PATTERNS IN SINAI'S WALK

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Sinai's random walk in random environment shows interesting patterns on the exponential time scale. We characterize the patterns that appear on infinitely many time scales after appropriate rescaling (a functional law of iterated logarithm). The curious rate function captures the difference between one-sided and two-sided behavior.

1. Introduction. For every integer k, pick p_k independently from a fixed probability measure on $[0, 1]$. Then, keeping the p_k's fixed, consider a nearest-neighbor random walk $S(n)$ on $\mathbb{Z}$, with $S(0) = 0$ and with probabilities $p_k, 1 - p_k$ of going right and left from k, respectively. This model, introduced by Chernov (1967), is the most well-studied model of motion in random medium.

We will assume that the random variables $p_1, (1 - p_1)$ have some finite negative moment. The walk $S(n)$, pictured in Figure 1, is recurrent exactly when $\log \frac{1-p_1}{p_1}$ has mean zero; see Solomon (1975). The graph of the walk seems much more confined than the ordinary random walk. Indeed, when $\log \frac{1-p_1}{p_1}$ has finite and positive variance as well, the typical value of $|S(n)|$ is of the order of $\log^2 n$, much less than the usual $\sqrt{n}$ for simple random walk; see Sinaĭ (1982). The walk in this regime is called *Sinai's walk*.

The logarithmic behavior of $|S(n)|$ suggests that we may get a more enlightening picture by considering $S(n)$ on an exponential time scale, namely the process $t \mapsto S(e^t)$ with the argument rounded down to the next integer. Figure 2 shows that the walk tends to get trapped by the environment. Indeed, the stationary measure for $S(n)$ is given by the exponential of a function with increments $\log \frac{1-p_k}{p_k}$, that is, a random walk on $\mathbb{Z}$. So at distance n there are regions with stationary measure as large as $e^{\sqrt{n}}$, in which $S(n)$ gets trapped for a long time.

The pattern we see in Figure 2 suggests a natural question: what patterns can we get that way? The main goal of this paper is to answer a mathematically precise version of this question. We consider rescaled versions of the path of $S(e^t)$
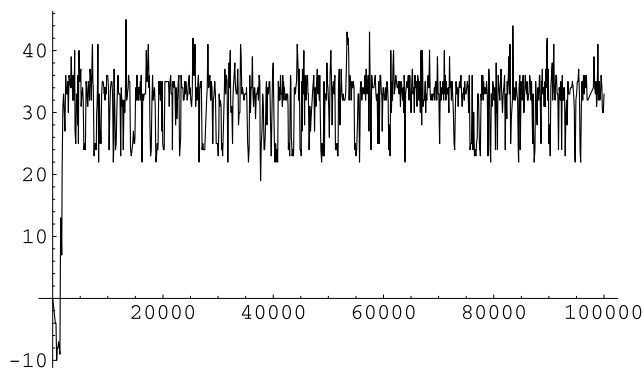
FIG. 1.   *Path of Sinai's walk $S(n)$.*

given by

$$\frac{S(e^{at})}{a^2 \log \log a}, \qquad t \geq 0,$$

and ask what are the possible limit points of the graph of this process as $a \to \infty$. For this, a topology on graphs has to be specified. As we will see, the spatial scaling factor $a^2 \log \log a$ is needed to ensure that the answer to our question is nontrivial.

Figure 2 suggests that we should consider a topology much weaker than the usual uniform-on-compacts convergence of functions: the process shows too many oscillations on this scale, and we do not even expect a function in the limit. Instead, we consider the graph occupation measure, and we view it as an element of the space of measures on $\mathbb{R}^+ \times \mathbb{R}$. For a measurable $\varphi \colon \mathbb{R}^+ \to \mathbb{R}$, its *graph occupation measure* is given by

$$m(\varphi)(A) := \mathrm{Leb}\{s \geq 0 \colon (s, \varphi(s)) \in A\}$$
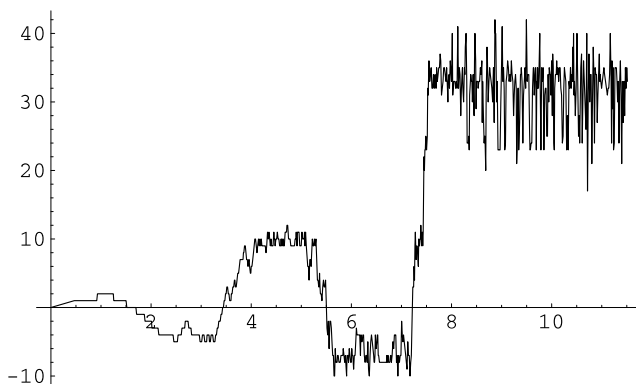

FIG. 2.   *The same path in exponential time, $S(e^t)$.*

for $A \subset [0, \infty) \times \mathbb{R}$ Borel set. Note that $m(\varphi)(\cdot \times \mathbb{R})$ is the Lebesgue measure.

The space of Borel measures on $\mathbb{R}^+ \times \mathbb{R}$ is equipped with the topology of local weak convergence. Then consider the following subset of that space:

$$\mathcal{M} = \left\{ \mu : \begin{array}{l} \mu(\cdot \times \mathbb{R}) \text{ is the Lebesgue measure,} \\ \exists f, g \geq 0 \text{ nondecreasing s.t. } \mathrm{supp}(\mu) \subset \overline{\mathrm{graph}(f)} \cup \overline{\mathrm{graph}(-g)} \end{array} \right\}.$$

For $\mu \in \mathcal{M}$, let f_μ and g_μ denote the unique minimal left-continuous choice of f, g in the above definition. Now μ projects to Lebesgue measure on $\mathbb{R}^+$, so μ restricted to the upper and lower half planes project to a partition of Lebesgue measure. Let $s_{\mu+}, s_{\mu-} \in [0, \infty]$ denote the supremum of the support of these projections, respectively. Let

$$(1) \quad I(\mu) := \frac{\pi^2}{2} \int_0^{s_{\mu+}} \frac{1}{t^2} \, d(f_\mu + g_\mu)(t) + \frac{\pi^2}{8} \int_{s_{\mu+}}^\infty \frac{1}{t^2} \, dg_\mu(t) \qquad \text{if } s_{\mu-} = \infty,$$

and if $s_{\mu-} < \infty$, in (1) we exchange f_μ, g_μ and replace $s_{\mu+}$ with $s_{\mu-}$. Note the striking difference between the parts with $\pi^2/8$ and $\pi^2/2$ coefficients—we will see that it is harder to be supported on the graph of two functions than on a single one.

THEOREM 1. *With probability* 1, *the* $a \to \infty$ *limit points of the graph occupation measures of the rescaled walk*

$$\frac{S(e^{at})}{a^2 \log \log a}, \qquad t \geq 0,$$

constitute the set

$$\mathcal{K} := \{\mu \in \mathcal{M} : I(\mu) \leq 1\}.$$

Also, there is at least one limit point along every sequence $a_n \to \infty$.

Our result is the analogue of Strassen's functional law of iterated logarithm for ordinary random walks [Strassen (1964)]. This is often stated in terms of the rescaled process restricted to a *finite* interval; such results easily follow from the full version. We also prove a version for the *Brox diffusion*, the continuous version of Sinai's walk; see Theorem 17 in Section 7. As discussed there, Theorem 1 extends to general environments that are close to Brownian motion. Our results are in agreement with Theorems 1.3, 8.1 of Hu and Shi (1998) about the one-point law,

$$\limsup_{a \to \infty} \frac{S(e^a)}{a^2 \log \log a} = \frac{8}{\pi^2}.$$

There are many ways in which Sinai's walk and the Brox diffusion are determined by their environment; see, for example, the results of Hu (2000), quoted as Theorem 16 in the present paper. In particular, the location of $S(n)$ is well predicted by $x(\log n)$, where x is the process of wells for the environment. For the

Brox diffusion, on the process level, the first author, Cheliotis (2008), showed that after some large random time, the path of the process $x(\log t)$ is close to that of the most-favorite-point process of the diffusion at time t.

In Section 2, we give a precise description of the process of wells x_B for the environment defined by two-sided Brownian motion B. Informally, consider the graph of B as a vessel in which water is poured gradually from the positive y axis. The water forms several increasing and merging puddles, also called wells. Then $x_B(h)$ is the x-coordinate of the bottom of the first-created well with depth at least h.

Our law of iterated logarithm is based on a similar theorem for the process of wells. This, in turn, is based on a large deviation principle for this process.

THEOREM 2. *The family of the laws of $\{m(x_B/M) : M > 0\}$, as $M \to \infty$, satisfies a large deviation principle on $\mathcal{M}$ with speed M and good rate function I.*

Interestingly, it is easier to avoid creating deep wells on one just of the axes than on both; this will be apparent from the proof of the theorem. This is the main reason for the two different factors $\pi^2/8$ and $\pi^2/2$.

In the flavor of the applications in Strassen (1964), we prove the following simple result about weighted integrals of $S(\cdot)$ along a geometric time scale.

COROLLARY 3. *For $r \geq 0$,*

$$\limsup_{a \to \infty} \frac{1}{a^2 \log\log a} \int_0^1 t^r S(e^{at}) \, dt = \frac{4}{\pi^2} \left(\frac{2}{r+3} \right)^{(r+3)/(r+1)}.$$

REMARK 4. There is a connection between our results and Chung's Law of iterated logarithm, which concerns the liminf behavior of the running maximum of random walk $\{S_n\}$ with increments of zero mean and variance 1. It states [see Jain and Pruitt (1975)]

$$\liminf_{n \to \infty} \log\log n \, \frac{\max_{1 \leq i \leq n} S_i^2}{n} = \frac{\pi^2}{8}.$$

Note the presence of the constant $\pi^2/8$ also here. The reason is that the only way Sinai's walk will take an unusually large value at a given time is if in a large interval the environment does not create large wells, which could delay the walk. This, in effect, confines the environment to a small interval for a long time. In this sense, our result is related to Wichura's theorem, a functional law of iterated logarithm for small values of the running absolute maximum of Brownian motion; see Mueller (1991).

Orientation. The structure of the paper is as follows. The first goal is to prove Theorem 2. Thus, Section 3 contains the large deviations upper bound, and Section 4 contains the lower bound. Section 5 combines these two results and exponential tightness to derive Theorem 2. Section 6 contains the proof of a functional law of the iterated logarithm for the environment, that is, for the family $(a^2 \log \log a)^{-1} x_B(a\cdot), a > e$. This is combined in Section 7 with a localization result to transfer the law to the motion. In Section 8 we estimate the probability that Brownian motion stays in certain sets for large intervals of time. The last section contains topological lemmas needed in Sections 3, 4 and 6.

2. The process of wells in the environment.

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. In Section 1, we introduced the process of wells by the following informal definition.

Consider the graph of f as a vessel in which water is poured gradually from the positive y axis. The water forms several increasing and merging puddles, also called wells. Then $x_f(h)$ is the x-coordinate of the bottom of the first-created well with depth at least h.

We now proceed to give a more detailed definition. For each point x_0 of local minimum for f, there are intervals $[a, c]$ containing x_0 with the property that $f(x_0)$ is the minimum value of f in $[a, c]$ and $f(a), f(c)$ are the maximum values of f on the intervals $[a, x_0], [x_0, c]$, respectively. Let $[a_{x_0}, c_{x_0}]$ be the maximal such interval. We call $f|[a_{x_0}, c_{x_0}]$ the *well* of x_0 and the number

$$\min\{f(a_{x_0}) - f(x_0), f(c_{x_0}) - f(x_0)\}$$

the *depth* of the well. We order wells by inclusion.

For $h > 0$, if there is a minimal well of depth at least h containing zero in its domain, we define $x_f(h)$ to be the smallest point in the domain of the well where f attains its minimum value on the well. If there is none, we let $x_f(h) = 0$. Finally, we let $x_f(0) = 0$.

For almost all two-sided Brownian paths B, for all $h > 0$, there is a unique point where B attains its minimum in the minimal well of depth at least h containing 0, and there is such a well. For such paths, x_B is a left-continuous step function. Moreover, x_B has the following monotonicity property: if $h_1 < h_2$ and $x_B(h_1), x_B(h_2)$ have the same sign, then $|x_B(h_1)| \leq |x_B(h_2)|$.

Finally, x_B inherits a scaling property from Brownian motion, namely for $a > 0$,

$$(2) \qquad\qquad (x_B(as))_{s \geq 0} \overset{\mathcal{L}}{=} (a^2 x_B(s))_{s \geq 0}.$$

The first step toward the proof of Theorem 2 is to study the behavior of the function x_B. More precisely, we will try to understand the probability that x_B is close to a particular step function.

Toward this end, let $\mathcal{S}$ be the set of all pairs of finite sequences

$$(\mathbf{h}, \mathbf{x}) \qquad \text{where } \mathbf{h} := (h_1, \ldots, h_N), \mathbf{x} := (x_1, \ldots, x_N)$$

for some $N \geq 1$, with the properties

$$0 < h_1 < h_2 < \cdots < h_N, \qquad x_1, x_2, \ldots, x_N \in \mathbb{R} \setminus \{0\},$$

and whenever $i < j$ and x_i, x_j have the same sign, then $|x_i| \leq |x_j|$.

For notational convenience, we will also use the indices $0, N + 1, \infty$, and set

$$x_0 := 0, \qquad x_\infty := -x_1 \quad \text{and} \quad h_0 := 0, \qquad h_{N+1} := h_\infty := 2h_N.$$

The mesh of the partition of $[0, h_N]$ induced by $\mathbf{h}$ is the number

$$\mathrm{mesh}(\mathbf{h}) := \min\{h_i - h_{i-1} : 1 \leq i \leq N\}.$$

For a pair $(\mathbf{h}, \mathbf{x})$ as above, we let $\mathcal{I} := \{1, \ldots, N\}$ and $\mathcal{I}_\infty \subset \mathcal{I}$ the largest set of consecutive integers in $\mathcal{I}$ containing N and for which all x_i for $i \in \mathcal{I}_\infty$ have the same sign.

For an index $i \in \mathcal{I}$, let i^- denote the greatest index $j \in \mathcal{I}$ less than i so that x_i and x_j have the same sign, and let $i^- = 0$ if there is no such index. Similarly, let i^+ denote the least index $j \in \mathcal{I}$ greater than i so that x_i and x_j have the same sign, and let $i^+ = \infty$ if there is no such index. In particular, $N^+ = \infty$. Also let α, β denote the first index i with positive and negative x_i, respectively, again with the value ∞ if there is no such index.

