ON A REFINEMENT OF THE GENERALIZED CATALAN NUMBERS FOR WEYL GROUPS

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ABSTRACT. Let Φ be an irreducible crystallographic root system with Weyl group W, coroot lattice $\check{Q}$ and Coxeter number h, spanning a Euclidean space V, and let m be a positive integer. It is known that the set of regions into which the fundamental chamber of W is dissected by the hyperplanes in V of the form $(\alpha, x) = k$ for $\alpha \in \Phi$ and $k = 1, 2, \ldots, m$ is equinumerous to the set of orbits of the action of W on the quotient $\check{Q}/(mh+1)\check{Q}$. A bijection between these two sets, as well as a bijection to the set of certain chains of order ideals in the root poset of Φ, are described and are shown to preserve certain natural statistics on these sets. The number of elements of these sets and their corresponding refinements generalize the classical Catalan and Narayana numbers, which occur in the special case $m = 1$ and $\Phi = A_{n-1}$.

1. INTRODUCTION AND RESULTS

Let V be an ℓ-dimensional Euclidean space, with inner product $(\ ,\ )$, and let Φ be an irreducible crystallographic root system spanning V. Let m be a fixed nonnegative integer. We denote by Cat_Φ^m the collection of hyperplanes in V defined by the affine equations $(\alpha, x) = k$ for $\alpha \in \Phi$ and $k = 0, 1, \ldots, m$. Thus Cat_Φ^m is a deformation of the Coxeter arrangement $\mathcal{A}_\Phi$, in the sense of [1, 16], known as an *extended Catalan arrangement*. It is invariant under the action of the Weyl group W associated to Φ and reduces to $\mathcal{A}_\Phi$ for $m = 0$. The Weyl group W acts on the coroot lattice $\check{Q}$ of Φ and its dilate $(mh+1)\check{Q}$, where h denotes the Coxeter number of Φ. Hence W acts also on the quotient

$$\check{Q}/(mh+1)\check{Q}.$$

We denote this quotient by T^m. For a fixed choice of a positive system $\Phi^+ \subseteq \Phi$, consider the partial order on Φ^+ defined by letting $\alpha \le \beta$ if $\beta - \alpha$ is a nonnegative linear combination of positive roots, known as the *root poset* of Φ. A (dual) *order ideal* in Φ^+ is a subset I of Φ^+ such that $\alpha \in I$ and $\alpha \le \beta$ in Φ^+ imply $\beta \in I$.

Our starting point is the following theorem, on the origins of which we comment later in this section.

Theorem 1.1 ([2, 10, 21]). *Let Φ be an irreducible crystallographic root system with Weyl group W, and let m be a nonnegative integer.*

The number of regions into which the fundamental chamber of $\mathcal{A}_\Phi$ is dissected by the hyperplanes of Cat_Φ^m *is equal to the number of orbits of the action of W on T^m and is given by the product*

$$(1.1) \qquad\qquad \prod_{i=1}^{\ell} \frac{e_i + mh + 1}{e_i + 1},$$

where ℓ is the rank, h is the Coxeter number and $e_1, e_2, \ldots, e_\ell$ are the exponents of Φ. Moreover, for $m = 1$ this number is equal to the number of ideals in the root poset of Φ.

To state our main result we need to introduce some more notation and definitions. For $y \in T^m$, consider the stabilizer of y with respect to the W-action on T^m. This is a subgroup of W generated by reflections. We call the minimum number of reflections needed to generate this subgroup its *rank* and denote it by $r(y)$. We may use the notation $r(x)$ for a W-orbit x in T^m, since stabilizers of elements of T^m in the same W-orbit are conjugate subgroups of W and hence have the same rank. We denote by $H_{\alpha,k}$ the affine hyperplane in V defined by the equation $(\alpha, x) = k$, and by $A_\circ$ the fundamental alcove of the affine Weyl arrangement corresponding to Φ. A *wall* of a region R of Cat_Φ^m is a hyperplane in V which supports a facet of R. We call a decreasing chain $\Phi^+ = I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots \supseteq I_m$ of ideals in Φ^+ a *co-filtered chain of ideals* of length m if

$$(1.2) \qquad\qquad (I_i + I_j) \cap \Phi^+ \subseteq I_{i+j}$$

holds for all indices i, j, where $I_i = I_m$ for $i > m$, and

$$(1.3) \qquad\qquad (J_i + J_j) \cap \Phi^+ \subseteq J_{i+j}$$

holds for all indices $i, j \geq 1$ with $i + j \leq m$, where $J_i = \Phi^+ \setminus I_i$. A positive root α is *indecomposable* of *rank m* with respect to this chain if $\alpha \in I_m$ and it is not possible to write $\alpha = \beta + \gamma$ with $\beta \in I_i$ and $\gamma \in I_j$ for indices $i, j \geq 0$ with $i + j = m$. The following theorem gives a refinement of Theorem 1.1.

Theorem 1.2. *Let Φ be an irreducible crystallographic root system with Weyl group W, let m be a positive integer and let $O^m(\Phi)$ be the set of orbits of the action of W on T^m. For any nonnegative integer i, the following are equal:*

- (i) *the number of regions R of Cat_Φ^m in the fundamental chamber of $\mathcal{A}_\Phi$ such that i walls of R of the form $H_{\alpha,m}$ separate R from $A_\circ$,*
- (ii) *the number of orbits $x \in O^m(\Phi)$ with $r(x) = i$, and*
- (iii) *the number of co-filtered chains of ideals in the root poset Φ^+ of length m having i indecomposable elements of rank m.*

In particular, the number of co-filtered chains of ideals in Φ^+ of length m is equal to the product (1.1).

In the special case $m = 1$ a co-filtered chain of ideals consists of a single ideal I in Φ and the set of rank one indecomposable elements is the set of minimal elements of I, which is an antichain in Φ^+. The arrangement Cat_Φ^m in this case consists of the hyperplanes H_α and $H_{\alpha,1}$ for all $\alpha \in \Phi$ and is known as the *Catalan arrangement* associated to Φ, denoted Cat_Φ. The equality between the quantities in (i) and (iii) of the following corollary of Theorem 1.2 was conjectured by S. Fomin and J.R. Stembridge and independently by F. Chapoton. It was obtained earlier in

an equivalent form (via Lemma 4.1) by Panyushev [14] and is also implicit in the recent work of Sommers [22] (see Proposition 6.2 there).

