f)) \quad \forall x \in [0, l(f)] \end{array} \right\}.$$

On $\mathcal{S}$ we define a topology for which the base of neighborhoods of an element $f \in \mathcal{S}$ is the collection of all sets of the form

$$\{g \in \mathcal{S} : |l(g) - l(f)| < \varepsilon \text{ and } |f(t \, l(f)) - g(t \, l(g))| < \varepsilon \text{ for all } t \in [0, 1]\}.$$

With this topology, $\mathcal{S}$ is a Polish space. Equip $\mathcal{S}$ with the Borel σ-algebra, and define the measures m_x^r, m_x^c the first to be the distribution of $\theta(|(w - w(x_1))||[x_1, x_2])$

and the second to be the distribution of $\theta(|(w - w(x_0))|||[x_0, x_1])$ (the superscripts r and c standing for renewal and central).

In the remaining part of this section we compute the distribution of the length and excess height of a slope picked from m_x^r or m_x^c. We assume $x = 1$ since the scaling property of Brownian motion gives the corresponding results for the case $x \neq 1$.

Let T be a slope picked from m_1^r. Earlier we described the way an upward slope is formed. Its length is the sum of two independent independent random variables Z_1, Z_2 with $Z_1 \overset{law}{=} \sigma, Z_2 \overset{law}{=} \tau - \sigma$ (where we take $x = 1$ in their definitions just before Fact 3). But from (i) of Fact 3, σ and $\tau - \sigma$ are independent. Thus, $l(T) \overset{law}{=} \tau$. By definition, τ is the time where the reflected process $M - w$ hits one. This reflected process has the same law as $|w|$, and the Laplace transform of the time it first hits one is known as

$$(6) \qquad\qquad \mathbb{E}[e^{-\lambda l(T)}] = (\cosh \sqrt{2\lambda})^{-1} \text{ for } \lambda > 0.$$

Also $\mathbb{E}(l(T)) = \mathbb{E}(\tau) = 1$. Using the Laplace inversion formula (see Marsden and Hoffman (1987), pg 531) we find that the density of $l = l(T)$ is

$$(7) \qquad f_l(x) = \frac{\pi}{2} \sum_{k \in \mathbb{Z}} (-1)^k (k + \frac{1}{2}) \exp[-\frac{\pi^2}{2}(k + \frac{1}{2})^2 \ x \ ] \quad , \ x > 0.$$

We note for future reference that, by (ii) of Fact 3, for any $a > 0$, the excess height of a slope picked from m_a^r is exponential with mean a; i.e., it has density

$$(8) \qquad\qquad p_a(x) = a^{-1} e^{-x/a} \qquad , \ x > 0.$$

Now let T_0 be a slope picked from m_1^c. More specifically, take T_0 to be the central 1-slope. Observe that, by virtue of Fact 2, we can "start a renewal process at $-\infty$" with i.i.d. alternating upward and downward slopes, and ask what the characteristics are of the slope covering zero. The renewal theorem says that the length of the slope covering zero is picked from the distribution of l given in (7) with size-biased sampling. Once the length, say z, is picked, we expect that the remaining characteristics of the slope, ignoring direction (upward or downward), are determined by the law of $T|l(T) = z$ under m_1^r. We give a formal proof of this. Notice that a regular conditional distribution for the random variable T given the σ-field $\sigma(l(T))$ exists because the space $\mathcal{S}$ is Polish.

Lemma 1. *For any measurable subset A of $\mathcal{S}$, it holds*

$$(9) \qquad\qquad \mathbb{P}(|\theta(T_0)| \in A) = \int_0^\infty \mathbb{P}(T \in A \mid l(T) = z) z f_l(z) \ dz,$$

where T has under $\mathbb{P}$ distribution m_1^r.

Proof. Let F_l be the distribution function of $l = l(T)$, and for $t \in \mathbb{R}$ let $T(t)$ be the 1-slope around t. That is, the slope whose domain contains t. Then $\mathbb{P}(|\theta(T_0)| \in A) = \mathbb{P}(|\theta(T(t))| \in A)$ for all $t > 0$ because $\theta(T_0)$ is the same as the image under θ of the slope around t for $(w_{s-t} - w_{-t} : s \in \mathbb{R})$, and the latter process is again a standard two sided Brownian motion. Now let $(Y_n)_{n \geq 0}$ be an independent sequence of slopes with $Y_n \overset{law}{=} (-1)^{n+1} T$. Glue them sequentially to get a function f in $C([0, +\infty))$ with $f(0) = 0$, and denote by $\tilde{T}(t)$ the slope around t, for $t > 0$.