Consider the function $\Phi_{\mathbf{h},\mathbf{x}}$ with domain $[0, \infty)$ and value $x_0 = 0$ on the interval $[0, h_1]$, x_i on the interval $(h_i, h_{i+1}]$ for $i \in \{1, \ldots, N-1\}$ and x_N on (h_N, ∞); see Figure 3. Recall the definition of the graph occupation measure $m(\cdot)$ from the Introduction, and let

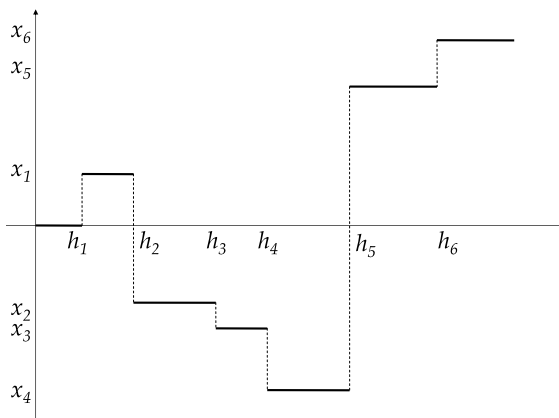$$\mu_{\mathbf{h},\mathbf{x}} := m(\Phi_{\mathbf{h},\mathbf{x}}).$$

FIG. 3. *The step function $\Phi_{\mathbf{h},\mathbf{x}}$.*

We will use the shorthand notation $I(\mathbf{h}, \mathbf{x})$ for the rate corresponding to this measure, namely

$$(3) \qquad I(\mathbf{h}, \mathbf{x}) := I(\mu_{\mathbf{h},\mathbf{x}}) = \frac{\pi^2}{2} \sum_{i \in \mathcal{I} \setminus \mathcal{I}_\infty} \frac{|x_i - x_{i^-}|}{h_i^2} + \frac{\pi^2}{8} \sum_{i \in \mathcal{I}_\infty} \frac{|x_i - x_{i^-}|}{h_i^2}.$$

3. Confining Brownian motion—the upper bound. The goal of this section is to prove the core of the large deviations upper bound of Section 5.

In what follows, m denotes the graph occupation measure, B a standard two-sided Brownian motion and x the process-of-wells mapping.

PROPOSITION 5 (Large deviation upper bound). *For each $\mu \in \mathcal{M}$ and $A < I(\mu)$, there exists an open neighborhood $\mathcal{U}$ of μ so that for all sufficiently large M, we have*

$$\mathbf{P}\big(m(x_B/M) \in \mathcal{U}\big) \leq e^{-AM}.$$

The proof of this proposition is given in Lemmas 6 and 7. We first define neighborhoods that will be easy to handle. Using the notation of Section 2, for $(\mathbf{h}, \mathbf{x}) \in \mathcal{S}$ and $\varepsilon > 0$, we define the following open set of measures:

$\mathcal{U}(\mathbf{h}, \mathbf{x}, \varepsilon)$

$$(4)$$
$$:= \left\{ \nu \in \mathcal{M} : \begin{array}{ll} \nu\big((h_i - \varepsilon, h_i + \varepsilon) \times (x_i, \infty)\big) > 0 & \text{for } i \in \mathcal{I}, x_i > 0 \\ \nu\big((h_i - \varepsilon, h_i + \varepsilon) \times (-\infty, x_i)\big) > 0 & \text{for } i \in \mathcal{I}, x_i < 0 \end{array} \right\}.$$

We claim that these neighborhoods cover everything efficiently, even for small ε.

LEMMA 6. *For each $\mu \in \mathcal{M}$ and $A < I(\mu)$, there exists $(\mathbf{h}, \mathbf{x}) \in \mathcal{S}$ so that $I(\mathbf{h}, \mathbf{x}) > A$ and $\mathcal{U}(\mathbf{h}, \mathbf{x}, \varepsilon) \ni \mu$ for all $\varepsilon > 0$.*

The proof of this topological lemma is standard, but a bit technical. We postpone it to Section 9.

LEMMA 7. *For $(\mathbf{h}, \mathbf{x}) \in \mathcal{S}$, $A < I(\mathbf{h}, \mathbf{x})$ and all small enough $\varepsilon > 0$, there is an integer M_ε so that*

$$(5) \qquad \mathbf{P}\big(m(x_B/M) \in \mathcal{U}(\mathbf{h}, \mathbf{x}, \varepsilon)\big) \leq e^{-AM} \qquad \text{for all } M \geq M_\varepsilon.$$

For a two-sided Brownian motion B, we define its reflection from its past minimum as the process

$$(6) \qquad R(t) := B(t) - \inf\{B(s) : s \text{ between } 0 \text{ and } t\}$$

for all $t \in \mathbb{R}$. This process appears naturally in the study of the wells created by B.

PROOF OF LEMMA 7.   For a locally bounded function $Q : \mathbb{R} \to \mathbb{R}$, define

$$\underline{Q}(t) := \inf\{Q(s) : s \text{ between } 0 \text{ and } t\}, \tag{7}$$

$$\overline{Q}(t) := \sup\{Q(s) : s \text{ between } 0 \text{ and } t\} \tag{8}$$

for all $t \in \mathbb{R}$.

Recall the mapping $\mu \mapsto (f_\mu, g_\mu)$ defined in the Introduction. For almost all Brownian paths B, the measure $\mu := m(x_B/M)$ satisfies

$$M f_\mu = f_{m(x_B)} = \overline{x_B}, \qquad M g_\mu = g_{m(x_B)} = \overline{(x_B)^-}.$$

Also, $\mu \in \mathcal{U}(\mathbf{h}, \mathbf{x}, \varepsilon)$ implies $f_\mu(h_i + \varepsilon) > x_i$ for $x_i > 0$, and similarly for $x_i < 0$. Then, for $i \in \mathcal{I}$ with $x_i > 0$, we have

$$\overline{x_B}(h_i + \varepsilon) > M x_i \quad \Rightarrow \quad \overline{R}(M x_i) < h_i + \varepsilon,$$

because otherwise an ascent on the right with height at least $h_i + \varepsilon$ is created before $M x_i$. This can be paired with an ascent on the negative axis of height at least $h_i + \varepsilon$, and the two will make $\overline{x_B}(h_i + \varepsilon)$ to be located in $(-\infty, M x_i]$, a contradiction. This and the symmetric argument for negative x_i shows that, on the event in the statement of the lemma, we have

$$\overline{R}(M x_i) < h_i + \varepsilon \qquad \text{for all } i \in \mathcal{I}.$$

For $M = 1$, a realization of the process R satisfying these restrictions is depicted in Figure 4.

There is one more piece of information we have for the path at the points $M x_i$ for all indices $i \in \mathcal{I} \setminus \mathcal{I}_\infty$ when this set is nonempty. That is,

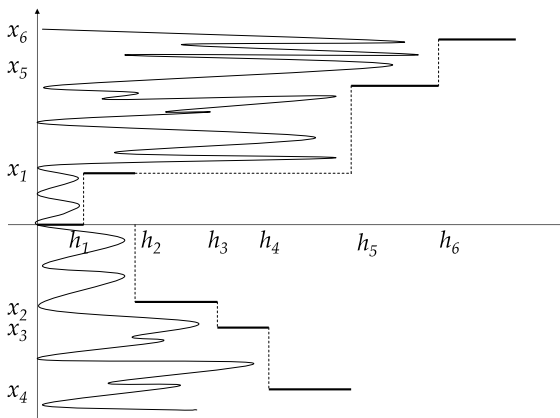$$\underline{B}(M x_i) \geq -h_N - \varepsilon.$$

FIG. 4.   *The restrictions on the reflected process.*

To see this, assume without loss of generality that $x_i > 0$. Since $\mu \in \mathcal{U}(\mathbf{h}, \mathbf{x}, \varepsilon)$, we have

$$\mu\big((h_i - \varepsilon, h_i + \varepsilon) \times (x_i, \infty)\big) > 0,$$

and thus $x_B(h') > Mx_i > 0$ for some $h' \in (h_i - \varepsilon, h_i + \varepsilon)$. Let

$$h^* := \inf\{h > h' : x_B(h) < 0\}.$$

We first argue that h^* is well defined, that is, the above set is not empty. Since $i \in \mathcal{I} \setminus \mathcal{I}_\infty$, there is $j > i$ with $x_j < 0$, and thus $x_B(h) < 0$ for some $h \in (h_j - \varepsilon, h_j + \varepsilon)$. Since $\varepsilon < \text{mesh}(\mathbf{h})/2$, we have $h_j - \varepsilon > h_i + \varepsilon$, and so indeed we have $h > h'$. Also, $h^* < h_N + \varepsilon$.

At h^*, x_B is positive because it is left continuous, but just after that it is negative. This means that the well of $x_B(h^*)$ has depth exactly h^*. But $B(x_B(h^*)) = \underline{B}(x_B(h^*))$, and combining this with $x_B(h^*) \geq x_B(h') > Mx_i$, we obtain

$$-h_N - \varepsilon < -h^* \leq \underline{B}(x_B(h^*)) \leq \underline{B}(Mx_i).$$

Thus the event of the lemma is contained on the event

$$C_M := \left\{ \begin{array}{ll} \overline{R}(Mx_i) < h_i + \varepsilon & \text{for } i \in \mathcal{I}, \\ \underline{B}(Mx_i) \geq -h_N - \varepsilon & \text{for } i \in \mathcal{I} \setminus \mathcal{I}_\infty \end{array} \right\}.$$

Recall from Section 2 that i^- refers to the index preceding i so that x_i and x_{i^-} have the same sign. Let $\mathbf{P}_{r,y}$ denote the law of the Markov process $(\overline{R}, \underline{B})$ started at the point (r, y). By the Markov property applied consecutively at Mx_{i^-} for $i \in \mathcal{I}$, we get

$$\mathbf{P}(C_M) \leq \prod_{i \in \mathcal{I} \setminus \mathcal{I}_\infty} \sup_{r \geq 0, y \leq 0} \mathbf{P}_{r,y}\big(\overline{R}(M(x_i - x_{i^-})) < h_i + \varepsilon,$$

$$\underline{B}(M(x_i - x_{i^-})) \geq -h_N - \varepsilon\big)$$

$$\times \prod_{i \in \mathcal{I}_\infty} \sup_{r \geq 0, y \leq 0} \mathbf{P}_{r,y}\big(\overline{R}(M(x_i - x_{i^-})) < h_i + \varepsilon\big).$$

As usual, the product over an empty index set is 1. Note that the process $(\overline{R}, -\underline{B})$ is nondecreasing in both coordinates of its starting point $(r, -y)$. Therefore we have the upper bound

$$\prod_{i \in \mathcal{I} \setminus \mathcal{I}_\infty} \mathbf{P}_{0,0}\big(\overline{R}(M(x_i - x_{i^-})) < h_i, \underline{B}(M(x_i - x_{i^-})) \geq -h_N - \varepsilon\big)$$

$$\times \prod_{i \in \mathcal{I}_\infty} \mathbf{P}_{0,0}\big(\overline{R}(M(x_i - x_{i^-})) < h_i + \varepsilon\big).$$

Then Lemma 20 implies that

$$\lim_{M \to \infty} \frac{\log \mathbf{P}(C_M)}{M} \leq -I(\mathbf{h}, \mathbf{x}) - o(\varepsilon),$$

where $o(\varepsilon)$ depends on $(\mathbf{h}, \mathbf{x})$ only. The claim follows. $\square$

4. Making a vessel—the lower bound. The goal of this section is to prove the large deviation lower bound. This can be formulated as follows:

PROPOSITION 8 (Large deviation lower bound). *For every open set $G \subset \mathcal{M}$, every $A > \inf_G I$ and for all sufficiently large M, we have*

$$\mathbf{P}\big(m(x_B/M) \in G\big) \geq e^{-AM}.$$

Again, we proceed in two steps. We will first define a convenient set of Brownian paths, $\mathcal{R}(\mathbf{h}, \mathbf{x}, \varepsilon, \delta)$, and then prove a topological lemma that reduces the problem to showing that these paths have high probability.

LEMMA 9. *For every open $G \subset \mathcal{M}$, and every $A > \inf_G I$, there exists $(\mathbf{h}, \mathbf{x}) \in \mathcal{S}$ so that $I(\mathbf{h}, \mathbf{x}) < A$ and*

$$\{m(x_B): B \in \mathcal{R}(\mathbf{h}, \mathbf{x}, \varepsilon, \varepsilon)\} \subset G$$

for all small enough $\varepsilon > 0$.

The proof of this topological lemma is postponed to Section 9. In light of this lemma, it suffices to give a lower bound on the probability that for B two-sided Brownian motion, $m(x_B)$ is close to $\mu_{\mathbf{h}, \mathbf{x}}$ in the large deviation regime. In fact, we will do this for a more restrictive set, namely for the event that x_B is close in the Skorokhod topology to $\Phi_{\mathbf{h}, \mathbf{x}}$.

Recall the Skorokhod topology on left continuous paths on $[0, \infty)$ with right limits. We call a set A of these paths an $[a, b]$-Skorokhod neighborhood of f if it is the inverse image of a Skorokhod neighborhood of $f|[a, b]$ under the restriction map.

PROPOSITION 10. *Let $(\mathbf{h}, \mathbf{x}) \in \mathcal{S}$. Then every $[0, 2h_N]$-Skorokhod neighborhood of $\Phi_{\mathbf{h}, \mathbf{x}}$ contains the image of the set $\mathcal{R}(\mathbf{h}, \mathbf{x}, \delta, \varepsilon)$ under the map x for all δ, ε small enough, and Brownian motion B satisfies*

$$(9) \qquad \lim_{M \to \infty} \frac{1}{M} \log \mathbf{P}\big(B(M\cdot) \in \mathcal{R}(\mathbf{h}, \mathbf{x}, \delta, \varepsilon)\big) = -I(\mathbf{h}, \mathbf{x}) + O_{\mathbf{h}, \mathbf{x}}(\delta, \varepsilon).$$

The rest of this section contains the proof of the proposition. First, we construct the desired set of paths, then we show that they can be arbitrarily close to $\Phi_{\mathbf{h}, \mathbf{x}}$, and finally we prove the desired probability decay.

Each path in $\mathcal{R}(\mathbf{h}, \mathbf{x}, \delta, \varepsilon)$ forms a "vessel," in which when water is poured from the y axis, the process of wells is close to $\Phi_{\mathbf{h}, \mathbf{x}}$ almost until depth $2h_N$ is reached.

Construction.   We define the following events, that is, sets of continuous functions $f : \mathbb{R} \to \mathbb{R}$, for $\varepsilon, \delta \in (0, 1)$, $h > 0$ and $x, y \in \mathbb{R}$ with $0 \le x < y$ or $y < x \le 0$. For all events, we require that, in the second endpoint of the interval mentioned, f takes a value in $[0, h - \varepsilon h]$; this is to make the building blocks fit together well. In addition, we require:

Confinement, $C(x, y, h)$: between times $x(1 + \delta)$, $y(1 - \delta)$, f stays in $[-\varepsilon^2 h, h]$.
Hole, $H(y, h)$: between times $y(1 - \delta)$, y, f stays in $[-\varepsilon h, h]$, visits below $-\varepsilon h + \varepsilon^2 h$.
Hole^R, $H^R(y, h)$: between times $y(1 - \delta)$, y, f stays in $[0, h]$, visits 0.
Barrier, $B(y, h)$: between times y, $y(1 + \delta)$, f stays in $[-\varepsilon^2 h, h + \varepsilon h]$, visits above h.

We omit the dependence on ε, δ from the notation.

Our basic restriction set, $\mathcal{R}(\mathbf{h}, \mathbf{x}, \delta, \varepsilon)$, is defined as the intersection of the following sets E_i, for $i \in \mathcal{I} \cup \{0\}$. Our goal is to ensure that functions f in these sets will have the property that x_f is $[0, 2h_N]$-Skorokhod-close to $\Phi_{\mathbf{h}, \mathbf{x}}$. We will comment on the importance of the individual sets E_i after their definition.