Corollary 1.3. *Let Φ be an irreducible crystallographic root system with Weyl group W, and let $O(\Phi)$ be the set of orbits of the action of W on $T = \check{Q}/(h+1)\,\check{Q}$. For any nonnegative integer i, the following are equal:*

  (i) *the number of i-element antichains in the root poset of Φ,*
  (ii) *the number of regions R of Cat_Φ in the fundamental chamber of $\mathcal{A}_\Phi$ such that i walls of R separate R from $A_\circ$, and*
  (iii) *the number of orbits $x \in O(\Phi)$ with $r(x) = i$.*

A few comments on Theorem 1.1 are in order. Formula (1.1) was proved for the number of regions of Cat_Φ^m in the fundamental chamber by the author [2, Corollary 1.3] (see also [1, Theorem 4.6]) and for the number of W-orbits of $\check{Q}/k\check{Q}$ (with $mh+1$ replaced by k) by Haiman [10, Theorem 7.4.2] under a hypothesis on k which includes the cases $k = mh + 1$. The same formula for the number of such orbits was earlier proved by Djoković [7, page 183] under a more restrictive assumption, namely that k is relatively prime to the order of W. The number of ideals in Φ^+ was computed case by case by Shi [21], and by A. Postnikov (unpublished) in the cases of the classical root systems, and was observed in [17] to fit the uniform formula (1.1) for $m = 1$. In his study of sign types for affine Weyl groups [20, 21], Shi also showed that the set of ideals in Φ^+ is in bijection with the set of positive sign types for W and with the set of regions into which the fundamental chamber is dissected by the hyperplanes of Cat_Φ. A bijection between ideals in Φ^+ and W-orbits of T was later described by Cellini and Papi [5] and is based on the construction of a map which assigns an element of the affine Weyl group W_a to each ideal in Φ^+ [4].

In the special case $m = 1$, the expression (1.1) is refered to as the *Catalan number* associated to Φ [2, 17], since it reduces to the familiar nth Catalan number for the root system A_{n-1}. It has a variety of interesting combinatorial interpretations apart from those given in Theorem 1.1, for instance as the number of noncrossing partitions associated to W [3, Proposition 5.2.1], [15, Section 4], as the number of vertices of the simple generalized associahedron associated to Φ [8, Theorem 1.9], and as the number of clusters in the corresponding cluster algebra of finite type [9]. No case-free proofs of any of these facts are known. The numbers which appear in Corollary 1.3 refine this Catalan number and are known as the *Narayana numbers* [12] in the case of the root system A_{n-1}. Therefore they may be refered to as the Narayana numbers associated to Φ. It has been verified case by case that the same numbers appear as the rank numbers of the poset of noncrossing partitions associated to W and as the entries of the h-vector of the simplicial generalized associahedron associated to Φ [6, 8] (and hence as the dimensions of the real cohomology groups of the associated complex projective toric variety); see, for instance, [18, Section 5.2]. These Catalan and Narayana numbers are generalized by the numbers which appear in Theorems 1.1 and 1.2, respectively.

Theorem 1.2 is proved in Sections 3 and 4 by means of two bijections. The first maps the set of co-filtered chains of ideals in Φ^+ of length m to the set of regions of Cat_Φ^m in the fundamental chamber (Theorem 3.6). It specializes to the bijection of Shi [21], mentioned earlier, when $m = 1$. The second maps this set of regions to the set of W-orbits of T^m (see Lemma 4.1 and Theorem 4.2). Our proof in fact yields a refinement of Theorem 1.2 (see Corollaries 3.12 and 4.3). More precisely,

it is shown that for any fixed k with $1 \le k \le m$ and for corresponding chain $\mathcal{I}$, region R and orbit x under the bijections, indecomposable elements of $\mathcal{I}$ of rank k (see Definition 3.8) correspond to facets of R of the form $H_{\alpha,k}$ separating R from $A_\circ$ and to facets of a certain simplex Σ_k^m whose supporting hyperplanes contain the point of $\check{Q} \cap (mh+1) \overline{A_\circ}$ which represents the orbit x. The simplices Σ_k^m are translated dilates of $\overline{A_\circ}$, one nested inside the other. In the case $m = 1$, it turns out that the composite of our two bijections coincides with the one constructed in [5]. However, our motivation for constructing the second bijection comes from the paper by Shi [20], in which an element of the affine Weyl group is associated to each sign type of Φ in order to count the total number of sign types. The existence of such an element (for positive sign types) is generalized to arbitrary m in Section 3 and plays an important role in our proof of Theorem 1.2. Section 2 includes notation and terminology related to root systems and their (affine) Weyl groups, as well as some basic facts that are needed in the following sections. In Section 5 we compute the numbers which appear in Theorem 1.2 in the cases of the classical root systems.

2. Preliminaries

In this section we introduce notation and terminology, and state a few elementary facts related to root systems and Weyl groups. We refer the reader to the text by Humphreys [11] for basic definitions and background.

Let V be an ℓ-dimensional Euclidean space with inner product $(\, , \,)$. Given a hyperplane arrangement $\mathcal{A}$ in V, meaning a discrete set of affine subspaces of V of codimension one, the *regions* of $\mathcal{A}$ are the connected components of the space obtained from V by removing the hyperplanes in $\mathcal{A}$. Let Φ be an irreducible, crystallographic root system spanning V. We fix a positive system $\Phi^+ \subseteq \Phi$ and the corresponding (ordered) set of simple roots $\Delta = (\sigma_1, \dots, \sigma_\ell)$. Part (i) of the following lemma is well known, while part (ii) (which is stronger) follows from the results of Sommers in [22, Section 3] (see also [14, Lemma 2.3]).