If we take σ as defined just before Fact 3 with $x = 1$ in the definition of τ there, then it holds

$$\mathbb{P}(|\theta(T_0)| \in A) = \mathbb{P}(|\theta(T(t))| \in A) = \mathbb{P}(|\theta(T(t))| \in A, \sigma < t) + \mathbb{P}(|\theta(T(t))| \in A, \sigma > t)$$

and

$$\mathbb{P}(|\theta(T(t))| \in A, \sigma < t) = \int_0^t \mathbb{P}(|\theta(\tilde{T}(t - y))| \in A) f_\sigma(y) \, dy,$$

where f_σ is the density of σ. We will take $t \to +\infty$ and finish with the proof by showing that the limit $\lim_{t \to +\infty} P(|\theta(\tilde{T}(t))| \in A)$ exists and

$$\lim_{t \to +\infty} \mathbb{P}(|\theta(\tilde{T}(t))| \in A) = \int_0^\infty \mathbb{P}(T \in A \mid l(T) = z) z f_l(z) \, dz.$$

To see this, define $g(t) := \mathbb{P}(|\theta(\tilde{T}(t))| \in A)$ for $t \geq 0$. Then

$$g(t) = \mathbb{P}(T \in A, \ l(T) > t) + \int_0^t \mathbb{P}(|\theta(\tilde{T}(t - s))| \in A) \, dF_l(s)$$

$$= \mathbb{P}(T \in A, \ l(T) > t) + \int_0^t g(t - s) \, dF_l(s).$$

The distribution of l is nonarithmetic with mean value 1. By the renewal theorem (Durrett (1996), Chapter 3, statement (4.9)), it follows that the $\lim_{t \to +\infty} g(t)$ exists and

$$\lim_{t \to +\infty} g(t) = \int_0^\infty \mathbb{P}(T \in A, \ l(T) > s) \, ds = \int_0^\infty \int_s^\infty \mathbb{P}(T \in A \mid l(T) = z) f_l(z) \, dz ds$$

$$= \int_0^\infty \int_0^z \mathbb{P}(T \in A \mid l(T) = z) f_l(z) \, ds dz = \int_0^\infty \mathbb{P}(T \in A \mid l(T) = z) z f_l(z) \, dz.$$

$\blacksquare$

Now we apply Lemma 1 to obtain the distribution of the length and height of the central 1-slope.

• Regarding the length of T_0, observe that for $x > 0$ the set $A := \{T \in \mathcal{S} : l(T) < x\}$ is open and

$$\mathbb{P}(l(T_0) < x) = \int_0^\infty \mathbb{P}(l(T) < x \mid l(T) = z) z f_l(z) \, dz = \int_0^x z \, f_l(z) \, dz.$$

So that $l(T_0)$ has the density

$$(10) \qquad\qquad f_{l(T_0)}(x) = x f_l(x) \qquad \text{for } x \geq 0,$$

which is the size-biased sampling formula from renewal theory.

• Regarding the excess height of T_0, observe that for $x > 0$ the set $A := \{T \in \mathcal{S} : \eta(T) < x\}$ is open and

$$\mathbb{P}(\eta(T_0) < x) = \int_0^\infty \mathbb{P}(\eta(T) < x \mid l(T) = z)z f_l(z)\, dz = \int_0^\infty \mathbb{P}(\eta(T) < x,\ l(T) = z)z\, dz.$$

Differentiating with respect to x we get the density of $\eta(T_0)$ as

$$\int_0^\infty \mathbb{P}(\eta(T) = x,\ l(T) = z)z\, dz = \mathbb{E}\left(l(T) \mid \eta(T) = x\right)e^{-x}.$$

From (iii), (iv) of Fact 3

$$\mathbb{E}(e^{-tl(T)} \mid \eta(T) = x\ ) = \psi(t)\ e^{-x\phi(t)}.$$

After some calculations, $\frac{\partial}{\partial t}\psi(t)\ e^{-x\phi(t)}|_{t=0} = -(2x + 1)/3$. The derivative can move inside the expectation on the left hand side of the above relation, giving $-\mathbb{E}(l(T) \mid \eta(T) = x)$, due to the monotone convergence theorem; which applies because $l(T) > 0$ and the function $(x \mapsto (1-e^{-ax})/x)$ is nonnegative and decreasing in $\mathbb{R}$ for any $a > 0$. Therefore the density of $\eta(T_0)$ is

$$(11) \qquad\qquad f_{\eta(T_0)}(x) = \frac{(2x + 1)e^{-x}}{3} \qquad \text{for } x > 0.$$

The last bit of information needed to achieve our goal, stated at the first sentence of this section, is the direction of the central 1-slope (i.e., upward or downward) and its location with respect to zero. By symmetry, T_0 is an upward slope with probability $1/2$, and from exercise 3.4.7 of Durrett (1996) it follows easily that given the length l of the slope $(w - w(x_0))|[x_0,\ x_1]$ around zero, the distance of zero from x_0 is uniformly distributed in $[0, l]$.

3. Proof of the Theorem and the Corollaries

For $x > 0$ and $w \in \mathcal{W}_1$ with set of x-extrema $R_x(w) = \{x_k : k \in \mathbb{Z}\}$ define

$$A_x(w) := \left\{\left(w - w(x_k)\right)|[x_k, x_{k+1}] : k \in \mathbb{Z}\right\}.$$

We refer to the parameter x as time since we are going to study the evolution of $A_x(w)$ as x increases. $A_x(w)$ is the set of slopes at time x. Roughly, as x increases, the slopes that have height smaller than x are absorbed into greater ones.