The beginning.

(10)
$$E_0 := C\big(0, x_\alpha \delta/(1 - \delta), h_\alpha\big) \cap B(x_\alpha \delta, h_\alpha)$$
$$\cap C\big(0, x_\beta \delta/(1 - \delta), h_\beta\big) \cap B(x_\beta \delta, h_\beta).$$

For $f \in E_0$, we have

$$x_f(h) \in \big(\delta(1 + \delta)x_\beta, \delta(1 + \delta)x_\alpha\big) \qquad \text{for } h \in [0, h_1],$$

because the two barrier sets create a well around zero of depth at least h_1.

The indices in $\mathcal{I} \setminus \mathcal{I}_\infty$.   For each index $i \in \mathcal{I} \setminus \mathcal{I}_\infty$, we define the set

$$E_i := C(w_i, x_i, h_i) \cap H(x_i, h_i) \cap B(x_i, h_{i+}),$$

where for all $i \in \mathcal{I}$, we let

$$w_i := \begin{cases} x_{i-}, & i \ne \alpha, \beta, \\ x_i \delta, & i = \alpha \text{ or } \beta. \end{cases}$$

The purpose of E_i is to guarantee that for $f \in E_0 \cap \cdots \cap E_i$, the value of $x_f(\cdot)$ will be near x_i for a time interval very close to $[h_i, h_{i+1}]$.

More precisely, assume that $x_i > 0$, and focus on the positive half of the path $f|[0, \infty)$. See Figure 5 for the case $i = 1$. What E_i adds to the intersection is that $f|[0, \infty)$ up to the point $x_i(1 - \delta)$ does not reach a new minimum or maximum. Then it creates a new minimum (hole) near x_i, and then a barrier of height h_{i+} ahead of it. But E_{i-} has already created a barrier of height about h_i. Moreover, on the negative side there is a barrier of height at least h_{i+1} following a minimum for
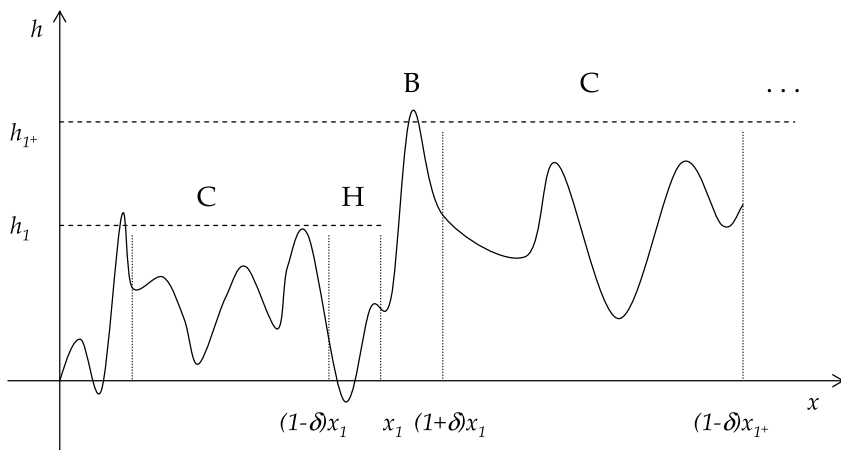
FIG. 5.   *The first positive blocks of the construction.*

$f|(-\infty, 0]$ which is not deeper than the hole in the event E_i. So indeed, the value of $x_f(\cdot)$ will be near x_i at least for a time interval almost equal to $[h_i, h_{i+1}]$.

Note also that the barrier created by E_i makes sure that from height about h_i until height about h_{i+}, the process x_f either stays constant or jumps to negative values; that is, it does not advance to another positive value.

The indices in $\mathcal{I}_\infty$.   By symmetry, we may assume that the x_i's for $i \in \mathcal{I}_\infty$ are positive. For a locally bounded function f defined in $\mathbb{R}$, and z fixed, let $R_z f : [z, \infty) \to [0, \infty)$ denote f reflected from its running minimum after z, namely

$$(11) \qquad R_z f(x) := f(x) - \inf_{s \in [z, x]} f(s).$$

Let $q = \min \mathcal{I}_\infty$. For $i \in \mathcal{I}_\infty$, define the set E_i of paths f so that $R_{w_q(1+\delta)} f$ is in

$$C(w_i, x_i, h_i) \cap H^R(x_i, h_i) \cap B(x_i, h_{i+}),$$

and f satisfies

$$(12) \qquad f(x) - f\big(w_i(1 + \delta)\big) \leq \varepsilon^2 \qquad \text{for } x \in [w_i(1 + \delta), x_i(1 + \delta)].$$

Note that $i^+ = i + 1$ unless $i = N$.

In order to understand these events E_i, we first consider the effect of the preceding events $E_0 \cap \cdots \cap E_{q-1}$. See Figure 6.

Assume first that $\mathcal{I}_\infty \neq \mathcal{I}$. In this case, $E_0 \cap \cdots \cap E_{q-1}$ puts restriction on the path on the interval $[x_{q-1}(1 + \delta), w_q(1 + \delta)]$. The minimum value of f there is negative of order ε, the maximum is attained in the interval $[x_{q-1}(1 + \delta), x_{q-1}]$, where the path goes over $2h_N$ because $(q - 1)^+ = \infty$ and $h_\infty = 2h_N$.
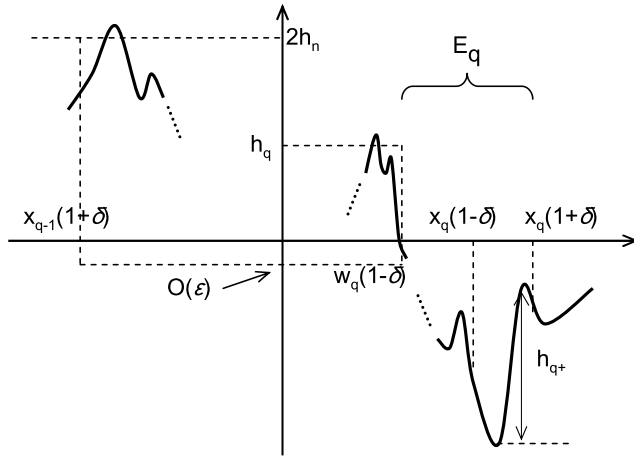
FIG. 6.   *The last part.*

When $\mathcal{I}_\infty = \mathcal{I}$, the event E_0 puts restriction on the path on the interval $[-x_1\delta(1 + \delta), w_q(1 + \delta)]$. The minimum value of f there is negative of order ε, the maximum is attained on $[-x_1\delta(1 + \delta), -x_1\delta]$, where the path goes over $2h_N$.

In both cases, the maximum on $[0, w_q(1 + \delta)]$ is attained in the interval $[w_q, w_q(1 + \delta)]$, where the path goes a bit over h_q and ends up in $[0, h_q - \varepsilon h_q]$.

Then the set E_q requires from f up to the point $x_q(1 - \delta)$ not to create an ascent of height larger than h_q (see Figure 6) and then to create one with lowest point having x-coordinate in $(x_q(1 - \delta), x_q(1 + \delta))$ and height around h_{q+}. The goal of (12) is to force f not to go above h_q, and this is obtained because as we noted $f(w_q(1 + \delta)) \le h_q - \varepsilon h_q$, and from that point on f stays below $h_q - \varepsilon h_q + \varepsilon^2$. These, together with the barrier of height $2h_N$ on the negative axis, guarantee that $x_f(h)$ is around x_q at least for h in an interval very close to $[h_q, h_{q+}]$.

The other E_i's with $i \in \mathcal{I}_\infty$ work in the same way.

The behavior of x_f for $f \in \mathcal{R}(\mathbf{h}, \mathbf{x}, \delta, \varepsilon)$.   We will now examine more precisely how x_f behaves when $f \in \mathcal{R}(\mathbf{h}, \mathbf{x}, \delta, \varepsilon)$. For $x, y \in \mathbb{R}$, define

$$f^\#(x, y) = \begin{cases} \sup\{f(t) - f(s) : x \le s \le t \le y\}, & x \le y, \\ \sup\{f(t) - f(s) : y \le t \le s \le x\}, & x \ge y. \end{cases}$$

Also let

$$\tilde{h}_1 := \max\{f(x) : x \text{ between } 0 \text{ and } x_1\delta(1 + \delta)\},$$

and let z_1 be the closest to zero point between 0 and $x_{\alpha \vee \beta}\delta(1 + \delta)$ where f takes the value $\tilde{h}_1$. In the interval between z_1 and $x_1\delta(1 + \delta)$, we have a well of depth around h_1. Its exact depth is

$$v_1 := \tilde{h}_1 - \min\{f(x) : x \text{ between } z_1 \text{ and } x_1\delta(1 + \delta)\}.$$

Also define

$$v_i := \begin{cases} f^{\#}(x_{i-1}, x_i\delta(1+\delta)), & \text{if } i = \alpha \vee \beta \neq \infty, \\ f^{\#}(x_{i-1}(1-\delta), x_{i-1}(1+\delta)), & \text{if } i - 1 = i^- \in \mathcal{I} \setminus \mathcal{I}_\infty, \\ f^{\#}(x_{i-1}, x_{i^-}(1+\delta)), & \text{if } i - 1 \neq i^- \in \mathcal{I} \setminus \mathcal{I}_\infty. \end{cases}$$

Now for $i^- \in \mathcal{I}_\infty$ let

$$v_i := \sup\{R_{w_q(1+\delta)} f(x) : x \in [x_{i^-}, x_{i^-}(1+\delta)]\},$$

and finally let $v_{N+1} = 2h_N$. From the above discussion and the definition of $\mathcal{R}(\mathbf{h}, \mathbf{x}, \delta, \varepsilon)$, we conclude that

$$(13) \quad v_i \in \begin{cases} [h_1, h_1 + \varepsilon(h_1 + \varepsilon h_{\alpha \vee \beta})], & \text{if } i = 1, \\ [h_i + (\varepsilon - \varepsilon^2)h_{i-1}, h_i + \varepsilon(h_i + h_{i-1})], & \text{if } i = \alpha \vee \beta \neq \infty \\ & \quad \text{or } i^- \in \mathcal{I} \setminus \mathcal{I}_\infty, \\ (h_i, h_i + \varepsilon h_i], & \text{if } i^- \in \mathcal{I}_\infty, \end{cases}$$

and

$$(14) \quad \begin{aligned} |x_f(h)| &\leq \delta(\delta + 1)(x_\alpha \vee |x_\beta|) \qquad \text{for } h \in [0, v_1], \\ x_f(h) &\text{ is between } x_i(1-\delta), x_i \qquad \text{for } h \in (v_i, v_{i+1}], i \in \mathcal{I}. \end{aligned}$$

We assumed that ε is small enough so that $v_1 < v_2 < \cdots < v_N$, and $-\varepsilon h_i + \varepsilon^2 h_i < -\varepsilon h_{i-1}$ for $i \in \mathcal{I} \setminus \mathcal{I}_\infty$. Informally, the second requirement guarantees that, for these i's, the set $H(x_i, h_i)$ creates a new, deeper minimum, and this is used for (14).

Relations (13), (14) show that any $[0, 2h_N]$-Skorokhod neighborhood of $\Phi_{\mathbf{h}, \mathbf{x}}$ contains $\{x_f : f \in \mathcal{R}(\mathbf{h}, \mathbf{x}, \delta, \varepsilon)\}$ if ε, δ are small enough.

The asymptotic probability of $\mathcal{R}(\mathbf{h}, \mathbf{x}, \delta, \varepsilon)$. It remains to prove (9).

We apply the Markov property and use Lemma 21. Note that for any two restriction sets concerning contiguous intervals, say $[x, y], [y, z]$, with $0 < x < y < z$, the allowed values for $f(y-)$ are the same as the ones on which we condition in the first three relations of Lemma 21. If the second set corresponds to an index in $\mathcal{I}_\infty$, the ending point of the first block is irrelevant.

It is also important that the limits computed in that lemma are uniform over the starting points of the processes involved. These observations allow us to conclude that the left-hand side of (9) equals

$$-\frac{\pi^2}{2} \sum_{i=\alpha,\beta} \frac{|x_i|\delta}{h_i^2}\left(\frac{1}{(1+\varepsilon^2)^2} + \frac{\delta}{(1+\varepsilon+\varepsilon^2)^2}\right)$$

$$-\frac{\pi^2}{2} \sum_{i \in \mathcal{I} \setminus \mathcal{I}_\infty}\left(\frac{|x_i - w_i - \delta(x_i + w_i)|}{h_i^2(1+\varepsilon^2)^2} + \frac{\delta|x_i|}{h_i^2(1+\varepsilon)^2} + \frac{\delta|x_i|}{h_{i^+}^2(1+\varepsilon+\varepsilon^2)^2}\right)$$

$$-\frac{\pi^2}{8} \sum_{i \in \mathcal{I}_\infty}\left(\frac{|x_i - w_i - \delta w_i|}{h_i^2} + \frac{\delta|x_i|}{h_{i^+}^2(1+\varepsilon)^2}\right),$$

which is $-I(\mathbf{h}, \mathbf{x}) + O_{\mathbf{h},\mathbf{x}}(\delta, \varepsilon)$. Note that for $\delta \searrow 0$, only the confinement sets appearing in E_i, for $i \in \mathcal{I}$, contribute to the rate of decay. The reason is that all other sets put restrictions on intervals of size proportional to δ. Similarly, the first restriction set E_0 does not contribute.

5. The large deviation principle for the process of wells.   In this section, we complete the proof of the large deviation principle for the family $\{m(x_B/M) : M > 0\}$ stated in Theorem 2.

Recall that a family $\{\mu_M : M > 0\}$ of Borel measures on a topological space $\mathcal{M}$ satisfies the *large deviation principle* with rate $I : \mathcal{M} \to [0, \infty]$ as $M \to \infty$ if for every measurable set $\mathcal{B} \subset \mathcal{M}$ and every $A > \inf_{\mathcal{B}^\circ} I$ and $A' < \inf_{\bar{\mathcal{B}}} I$ we have

$$(15) \qquad -A \leq \frac{\log \mu_M(\mathcal{B})}{M} \leq -A'$$

for all sufficiently large M, where $\mathcal{B}^\circ, \bar{\mathcal{B}}$ denote the interior and the closure of $\mathcal{B}$, respectively.

Recall also that I is a *good rate function* if $I^{-1}[0, A]$ is compact for all finite A. In particular, these sets are closed, which is equivalent to I being lower semicontinuous.

We have established in Propositions 5, 8 the core upper and lower bounds. Next, we prove exponential tightness. Recall that a family of measures $\{\mu_M : M > 0\}$ as above is *exponentially tight* as $M \to \infty$ if for every $A > 0$ there is a compact set $\mathcal{Q} \subset \mathcal{M}$ so that $\mu_M(\mathcal{Q}^c) < e^{-AM}$ for all large enough M.