Lemma 2.1. (i) *If $\alpha_1, \alpha_2, \dots, \alpha_r \in \Phi^+$ with $r \ge 2$ and $\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_r \in \Phi^+$, then there exists i with $1 \le i \le r$ such that $\alpha - \alpha_i \in \Phi^+$.*

(ii) ([14, 22]) *If $\alpha_1, \alpha_2, \dots, \alpha_r \in \Phi$ and $\alpha_1 + \alpha_2 + \cdots + \alpha_r = \alpha \in \Phi$, then $\alpha_1 = \alpha$ or there exists i with $2 \le i \le r$ such that $\alpha_1 + \alpha_i \in \Phi \cup \{0\}$.*

Proof. For reasons of self-containment we give a proof of part (ii). The statement is trivial for $r \le 2$, so let $r \ge 3$ and proceed by induction on r. Clearly either $\alpha_1 = \alpha$ or $(\alpha - \alpha_1, \alpha_2 + \alpha_3 + \cdots + \alpha_r) > 0$, whence we conclude that $(\alpha_1, \alpha_i) < 0$ or $(\alpha, \alpha_i) > 0$ for some $i \ge 2$. In the former case it follows that $\alpha_1 + \alpha_i \in \Phi \cup \{0\}$, as desired. In the latter case it follows that $\alpha - \alpha_i \in \Phi \cup \{0\}$ and hence, assuming $i = r$ without loss of generality, that either $\alpha_1 + \alpha_2 + \cdots + \alpha_{r-1} = \alpha - \alpha_r \in \Phi$ or $\alpha_1 + \alpha_2 + \cdots + \alpha_{r-2} = -\alpha_{r-1} \in \Phi$. Induction completes the proof in both cases. $\quad\square$

We denote by $s_1, s_2, \dots, s_\ell$, respectively, the reflections in the linear hyperpanes orthogonal to the elements of Δ, called *simple reflections*. We let $\tilde{\alpha}$ be the highest root, $e_1, e_2, \dots, e_\ell$ be the exponents and h be the Coxeter number of Φ, and we set $p = mh + 1$, where m is a fixed positive integer. The following fact can be checked directly, for instance, from the tables given in [11, Sections 3.18 and 4.9].

Lemma 2.2. *If $\tilde{\alpha} = \sum_{i=1}^{\ell} c_i \, \sigma_i$, then $\sum_{i=1}^{\ell} c_i = h - 1$.*

We denote by $\mathcal{A}_\Phi$ the *Coxeter arrangement* associated to Φ, i.e., the collection of linear hyperplanes in V which are orthogonal to the roots, and by W the corresponding *Weyl group*, generated by the reflections in these hyperplanes. Thus W is finite and minimally generated by the set of simple reflections; it leaves Φ invariant and acts simply transitively on the set of regions of $\mathcal{A}_\Phi$, called *chambers*. The *fundamental chamber* is the region defined by the inequalities $0 < (\alpha, x)$ for $\alpha \in \Phi^+$. The set $Z(\Phi)$ of vectors $x \in \mathbb{R}^\ell$ satisfying $(\alpha, x) \in \mathbb{Z}$ for all $\alpha \in \Phi$ is the *coweight lattice* associated to Φ. The *coroot lattice* $\check{Q}$ of Φ is the $\mathbb{Z}$-span of the set of coroots

$$\Phi^\vee = \left\{ \frac{2\alpha}{(\alpha, \alpha)} \ : \ \alpha \in \Phi \right\}.$$

Since Φ is crystallographic, we have $\check{Q}(\Phi) \subseteq Z(\Phi)$. The group W acts on the lattice $\check{Q}$ and on its sublattice $p\check{Q}$; hence it also acts on the quotient $T^m = \check{Q}/p\check{Q}$. We denote by $O^m(\Phi)$ the set of orbits of the W-action on T^m, and we use the notation T and $O(\Phi)$ when $m = 1$.

For any real k and $\alpha \in \Phi$ we denote by $H_{\alpha,k}$ the hyperplane in V defined by the equation $(\alpha, x) = k$, and we set $H_\alpha = H_{\alpha,0}$. We denote by $\widetilde{\mathcal{A}}_\Phi$ the *affine Coxeter arrangement*, which is the infinite hyperplane arrangement in V consisting of the hyperplanes $H_{\alpha,k}$ for $\alpha \in \Phi$ and $k \in \mathbb{Z}$, and by W_a the *affine Weyl group*, generated by the reflections in the hyperplanes of $\widetilde{\mathcal{A}}_\Phi$. The group W_a is the semidirect product of W and the translation group in V corresponding to the coroot lattice $\check{Q}$, and is minimally generated by the set $\{s_0, s_1, \ldots, s_\ell\}$ of *simple affine reflections*, where s_0 is the reflection in the hyperplane $H_{\widetilde{\alpha},1}$. For $w \in W_a$ and $0 \leq i \leq \ell$, the reflection s_i is a *right descent* of w if $\ell(ws_i) < \ell(w)$, where $\ell(w)$ is the length of the shortest expression of w as a product of simple affine reflections. The action of W_a on $\widetilde{\mathcal{A}}_\Phi$ is determined by the following elementary lemma.

Lemma 2.3 ([11, Section 4.1]). *For $w \in W$, $\alpha \in \Phi$, $\lambda \in \check{Q}$ and $k \in \mathbb{R}$ we have:*

  (i) $wH_{\alpha,k} = H_{w\alpha,k}$.
  (ii) $H_{\alpha,k} + \lambda = H_{\alpha,k+(\alpha,\lambda)}$.

The group W_a acts simply transitively on the set of regions of $\widetilde{\mathcal{A}}_\Phi$ called *alcoves*. The *fundamental alcove* of $\widetilde{\mathcal{A}}_\Phi$ can be defined as

$$A_\circ = \{x \in V : 0 < (\sigma_i, x) \text{ for } 1 \leq i \leq \ell \text{ and } (\tilde{\alpha}, x) < 1\}.$$

Note that every alcove can be written as $wA_\circ$ for a unique $w \in W_a$. Moreover, given $\alpha \in \Phi^+$, there exists a unique integer k, denoted $k(w, \alpha)$, such that $k < (\alpha, x) < k+1$ for all $x \in wA_\circ$. The next lemma is the main result of [19].

Lemma 2.4 ([19, Theorem 5.2]). *Let k_α be an integer for each $\alpha \in \Phi^+$. There exists $w \in W_a$ such that $k(w, \alpha) = k_\alpha$ for each $\alpha \in \Phi^+$ if and only if*

$$(2.1) \qquad\qquad k_\alpha + k_\beta \leq k_{\alpha+\beta} \leq k_\alpha + k_\beta + 1$$

for all $\alpha, \beta \in \Phi^+$ with $\alpha + \beta \in \Phi^+$.