For any Lebesgue measurable set $S \subset [0, +\infty)$, $x \geq 1$, and $k \in \mathbb{N}$ we define

$$\mathcal{U}(x, S, k) = \mathbb{P}\left(\begin{array}{c} \text{In } A_x(w) \text{ the central slope has excess height } y \in S \\ \text{and } b_\cdot(w) \text{ has changed sign } k \text{ times in } [1, x] \end{array}\right).$$

It is important to note that the values of b up to time x are "encoded" in the central slope of $A_x(w)$. So the number of sign changes of $b_\cdot(w)$ in $[1, x]$ can be inferred by that slope.

For x, k fixed, $\mathcal{U}$ is a measure that satisfies

$$\mathcal{U}(x, S, k) \leq \mathbb{P}\big(\text{In } A_x(w) \text{ the central slope has excess height } y \in S\big),$$

and the right-hand side, considered as a function of S, is a measure absolutely continuous with respect to the Lebesgue measure with density $(1/x)f_{\eta(T_0)}(\cdot/x)$, where $f_{\eta(T_0)}$ is given in (11). Therefore the measure on the left-hand side has a
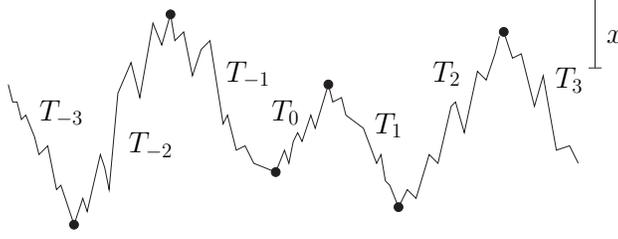
FIGURE 3.   The decomposition of a piece of the Brownian path in x-slopes. The dots mark the points of x-extrema. The length x is shown on the side

density also (an element of $L^1([0, +\infty)))$, call it $u(x, y, k)$, and for Lebesgue almost all y it holds

$$(12) \qquad \sum_{k=0}^{+\infty} u(x, y, k) = \frac{1}{x} f_{\eta(T_0)}\left(\frac{y}{x}\right).$$

Define $U : [1, +\infty) \times [0, +\infty) \times \mathbb{N} \to [0, +\infty)$ with $U(x, y, k) := \mathcal{U}(x, [y, +\infty), k)$. U is continuous as is proved in Lemma 7 in section 4. We plan to establish a PDE for U. To do this, we look at $A_x(w)$ and try to predict how $A_{x+\varepsilon}(w)$ should look like.

In the transition from $A_x(w)$ to $A_{x+\varepsilon}(w)$, a slope around a point remains the same if the excess height of this and the two neighboring slopes are greater than ε. In case one slope has excess height in $[0, \varepsilon)$, it does not appear in $A_{x+\varepsilon}(w)$. For example, in Figure 3 the slope T_0 has a height say $x + h_0$ with $h_0 \in [0, \varepsilon)$, and the two neighboring slopes T_{-1}, T_1, have height greater than $x + \varepsilon$. Assume that T_{-1}, T_1, have heights $x + v_1$, $x + v_2$ respectively. In $A_{x+\varepsilon}(w)$ we know that the slopes T_{-1}, T_0, T_1 will merge to constitute a new slope with height $x + v_1 + x + v_2 - x - h_0 = x + v_1 + v_2 - h_0$; i.e., with excess height $v_1 + v_2 - h_0 - \varepsilon$. The slopes T_{-2}, T_2 can stay as they are in the transition from $A_x(w)$ to $A_{x+\varepsilon}$ or they can be extended if some of T_{-3}, T_3 has excess height in $[0, \varepsilon)$. In any case, they don't interfere with T_{-1}, T_0, T_1. This simple observation, combined with the renewal structure of the sets $A_x(w)$, is the basis for the next lemma, which is the first step towards establishing a PDE that U solves. We denote by $\partial_y U$ the y derivative of $U(x, y, k)$, and recall that $p_x(v)$ was defined in (8) as the density of an exponential with mean x.

Lemma 2. *For $x \geq 1$, $y \geq 0$, $\varepsilon > 0, k \in \mathbb{N}$ we have*

$$(13) \quad U(x + \varepsilon, y, k) = U(x, y + \varepsilon, k)e^{-2\varepsilon/x} + \frac{2\varepsilon}{x} \int_0^{+\infty} p_x(v)\, U\big(x, (y - v)^+, k\big)\, dv -$$

$$1_{\{k \geq 1\}}\varepsilon\, \partial_y U(x, 0, k - 1)(y/x + 1)e^{-y/x} + o(\varepsilon),$$

where for $k \geq 1$ we assume that $U(x, y, k - 1)$ is differentiable in y with continuous derivative. The term $o(\varepsilon)$ depends on x, y, k.

Proof. The left-hand side of the equation is the probability of an event referring to $A_{x+\varepsilon}(w)$, and we express it in terms of probabilities referring to $A_x(w)$. In $A_x(w)$ we focus our attention on the seven x-slopes closest to zero. Denote them by T_i, $i = -3, \ldots, 3$ in the order they appear in the path of w from left to right,

with T_0 being the central one. The slopes $|\theta(T_i)|$, $i = -3, \ldots, 3$ are independent having for $i \neq 0$ law m_x^r, and $|\theta(T_0)|$ having law m_x^c. The probability of the event that at least two of them have excess height in $[0, \varepsilon)$ is bounded by $21\varepsilon^2/x^2$, and this is accounted for in the $o(\varepsilon)$ term in (13). In the complement of this event, the event whose probability appears in the left-hand side of (13) happens if and only if in $A_x(w)$ one of the following three holds (considering what slope, if any, among the seven has excess height in $[0, \varepsilon)$).