LEMMA 11.   *The family $\{m(x_B/M) : M > 0\}$ is exponentially tight.*

PROOF.   By the definition of $\mathcal{M}$, for every $a > 0$, the set

$$(16) \qquad \mathcal{Q}_a := \left\{ \mu \in \mathcal{M} : \operatorname{supp}(\mu) \subset \bigcup_{k=1}^{\infty} [k-1, k] \times [-ak^3, ak^3] \right\}$$

is compact in $\mathcal{M}$. Recall definitions (6) and (8). We have

$$(17) \qquad \begin{aligned} \mathbf{P}\big(m(x_B/M) \in \mathcal{Q}_a^c\big) &\leq \sum_{k=1}^{\infty} \mathbf{P}\big(\max\{\overline{x_B}(k), \overline{(x_B)^-}(k)\} \geq aMk^3\big) \\ &\leq 2 \sum_{k=1}^{\infty} \mathbf{P}\big(\overline{x_B}(1) \geq aMk\big) \end{aligned}$$

using the scaling and symmetry properties of x_B. But for all $x > 0$, we have that $\overline{x_B}(1) \geq x$ implies $\bar{R}(x) < 1$, which has probability at most

$$\mathbf{P}\big(B[0, x] \subset (-1, 1)\big) \leq C e^{-x\pi^2/8}$$

with C a constant. Here we used the fact that R has the same law as $|B|$ and relation (36). Consequently,

$$(18) \qquad \mathbf{P}\big(m(x_B/M) \in \mathcal{Q}_a^c\big) \le C' e^{-Ma\pi^2/8}$$

for a constant C'. Since a was arbitrary, exponential tightness follows. □

PROOF OF THEOREM 2, THE LARGE DEVIATION PRINCIPLE. The first inequality in (15) is a reformulation of Proposition 8 applied to the open set $\mathcal{B}^\circ$. For the second, let $A < \inf_{\bar{\mathcal{B}}} I$. By exponential tightness (Lemma 11) there exists a compact set $\mathcal{Q}$ so that

$$P\big(m(x_B/M) \in \mathcal{Q}^c\big) \le e^{-AM}$$

for all sufficiently large M. Each point in $\mathcal{Q} \cap \bar{\mathcal{B}}$ can be covered with an open set satisfying the same asymptotic bound by Proposition 5. To get the second inequality of (15), take a finite subcover of the compact set $\mathcal{Q} \cap \bar{\mathcal{B}}$, and use the union bound.

Finally, we show that I is a good rate function. Take $A > 0$ and $\mu \in I^{-1}(A, \infty]$. By Proposition 5, μ has an open neighborhood $\mathcal{U}$ so that

$$-A > \limsup_{M \to \infty} \frac{\log P(m(x_B/M) \in \mathcal{U})}{M}.$$

The right-hand side is bounded below by $-\inf_{\mathcal{U}} I$ because of the large deviation lower bound. Thus $\mathcal{U} \subset I^{-1}(A, \infty]$, which shows that the latter set must be open. Thus $I^{-1}[0, A]$ is closed. On the other hand, exponential tightness and the large deviation lower bound gives $\inf_{\mathcal{Q}^c} I > A$ for some compact $\mathcal{Q}$, so $I^{-1}[0, A] \subset \mathcal{Q}$ must also be compact. □

6. The limit points of the environment. For $a > 1$, and $B: \mathbb{R} \to \mathbb{R}$ a continuous path, we define the function $Z_a: [0, \infty) \to \mathbb{R}$ by

$$(19) \qquad Z_a(s) := \frac{x_B(sa)}{a^2 \log \log a}$$

for all $s \ge 0$. We will determine the limit points of the family of measures $(m(Z_a))_{a>e}$, as $a \to \infty$, with respect to the topology of local weak convergence, when B is a two-sided Brownian path. Then Theorem 1 will follow from localization results connecting x_B with the Sinai walk. We start by doing this along geometric sequences.

Geometric sequences. In the next proposition, we show that all limit points of $(m(Z_a))_{a>e}$ along geometric sequences fall into a certain set $\mathcal{K}$. Then Proposition 13 shows that in fact along any geometric sequence, all points of $\mathcal{K}$ are limit points.

PROPOSITION 12. *If $c > 1$, then with probability one, every subsequence of $\{m(Z_{c^n}) : n > 1/\log c\}$ has a further convergent subsequence, and the limit points of the original sequence are contained in the set*

$$\mathcal{K} := \{\mu \in \mathcal{M} : I(\mu) \le 1\}.$$

PROOF. By the scaling property of x_B and (18),

$$\mathbf{P}\big(m(Z_{c^n}) \in \mathcal{Q}_a^c\big) = \mathbf{P}\big(m(x_B/\log\log c^n) \in \mathcal{Q}_a^c\big) \le C' \exp\Big(-\frac{a\pi^2}{8}\log\log c^n\Big).$$

For $a > 8/\pi^2$, the first Borel–Cantelli lemma implies that $m(Z_{c^n}) \in \mathcal{Q}_a$ eventually, and the first claim follows by the compactness of $\mathcal{Q}_a$.

For each point in $\mathcal{K}^c$, Proposition 5 provides an open set $\mathcal{U}$ containing it so that for some $A(\mathcal{U}) > 1$ and for all large enough n, we have

(20) $$\mathbf{P}\big(m(Z_{c^n}) \in \mathcal{U}\big) \le \exp\big(-A(\mathcal{U})\log\log c^n\big).$$

Now each such set $\mathcal{U}$ can be written as a union of elements of a fixed countable base. Thus K^c can be covered with a countable collection of open sets $\mathcal{U}_k$ satisfying (20). By the first Borel–Cantelli Lemma and the union bound, no $\mathcal{U}_k$ contains a limit point a.s. $\square$

The promised complement of Proposition 12 is as follows.

PROPOSITION 13. *If $c > 1$, then with probability one, the limit points of $\{m(Z_{c^n}) : n > 1/\log c\}$ include the points of the set*

$$\mathcal{K} := \{\mu \in \mathcal{M} : I(\mu) \le 1\}.$$

PROOF. Note that it suffices to prove that every open set $\mathcal{U}$ intersecting $\mathcal{K}$ contains a limit point with probability one. Using this claim for all such elements $\mathcal{U}$ of a countable base for $\mathcal{M}$, we conclude that the limit points are a.s. dense in $\mathcal{K}$. Since they form a closed set, this set must contain $\mathcal{K}$.

By Lemma 26 the minimum of I on an open set is either 0, ∞ or is not achieved. Therefore every open set $\mathcal{U}$ intersecting $\mathcal{K}$ has $\inf_{\mathcal{U}} I < 1$. By Lemma 9, there exists $(\mathbf{h}, \mathbf{x}) \in \mathcal{S}$ so that $I(\mathbf{h}, \mathbf{x}) < 1$ and

$$\{m(x_B) : B \in \mathcal{R}(\mathbf{h}, \mathbf{x}, \varepsilon, \varepsilon)\} \subset \mathcal{U}$$

for all small enough $\varepsilon > 0$.

Define $n_0 = \lfloor 1/\log c \rfloor + 2$, and for $n \ge n_0$, let A_n be the set of paths B so that the rescaling satisfies

$$\frac{B(c^{2n}\log\log c^n \times \cdot)}{c^n} \in \mathcal{R}(\mathbf{h}, \mathbf{x}, \varepsilon, \varepsilon).$$

Note that if a path B belongs to A_n, then the corresponding path Z_{c^n} from (19) satisfies $m(Z_{c^n}) \in \mathcal{U}$. Thus it suffices to show that A_n i.o. a.s.

Since

$$(21) \qquad\qquad A_n = \mathcal{R}(\mathbf{h}c^n, \mathbf{x}c^{2n} \log\log c^n, \varepsilon, \varepsilon),$$

the scaling property of Brownian motion implies

$$\mathbf{P}(A_n) = \mathbf{P}(\mathcal{R}(\mathbf{h}, \mathbf{x} \log\log c^n, \varepsilon, \varepsilon)).$$

Then Proposition 10 gives

$$\liminf_{n\to\infty} \frac{\log \mathbf{P}(A_n)}{\log\log c^n} \geq -I(\mathbf{h}, \mathbf{x}) - O_{\mathbf{h},\mathbf{x}}(\varepsilon) > -(1 - \delta_1)$$

for some $\delta_1 \in (0, 1 - I(\mathbf{h}, \mathbf{x}))$ and all small enough ε. Consequently, there is an n_1 so that for $n \geq n_1$ we have $\mathbf{P}(A_n) \geq n^{-1+\delta_1}$. Then for all n,

$$(22) \qquad\qquad \sum_{k=n_0}^{n} \mathbf{P}(A_k) > Cn^{\delta_1}$$

for an appropriate constant $C > 0$. In particular, $\sum \mathbf{P}(A_k) = \infty$. In order to conclude that A_n i.o., we use a correlation bound given in the upcoming Lemma 14. Let $\Sigma_n := 1_{A_{n_0}} + \cdots + 1_{A_n}$. We write

$$\mathbf{E}(\Sigma_n^2) = \mathbf{E}\Sigma_n + 2 \sum_{\substack{k=n_0 \\ l>k}}^{n} \mathbf{P}(A_k \cap A_l).$$

Let Δ, C_0 be as in Lemma 14, and $d_k := \Delta + (\log\log k)/(2\log c)$. We bound the probabilities $\mathbf{P}(A_k \cap A_l)$, $n_0 \leq k < l \leq n$, in one of two ways, according whether $|k - \ell| \leq d_k$, thus getting for their sum the upper bound

$$C_0 \sum_{\substack{k=n_0 \\ l-k>d_k}}^{n} \mathbf{P}(A_k)\mathbf{P}(A_l) + d_n \sum_{k=n_0}^{n} \mathbf{P}(A_k) \leq \left(\frac{C_0}{2} + \frac{d_n}{\mathbf{E}(\Sigma_n)}\right)(\mathbf{E}\Sigma_n)^2.$$

But $d_n/\mathbf{E}(\Sigma_n) \to 0$ by (22), so that the Kochen–Stone lemma [Durrett (2010), Exercise 2.3.20] gives

$$(23) \qquad\qquad \mathbf{P}(A_n \text{ i.o.}) \geq \limsup_{n\to\infty} \frac{(\mathbf{E}\Sigma_n)^2}{\mathbf{E}(\Sigma_n^2)} \geq 1/C_0.$$

To prove that $\{A_n \text{ i.o.}\}$ holds a.s., we will prove that it is a tail event, that is, that it belongs to the σ-algebra $\bigcap_{t>0} \sigma(B(s): |s| \geq t)$, and we will apply Theorem 8.2.7 from Durrett (2010).

To see this, fix $t_0 > 0$. A function $f: \mathbb{R} \to \mathbb{R}$ belongs to A_n if its values on a certain interval around zero satisfy certain conditions imposed by the sets whose

intersection defines A_n. For n that satisfies $t_0 < \varepsilon(x_\alpha \wedge |x_\beta|)c^{2n} \log \log c^n$, we isolate the conditions concerning the values of f on $[-t_0, t_0]$ and write

$$
\begin{aligned}
A_n &= \mathcal{R}(\mathbf{h}c^n, \mathbf{x}c^{2n} \log \log c^n, \varepsilon, \varepsilon) \\
&= C\big(0, c^{2n}(\log \log c^n)x_\alpha \varepsilon/(1-\varepsilon), c^n h_\alpha\big) \\
&\quad \cap C\big(0, c^{2n}(\log \log c^n)x_\beta \varepsilon/(1-\varepsilon), c^n h_\beta\big) \cap C_n \\
&= C\big(0, t_0/(1-\varepsilon), c^n h_\alpha\big) \cap C\big(t_0/(1+\varepsilon), c^{2n}(\log \log c^n)x_\alpha \varepsilon/(1-\varepsilon), c^n h_\alpha\big) \\
&\quad \cap C\big(0, -t_0/(1-\varepsilon), c^n h_\beta\big) \\
&\quad \cap C\big(-t_0/(1+\varepsilon), c^{2n}(\log \log c^n)x_\beta \varepsilon/(1-\varepsilon), c^n h_\beta\big) \cap C_n \\
&= C\big(0, t_0/(1-\varepsilon), c^n h_\alpha\big) \cap C\big(0, -t_0/(1-\varepsilon), c^n h_\beta\big) \cap A'_n,
\end{aligned}
$$

where C_n is the intersection of the remaining sets involved in the definition of A_n, and

$$
\begin{aligned}
A'_n &:= C\big(t_0/(1+\varepsilon), c^{2n}(\log \log c^n)x_\alpha \varepsilon/(1-\varepsilon), c^n h_\alpha\big) \\
&\quad \cap C\big(-t_0/(1+\varepsilon), c^{2n}(\log \log c^n)x_\beta \varepsilon/(1-\varepsilon), c^n h_\beta\big) \cap C_n.
\end{aligned}
$$

Now $\{A_n \text{ i.o.}\} \subset \{A'_n \text{ i.o.}\}$, but also $\{A'_n \text{ i.o.}\} \subset \{A_n \text{ i.o.}\}$ since every function f in the first set belongs to $C(0, t_0/(1-\varepsilon), c^n h_\alpha) \cap C(0, -t_0/(1-\varepsilon), c^n h_\beta)$ provided that $\max\{|f(s)| : |s| \leq t_0\} < \varepsilon^2(h_\alpha \vee h_\beta)c^n$. And the last inequality holds for all large n because $c > 1$, and f is bounded on $[-t_0, t_0]$, being continuous. Since for all large n we have $A'_n \in \sigma(B(s) : |s| \geq t_0)$, it follows that $\{A_n \text{ i.o.}\} \in \sigma(B(s) : |s| \geq t_0)$, and this proves our assertion. □

The next lemma shows a version of near independence for the family of sets $\{A_n : n \geq 1\}$, defined in the proof of Proposition 13, and uses the notation set up in that proof.

LEMMA 14.   *There are $\Delta, C_0 \in (0, \infty)$ depending on $\mathbf{h}, \mathbf{x}, \varepsilon$ such that*

$$
\mathbf{P}(A_k \cap A_l) \leq C_0 \mathbf{P}(A_k)\mathbf{P}(A_l)
$$

for $k \geq n_0$ and

(24)
$$
l - k > \Delta + \frac{1}{2}\frac{\log \log k}{\log c}.
$$

PROOF.   For the pair $(\mathbf{h}, \mathbf{x}) \in \mathcal{S}$, we will use the notation of Section 2.

We assume that $x_N > 0$. Let p equal $\max(\mathcal{I} \setminus \mathcal{I}_\infty)$ if the set is nonempty, and ∞ otherwise. For any integer $n \geq n_0$, define

$$
J_n := c^{2n} \log \log c^n [x_p(1+\varepsilon), x_N(1+\varepsilon)].
$$

Recall that $x_\infty = -x_1$. J_n is the interval where A_n imposes restrictions on B.