We say that two open regions in V are *separated* by a hyperplane $H \in \widetilde{\mathcal{A}}_\Phi$ if they lie in different half-spaces relative to H. If R is a region of a subarrangement of $\widetilde{\mathcal{A}}_\Phi$ or the closure of such a region (in particular, if R is a chamber or an alcove), we refer to the hyperplanes of $\widetilde{\mathcal{A}}_\Phi$ which support facets of R as the *walls* of R. Let

$H_0 = H_{\tilde{\alpha},1}$ and $H_i = H_{\sigma_i}$ for $1 \le i \le \ell$ be the walls of $A_\circ$. The next lemma follows, for instance, from the content of [11, Section 4.5].

Lemma 2.5. *For $w \in W_a$ the number of walls of the alcove $wA_\circ$ separating $wA_\circ$ from $A_\circ$ is equal to the number of right descents of w.*

3. Ideals, regions and the affine Weyl group

In this section we generalize some of the work of Shi [20, 21] and Cellini and Papi [4] on regions of Cat_Φ and order ideals in Φ^+ to regions of Cat_Φ^m and co-filtered chains of ideals, and we prove part of Theorem 1.2. More specifically, we establish the equality of the numbers appearing in (i) and (iii) in the statement of the theorem.

Let $\mathcal{I}$ be a co-filtered chain of ideals

$$\Phi^+ = I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots \supseteq I_m$$

in Φ^+ of length m, so that (1.2) and (1.3) hold, where $J_i = \Phi^+ \setminus I_i$. For any $\alpha \in \Phi^+$ we define

$$k_\alpha(\mathcal{I}) = \max\{k_1 + k_2 + \cdots + k_r : \alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_r \text{ with } \alpha_i \in I_{k_i} \text{ for all } i\}.$$

This quantity will play a major role in the sequel. Observe that $k_\alpha(\mathcal{I}) = 0$ if and only if $\alpha \in J_1$. Moreover, if $k_\alpha(\mathcal{I}) \ge 1$, then, by applying Lemma 2.1 (ii) repeatedly, we can write $\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_r$ with $\alpha_i \in I_{k_i}$, $k_\alpha(\mathcal{I}) = k_1 + k_2 + \cdots + k_r$ and $k_i \ge 1$ for all i. We begin with a few lemmas.

Lemma 3.1. *If $\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_r \in \Phi^+$ and $\alpha_i \in \Phi^+$ for all i, then*

$$k_\alpha(\mathcal{I}) \ge \sum_{i=1}^{r} k_{\alpha_i}(\mathcal{I}).$$

Proof. This is clear from the definition. $\square$

Lemma 3.2. *Let $\alpha \in \Phi^+$ and $k_\alpha(\mathcal{I}) = k$.*

(i) *We have $\alpha \in I_{\min\{k,\,m\}}$.*
(ii) *If $k > m$, then there exist $\beta, \gamma \in I_1$ with $\alpha = \beta + \gamma$ and $k = k_\beta(\mathcal{I}) + k_\gamma(\mathcal{I})$. Moreover, we may choose β so that $\beta \in I_t$, where $k_\beta(\mathcal{I}) = t \le m$.*

Proof. Let $\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_r$ with $\alpha_i \in I_{k_i}$ for all i and $k = k_1 + k_2 + \cdots + k_r$. We may assume that $k \ge 1$ and hence that $k_i \ge 1$ for all i. Conditions (1.2) and Lemma 2.1 (i) imply part (i) by induction on r. If $k > m$, then clearly $r \ge 2$, and by Lemma 2.1 (i) again there exists an index j such that $\alpha - \alpha_j \in \Phi^+$. Let $\beta = \alpha_j$, $\gamma = \alpha - \alpha_j$ and $t = k_j$, and observe that $k_\beta(\mathcal{I}) \ge t$ and $k_\gamma(\mathcal{I}) \ge k - t$. Since $k_\beta(\mathcal{I}) + k_\gamma(\mathcal{I}) \le k$ by Lemma 3.1, we must have $k_\beta(\mathcal{I}) = t$ and $k_\gamma(\mathcal{I}) = k - t$. This proves (ii). $\square$

The next lemma generalizes Lemma 2.3 in [4].

Lemma 3.3. *If $\alpha, \beta, \alpha + \beta \in \Phi^+$ and $a, b \ge 0$ are integers such that $k_{\alpha+\beta}(\mathcal{I}) \ge a + b$, then $k_\alpha(\mathcal{I}) \ge a$ or $k_\beta(\mathcal{I}) \ge b$.*

Proof. We proceed by induction on $k = k_{\alpha+\beta}(\mathcal{I})$. If $k \leq m$ then $\alpha + \beta \in I_k$ by Lemma 3.2 (i), and hence $\alpha + \beta \in I_{a+b}$. In view of (1.3) this implies that $\alpha \in I_a$ or $\beta \in I_b$, from which the result is clear. If $k > m$, by Lemma 3.2 (ii) we can write $\alpha + \beta = \eta + \theta$, where $\eta, \theta \in \Phi^+$, $k_\eta(\mathcal{I}) = t$, $k_\theta(\mathcal{I}) = k - t$ and $0 < t < k$. If $\alpha = \eta$ and $\beta = \theta$ or vice versa, then $k_\alpha(\mathcal{I}) + k_\beta(\mathcal{I}) = k_\eta(\mathcal{I}) + k_\theta(\mathcal{I}) = k \geq a + b$, and the result is clear. If not, since

$$(\alpha, \eta) + (\alpha, \theta) + (\beta, \eta) + (\beta, \theta) = (\alpha + \beta, \alpha + \beta) > 0,$$

one of the four summands on the left side must be positive. This implies that at least one of $\alpha - \eta = \theta - \beta$ and $\alpha - \theta = \eta - \beta$ is in Φ. By exchanging the roles of η and θ, if necessary, we may assume that $\alpha - \eta \in \Phi$. By exchanging the roles of α and β too, if necessary, we may assume further that $\alpha - \eta \in \Phi^+$. Let $\alpha = \eta + \mu$ with $\mu \in \Phi^+$, so that $\beta = \theta - \mu$. If $k_\alpha(\mathcal{I}) \geq a$, then there is nothing to prove. If $k_\alpha(\mathcal{I}) < a$, then $k_\eta(\mathcal{I}) + k_\mu(\mathcal{I}) < a$ by Lemma 3.1, so that $k_\mu(\mathcal{I}) < a - t$. Since $\theta = \beta + \mu$ and $k_\theta(\mathcal{I}) = k - t < k$, it follows by induction that $k_\beta(\mathcal{I}) \geq k - a$ and hence that $k_\beta(\mathcal{I}) \geq b$, as desired. $\qquad\square$

Corollary 3.4. *We have*

$$k_\alpha(\mathcal{I}) + k_\beta(\mathcal{I}) \ \leq \ k_{\alpha+\beta}(\mathcal{I}) \ \leq \ k_\alpha(\mathcal{I}) + k_\beta(\mathcal{I}) + 1$$

whenever $\alpha, \beta, \alpha + \beta \in \Phi^+$.