- At most one of T_{-3}, T_{-2}, T_2, T_3 has excess height in $[0, \varepsilon)$, the excess height of each of T_{-1}, T_0, T_1 is at least ε, and b has changed sign k times in $[1, x]$. In this case the slope around zero is the same for both $A_x(w), A_{x+\varepsilon}(w)$.
- Exactly one of T_{-1}, T_1 has excess height in $[0, \varepsilon)$, T_0 has excess height at least ε, and b has changed sign k times in $[1, x]$. In this case $b_{x+\varepsilon}$ has the same sign as b_x. For example, assume that T_1 has excess height in $[0, \varepsilon)$. If $b_x > 0$ then $b_{x+\varepsilon} > b_x$, while if $b_x < 0$ then $b_{x+\varepsilon} = b_x$ and simply the central slope in $A_{x+\varepsilon}(w)$ results from merging T_0, T_1, T_2.
- $k \geq 1$, T_0 has excess height in $[0, \varepsilon)$, and b has changed sign $k - 1$ times in $[1, x]$. In this case $b_{x+\varepsilon}b_x < 0$ because in $A_{x+\varepsilon}(w)$ the central slope results from merging T_{-1}, T_0, T_1 and it has different direction than T_0 (i.e., if for example T_0 is an upward slope, then the central slope in $A_{x+\varepsilon}(w)$ is a downward slope).

The first case has probability

$$U(x, y + \varepsilon, k)\, \mathbb{P}(\eta_{-1} \in [\varepsilon, +\infty))\, \mathbb{P}(\eta_1 \in [\varepsilon, +\infty)) = U(x, y + \varepsilon, k) \exp(-2\varepsilon/x)$$

with η_{-1} (resp. η_1) denoting the excess height of T_{-1} (resp. T_1). This follows by the independence mentioned above and by the fact that η_{-1} and η_1 have exponential distribution with mean x, and it expresses the demand that both of the x-slopes neighboring the central x-slope have excess height greater than ε. In this case, the excess heights of T_{-3}, T_{-2}, T_2, T_3 do not matter. They can't influence the central slope in $A_{x+\varepsilon}(w)$.

The second case has probability

$$2 \int_0^\varepsilon \int_\varepsilon^{+\infty} p_x(v_1)\, p_x(v_2)\, U\big(x, \varepsilon \vee (y + \varepsilon - v_2 + v_1), k\big)\, dv_2 dv_1$$

because, say in the event that T_1 has excess height in $[0, \varepsilon)$, the central slope in $A_{x+\varepsilon}(w)$ comes from merging T_0, T_1, T_2 of $A_x(w)$. Assume that they have excess heights u, v_1, v_2 respectively. Then the central slope in $A_{x+\varepsilon}(w)$ will have excess height $u - v_1 + v_2 - \varepsilon$, and the requirement that this is greater than y translates to u being greater than $(y + v_1 - v_2 + \varepsilon)^+$. And of course, by assumption, T_0 and T_2 have excess height greater than ε.

U is continuous as it is proved in Lemma 7. Thus, dividing with ε the previous integral and taking $\varepsilon \to 0$ we get as limit

$$\frac{2}{x} \int_0^{+\infty} p_x(v_2)\, U\big(x, (y - v_2)^+, k\big)\, dv_2.$$

From this procedure we pick up another $o(\varepsilon)$ term.

The last case has probability

$$- \int_0^\varepsilon \int_\varepsilon^{+\infty} \int_\varepsilon^{+\infty} \partial_y U(x, z, k - 1)\, p_x(v_1)\, p_x(v_2)\, 1_{\{v_1 + v_2 \geq y + z + \varepsilon\}}\, dv_1 dv_2 dz.$$

By assumption, $-\partial_y U(x, y, k-1)$ exists and it is the density of the measure $\mathcal{U}(x, \cdot, k)$. The dummy variables v_1, v_2, z stand for the excess heights of T_{-1}, T_1, T_0 respectively, and in this case the central slope in $A_{x+\varepsilon}(w)$ has excess height $v_1 + v_2 - z - \varepsilon$, giving the restriction $v_1 + v_2 - z - \varepsilon \geq y$. Since $\partial_y U(x, \cdot, k-1)$ is continuous, dividing by ε and taking $\varepsilon \to 0$ we get

$$-\partial_y U(x, 0, k-1) \int_0^{+\infty} \int_0^{+\infty} 1_{\{v_1+v_2\geq y\}}\, p_x(v_1)\, p_x(v_2)\, dv_1 dv_2.$$

And again we pick up an $o(\varepsilon)$ term. The double integral equals $(y/x+1)\exp(-y/x)$.
∎

Before getting to the actual proof of our theorem, we give a non-rigorous short derivation to illustrate its main steps. The main problem is that we don't know if U is differentiable in the x, y variables for every $k \in \mathbb{N}$. Assume for the moment that it is. For $f, g \in L^1([0, +\infty))$, as usual, we define $f * g \in C([0, +\infty))$ by $(f * g)(x) := \int_0^x f(x-y)g(y)\, dy$ for all $x \in [0, +\infty)$.