A_l is the intersection of several requirements the first of which [the two confinement sets of E_0 in (10)] refers to the time interval

$$F_l := c^{2l} \log \log c^l [\varepsilon x_\beta, \varepsilon x_\alpha].$$

We would like to have l so large that J_k will be in the interior of F_l, so that knowing that A_k happened does not influence much the probability of A_l. We ensure that $J_k \subset F_l/2$ by assuming that

$$(25) \qquad l > k + \frac{1}{2 \log c} \left\{ \log\left(\frac{|x_p|}{|x_\beta|} \vee \frac{x_N}{x_\alpha} \right) + \log \frac{2(1+\varepsilon)}{\varepsilon} \right\}$$

for the rest of the proof. Note that (25) is implied by (24) with an appropriate choice of Δ. We let

$$A^+ = \{ B|[0, \infty) : B \in A \}, \qquad A^- = \{ B|(-\infty, 0] : B \in A \}.$$

It is enough to prove the claim of the lemma for the pairs $\{A_k^+, A_l^+\}$, $\{A_k^-, A_l^-\}$ as they are independent. We will do it for the first. Let

$$\Theta_l := h_\alpha c^l [-\varepsilon^2, 1] \supset h_\alpha c^l \left[-\frac{\varepsilon^2}{2}, 1 - \frac{\varepsilon^2}{2} \right] =: \tilde{\Theta}_l$$

and

$$A_{k,l}^+ := A_k^+ \cap \{ B(s) \in \Theta_l \text{ for } 0 < s \in J_k \}.$$

Paths B in A_l satisfy $B(F_l \cap [0, \infty)) \subset \Theta_l$. So that $A_k^+ \cap A_l^+ = A_{k,l}^+ \cap A_l^+$ since $J_k \subset F_l$. Let $A_l^+(J_k)$ denote the paths that satisfy the restrictions put by A_l^+ for the time interval J_k, and define $A_l^+(J_l \setminus J_k)$ analogously. Denote by j_k the right endpoint of J_k, and let

$$q(x) := \mathbf{P}\big(A_l^+(J_l \setminus J_k) | B_{j_k} = x \big).$$

We have

$$A_k^+ \cap A_l^+ = A_{k,l}^+ \cap A_l^+ \subset A_{k,l}^+ \cap A_l^+(J_l \setminus J_k),$$

and the probability of the right-hand side can be written as

$$\mathbf{E}[\mathbf{1}_{A_{k,l}^+} q(B_{j_k})] \leq \mathbf{P}(A_k^+) \max_{\Theta_l} q.$$

On the other hand,

$$\mathbf{P}(A_l^+) = \mathbf{P}\big(A_l^+(J_l \setminus J_k) \cap A_l^+(J_k) \big) = \mathbf{E}\big[\mathbf{1}_{A_l^+(J_k)} q(B_{j_k}) \big]$$

$$\geq \mathbf{P}\big(A_l^+(J_k) \text{ and } B_{j_k} \in \tilde{\Theta}_l \big) \min_{\tilde{\Theta}_l} q.$$

So that

$$(26) \qquad \frac{\mathbf{P}(A_k^+ \cap A_l^+)}{\mathbf{P}(A_k^+)\mathbf{P}(A_l^+)} \leq \frac{\max_{\Theta_l} q}{\min_{\tilde{\Theta}_l} q} \mathbf{P}\big(A_l^+(J_k) \text{ and } B_{j_k} \in \tilde{\Theta}_l \big)^{-1}.$$

To bound the last term, note that $B|[0, \infty) \in A_l^+(J_k)$ follows from $B([0, j_k]) \subset \tilde{\Theta}_\ell$. The restriction (24) on $l - k$ shows that

$$c^{2k-2l} \log \log c^k < c^{-2\Delta} \left(1 + \frac{\log \log c}{\log 2} \right) =: c_1.$$

This and Brownian scaling yield the lower bound

$$\mathbf{P}\big(A_l^+(J_k) \text{ and } B_{j_k} \in \tilde{\Theta}_l\big) \geq \mathbf{P}\left(B([0, c_1(1 + \varepsilon)x_N]) \subset h_\alpha \left[-\frac{\varepsilon^2}{2}, 1 - \frac{\varepsilon^2}{2} \right] \right),$$

which is positive and does not depend on k, l.

To bound the fraction in the right-hand side of (26), note that with f_l the right endpoint of F_l, the event $A_l^+(F_l \setminus J_k)$ is equivalent to $B([j_k, f_l]) \subset \Theta_\ell$ and $B_{f_l} \in [0, (1 - \varepsilon)h_\alpha c^l]$, by the definition of the confinement set $C(0, x_\alpha \varepsilon/(1 - \varepsilon), h_\alpha)$. Let $r(x, y)$ denote the density of $B(f_l)$ for Brownian motion started from x at time j_k restricted to this event. By the Markov property, we have

$$q(x) = \int r(x, y) \mathbf{P}(A_l^+(J_l \setminus F_l) | B(f_l) = y) \, dy,$$

which gives the bound

$$\frac{\max_{\Theta_l} q}{\min_{\tilde{\Theta}_l} q} \leq \max \left\{ \frac{r(x_1, y)}{r(x_2, y)} : x_1 \in \Theta_l, x_2 \in \tilde{\Theta}_l, y \in [0, (1 - \varepsilon)h_\alpha c^l] \right\}.$$

With the notation introduced in the beginning of Section 8, we have

$$r(x, y) = Q^{(1+\varepsilon^2)h_\alpha c^l}(f_l - j_k, x + \varepsilon^2 h_\alpha c^l, y + \varepsilon^2 h_\alpha c^l),$$

and (36), (37) give that the above maximum is bounded above by a constant (that depends only on ε) as long as

$$\frac{f_l - j_k}{(1 + \varepsilon^2)^2 h_\alpha^2 c^{2l}} \geq t_0(\varepsilon).$$

This holds for k, l satisfying (24) provided that Δ is large enough.   □

From geometric sequences to the full family. We will now show that Propositions 12, 13 imply the result for the full family. As noted in Vervaat (1990), this can be done easily using the scaling properties of the rate function I and the regular variation of the scaling factor $a^2 \log \log a$ in (19).

PROPOSITION 15. *With probability one, every sequence $(m(Z_{t_k}))_{k \geq 1}$ with $t_k \to \infty$ has a convergent subsequence, and the set of all possible limit points is exactly*

$$\mathcal{K} := \{ \mu \in \mathcal{M} : I(\mu) \leq 1 \}.$$

PROOF. For a measure $\mu \in \mathcal{M}$ and $a > 0$, let $\mu_a \in \mathcal{M}$ denote the rescaled version of μ defined on every product of measurable sets $H \times X \subset [0, \infty) \times \mathbb{R}$ as

$$\mu_a(H \times X) = a\mu(a^{-1}H \times a^{-2}X).$$

Then for all continuous functions $\psi : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ with compact support,

$$(27) \qquad \int \psi(h, x) \, d\mu_a(h, x) = a \int \psi(ah, a^2 x) \, d\mu(h, x).$$

Also, $f_{\mu_a}(t) = a^2 f_\mu(t/a)$ for all $t \geq 0$, and the analogous statement holds for g_{μ_a}. Consequently, $I(\mu_a) = I(\mu)$.

For $t > 1$, let $\mu^{(t)} = m(Z_t)$. We will use the following claim.

CLAIM. *If $\mu \in \mathcal{M}$, and for two sequences $(t_k)_{k \geq 1}, (p_k)_{k \geq 1}$ with $\lim_{k \to \infty} t_k = \infty$, $\lim_{k \to \infty} t_k/p_k = a \in (0, \infty)$ we have $\mu^{(p_k)} \to \mu$, then $\mu^{(t_k)} \to \mu_{1/a}$.*

PROOF. Take $\psi : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ continuous of compact support. For $k \geq 1$, we may assume that $p_k, t_k \geq 1$, and let

$$a_k := \frac{t_k}{p_k}, \qquad \beta_k := \frac{\log \log p_k}{\log \log t_k}.$$

Then $\int \psi(h, x) \, d\mu^{(t_k)}(h, x)$ equals

$$\int_0^\infty \psi\left(t, \frac{x_B(a_k p_k t)}{a_k^2 p_k^2 \log \log t_k}\right) dt = a_k^{-1} \int_0^\infty \psi\left(a_k^{-1}s, a_k^{-2}\beta_k \frac{x_B(p_k s)}{p_k^2 \log \log p_k}\right) ds$$

$$= a_k^{-1} \int \psi(a_k^{-1}h, a_k^{-2}\beta_k x) \, d\mu^{(p_k)}(h, x)$$

$$= a a_k^{-1} \int \psi(a a_k^{-1}h, a^2 a_k^{-2}\beta_k x) \, d\mu_{1/a}^{(p_k)}(h, x).$$

The last equality follows form (27). Since $a a_k^{-1} \to 1$ and $\mu_{1/a}^{(p_k)} \to \mu_{1/a}$, it suffices to prove that

$$\int \psi(a a_k^{-1}h, a^2 a_k^{-2}\beta_k x) \, d\mu_{1/a}^{(p_k)}(h, x) - \int \psi(h, x) \, d\mu_{1/a}^{(p_k)}(h, x) \to 0$$

as $k \to \infty$.

Let C be the set of points in $[0, \infty) \times \mathbb{R}$ with Euclidean distance at most 1 from the support of ψ, and h^* the maximum of the first projection of C. For $\varepsilon > 0$, using the compactness of C, the uniform continuity of ψ, and the fact that $\mu_{1/a}^{(p_k)} \in \mathcal{M}$, we obtain that for large k, the absolute value of the last difference is bounded from above by

$$\int_C \varepsilon d\mu_{1/a}^{(p_k)}(h, x) \leq \varepsilon h^*.$$

This proves the claim. □

Now let $t_k \to \infty$ be a sequence. We fix a $c > 1$, and write this sequence uniquely as $t_k = a_k c^{i_k}$ with integers i_k and real numbers $a_k \in [1, c)$.

Regarding the first assertion of the proposition, note that by Proposition 12, $m(Z_{c^{i_k}})$ has limit points in $\mathcal{K}$. Pick one, say μ, and then passing to a further subsequence along which a_k converges to some limit $a(c) \in [1, c]$, we see using the claim above that $\mu_{1/a(c)}$ is a limit point along the sequence t_k.

For the second assertion, we have by Propositions 12, 13 that almost surely, all limit points along c^k are exactly the elements of the set $\mathcal{K}$. It remains to show that along the above sequence $(t_k)_{k \geq 1}$, we do not get limit points outside $\mathcal{K}$. If μ' is a limit point along t_k, then as above, we pass to a further subsequence along which a_k converges to some limit $a(c) \in [1, c]$. It follows again from the claim above that along c^{i_k}, $\mu'_{a(c)}$ is a limit point. By Proposition 12, we have $I(\mu'_{a(c)}) \leq 1$. So that $I(\mu') = I(\mu'_{a(c)}) \leq 1$, that is, $\mu' \in \mathcal{K}$. $\quad\square$

7. The limit points of the motion.

The continuous time and space analogue of Sinai's walk is diffusion in random environment, that is, the diffusion X with $X(0) = 0$ that satisfies the formal differential equation

$$(28) \qquad dX(t) = d\beta(t) - \tfrac{1}{2} V'(X(t)) \, dt.$$

Here, β is standard Brownian motion, and V, the environment, is a random function we pick before running the diffusion. For the rigorous definition of this diffusion, as well as its relation with Sinai's walk; see Shi (2001), Seignourel (2000).

In this work, we will consider diffusions run in a Brownian-like environment. That is, we require from the measure governing V to be such that there is, on a possibly enlarged probability space, a standard two-sided Brownian motion B such that for all $n \geq 1$, we have

$$(29) \qquad \mathbf{P}\Big(\sup_{|x| \leq n} |V(x) - B(x)| \geq C_1 \log n\Big) \leq \frac{1}{n^{C_2}}$$

for some constants C_1, C_2. For these environments, the diffusion does not explode in finite time. Moreover, its behavior is dominated by the environment, and one aspect of this phenomenon is captured by the following result. Recall the definition of x_B from Section 2.

THEOREM 16 [Hu (2000), Theorem 1.1]. *Assume that V satisfies* (29). *For every $\delta_1 > 0$, there exists $C, t_0 > 0$ so that for $t \geq t_0$ and $\lambda \geq 1$, we have*

$$(30) \qquad \mathbf{P}(|X(t) - x_B(\log t)| > \lambda) \leq C\Big(\frac{\log \log t}{\sqrt{\lambda}} + \frac{1}{(\log t)^{1 - \delta_1}}\Big).$$

The limit points of the diffusion. For the diffusion defined by (28), where V satisfies (29), we have the following analog of Theorem 1.

THEOREM 17. *With probability 1, the limit points, as $a \to \infty$, in the topology of local weak convergence of the graph occupation measures of the random functions*

$$y_a := \frac{X(e^{at})}{a^2 \log \log a}, \qquad t \geq 0,$$

constitute the set

$$\mathcal{K} := \{\mu \in \mathcal{M} : I(\mu) \leq 1\}.$$

Also, there is at least one limit point along every sequence $a_n \to \infty$.

PROOF. Note that if f_n, g_n is a sequence of functions on $[a, b]$ with $m(f_n) \to \mu$ and $f_n - g_n \to 0$ in (Lebesgue) measure, then $m(g_n) \to \mu$. Indeed, assuming that for bounded, uniformly continuous φ we have

$$\int_a^b \varphi(t, f_n(t)) \, dt \to \int_{[a,b] \times \mathbb{R}} \varphi \, d\mu,$$

we break down the integral to the set where $|f_n(s) - g_n(s)| < \varepsilon$ and its complement. Since φ is uniformly continuous, on this set the integrand is close to $\varphi(t, g_n(t))$, while the measure of the complement of this set is small. Thus $m(g_n) \to \mu$.

Using this observation, Proposition 15 and the definition of the topology of $\mathcal{M}$, it is clear that to prove Theorem 17, it suffices to show that for every $0 < \varepsilon < M < \infty$, as $a \to \infty$ we have

$$\mathcal{L}\{s \in [\varepsilon, M] : |X(e^{as}) - x_B(as)| > \varepsilon a^2 \log \log a\} \to 0.$$

We prove this for $0 < \varepsilon < M = 1$, as this is in no way different than the general M case. By changing variables $w = as$, the above quantity will become

$$\int_{a\varepsilon}^a \frac{\mathbf{1}(|X(e^w) - x_B(w)| > \varepsilon a^2 \log \log a)}{a} \, dw$$

$$\leq \int_{a\varepsilon}^\infty \frac{\mathbf{1}(|X(e^w) - x_B(w)| > \varepsilon w^2 \log \log w)}{w} \, dw.$$

If the last integral is finite for some $a > 0$, then it converges to 0 as $a \to \infty$. Its expectation is bounded using Theorem 16, provided a satisfies $\varepsilon a > \log t_0$ and $(\varepsilon a)^2 \log \log (\varepsilon a) > 1$, by

$$\int_{a\varepsilon}^\infty \frac{1}{w} \mathbf{P}(|X(e^w) - x_B(w)| > \varepsilon w^2 \log \log w) \, dw$$

$$\leq \int_{a\varepsilon}^\infty \frac{c}{w} \left(\frac{\log w}{\sqrt{w^2 \log \log w}} + \frac{1}{w^{1-\delta_1}} \right) dw < \infty.$$

So that the integral is finite with probability 1. $\quad\square$

The limit points of the walk. To prove Theorem 1, we will embed the walk it in a diffusion generated by an appropriate random environment V.