Proof. The first inequality is a special case of Lemma 3.1. Letting $a = k_\alpha(\mathcal{I}) + 1$ and $b = k_{\alpha+\beta}(\mathcal{I}) - a$, the second inequality follows from Lemma 3.3 in the case $b \geq 0$, and is obvious otherwise. $\qquad\square$

To define a region R of Cat_Φ^m in the fundamental chamber associated to $\mathcal{I}$, let $R_\mathcal{I}$ be the set of points $x \in V$ which satisfy

$$\begin{aligned} (\alpha, x) > k, &\qquad \text{if} \ \ \alpha \in I_k \ , \\ 0 < (\alpha, x) < k, &\qquad \text{if} \ \ \alpha \in J_k, \end{aligned}$$

for $0 \leq k \leq m$.

Proposition 3.5. *There exists $w \in W_a$ such that $k(w, \alpha) = k_\alpha(\mathcal{I})$ for $\alpha \in \Phi^+$. Moreover, $wA_\circ \subseteq R_\mathcal{I}$. In particular, $R_\mathcal{I}$ is nonempty.*

Proof. The first statement follows from Lemma 2.4 and Corollary 3.4. For the second statement, let $\alpha \in \Phi^+$ and $0 \leq k \leq m$. Part (i) of Lemma 3.2 implies that $k_\alpha(\mathcal{I}) \geq k$ if and only if $\alpha \in I_k$. Hence from the inequalities

$$k_\alpha(\mathcal{I}) \ < \ (\alpha, x) \ < \ k_\alpha(\mathcal{I}) + 1,$$

which hold for $x \in wA_\circ$, we conclude that $wA_\circ \subseteq R_\mathcal{I}$. $\qquad\square$

Let ψ be the map which assigns the set $R_\mathcal{I}$ to a co-filtered chain of ideals $\mathcal{I}$ in Φ^+ of length m. Conversely, given a region R of Cat_Φ^m in the fundamental chamber of $\mathcal{A}_\Phi$, let $\phi(R)$ be the sequence $\Phi^+ = I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots \supseteq I_m$, where I_k is the set of $\alpha \in \Phi^+$ for which $(\alpha, x) > k$ in R. The next theorem generalizes [21, Theorem 1.4].

Theorem 3.6. *The map ψ is a bijection from the set of co-filtered chains of ideals in Φ^+ of length m to the set of regions of Cat_Φ^m which lie in the fundamental chamber of $\mathcal{A}_\Phi$, and the map ϕ is its inverse.*
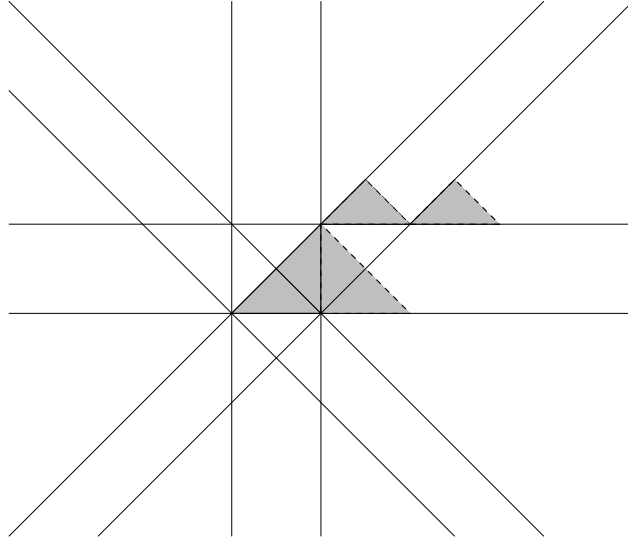
FIGURE 1. The representing alcoves of the regions in the funda-
mental chamber for $\Phi = B_2$ and $m = 1$.

Proof. That the map ψ is well defined follows from Proposition 3.5, which guar-
antees that the set $R_{\mathcal{I}}$ is nonempty. To check that ϕ is well defined, observe that
if R is a region of Cat^m_Φ in the fundamental chamber of $\mathcal{A}_\Phi$ and if $(\alpha, x) > i$ and
$(\beta, x) > j$ hold for $x \in R$, then $(\alpha + \beta, x) > i + j$ must hold for $x \in R$, so that $\phi(\mathcal{I})$
satisfies (1.2). Similarly, $\phi(\mathcal{I})$ satisfies (1.3). It is clear that ψ and ϕ are inverses
of each other. $\square$

Let $R = R_{\mathcal{I}}$ be a region of Cat^m_Φ in the fundamental chamber of $\mathcal{A}_\Phi$, where
$\mathcal{I} = \phi(R)$. Let w_R denote the element of the affine Weyl group W_a which is
assigned to $\mathcal{I}$ in Proposition 3.5. The following proposition implies that $w_R A_\circ$ is
the alcove in R which is closest to $A_\circ$; see Figures 1 and 2 (§4). In the special case
$m = 1$ the existence of such an alcove was established by Shi [20, Proposition 7.2]
for all regions of the arrangement in V consisting of the hyperplanes H_α and $H_{\alpha,1}$
for $\alpha \in \Phi^+$.

Proposition 3.7. *Let R be a region of Cat^m_Φ in the fundamental chamber of $\mathcal{A}_\Phi$.
The element w_R is the unique $w \in W_a$ such that $w A_\circ \subseteq R$ and whenever $\alpha \in \Phi^+$,
$k \in \mathbb{Z}$ and $(\alpha, x) < k$ for some $x \in R$ we have $(\alpha, x) < k$ for all $x \in w A_\circ$.*

Proof. Suppose that $\alpha \in \Phi^+$, and $(\alpha, x) < k$ holds for some $x \in R = R_{\mathcal{I}}$ and $k \in \mathbb{Z}$.
To show that $(\alpha, x) < k$ holds for $x \in w_R A_\circ$ it suffices to show that $k_\alpha(\mathcal{I}) < k$. Let
us write $\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_r$ with $\alpha_i \in I_{k_i}$ for all i and $k_1 + k_2 + \cdots + k_r = k_\alpha(\mathcal{I})$.
We have $(\alpha_i, x) > k_i$ for all $x \in R$, and hence $(\alpha, x) > k_\alpha(\mathcal{I})$ for all such x. Thus
if $(\alpha, x) < k$ for some $x \in R$, we must have $k_\alpha(\mathcal{I}) < k$. Uniqueness is clear. $\square$

We now introduce the elements of Φ^+ which will play the role that the minimal
elements of the ideal I play in the case $m = 1$.