The above lemma would give for U the PDE

(14) $\quad (\partial_x - \partial_y)\, U(x, y, k) = -\dfrac{2}{x}\, U(x, y, k) + \dfrac{2}{x^2}\, \big(U(x, \cdot, k) * e^{-\frac{\cdot}{x}}\big)(y)$

$$+ \dfrac{2}{x} e^{-\frac{y}{x}} U(x, 0, k) - 1_{\{k\geq 1\}} \partial_y U(x, 0, k-1)(y/x+1)\exp(-y/x).$$

Let $f(x, y, k) = U(x, yx, k)$; i.e., $U(x, y, k) = f(x, y/x, k)$.
Then f should satisfy

(15) $\quad (x\,\partial_x - (1+y)\,\partial_y + 2)f(x, y, k) = 2\big(f(x, \cdot, k) * e^{-\cdot}\big)(y) + 2\,e^{-y} f(x, 0, k)$

$$- 1_{\{k\geq 1\}}\,(y+1)\,e^{-y}\,\partial_y f(x, 0, k-1),$$

while the conditions at $x = 1$ are

(16) $\qquad f(1, y, 0) = (2y/3+1)\,e^{-y}$ $\qquad\qquad$ for $y \geq 0$,

(17) $\qquad f(1, y, k) = 0$ $\qquad\qquad\qquad$ for $y \geq 0, k \geq 1$.

The first equation comes from (11), the second is clear.

For $z \in D := \{z \in \mathbb{C} : |z| < 1\}$, the generating function $M(x, y, z) := \sum_{k=0}^{\infty} f(x, y, k)\, z^k$ is well defined. Assuming that M is differentiable with respect to x, y and its x, y derivatives are obtained with term by term differentiation, we see that M satisfies the PDE problem

$(x\,\partial_x - (1+y)\,\partial_y + 2)\, M(x, y, z) = 2\big(M(x, \cdot, z) * e^{-\cdot}\big)(y)$

(18) $\qquad\qquad +2\,e^{-y} M(x, 0, z) - (y+1)\,e^{-y}\, z\partial_y M(x, 0, z)$ $\quad$ in $(1, \infty) \times (0, \infty)$,

(19) $\qquad\qquad\qquad M(1, y, z) = (2y/3+1)\,e^{-y}$ $\quad$ for $\quad y \geq 0$.

We try for a solution of the form

$$M(x, y, z) = [a(x, z) + b(x, z)\, y]\, e^{-y}.$$

Substituting this into (18) we see that e^{-y} factors out in both sides, and after cancellation, we arrive in an equality of two first degree polynomials in y with

coefficients depending on x, z. Equating the coefficients in equal powers of y in the two sides of the equation, we arrive at the following system of ODEs for a, b.

$$x\,\partial_x\,a(x, z) + (z - 1)\,(b(x, z) - a(x, z)) = 0,$$
$$x\,\partial_x\,b(x, z) + (2 + z)\,b(x, z) - (z + 1)\,a(x, z) = 0.$$

The initial condition (19) for M, expressed in terms of a, b, becomes $a(1, z) = 1$, $b(1, z) = 2/3$ for all $z \in D$. We easily see that the only solution of the system satisfying these conditions is

$$a(x, z) = c_1(z)\,x^{\lambda_1(z)} + c_2(z)\,x^{\lambda_2(z)},$$

$$b(x, z) = c_1(z)\left(1 + \frac{\lambda_1(z)}{1 - z}\right)x^{\lambda_1(z)} + c_2(z)\left(1 + \frac{\lambda_2(z)}{1 - z}\right)x^{\lambda_2(z)},$$

where

$$\lambda_1(z) = \frac{-3 + \sqrt{5 + 4z}}{2}, \quad \lambda_2(z) = \frac{-3 - \sqrt{5 + 4z}}{2},$$

and

$$c_1(z) = \big((z-1)/3 - \lambda_2(z)\big)/(\lambda_1(z) - \lambda_2(z)), \quad c_2(z) = \big(-(z-1)/3 + \lambda_1(z)\big)/(\lambda_1(z) - \lambda_2(z)).$$

Now $E(z^{k(x)}) = M(x, 0, z) = a(x, z)$, which is what we want.

Proof of the Theorem: The proof is done by taking the steps of the above "proof" in reverse order. This time all the steps can be justified. We will need three lemmata. Two of them are non-trivial, and their proof is given in section 4.

Lemma 3. *The solution M of (18), (19) obtained above is analytic as a function of z in D. The coefficients of its development as a power series around zero are differentiable with respect to x and y in $(1, +\infty) \times (0, +\infty)$, and its x, y-derivatives are continuous and can be found with term by term differentiation.*

Using this lemma, we write M as

$$M(x, y, z) := \sum_{k=0}^{\infty} g(x, y, k)\,z^k.$$

Differentiating M term by term and equating the coefficients of equal powers of z in the two sides of (18), we see that the sequence of functions $\big(g(\cdot, \cdot, k)\big)_{k \geq 0}$ satisfies the PDEs (15) with conditions at $x = 1$ given by (16), (17).