Let $(S_n)_{n \geq 1}$ be Sinai's walk with $\operatorname{Var}(\log((1 - p_1)/p_1)) = 1$. Define the step potential V as follows: $V(0) = 0$, and for every $n \in \mathbb{Z}$, V is constant in $[n - 1, n)$, and jumps at n by $V(n) - V(n-) = \log((1 - p_n)/p_n)$. This potential can be placed on a possibly enlarged probability space with a two-sided Brownian motion B so that (29) is satisfied. This follows from the strong approximation theorem of Komlós–Major–Tusnády [Theorem 1 in Komlós, Major and Tusnády (1976)]. The theorem requires that $Y := \log((1 - p_1)/p_1)$ has $\mathbf{E}(e^{\lambda Y}) < \infty$ for λ in a neighborhood of 0, which is exactly the assumption we made for the law of p in the Introduction.

The walk can be embedded in the diffusion X run in the environment V as follows. Let $t_0 = 0$ and $t_n = \inf\{t > t_{n-1} : |X(t) - X(t_{n-1})| = 1\}$ for $n \geq 1$.

THEOREM 18 [Hu and Shi (1998), Proposition 9.1]. $(X(t_n))_{n \geq 1}$ *has the same law as* $(S_n)_{n \geq 1}$. *Moreover,* $\{t_{n+1} - t_n : n \geq 1\}$ *are i.i.d. with distribution that of the first hitting time* T *of* 1 *for reflected standard Brownian motion.*

We will need the fact that the law of $1/T$ has exponential tails. This holds since if $1/T > x > 0$, then the maximum or the negative of the minimum of Brownian motion on $[0, 1/x]$ is at least 1. Since the maximum has the same distribution as $|B(x)|$, we have

$$(31) \qquad P(1/T > x) \leq 2P(|B(1/x)| > 1) = 4P(B(1) > \sqrt{x}) \leq ce^{-x/2}.$$

PROOF OF THEOREM 1. Let $t(\cdot)$ be the the piecewise linear continuous extension of t_n so that $t(n) = t_n$. To prove the theorem, it suffices to show that for every $0 < \varepsilon < M < \infty$, as $a \to \infty$,

$$\frac{S(e^{as}) - x_B(as)}{a^2 \log \log a} \to 0 \qquad \text{in measure on } [\varepsilon, M], \text{ a.s.}$$

Since $|S(s) - X(t(s))| \leq 1$, it suffices to show the previous claim with $X(t(e^{as}))$ instead of $S(e^{as})$. We break this down into two parts, namely

$$(32) \quad \frac{X(t(e^{as})) - x_B(\log t(e^{as}))}{a^2 \log \log a} \to 0, \qquad \frac{x_B(as) - x_B(\log t(e^{as}))}{a^2 \log \log a} \to 0$$

in measure on the interval $[\varepsilon, M]$ as $a \to \infty$. We assume for simplicity that $M = 1$. Proceeding the same way as for the diffusion, for the first claim it suffices to prove that

$$\int_{\varepsilon a}^{\infty} \frac{1}{w} \mathbf{1}\big(|X(t(e^w)) - x_B(\log t(e^w))| > \varepsilon w^2 \log \log w\big) \, dw < \infty$$

for some $a > 0$. The function t has derivative equal to $t_n - t_{n-1}$ in $(n - 1, n)$, and undefined in n for every positive integer n. The integral over the w's with $1/t'(e^w) > \log w$ has expectation bounded above by

$$\mathbf{E} \int_{\varepsilon a}^{\infty} \frac{1}{w} \mathbf{1}(1/t'(e^w) > \log w) \, dw = \int_{\varepsilon a}^{\infty} \frac{1}{w} \mathbf{P}(1/t'(e^w) > \log w) \, dw,$$

which is finite because $1/t'$ has the same distribution as $1/T$ in (31). For the rest of the integral, we change variables $r = \log t(e^w)$ and reduce the problem to the finiteness of

$$\int_{a\varepsilon}^{\infty} \frac{1}{w} \frac{t(e^w)}{e^w} \frac{\mathbf{1}(1/t'(e^w) \le \log w)}{t'(e^w)} \mathbf{1}(|X(e^r) - x_B(r)| > \varepsilon w^2 \log\log w) \, dr.$$

By the law of large numbers and the fact that $\mathbf{E}T = 1$, we have $t(s)/s \to 1$. So $t(e^w)/e^w \to 1$ as $r \to \infty$, which shows that $w/r \to 1$ as well. Thus the above integral is finite if

$$\int_{a\varepsilon}^{\infty} \frac{\log r}{r} \mathbf{1}(|X(e^r) - x_B(r)| > (\varepsilon/2) r^2 \log\log r) \, dr$$

is finite. This follows by taking expectations and using Theorem 16.

For the second convergence claim in (32), it suffices to show that

$$\int_{2}^{\infty} \frac{1}{w} \mathbf{1}(x_B(w) \ne x_B(\log t(e^w))) \, dw < \infty.$$

Note that $x_B(w) \ne x_B(\log t(e^w))$ implies that x_B has a jump between w and $\log t(e^w)$. By the law of large numbers, for all w large, this interval is contained in $(w - 1, w + 1)$. So it suffices to show the finiteness of the integral

$$\int_{2}^{\infty} \frac{1}{w} \mathbf{1}(x_B \text{ has a jump in } (w - 1, w + 1)) \, dw.$$

Applying Lemma 19, we bound its expectation from above by

$$c \int_{2}^{\infty} \frac{1}{w} \log \frac{w + 1}{w - 1} \, dw < \infty. \qquad \square$$

In the proof of Theorem 1, we use the next lemma, which gives a bound on the probability that x_B jumps on an interval.

LEMMA 19. *The process x_B satisfies $\mathbf{P}(x_B(s) \ne x_B(t)) \le c|\log(t/s)|$ for some finite constant c and all $t, s > 0$.*

PROOF. This holds because the jumps of $x_B(e^t)$ form a translation-invariant point process on $\mathbb{R}$ with finite mean density c. Rather than proving this, we will invoke the exact formula for the above probability. Assuming that $s < t$, and using

the scaling property of x_B [see (2)], the probability in question equals $\mathbf{P}(x_B(1) \neq x_B(t/s))$. However,

$$\mathbf{P}(x_B(1) = x_B(t/s)) = \left(\frac{t}{s}\right)^{-2} \frac{5 - 2e^{-(t/s)+1}}{3}$$

as is shown in the proof of Theorem 2.5.13 in Zeitouni (2004). And this gives easily the required bound. □

PROOF OF COROLLARY 3.   In fact, we will prove that if $\gamma : [0, 1] \to [0, \infty)$ is differentiable with $(t \mapsto t^3 \gamma(t))$ nondecreasing, then

(33) $$\limsup_{a \to \infty} \frac{1}{a^2 \log \log a} \int_0^1 \gamma(t) S(e^{at}) \, dt = \frac{4}{\pi^2} s_0^3 a(s_0),$$

where s_0 is any root of $2 \int_s^1 \gamma = s\gamma(s)$ in $(0, 1)$.

For $H : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ with compact support and whose projection of the set of the discontinuity points in the x-axis has Lebesgue measure zero, the map $(\mathcal{M} \ni \mu \mapsto \int H \, d\mu)$ is continuous in the weak topology, because any $\mu \in \mathcal{M}$ has first projection Lebesgue measure. Combining this with the definition of the graph occupation measure, we get

(34)
$$\limsup_{a \to \infty} \int_0^\infty H\left(t, \frac{S(e^{at})}{a^2 \log \log a}\right) dt$$
$$= \sup\left\{\int H(x, y) \, d\mu(x, y) : \mu \in \mathcal{M}, I(\mu) \leq 1\right\}.$$

Everywhere below, we use the abbreviation $A := 8/\pi^2$.

For the choice $H(x, y) := \gamma(x)y\mathbf{1}_{x \in [0,1], |y| \leq A+1}$ the limits in equations (33), (34) agree because by Theorem 1.3 in Hu and Shi (1998), it holds

$$\overline{\lim_{n \to \infty}} \frac{\max_{1 \leq k \leq n} |S_k|}{(\log n)^2 \log \log \log n} = A.$$

It remains to evaluate the supremum in (34) for this choice of H. If γ is identically zero, the corollary holds trivially. So we assume that γ is positive somewhere in $[0, 1]$. Since γ is nonnegative, it follows from the form of the rate function I that the above supremum equals

$$\sup\left\{\int_0^1 \gamma(t) f(t) \, dt : f(0) = 0, f \text{ nondecreasing}, \int_0^1 t^{-2} \, df(t) \leq A\right\}.$$

We did not include the factor $\mathbf{1}_{|f(t)| \leq A+1}$ inside the integral because the conditions on f imply that $0 \leq f(t) = \int_0^t df(s) \leq \int_0^1 s^{-2} \, df(s) \leq A$.

For a given nondecreasing $f \geq 0$, define $F(t) := \int_0^t s^{-2} \, df(s)$, so that $f(t) = \int_0^t s^2 \, dF(s)$. We use this representation of f and apply first Fubini's theorem and

then integration by parts in $\int_0^1 \gamma(t) f(t)\, dt$ to write it as

$$\int_0^1 F(s) r'(s)\, ds,$$

where $r(s) := -s^2 \int_s^1 \gamma(t)\, dt$. Using the fact that $(t \mapsto t^3 \gamma(t))$ is nondecreasing in $[0, 1]$, we find that r' is nonpositive before s_0 and nonnegative after s_0. And since $F \leq A$, the above integral is bounded above by

$$-Ar(s_0) = \frac{A}{2} s_0^3 \gamma(s_0).$$

The last equality follows from $r'(s_0) = 0$.

For the choice $f^\gamma(t) = s_0^2 A \mathbf{1}_{(s_0, \infty)}$, we get $\int_0^1 \gamma(t) f^\gamma(t)\, dt = A s_0^2 \int_{s_0}^1 \gamma(t)\, dt = A s_0^3 \gamma(s_0)/2$, so that the supremum is achieved. Clearly, the only measure that achieves the supremum is the element of $\mathcal{M}$ that puts all its mass on the graph of f^γ.

Finally, for the case of the corollary, $\gamma(t) = t^r$, we compute $s_0 = \eta_r := (\frac{2}{r+3})^{1/(r+1)}$, while the value of the supremum is $(A/2)\eta_r^{r+3}$. □

8. The probability of confinement. In this section, we compute the asymptotic decay of the probabilities that Brownian motion or Brownian motion reflected from its running minimum stay on certain bounded sets for large intervals of time.

Fix $h > 0$, $x \in (0, h)$, $t > 0$, and let $Q^h(t, x, \cdot)$ be the density of the measure

$$S \mapsto \mathbf{P}_x\big(B_t \in S, B[0, t] \subset (0, h)\big).$$

Proposition 8.2 in Port and Stone (1978) gives

$$(35) \qquad Q^1(t, x, y) = 2 \sum_{n=1}^\infty e^{-n^2 \pi^2 t/2} \sin(n\pi x) \sin(n\pi y).$$

Using this and Brownian scaling, we get that there exists a universal constant c_2 so that for all $t \geq 1$, $x, y \in [0, h]$, we have

$$(36) \qquad Q^h(t, x, y) \leq c_2 h^{-1} \exp\left(-\frac{\pi^2}{2} \frac{t}{h^2}\right).$$

Moreover, for every $\varepsilon > 0$ there exists a constant $c_1 = c_1(\varepsilon)$ so that for all $t \geq 1$, and $x, y \in [\varepsilon h, (1 - \varepsilon)h]$, we have

$$(37) \qquad Q^h(t, x, y) \geq c_1(\varepsilon) h^{-1} \exp\left(-\frac{\pi^2}{2} \frac{t}{h^2}\right).$$

Recall from (7) the notation for the past minimum of a given process, and from (6) the process $R = B - \underline{B}$. The probability that R stays confined in an interval for a large time interval $[0, t]$ decays exponentially in t. In the next lemma, we compute the exact rate of decay.

LEMMA 20.   *For $K > 0$, $\varepsilon \in [0, 1/2)$, $w \in [0, 1)$ and $z \in (0, 1)$,*

(a) $\displaystyle\lim_{t\to\infty} \frac{1}{t} \log \mathbf{P}_z\big(B([0, t]) \subset [0, 1], B(t) \in [\varepsilon, 1 - \varepsilon]\big) = -\frac{\pi^2}{2}$,

(b) $\displaystyle\lim_{t\to\infty} \frac{1}{t} \log \mathbf{P}\big(R([0, t]) \subset [0, 1], R(t) \in [0, 1 - \varepsilon]|R(0) = w\big) = -\frac{\pi^2}{8}$,

(c) $\displaystyle\lim_{t\to\infty} \frac{1}{t} \log \mathbf{P}_0\big(R([0, t]) \subset [0, 1], \underline{B}(t) \geq -K\big) = -\frac{\pi^2}{2}$.

For $\varepsilon_1 \in (0, 1/2)$ fixed, the convergence in (a) is uniform over $z \in [\varepsilon_1, 1 - \varepsilon_1]$, and the convergence in (b) is uniform over $w \in [0, 1 - \varepsilon_1]$.

Comparing (b) and (c), note the drastic effect of the restriction $\underline{B}(t) \geq -K$. The process $(R, -\underline{B})$ has the same law as $(|B|, L)$ where L is the process of local time at zero for the Brownian motion B. Phrased in terms of $(|B|, L)$, the first event requires $B([0, t]) \subset [-1, 1]$, the other requires additionally that $L_t^0 \leq K$, that is, B does not hit zero many times. This restriction makes the second event more like $B([0, t]) \subset (0, 1]$, that is, B is essentially restricted to an interval of half size than before.

PROOF.   (a) Follows by integrating (36) and (37) over y.

(b) Since R has the same law as the absolute value of Brownian motion, the claim follows by the scaling property of Brownian motion and (a).

(c) *Lower bound:* Pick an open interval V of length 1 around 0 that does not contain $-K$. Then

$$B([0, t]) \subset V \quad \text{implies} \quad R([0, t]) \subset [0, 1], \qquad \underline{B}_t \geq -K.$$

By (a) applied with $\varepsilon = 0$, the probability of the first event decays like $\exp(-t[\pi^2/2 + o(1)])$ as $t \to \infty$.

(c) *Upper bound:* Let A_t denote the event in question. Subdivide the rectangle $[0, t] \times [-K, 0]$ into $n \times n$ small isomorphic rectangles. Each rectangle is a product of a time interval $\mathcal{T}_i := [(i - 1)t/n, it/n]$ with $i = 1, 2, \ldots, n$ and a space interval. Consider the graph of the process $\underline{B}$, and let J be the union of the subdivision rectangles it intersects. Fix $m \geq 1$. When $\underline{B}(t) \geq -K$, we have:

- $J = \bigcup_{i=1}^n \mathcal{T}_i \times \mathcal{B}_i$ for some space intervals $\mathcal{B}_i$.
- Let $\mathcal{N} = \{i : \text{length } \mathcal{B}_i \leq (m + 2)K/n\}$. Then $|\mathcal{N}| \geq (1 - 1/m)n$.