Definition 3.8. Given $1 \leq k \leq m$, a root $\alpha \in \Phi^+$ is indecomposable of rank k with respect to $\mathcal{I}$ if $\alpha \in I_k$ and

  (i) $k_\alpha(\mathcal{I}) = k$,
  (ii) it is not possible to write $\alpha = \beta + \gamma$ with $\beta \in I_i$ and $\gamma \in I_j$ for indices $i, j \geq 0$ with $i + j = k$, and
  (iii) if $\alpha + \beta \in I_t$ for some $\beta \in \Phi^+$ and $t \leq m$ and if $k_{\alpha+\beta}(\mathcal{I}) = t$, then $\beta \in I_{t-k}$.

Observe that, by part (i) of Lemma 3.2, the assumption $\alpha \in I_k$ in this definition is actually implied by condition (i). For $k = m$ the definition is equivalent to the one proposed in Section 1, as the following lemma shows.

Lemma 3.9. *A positive root α is indecomposable of rank m with respect to $\mathcal{I}$ if and only if $\alpha \in I_m$ and it is not possible to write $\alpha = \beta + \gamma$ with $\beta \in I_i$ and $\gamma \in I_j$ for indices $i, j \geq 0$ with $i + j = m$.*

Proof. Suppose that $\alpha \in I_m$ satisfies the condition in the statement of the lemma. Since condition (iii) in Definition 3.8 is trivial for $k = m$, to show that α is indecomposable of rank m it suffices to show that $k_\alpha(\mathcal{I}) = m$. Indeed, if $k_\alpha(\mathcal{I}) = k > m$, then, by part (ii) of Lemma 3.2, we can write $\alpha = \beta + \gamma$ with $\beta \in I_t$ for some $1 \leq t \leq m$, $\gamma \in I_1$ and $k_\gamma(\mathcal{I}) = k - t$. Then $k_\gamma(\mathcal{I}) > m - t$, and part (i) of the same lemma implies that $\gamma \in I_{m-t}$, contrary to the hypothesis. The converse is trivial. $\square$

Lemma 3.10. *Suppose that α is indecomposable with respect to $\mathcal{I}$.*
  (i) *We have $k_\alpha(\mathcal{I}) = k_\beta(\mathcal{I}) + k_\gamma(\mathcal{I}) + 1$ whenever $\alpha = \beta + \gamma$ with $\beta, \gamma \in \Phi^+$.*
  (ii) *We have $k_\alpha(\mathcal{I}) + k_\beta(\mathcal{I}) = k_{\alpha+\beta}(\mathcal{I})$ whenever $\beta, \alpha + \beta \in \Phi^+$.*

Proof. Let $k_\alpha(\mathcal{I}) = k$, so that $\alpha \in I_k$.

(i) Suppose that $\alpha = \beta + \gamma$ with $\beta, \gamma \in \Phi^+$, and let $k_\beta(\mathcal{I}) = i$ and $k_\gamma(\mathcal{I}) = j$. By Lemma 3.2 (i) we have $\beta \in I_i$ and $\gamma \in I_j$, so condition (ii) in Definition 3.8 does not allow that $k = i + j$. By Corollary 3.4 we must have $k = i + j + 1$.

(ii) Suppose that $\beta, \alpha + \beta \in \Phi^+$ and write $\alpha + \beta = \alpha_1 + \alpha_2 + \cdots + \alpha_r$ with $\alpha_i \in I_{k_i}$ for all i and $k_1 + k_2 + \cdots + k_r = k_{\alpha+\beta}(\mathcal{I})$. Letting $t = k_{\alpha+\beta}(\mathcal{I})$, by Corollary 3.4 it suffices to show that $k_\beta(\mathcal{I}) \geq t - k$. If $r = 1$, then $\alpha + \beta \in I_t$, and condition (iii) in Definition 3.8 implies that $\beta \in I_{t-k}$, so the desired inequality is clear. Suppose $r \geq 2$. We have $-\alpha + \alpha_1 + \cdots + \alpha_r = \beta \in \Phi$, and hence $-\alpha + \alpha_i \in \Phi \cup \{0\}$ for some i by Lemma 2.1 (ii), say for $i = 1$. Thus either $\alpha = \alpha_1$ or $\alpha = \alpha_1 + \mu$ or $\alpha_1 = \alpha + \mu$ with $\mu \in \Phi^+$. In the first case we have $\beta = \alpha_2 + \cdots + \alpha_r$, and hence $k_\beta(\mathcal{I}) = t - k_1 = t - k$. In the second case we have $\beta + \mu = \alpha_2 + \cdots + \alpha_r$, and hence $k_{\beta+\mu}(\mathcal{I}) = k_2 + \cdots + k_r = t - k_1$. Since α is indecomposable, we have $k = k_\alpha(\mathcal{I}) = k_{\alpha_1}(\mathcal{I}) + k_\mu(\mathcal{I}) + 1 = k_1 + k_\mu(\mathcal{I}) + 1$ by Lemma 3.10 (i), and hence

$$k_\beta(\mathcal{I}) \geq k_{\beta+\mu}(\mathcal{I}) - k_\mu(\mathcal{I}) - 1 = k_{\beta+\mu}(\mathcal{I}) + k_1 - k = t - k.$$

In the last case we have $k_\mu(\mathcal{I}) \geq k_1 - k$ by the case $r = 1$ treated earlier and $\beta = \mu + \alpha_2 + \cdots + \alpha_r$, so

$$k_\beta(\mathcal{I}) \geq k_\mu(\mathcal{I}) + k_2 + \cdots + k_r \geq t - k.$$

This completes the proof. $\square$

The following theorem explains the connection between indecomposable elements of $\mathcal{I}$ and separating walls of $R_\mathcal{I}$. Recall from the proof of Proposition 3.7 that $(\alpha, x) > k_\alpha(\mathcal{I})$ holds for all $\alpha \in \Phi^+$ and $x \in R_\mathcal{I}$.