For $k \in \mathbb{N}$ define $\tilde{g}(\cdot, \cdot, k) : [1, +\infty) \times [0, +\infty) \to \mathbb{R}$ by $\tilde{g}(x, y, k) = g(x, y/x, k)$ for $(x, y, k) \in [1, +\infty) \times [0, +\infty)$. The sequence of functions $\big(\tilde{g}(\cdot, \cdot, k)\big)_{k \geq 0}$ satisfies the PDEs (14) with conditions at $x = 1$ given by (16), (17).

The proof is finished by showing that the sequence $\big(U(\cdot, \cdot, k)\big)_{k \in \mathbb{N}}$ satisfies a weak form of these PDEs, and then a uniqueness result will identify U as $\tilde{g}$.

For fixed $c > 1$, define $g_{c,k}(x) = U(x, c - x, k)$ for $x \in [1, c]$. We state as a lemma an equation that $g_{c,k}$ satisfies. The proof is straightforward from Lemma 2.

Lemma 4. *The function $g_{c,k}$ is differentiable in $(1, c)$ and satisfies*

$$(20) \quad g'_{c,k}(x) = -\frac{2}{x}\,g_{c,k}(x) + \frac{2}{x^2}\,\big(U(x, \cdot, k) * e^{-\frac{\cdot}{x}}\big)(c - x)$$

$$+ \frac{2}{x}\,e^{-\frac{c-x}{x}}\,U(x, 0, k) - 1_{\{k \geq 1\}}\,\partial_y U(x, 0, k - 1)\,(c/x)\,e^{-(c-x)/x},$$

where for $k \geq 1$ we assume that $U(x, y, k-1)$ is differentiable in y with continuous derivative.

And the promised uniqueness result is the following.

Lemma 5 (Uniqueness). *Let $f : [1, +\infty) \times [0, +\infty) \to [0, +\infty)$ be a continuous function such that for every $c > 1$ the function $g_c : [1, c] \to \mathbb{R}$ with $g_c(x) := f(x, c - x)$ is continuous on $[1, c]$ and differentiable on $(1, c)$, with*

$$(21) \qquad g_c'(x) = -\frac{2}{x} \, g_c(x) + \frac{2}{x^2} \, \left(f(x, \cdot) * e^{-\frac{\cdot}{x}}\right)(c - x) + \frac{2}{x} e^{-\frac{c-x}{x}} f(x, 0)$$

and $g_c(1) = 0$. Then $f \equiv 0$.

Now using induction we show that

$$U(\cdot, \cdot, k) = \tilde{g}(\cdot, \cdot, k) \qquad \forall k \in \mathbb{N}.$$

The function $U(\cdot, \cdot, 0) - \tilde{g}(\cdot, \cdot, 0)$ satisfies the assumptions of Lemma 5 because of the PDE problem that $\tilde{g}(\cdot, \cdot, 0)$ solves and Lemma 4. For $k \geq 1$, assuming the statement true for $k-1$, the same argument works for $U(\cdot, \cdot, k) - \tilde{g}(\cdot, \cdot, k)$, where now the assumption on $U(\cdot, \cdot, k-1)$ required by Lemma 4 is provided by the inductive hypothesis.

Therefore, $\sum_{k=0}^{+\infty} U(x, xy, k) \, z^k = M(x, y, z)$ for $(x, y, z) \in [1, +\infty) \times [0, +\infty) \times D$ and $\mathbb{E}(z^{k(x)}) = M(x, 0, z) = a(x, z)$, proving (3) for $z \in D$. ∎

Proof of Corollary 2: Our theorem gives that $\mathbb{P}(b$ doesn't change sign in $[1, x]) = a(x, 0)$, where the function a is defined on page 14, and by scaling

$$\mathbb{P}(b \text{ doesn't change sign in } [x, y]) = a(y/x, 0) \qquad \text{for } 0 < x < y.$$

Consequently, the density of the last point before y that we have sign change is $-yx^{-2} \partial_x a(y/x, 0)$, where we use ∂_x here and below to denote derivative with respect to the first argument. Differentiating with respect to y, we get the density of the event that x, y are consecutive times of sign change as

$$x^{-2} \partial_x a(y/x, 0) + yx^{-3} \partial_{xx} a(y/x, 0),$$

which, after using the expression for $a(x, 0)$, becomes

$$\frac{1}{3(\lambda_1 - \lambda_2)xy} \left((y/x)^{\lambda_1} - (y/x)^{\lambda_2}\right).$$

The event that b changes sign at x translates to the central x-slope having excess height 0. This has density $1/(3x)$ due to the scaling property of the x-slopes and relation (11), which refers to the central 1-slope. Thus, the density of the time Y where the next sign change after x happens, given that there was a sign change at x, is

$$(22) \qquad h(y) = \begin{cases} \frac{1}{(\lambda_1 - \lambda_2)y} \left((y/x)^{\lambda_1} - (y/x)^{\lambda_2}\right) & \text{if } y \geq x, \\ 0 & \text{otherwise.} \end{cases}$$

The quotient Y/x, given that there was a sign change at x, has the density of the r_i's given in (5). ∎

Proof of Corollary 4: Observe that the right-hand side of (3) is a function analytic in $\mathbb{C} \setminus (-\infty, -5/4]$. As for the left-hand side, we have the following lemma.