The first claim is clear. For the second, note that on the time intervals $\mathcal{T}_i$ for $i \notin \mathcal{N}$ the process $\underline{B}$ decreases by at least mK/n. But the total decrease is at most K, so there are at most n/m such indices i. We have

$$P(A_t) = P\big(\underline{B}(t) \geq -K, \text{ graph } B[0, t] \subset \text{graph } \underline{B}[0, t] + \{0\} \times [0, 1]\big)$$

$$\leq \sum_J \mathbf{P}(\text{graph } B[0, t] \subset J + \{0\} \times [0, 1]).$$

Here the sum is over all unions J of rectangles satisfying the conditions above. By the Markov property, we get the upper bound

$$\sum_J \prod_{i=1}^{n} \max_x \mathbf{P}_x\big(B(\mathcal{T}_i) \subset \mathcal{B}_i + [0,1]\big)$$

$$\leq 2^{(n^2)} \max_x \big(\mathbf{P}_x\big(B[0, t/n] \subset [0, 1 + (m+2)K/n]\big)\big)^{|\mathcal{N}|}.$$

The inequality follows by considering only the indices $i \in \mathcal{N}$. Brownian scaling, part (a) with $\varepsilon = 0$ and the fact $|\mathcal{N}| \geq n(1 - 1/m)$ gives

$$\limsup_{t \to \infty} \frac{1}{t} \log \mathbf{P}(A_t) \leq -\frac{1}{n}\left(\frac{\pi^2}{2} \frac{1}{(1 + (m+2)K/n)^2}\right) n(1 - 1/m).$$

Since this holds for m, n arbitrary, we let $n = m^2 \to \infty$ to get the desired upper bound. $\square$

Below, we will use the operator R_x of reflection from the past infimum, defined in (11), and the notation $R = R_0 B$ from (6).

For $0 \leq w \leq x < y$ and $0 < h_1 < h_2$, call $\Gamma(w, x, y, h_1, h_2)$ the set

$$\{R_w f \in C(x, y, h_1) \cap H^R(y, h_1) \cap B(y, h_2)\}$$

$$\cap \{f - f(x(1+\delta)) \leq \varepsilon^2 \text{ on } [x(1+\delta), y(1+\delta)]\},$$

which is involved in the definition of $\mathcal{R}(\mathbf{h}, \mathbf{x}, \delta, \varepsilon)$. More precisely, the set E_i corresponding to an index $i \in \mathcal{I}_\infty$, defined in Section 4, is exactly the set

$$\Gamma\big(w_q(1+\delta), w_i, x_i, h_i, h_{i+}\big).$$

The following lemma computes for large M the probability that the scaled Brownian motion $B(M\cdot)$ is in the special sets C, H, B, Γ, defined in Section 4 and above.

LEMMA 21. *Let* $0 \leq x < y$, $0 < h_1 < h_2$, $h > 0$, *and small enough* $\varepsilon, \delta > 0$. *Uniformly on* $z \in [0, h - \varepsilon h]$, $z_1 \in [0, h_1 - \varepsilon h_1]$, $z_2 \in \mathbb{R}$, *as* $M \to \infty$ *we have*:

(a) $\dfrac{1}{M} \log \mathbf{P}\big(B(M\cdot) \in C(x, y, h) | B(Mx(1+\delta)) = z\big) \to -\dfrac{\pi^2}{2} \dfrac{y - x - \delta(x+y)}{h^2(1+\varepsilon^2)^2}$;

(b) $\dfrac{1}{M} \log \mathbf{P}\big(B(M\cdot) \in H(y, h) | B(My(1-\delta)) = z\big) \to -\dfrac{\pi^2}{2} \dfrac{\delta y}{h^2(1+\varepsilon)^2}$;

(c) $\dfrac{1}{M} \log \mathbf{P}\big(B(M\cdot) \in B(y, h_2) | B(My) = z_1\big) \to -\dfrac{\pi^2}{2} \dfrac{\delta y}{h_2^2(1+\varepsilon+\varepsilon^2)^2}$;

(d) $\dfrac{1}{M} \log \mathbf{P}\left(B(M\cdot) \in \Gamma \left| \begin{array}{l} R(Mx(1+\delta)) = z_1 \\ B(Mx(1+\delta)) = z_2 \end{array}\right.\right)$

$$\to -\frac{\pi^2}{8}\left(\frac{y - x - \delta x}{h_1^2} + \frac{\delta y}{h_2^2(1+\varepsilon)^2}\right),$$

where $\Gamma := \Gamma(w, x, y, h_1, h_2)$ *and the events* C, H, B, Γ *depend on* $\varepsilon, \delta.$

PROOF. (a) It follows from Lemma 20(a) and the scaling property of Brownian motion.

(b) The exponential rate of decay of the event in question is the same as the one of confinement on $[-\varepsilon h, h]$ between times $y(1 - \delta)$, y and ending in $[0, (1 - \varepsilon)h]$. Because the difference of the two events is contained on the event of confinement on the smaller interval $[-\varepsilon h + \varepsilon^2 h, h]$, which decreases exponentially faster. Thus, the result follows again from Lemma 20(a).

(c) The same reasoning as in part (b) proves this claim too.

(d) We can assume that $w = 0$. Then we let $\Gamma(x, y, h_1, h_2) = \Gamma(0, x, y, h_1, h_2)$, and $\Gamma'(x, y, h_1, h_2)$ to be only the first set in the intersection defining $\Gamma(x, y, h_1, h_2)$; that is, we remove the restriction $f - f(x(1+\delta)) \le \varepsilon^2$ on $[x(1+\delta), y(1+\delta)]$. We first prove the claim with Γ' in place of Γ. To this aim, we observe that

$$(38) \quad \lim_{M \to \infty} \frac{1}{M} \log \mathbf{P}\big(R(M \cdot) \in C(x, y, h_1) | R\big(Mx(1 + \delta)\big) = z_1\big)$$

$$= -\frac{\pi^2}{8} \frac{y - x - \delta(x + y)}{h_1^2},$$

$$(39) \quad \lim_{M \to \infty} \frac{1}{M} \log \mathbf{P}\big(R(M \cdot) \in H^R(y, h_1) | R(My(1 - \delta)) = z_1\big) = -\frac{\pi^2}{8} \frac{\delta y}{h_1^2},$$

$$(40) \quad \lim_{M \to \infty} \frac{1}{M} \log \mathbf{P}\big(R(M \cdot) \in B(y, h_2) | R(My) = z_1\big) = -\frac{\pi^2}{8} \frac{\delta y}{h_2^2 (1 + \varepsilon)^2},$$

where the convergence is uniform over $z_1 \in [0, h_1 - \varepsilon h_1]$.

The first expression follows from Lemma 20(b). For the second, an upper bound is given by the same relation because the event requires confinement on $[0, h_1]$ for the time interval $[My(1 - \delta), My\delta]$. For a lower bound, we will consider two events whose intersection is inside the event of interest and whose probability we will estimate. The first event

$$\left\{ \begin{array}{l} \text{In the time interval } [My(1 - \delta), My(1 - \delta) + 1], \\ R \text{ visits } 0, \text{ stays in } [0, h_1], \text{ ends in } [0, h_1 - \varepsilon h_1] \end{array} \right\}$$

realizes the requirement of the visit to zero. Given that $R(My(1 - \delta)) = z_1 \in [0, h_1 - \varepsilon h_1]$, this event has a positive probability independent of M. The second event is

$$\left\{ \begin{array}{l} \text{In the time interval } [My(1 - \delta) + 1, My], \\ \quad R \text{ stays in } [0, h_1], \text{ ends in } [0, h_1 - \varepsilon h_1] \end{array} \right\}.$$

To compute the probability of the intersection, we apply the Markov property at time $My(1 - \delta) + 1$. Then the probability of the second event, conditioned on the value of R at $My(1 - \delta) + 1$, will decay exponentially as $M \to \infty$ with the

same rate as if R was staying in $[0, h_1]$ in the slightly larger time interval $[My(1 - \delta), My]$ and was ending in $[0, h_1 - \varepsilon h_1]$. So that the lower bound obtained for the left-hand side of (39) coincides with the upper bound. Equation (40) is proved in the same way.

Relation (d) with Γ' is place of Γ now follows by applying the Markov property and using (38), (39), (40).

To prove (d) itself, we note that the left-hand side increases if we put Γ' in place of Γ. This observation, together with the above, gives an upper bound, but we can show a lower bound, too.

Let Δ_M be the set of continuous functions f on $[0, \infty)$ with

$$f\big(x(1 + \delta) + M^{-1}\big) - f\big(x(1 + \delta)\big) \le -h_2(1 + \varepsilon),$$
$$f(s) - f\big(x(1 + \delta)\big) < \varepsilon^2, \qquad Rf(s) \le h_1 - \varepsilon h_1$$

for $s \in [x(1 + \delta), x(1 + \delta) + M^{-1}]$, and E_M the set

$$C\big(x + (M(1 + \delta))^{-1}, y, h_1\big) \cap H^R(y, h_1) \cap B(y, h_2).$$

Then

$$\{B(M\cdot) \in \Delta_M\} \cap \{R(M\cdot) \in E_M\} \subset \{B(M\cdot) \in \Gamma(x, y, h_1, h_2)\}.$$

To see this, note that the inclusion holds with Γ' in place of Γ. But because at $x(1 + \delta) + M^{-1}$ the process $B(M\cdot) - B(Mx(1 + \delta))$ takes a value less than $-h_2(1 + \varepsilon)$, and after that R stays below $h_2(1 + \varepsilon)$, it follows that $B(M\cdot) - B(Mx(1 + \delta))$ stays negative in the interval $[x(1 + \delta) + M^{-1}, y(1 + \delta)]$. And of course it stays below ε^2 in $[x(1 + \delta), x(1 + \delta) + M^{-1}]$ because of Δ_M.

By applying the Markov property at time $Mx(1 + \delta) + 1$, we get that the probability of the above intersection, conditioned on the values of $R(Mx(1 + \delta))$, $B(Mx(1 + \delta))$ as in (d), is at least the product of

$$(41) \qquad \mathbf{P}\left(B(M\cdot) \in \Delta_M \,\middle|\, \begin{matrix} R(Mx(1 + \delta)) = z_1 \\ B(Mx(1 + \delta)) = z_2 \end{matrix}\right)$$

and

$$(42) \qquad \inf_{z_3 \in [0, h_1 - \varepsilon h_1]} \mathbf{P}\big(R(M\cdot) \in E_M \,|\, R(Mx(1 + \delta) + 1) = z_3\big).$$

The probability in (41) is positive and does not depend on M. The asymptotic decay as $M \to \infty$ for the probability in (42) is computed as in the case of Γ'. The change in the restriction interval from $[Mx(1 + \delta), My(1 - \delta)]$ to $[Mx(1 + \delta) + 1, My(1 - \delta)]$ does not change the result. $\quad \square$

9. The topology of $\mathcal{M}$ and step functions. This section contains the proofs of the topological lemmas used in Sections 3, 4 and 6 for the large deviation principle and the functional law of the iterated logarithm for the environment.

Lemma 6 is a consequence of Lemmas 22 and 23 below.

LEMMA 22.   *Let $\mu \in \mathcal{M}$ and $(\mathbf{h}, \mathbf{x}) \in \mathcal{S}$. Assume that*

(43)   *there is $\mathbf{x}' \in \mathbb{R}^N$ so that $(h_i, x_i') \in \mathrm{supp}(\mu)$ for all $i \in \mathcal{I}$ and $x_i' > x_i$*
        if $x_i > 0$ and $x_i' < x_i$ if $x_i < 0$.

Then the set $\mathcal{U}(\mathbf{h}, \mathbf{x}, \varepsilon)$ defined in (4) is a neighborhood of μ for every $\varepsilon > 0$.

The proof is straightforward using the definition of weak convergence, so we omit it.

LEMMA 23.   *For each $\mu \in \mathcal{M}$ and $A < I(\mu)$, there is $(\mathbf{h}, \mathbf{x}) \in \mathcal{S}$ satisfying* (43) *so that $I(\mathbf{h}, \mathbf{x}) > A$.*

PROOF.   We will abbreviate f_μ, g_μ to f, g. We also remind the reader that for a bounded function F, a nondecreasing function α, both defined on a finite closed interval $[a, b]$, and a partition $\mathcal{P} = \{a =: t_0 < t_1 < \cdots < t_n := b\}$ of $[a, b]$, the lower Stieltjes sum $L(\mathcal{P}, F, \alpha)$ is defined as

$$\sum_{i=1}^{n} \inf\{F(t) : t \in [t_{i-1}, t_i]\}\big(\alpha(t_i) - \alpha(t_{i-1})\big).$$

We consider three cases for μ.

CASE 1. $0 < s_{\mu-} < s_{\mu+} = \infty$.

We can write $A = (A_1 + A_2)\pi^2/2 + A_3\pi^2/8$ for some A_i's with

$$\int_0^{s_{\mu-}} t^{-2}\,df(t) > A_1, \qquad \int_0^{s_{\mu-}} t^{-2}\,dg(t) > A_2, \qquad \int_{s_{\mu-}}^{H} t^{-2}\,df(t) > A_3,$$

where $H \in (s_{\mu-}, \infty)$ is large enough. Since f, g are left continuous at $s_{\mu-}$, we can find two finite subsets $\mathcal{P}_1, \mathcal{P}_2$ of $[0, s_{\mu-})$ so that $\mathcal{P}_1 \cap \mathcal{P}_2 = \{0\}$, and when considered as partitions of the intervals $[0, \max \mathcal{P}_1], [0, \max \mathcal{P}_2]$, the corresponding lower Stieltjes sums satisfy

(44) $$L(\mathcal{P}_1, t^{-2}, f) > A_1, \qquad L(\mathcal{P}_2, t^{-2}, g) > A_2.$$

We can also find a finite subset $\mathcal{P}_3$ of $[s_{\mu-}, H]$ containing $s_{\mu-}$, with

(45) $$L(\mathcal{P}_3, t^{-2}, f) > A_3.$$

We can assume that $f|\mathcal{P}_1 \cup \mathcal{P}_3, g|\mathcal{P}_2$ are strictly increasing. In particular, $f(\zeta_1), g(\zeta_2) > 0$, with $\zeta_i := \min(\mathcal{P}_i \setminus \{0\})$ for $i = 1, 2$. We can also assume that

(46) $(h, f(h)) \in \mathrm{supp}(\mu)$ for $h \in \mathcal{P}_1 \cup \mathcal{P}_3$,

               $(h, -g(h)) \in \mathrm{supp}(\mu)$ for $h \in \mathcal{P}_2$.

If, for example, this is not the case for an $h \in \mathcal{P}_1$, we go as follows. The point

$$h' := \sup\{\eta \leq h : (\eta, f(\eta)) \in \operatorname{supp}(\mu)\}$$

satisfies $h' < h$ by the assumption and the left continuity of f, $f(h') = f(h)$ by the minimality property in the definition of f, and $(h', f(h')) \in \operatorname{supp}(\mu)$. We can find an $h'' < h'$ near h' such that $(h'', f(h'')) \in \operatorname{supp}(\mu)$, $h'' \notin \mathcal{P}_2$, and $f(h'')$ is as close to $f(h)$ as we want, because f is left continuous. Finally, we replace h with h'' in $\mathcal{P}_1$. The lower Stieltjes sum over the new partition is larger than before because t^{-2} is decreasing.