Theorem 3.11. *If $\mathcal{I}$ is a co-filtered chain of ideals in Φ^+ of length m with corresponding region $R = R_{\mathcal{I}}$ and $1 \leq k \leq m$, then the following sets are equal:*

  (i) *the set of indecomposable roots $\alpha \in \Phi^+$ with respect to $\mathcal{I}$ of rank k,*
  (ii) *the set of $\alpha \in \Phi^+$ such that $H_{\alpha,k}$ is a wall of R separating R from $A_{\circ}$, and*
  (iii) *the set of $\alpha \in \Phi^+$ such that $H_{\alpha,k}$ is a wall of $w_R A_{\circ}$ separating $w_R A_{\circ}$ from $A_{\circ}$.*

Proof. Let $\Phi^+ = I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots \supseteq I_m$ be the chain $\mathcal{I}$, as usual. Let $w = w_R$ and let $F_k(\mathcal{I})$, $F_k(R)$ and $F_k(w)$ denote the sets refered to in (i), (ii) and (iii), respectively. To prove that these sets coincide, we will show that $F_k(R) \subseteq F_k(\mathcal{I}) \subseteq F_k(w) \subseteq F_k(R)$.

Suppose that $\alpha \in F_k(R)$. Since $(\alpha, x) > k$ holds for $x \in R$, we have $\alpha \in I_k$. To show that α is indecomposable of rank k with respect to $\mathcal{I}$, we need to check that the conditions in Definition 3.8 are satisfied. We first claim that it is not possible to write $\alpha = \beta + \gamma$ with $\beta, \gamma \in \Phi^+$ satisfying $k_\beta(\mathcal{I}) \geq i$ and $k_\gamma(\mathcal{I}) \geq j$ for some $i, j \geq 0$ with $i + j = k$. Indeed, if that were the case, then $(\beta, x) > i$ and $(\gamma, x) > j$ would be valid for $x \in R$. Since these inequalities imply the inequality $(\alpha, x) > k$, the hyperplane $H_{\alpha,k}$ would not support a facet of R. Clearly the claim implies condition (ii). In view of Lemma 3.2 (ii) it also implies that $k_\alpha(\mathcal{I}) \leq m$, so we have $k \leq k_\alpha(\mathcal{I}) \leq m$. If $k = m$, then we must have $k_\alpha(\mathcal{I}) = m$. If $k < m$, then $(\alpha, x) < k + 1$ holds for $x \in R$, and hence $\alpha \in J_{k+1}$. Lemma 3.2 (i) then forces that $k_\alpha(\mathcal{I}) = k$, and this verifies condition (i). Similarly, to verify (iii) suppose that $\alpha + \beta \in I_t$ for some $\beta \in \Phi^+$ and $k_{\alpha+\beta}(\mathcal{I}) = t$. Since $(\alpha, x) > k$ is implied by the inequalities $(\alpha + \beta, x) > t$ and $(\beta, x) < t - k$ and the first of these is valid for $x \in R$, the second cannot be valid for all such x. This means that $\beta \in I_{t-k}$.

Next suppose that $\alpha \in F_k(\mathcal{I})$, meaning that α is indecomposable of rank k with respect to $\mathcal{I}$. We have $(\alpha, x) > k$ for all $x \in R$ and hence all $x \in wA_{\circ}$. Recall that $k(w, \beta) = k_\beta(\mathcal{I})$ for all $\beta \in \Phi^+$. To show that $H_{\alpha,k}$ is a wall of $wA_{\circ}$, it suffices to show that there exists a $w' \in W_a$ with $k(w', \beta) = \bar{k}_\beta$ for all $\beta \in \Phi^+$, where

$$\bar{k}_\beta = \begin{cases} k_\beta(\mathcal{I}), & \text{if } \beta \neq \alpha, \\ k_\alpha(\mathcal{I}) - 1, & \text{if } \beta = \alpha. \end{cases}$$

In view of Lemma 2.4 and Corollary 3.4, this amounts to proving that $k_\alpha(\mathcal{I}) = k_\beta(\mathcal{I}) + k_\gamma(\mathcal{I}) + 1$ whenever $\alpha = \beta + \gamma$ with $\beta, \gamma \in \Phi^+$, and that $k_\alpha(\mathcal{I}) + k_\beta(\mathcal{I}) = k_{\alpha+\beta}(\mathcal{I})$ whenever $\beta, \alpha + \beta \in \Phi^+$. This is precisely the content of Lemma 3.10.

Finally, suppose that $\alpha \in F_k(w)$. If $w'A_{\circ}$ is the alcove of $\widetilde{\mathcal{A}}_\Phi$ which is the reflection of $wA_{\circ}$ on its wall $H_{\alpha,k}$, then $(\alpha, x) < k$ holds for all $x \in w'A_{\circ}$. Proposition 3.7 implies that $w'A_{\circ}$ is not contained in R. Hence $H_{\alpha,k}$ must be a wall of R; in other words, $\alpha \in F_k(R)$. $\square$

We denote by W_{Cat}^m the subset of W_a consisting of the elements w_R for the regions R of Cat_Φ^m in the fundamental chamber of $\mathcal{A}_\Phi$, and abbreviate this set as W_{Cat} in the case $m = 1$.

Corollary 3.12. *For any nonnegative integers $i_1, i_2, \ldots, i_m$ the following are equal:*

  (i) *the number of co-filtered chains of ideals in Φ^+ of length m having i_k indecomposable elements of rank k for each $1 \leq k \leq m$,*

(ii) *the number of regions R of Cat_Φ^m in the fundamental chamber of $\mathcal{A}_\Phi$ such that i_k walls of R of the form $H_{\alpha,k}$ separate R from $A_\circ$ for each $1 \leq k \leq m$, and*

(iii) *the number of $w \in W_{\mathrm{Cat}}^m$ such that i_k walls of $wA_\circ$ of the form $H_{\alpha,k}$ separate $wA_\circ$ from $A_\circ$ for each $1 \leq k \leq m$.*

Proof. Combine Theorems 3.6 and 3.11.   □

The following corollary is immediate.