Lemma 6. *For any $x > 1$, the power series $\sum_{n=0}^{+\infty} \mathbb{P}(k(x) = n) \, z^n$ defines an entire function.*

And from a basic property of analytic functions, it follows that the quantities $M(x, 0, z)$, $a(x, z)$ agree for all $z \in \mathbb{C} \smallsetminus (-\infty, -5/4]$. ■

Remark 4. Of course, Lemma 6 implies that the right hand side of (3) can be extended to an entire function. However, the way (3) is written doesn't allow as to claim that it holds for all $z \in \mathbb{C}$ because the function $(z \to \sqrt{4 + 4z})$ doesn't have an entire extension.

Proof of Corollary 5: We apply the Gartner-Ellis Theorem (Theorem 2.3.6 in Dembo and Zeitouni (1998)). The moment generating function of $k(e^t)/t$ is given for any $\lambda \in \mathbb{R}$ by $\Lambda_t(\lambda) := \log \mathbb{E}(\exp\{\lambda k(e^t)/t\})$. So $t^{-1}\Lambda_t(t\lambda) = t^{-1} \log \mathbb{E}(e^{\lambda k(e^t)}) = t^{-1} \log M(e^t, 0, e^\lambda)$, and using Corollary 4 we see that

$$\Lambda(\lambda) := \lim_{t \to \infty} \frac{1}{t}\Lambda_t(t\lambda) = \lambda_1(e^\lambda) = \frac{-3 + \sqrt{5 + 4e^\lambda}}{2}.$$

The Fenchel-Legendre transform Λ^* of Λ, defined by $\Lambda^*(x) = \sup_{\lambda \in \mathbb{R}}\{\lambda x - \Lambda(\lambda)\}$ for all $x \in \mathbb{R}$, is found to be the function I defined in the statement of the proposition. Also $D_\Lambda := \{\lambda \in \mathbb{R} : \Lambda(\lambda) < \infty\} = \mathbb{R}$, and Λ is strictly convex and differentiable in D_Λ. The result follows from the Gartner-Ellis theorem. ■

4. Proofs of the lemmata

Lemma 7. *For any $x \geq 1$, $y \geq 0$, $k \in \mathbb{Z}$, and $\varepsilon > 0$, we have*

$$|U(x + \varepsilon, y, k) - U(x, y + \varepsilon, k)| \leq 3\varepsilon/x,$$
$$|U(x, y, k) - U(x, y + \varepsilon, k)| \leq \varepsilon/x.$$

In particular, U is continuous.

Proof. Call A and B the two events whose probabilities are $U(x+\varepsilon, y, k)$ and $U(x, y + \varepsilon, k)$ respectively. Then $A \triangle B \subset [$ In $A_x(w)$ at least one of the three slopes neighboring zero has excess height $< \varepsilon]$. Denote by T_0, T_1 the central slope and the slope to the right to it in $A_x(w)$. Then

$$\mathbb{P}(A \triangle B) \leq 2\mathbb{P}(\ T_1 \text{ has excess height } < \varepsilon \ ) + \mathbb{P}(\ T_0 \text{ has excess height } < \varepsilon \ )$$

$$= 2 \int_0^{\varepsilon/x} e^{-z} \, dz + \int_0^{\varepsilon/x} (2z/3 + 1/3)e^{-z} \, dz < 3\varepsilon/x.$$

The other inequality follows because the density of the measure $\mathcal{U}(x, S, k)$ is bounded by $1/x$ (see (12)). ■

Proof of Lemma 3: Define $K : \mathbb{C} \times \mathbb{C} \times D \to \mathbb{C}$ with $K(z_1, z_2, z) := M(e^{z_1}, z_2, z)$ for $(z_1, z_2, z) \in \mathbb{C} \times \mathbb{C} \times D$. Clearly, K is a holomorphic function (choose an analytic branch of the square root function defined on $\mathbb{C}\backslash(-\infty, 0)$; the number $5+4z$ is there for $z \in D$) and has a power series development centered at zero that converges in $\mathbb{C} \times \mathbb{C} \times D$ (see, e.g., Krantz (1992), Proposition 2.3.16). The claims of the lemma follow by the relation $M(x, y, z) = K(\log x, y, z)$ and standard properties of power series. ■

Proof of Lemma 5: For $c > 1$ and $x \in [1, c]$, define $N(c, x) := \sup\{|f(z, c - z)| : z \in [1, x]\}$. N is well defined because f is bounded on compact sets.
From (21), for $x \in (1, c)$ one has

$$|g_c'(x)| \leq 2 N(c, x) + 4 \sup_{1 \leq d \leq c} N(d, x),$$

and since $g_c(1) = 0$, we get after integrating

$$|f(x, c - x)| \leq 6 \int_1^x \sup_{1 \leq d \leq c} N(d, t) dt,$$

which implies

$$N(c, x) \leq 6 \int_1^x \sup_{1 \leq d \leq c} N(d, t) dt$$

and

$$\sup_{1 \leq d \leq c} N(d, x) \leq 6 \int_1^x \sup_{1 \leq d \leq c} N(d, t) dt.$$