Also, for the $\eta_i := \max \mathcal{P}_i$ for $i = 1, 2$, we can arrange that $\eta_1 < \eta_2$ because $(s_{\mu-}, -g(s_{\mu-})) \in \operatorname{supp}(\mu)$.

Let $\mathbf{h} \in \mathbb{R}^N$ be the vector having coordinates the elements of the set $(\mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3) \setminus \{0\}$ ordered as $h_1 < \cdots < h_N$, and define the vector $\mathbf{x}(\varepsilon) \in \mathbb{R}^N$ as

$$x_i(\varepsilon) := \begin{cases} f(h_i) - \varepsilon, & \text{if } h_i \in \mathcal{P}_1 \cup \mathcal{P}_3, \\ -g(h_i) + \varepsilon, & \text{if } h_i \in \mathcal{P}_2, \end{cases}$$

for all $i \in \{1, \ldots, N\}$ and all $\varepsilon \in [0, f(\zeta_1) \wedge g(\zeta_2))$.

Using the notation of Section 2, we note that for ε small enough, as above, all the pairs $(\mathbf{h}, \mathbf{x}(\varepsilon))$ give rise to the same index set $\mathcal{I}_\infty$, and we have $\mathcal{I}_\infty = \{i : h_i \in \mathcal{P}_3\}$ because $h_i \leq \eta_1 < \eta_2$ for all i with $h_i \in \mathcal{P}_1$, and $\eta_2 = h_{i_0}$ with $x_{i_0}(\varepsilon) < 0$. Also

$$I(\mathbf{h}, \mathbf{x}(\varepsilon)) = \frac{\pi^2}{2} \sum_{i \in \mathcal{I} \setminus \mathcal{I}_\infty} \frac{|x_i(\varepsilon) - x_{i-}(\varepsilon)|}{h_i^2} + \frac{\pi^2}{8} \sum_{i \in \mathcal{I}_\infty} \frac{|x_i(\varepsilon) - x_{i-}(\varepsilon)|}{h_i^2}$$

and

$$\lim_{\varepsilon \to 0} I(\mathbf{h}, \mathbf{x}(\varepsilon)) = I(\mathbf{h}, \mathbf{x}(0)) > A.$$

The last inequality follows from (44), (45), the equalities

$$\tag{47} \sum_{i \in \mathcal{I} \setminus \mathcal{I}_\infty : x_i > 0} \frac{|x_i(0) - x_{i-}(0)|}{h_i^2} = L(\mathcal{P}_1, t^{-2}, f),$$

$$\tag{48} \sum_{i \in \mathcal{I} : x_i < 0} \frac{|x_i(0) - x_{i-}(0)|}{h_i^2} = L(\mathcal{P}_2, t^{-2}, g),$$

in which we use that $f(0) = g(0) = 0$, and the inequality

$$\sum_{i \in \mathcal{I}_\infty} \frac{|x_i(0) - x_{i-}(0)|}{h_i^2} \geq L(\mathcal{P}_3, t^{-2}, f).$$

The last inequality holds because the left-hand side equals exactly the Stieltjes sum in the right-hand side plus the term corresponding to $i := \min \mathcal{I}_\infty$.

Thus, for small ε, the pair $(\mathbf{h}, \mathbf{x}(\varepsilon))$ is in $\mathcal{S}$, satisfies assumption (43) because of (46), and it has $I(\mathbf{h}, \mathbf{x}(\varepsilon)) > A$.

CASE 2. $s_{\mu-} = s_{\mu+} = \infty$.

There is $H > 0$ finite with

$$(\pi^2/2) \int_0^H t^{-2} \, d(f+g)(t) > A.$$

Let A_1, A_2 be such that $A = (\pi^2/2)(A_1 + A_2)$ and

$$\int_0^H t^{-2} \, df(t) > A_1, \qquad \int_0^H t^{-2} \, dg(t) > A_2.$$

We find two finite subsets $\mathcal{P}_1, \mathcal{P}_2$ of $[0, H]$, so that $\mathcal{P}_1 \cap \mathcal{P}_2 = \{0\}$, and when considered as partitions of the intervals $[0, \max \mathcal{P}_1], [0, \max \mathcal{P}_2]$, the corresponding lower Stieltjes sums satisfy

(49) $$L(\mathcal{P}_1, t^{-2}, f) > A_1,$$

(50) $$L(\mathcal{P}_2, t^{-2}, g) > A_2.$$

Again, we can assume that $f|\mathcal{P}_1, g|\mathcal{P}_2$ are strictly increasing, $(h, f(h)) \in \mathrm{supp}(\mu)$ for $h \in \mathcal{P}_1$, $(h, -g(h)) \in \mathrm{supp}(\mu)$ for $h \in \mathcal{P}_2$, and $\eta_1 < \eta_2$, where $\eta_i := \max \mathcal{P}_i$ for $i = 1, 2$, as before.

Pick a number $\eta_1' > \eta_2$ with $(\eta_1', f(\eta_1')) \in \mathrm{supp}(\mu)$ (recall that $s_{\mu+} = \infty$), and let $\mathcal{P}_3 := \{\eta_2, \eta_1'\}$.

Let $\mathbf{h} \in \mathbb{R}^N$ be the vector having coordinates the elements of the set $(\mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3) \setminus \{0\}$ ordered as $h_1 < \cdots < h_N = \eta_1'$, and the proof continues as in the first case. Here we just note that in the resulting pairs $(\mathbf{h}, \mathbf{x}(\varepsilon))$, only one element belongs to the final index set $\mathcal{I}_\infty$, which is due to η_1'. The presence of η_1' is needed so that in the formula for $I(\mathbf{h}, \mathbf{x})$, all increments $(x_i - x_{i-})/h_i^2$ with $i \leq N - 1$ get coefficient $\pi^2/2$, and this is enough to make $I(\mathbf{h}, \mathbf{x})$ larger than A because of (49), (50).

CASE 3. $0 = s_{\mu-} < s_{\mu+} = \infty$.

In this case, we work only with the function f and one partition. The proof is similar to the previous case and easier.

Since the roles of f, g are symmetric, these are the only truly different cases. $\qquad \square$

Lemma 9 is an immediate consequence of Lemmas 24 and 25 below. The next lemma essentially shows that the pairs $(\mu_{\mathbf{h},\mathbf{x}}, I(\mathbf{h}, \mathbf{x}))$ are relatively dense in $\{(\mu, I(\mu)) : \mu \in \mathcal{M}\}$.

For $L > 0$, let $\mathcal{T}_L$ be the topology of weak convergence on compact subsets of $[0, L] \times \mathbb{R}$ for elements of $\mathcal{M}$. Note that the topology of $\mathcal{M}$ is the union of the family $\{\mathcal{T}_L : L > 0\}$, which is increasing.

LEMMA 24. *For every open $G \subset \mathcal{M}$, $\mu \in G$ with $I(\mu) < \infty$, and $\delta > 0$, there exists $(\mathbf{h}, \mathbf{x}) \in \mathcal{S}$ and $G_{\mathbf{h},\mathbf{x}} \in \mathcal{T}_{2h_N}$ so that $\mu_{\mathbf{h},\mathbf{x}} \in G_{\mathbf{h},\mathbf{x}} \subset G$, and $|I(\mu_{\mathbf{h},\mathbf{x}}) - I(\mu)| < \delta$.*

PROOF. Assume that $0 < s_{\mu-} < \infty$. For $0 < b < a$ and f, g increasing and left continuous on $[0, a]$, we will use the notation

$$I(f, g, a, b) = \frac{\pi^2}{2} \int_0^b \frac{1}{t^2} d(f + g)(t) + \frac{\pi^2}{8} \int_b^a \frac{1}{t^2} df(t).$$

By the definition of the topology of $\mathcal{M}$, there is an $L > s_{\mu-}$ and $U \in \mathcal{T}_L$ neighborhood of μ such that $U \subset G$. We can also assume that $I(f_\mu, g_\mu, L, s_{\mu-}) > I(\mu) - \delta$.

We can approximate in the Skorokhod topology the restrictions of f_μ, g_μ on $[0, L]$ by monotone left continuous step functions $f^{(n)}, g^{(n)}$ with finitely many steps so that $f^{(n)}, g^{(n)}$ are constant on $[L, \infty)$ and $[s_{\mu-} - 1/n, \infty)$ respectively [we use the left continuity of g_μ at $s_{\mu-}$ to satisfy that together with (51)], they do not have common jump times, and

(51) $\qquad\qquad I(f^{(n)}, g^{(n)}, L, s_{\mu-}) \to I(f_\mu, g_\mu, L, s_{\mu-}).$

We can also assume that $f^{(n)}$ has a jump in $(L/2, L)$.

Then we approximate the measure $\mu(\cdot \times \mathbb{R}^+)$ on $[0, L]$ by a sequence of measures on $[0, \infty)$ whose densities are right continuous step functions q_n with values $0, 1$, finitely many steps, $q_n = 1$ on $[s_{\mu-}, \infty)$, and $q_n = 0$ on an interval inside $(s_{\mu-} - 2/n, s_{\mu-} - 1/n)$. By introducing extra jumps in q_n, we can further ensure that

(52) $\qquad\qquad q_n(h) = \begin{cases} 1, & \text{if } h \text{ is a jump time of } f^{(n)}, \\ 0, & \text{if } h \text{ is a jump time of } g^{(n)}. \end{cases}$

If f_μ has a jump at $s_{\mu-}$, then we require in addition that

(53) $f^{(n)}$ is $1/n$-close to f_μ at time $s_{\mu-} - 1/n$, and $q_n = 1$ on $[s_{\mu-} - 1/n, \infty)$.

Define the step function

$$u_n(h) = \begin{cases} f^{(n)}(h), & \text{if } q_n(h) = 1, \\ -g^{(n)}(h), & \text{if } q_n(h) = 0 \end{cases}$$

at all points $h \in [0, \infty)$ where q_n does not jump, and extend it to the remaining finite set of points so that it is left continuous. Clearly this is a function of the form $\Phi_{\mathbf{h},\mathbf{x}}$ with $(\mathbf{h}, \mathbf{x}) \in \mathcal{S}$.

Let $\nu_n = m(u_n)$, the graph occupation measure of u_n. By our construction, $f_{\nu_n} = f^{(n)}$, $g_{\nu_n} = g^{(n)}$, because of (52) and the right continuity of q_n, $\nu_n|[0, L] \times \mathbb{R} \to \mu|[0, L] \times \mathbb{R}$, $s_{\nu_n+} = \infty$, and $s_{\nu_n-} \to s_{\mu-}$ because $s_{\mu-} - 2/n < s_{\nu_n-} < s_{\mu-}$.

Furthermore, by (51), the fact that $s_{\nu_n-} \to s_{\mu-}$, and that any possible jump of f_μ at $s_{\mu-}$ is treated appropriately through (53), we have

$$I(\nu_n) = I(f^{(n)}, g^{(n)}, L, s_{\nu_n-}) \to I(f_\mu, g_\mu, L, s_{\mu-}) \in (I(\mu) - \delta, I(\mu)].$$

Take now n sufficiently large so that $\nu_n \in U$, $I(\nu_n) > I(\mu) - \delta$, and let $(\mathbf{h}, \mathbf{x}) \in \mathcal{S}$ be such that $\nu_n = \mu_{\mathbf{h},\mathbf{x}}$. Finally, let $G_{\mathbf{h},\mathbf{x}} := U$. Since $f^{(n)}$ has a jump in $(L/2, L)$, it holds $2h_N > L$, and thus $U \in \mathcal{T}_L \subset \mathcal{T}_{2h_N}$.

The remaining truly different cases are $s_{\mu-} = 0$, $s_{\mu-} = s_{\mu+} = \infty$; the proof in these cases is similar and easier. □

LEMMA 25. *If* $(\mathbf{h}, \mathbf{x}) \in \mathcal{S}$, *and* $\mu_{\mathbf{h},\mathbf{x}} \in G_{\mathbf{h},\mathbf{x}} \in \mathcal{T}_{2h_N}$, *then for all sufficiently small* ε, *we have*

$$\{m(x_B) : B \in \mathcal{R}(\mathbf{h}, \mathbf{x}, \varepsilon, \varepsilon)\} \subset G_{\mathbf{h},\mathbf{x}}.$$

PROOF.   The paths B contained in $\mathcal{R}(\mathbf{h}, \mathbf{x}, \varepsilon, \varepsilon)$ have the property that x_B is a step function whose jump times and values are close to that of $\Phi_{\mathbf{h},\mathbf{x}}$ in the interval $[0, 2h_N]$; see (13) and (14). In particular, for any $\delta > 0$, there exists $\varepsilon_0 > 0$ so that for $\varepsilon \in (0, \varepsilon_0)$, $\{x_B : B \in \mathcal{R}(\mathbf{h}, \mathbf{x}, \varepsilon, \varepsilon)\}$ is contained in the $[0, 2h_N]$-Skorokhod ball of radius δ about $\Phi_{\mathbf{h},\mathbf{x}}$.

On the space of real left continuous functions on $[0, \infty)$ having right limits, consider for $L > 0$ the topology of Skorokhod convergence in $[0, L]$. Also let $\mathcal{T}'_L$ the topology of weak convergence on the compact subsets of $[0, L] \times \mathbb{R}$ for measures on $[0, \infty) \times \mathbb{R}$. The graph occupation measure is a continuous functions between the two spaces with the above topologies. Let $G' \in \mathcal{T}'_{2h_N}$ so that $G' \cap \mathcal{M} = G_{\mathbf{h},\mathbf{x}}$. By the continuity of m just mentioned, it follows that $m^{-1}(G')$ contains some $[0, 2h_N]$-Skorokhod ball around $\Phi_{\mathbf{h},\mathbf{x}}$, and therefore also the set $\{x_B : B \in \mathcal{R}(\mathbf{h}, \mathbf{x}, \varepsilon, \varepsilon)\}$ for all $\varepsilon > 0$ small enough. Since the image of $\{x_B : B \in \mathcal{R}(\mathbf{h}, \mathbf{x}, \varepsilon, \varepsilon)\}$ under m is also in $\mathcal{M}$, the claim follows. □

The following lemma is needed in Proposition 13, toward the proof of the functional law of iterated logarithm for the environment. In order to show that all rate-1 measures are limit points, we need to show that they can be approximated by lower-rate ones. This is implied by the following.

LEMMA 26.   *The minimum of the rate function I on an open set is either zero, infinity, or is not achieved.*

PROOF.   Let $\mu \in G$ with G open and $I(\mu)$ positive and finite. Recall that $I(\mu)$ is defined in (1) in terms of f, g whose graph μ is supported on. Let $\mu_\varepsilon(I \times J) = \mu(I \times (1 - \varepsilon)^{-1}J)$, that is, a scaled version of μ that is supported on the graph of $(1 - \varepsilon)f$ and $(1 - \varepsilon)g$. Then $I(\mu_\varepsilon) = (1 - \varepsilon)I(\mu)$. Also, $\mu_\varepsilon \to \mu$ locally weakly as $\varepsilon \to 0$, so for small enough ε we have $\mu_\varepsilon \in G$. □

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