The function $A(x) = \sup_{1 \leq d \leq c} N(d, x)$ $(x \in [1, c])$ is continuous (because f is) and has $A(1) = 0$. An application of Gronwall's lemma to A gives $N(d, x) = 0$ for $x \in [1, c]$, $d \in [1, c]$; i.e., $f(x, y) = 0$ for all $(x, y) \in [1, +\infty) \times [0, +\infty)$. ∎

Proof of Lemma 6: Let X_k be as in Corollary 2. Then

$$\mathbb{P}(k(x) = n + 1) \leq \mathbb{P}(k(x) \geq n + 1) = \mathbb{P}(X_{n+1} \leq x)$$
$$\leq \mathbb{P}(r_1 r_2 \cdots r_n \leq x) = \mathbb{P}(\log r_1 + \log r_2 + \cdots + \log r_n \leq \log x).$$

For $i \geq 1$ set $Y_i = \log r_i$, $S_i = Y_1 + Y_2 + \cdots + Y_i$, and let m_i be the distribution measure of S_i/i. By Cramer's theorem (Dembo and Zeitouni (1998) Theorem 2.2.3), the sequence $(m_i)_{i \geq 1}$ satisfies a large deviation principle with rate function $I(x) = \sup_{\lambda \in \mathbb{R}}\{\lambda x - \log \mathbb{E}(e^{\lambda \log r_1})\}$ for $x \in \mathbb{R}$.
Clearly, for any $\varepsilon > 0$ we have

$$\varlimsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}(\frac{S_n}{n} < \frac{\log x}{n}) < -I(\varepsilon).$$

And $\lim_{\varepsilon \to 0} I(\varepsilon) = +\infty$ because I is lower semicontinuous and $I(0) = +\infty$. To see the last point, observe that

$$I(0) = \sup_{\lambda \in \mathbb{R}}\{-\log \mathbb{E}(e^{\lambda \log r_1})\} \geq \varlimsup_{\lambda \to -\infty}\{-\log \mathbb{E}(e^{\lambda \log r_1})\} = +\infty$$

because $\lim_{\lambda \to -\infty} e^{\lambda \log r_1} = 0$ with probability one and the bounded convergence theorem applies. Thus $\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(k(x) = n + 1) = -\infty$, proving that the radius of convergence for the power series is infinite. ∎

Lemma 8. *For $\mathbb{P}$ and $\mathcal{W}_1$ as defined in the introduction, it holds that $\mathbb{P}(\mathcal{W}_1) = 1$.*

Proof. First we prove that for fixed $x > 0$, the set

$$C_x := \{w \in C(\mathbb{R}) : R_x(w) \text{ has the properties appearing in the definition of } \mathcal{W}_1\}$$

has $\mathbb{P}(C_x) = 1$. Observe that for z a point of x-minimum and $\alpha_z := \sup\{\alpha < z : w(\alpha) \geq w(z) + x\}$, $\beta_z := \inf\{\beta > z : w(\beta) \geq w(z) + x\}$ it holds that $\alpha_z < z < \beta_z$ because w is continuous at z. And there is no other point of x-minimum in (α_z, β_z) since if $\tilde{z}$ is such a point, say in (α_z, z), then assuming $\beta_{\tilde{z}} < z$ we get a contradiction with the definition of a_z while assuming $\beta_{\tilde{z}} > z$ we get that w takes the same value in two local minima, which has probability zero. Now assume

that there is a strictly monotone, say increasing, sequence $(z_n)_{n\geq 1}$ of x-minima converging to $z_\infty \in \mathbb{R}$. By the above observations we get $\overline{\lim}_{y,\tilde{y}\nearrow z_\infty}(w(y)-w(\tilde{y})) \geq x$ contradicting the continuity of w at z_∞. Similarly if $(z_n)_{n\geq 1}$ is decreasing. So in a set of w's in $C(\mathbb{R})$ with probability 1, it holds that the set of x-minima of w has no accumulation point. The same holds for the set of x-maxima and as a result also for $R_x(w)$. Since $\underline{\lim}_{|t|\to\infty} w_t = -\infty$, $\overline{\lim}_{|t|\to\infty} w_t = +\infty$, it follows that $\mathbb{P}(R_x(w)$ is unbounded above and below$) = 1$. It is clear that between two consecutive x-maxima (resp. minima) there is exactly one x-minimum (resp. x-maximum). Consequently, $\mathbb{P}(C_x) = 1$.

Finally, note that for all $n \in \mathbb{N} \setminus \{0\}$ we have $R_n(w) \subset R_x(w) \subset R_{1/n}(w)$ for $x \in [1/n, n]$, from which it follows that $\mathcal{W}_1 = \cap_{x\in(0,+\infty)}C_x = \bigcap_{n\in\mathbb{N}\setminus\{0\}}(C_n \cap C_{1/n})$. Thus, $\mathbb{P}(\mathcal{W}_1) = 1$. ∎

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