Corollary 3.13. *For any nonnegative integer i the quantities which appear in* (i) *and* (iii) *in the statement of Theorem 1.2 are both equal to the number of $w \in W_{\mathrm{Cat}}^m$ such that i walls of $wA_\circ$ of the form $H_{\alpha,m}$ separate $wA_\circ$ from $A_\circ$.*

We now consider the special case $m = 1$. A co-filtered chain of ideals $\mathcal{I}$ of length m in this case is simply a single ideal I in Φ^+ and, by Lemma 3.9, the indecomposable elements of $\mathcal{I}$ are exactly the minimal elements of I. These elements form an antichain in Φ^+, meaning a subset of pairwise incomparable elements. Conversely, any antichain $A \subseteq \Phi^+$ defines the ideal I_A which consists of all $\beta \in \Phi^+$ such that $\beta \geq \alpha$ for some $\alpha \in A$. The set of minimal elements of I_A is equal to A, and the map which assigns I_A to A is a bijection from the set of antichains to the set of ideals in Φ^+.

Corollary 3.14. *For any nonnegative integer i the following are equal:*

(i) *the number of i-element antichains in the root poset of Φ,*

(ii) *the number of regions R of Cat_Φ in the fundamental chamber of $\mathcal{A}_\Phi$ such that i walls of R separate R from $A_\circ$,*

(iii) *the number of $w \in W_{\mathrm{Cat}}$ such that i walls of $wA_\circ$ separate $wA_\circ$ from $A_\circ$, and*

(iv) *the number of elements $w \in W_{\mathrm{Cat}}$ having i right descents.*

Proof. This follows from the case $m = 1$ of Corollary 3.12 and Lemma 2.5.   □

Observe that the region of Cat_Φ which corresponds to the ideal I_A under the bijection ψ of Theorem 3.6 is the set R_A of points x in V which satisfy

$$(\beta, x) > 1, \qquad \text{if } \beta \in I_A,$$
$$0 < (\beta, x) < 1, \quad \text{if } \beta \in \Phi^+ \setminus I_A.$$

By Theorem 3.6, the map which sends an antichain A to the set R_A is a bijection from the set of antichains in Φ^+ to the set of regions of Cat_Φ in the fundamental chamber of $\mathcal{A}_\Phi$. This bijection was first established and exploited by Shi [21]. In view of Theorem 3.11 (applied for $k = m = 1$), its inverse sends a region R to the set of $\alpha \in \Phi^+$ such that $H_{\alpha,1}$ is a wall of R and $(\alpha, x) > 1$ holds for $x \in R$.

4. Proof of the main theorem

In this section we complete the proof of Theorem 1.2 (see Corollary 4.4). We begin with some more notation and known facts on the action of W on T^m.

By the reflection in W corresponding to a hyperplane $H_{\alpha,k}$ we mean the reflection in the linear hyperplane H_α. We let $p = mh + 1$, as in Section 2, and denote by D_Φ^m the set $\check{Q} \cap p\,\overline{A_\circ}$. The following lemma, which is part of [10, Lemma 7.4.1], implies that D_Φ^m is a set of representatives for the orbits of the W-action on T^m.
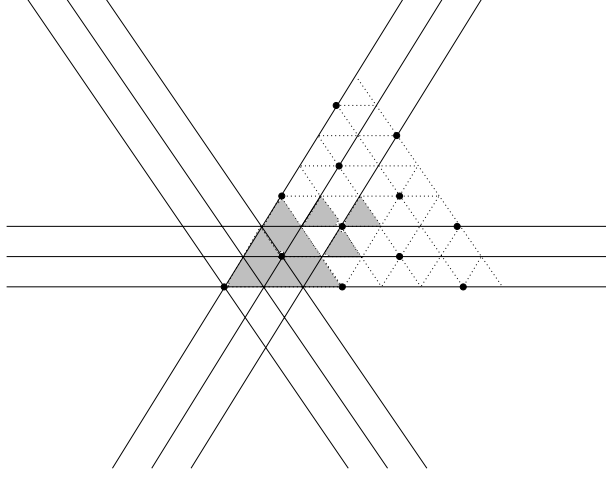
FIGURE 2. The representing alcoves of the regions in the funda-
mental chamber and the simplex $p\,\overline{A_\circ}$ for $\Phi = A_2$ and $m = 2$.

Lemma 4.1. *The natural inclusion map from D_Φ^m to the set $O^m(\Phi)$ of orbits of the W-action on T^m is a bijection.*

Moreover, if $y \in D_\Phi^m$, then the stabilizer of y with respect to the W-action on T^m is the subgroup of W generated by the reflections corresponding to the walls of $p\,\overline{A_\circ}$ which contain y. In particular, $r(y)$ is equal to the number of walls of $p\,\overline{A_\circ}$ which contain y.

Our objective is to define a bijection $\rho : W_{\mathrm{Cat}}^m \to D_\Phi^m$ such that for $w \in W_{\mathrm{Cat}}^m$, the number of walls of $wA_\circ$ of the form $H_{\alpha,m}$ which separate $wA_\circ$ from $A_\circ$ is equal to the number of walls of $p\,\overline{A_\circ}$ which contain $\rho(w)$. Let R_f be the region of Cat_Φ^m defined by the inequalities $m < (\sigma_i, x)$ for $1 \le i \le \ell$. Let $w_f = w_{R_f}$ be the unique element w of W_{Cat}^m such that $wA_\circ \subseteq R_f$. Note that w_f is not necessarily a translation in V (see, for instance, Figure 1). We define the map $\rho : W_{\mathrm{Cat}}^m \to \check{Q}$ by

$$\rho(w) = (w_f\, w^{-1}) \cdot 0$$

for $w \in W_{\mathrm{Cat}}^m$. Observe that, by Lemma 2.2, the alcove $w_f A_\circ$ can be described explicitly as the open simplex in V defined by the linear inequalities $m < (\sigma_i, x)$ for $1 \le i \le \ell$ and $(\tilde{\alpha}, x) < mh - m + 1$. For any $0 \le k \le m$ we define the simplex

$$\Sigma_k^m = \{x \in V : m - k \le (\sigma_i, x) \text{ for } 1 \le i \le \ell \text{ and } (\tilde{\alpha}, x) \le mh - m + k + 1\},$$

so that Σ_k^m reduces to $w_f\overline{A_\circ}$ and $p\,\overline{A_\circ}$ for $k = 0$ and $k = m$, respectively. For any ℓ-dimensional simplex Σ in V bounded by hyperplanes $H_{\alpha,k}$ in $\widetilde{\mathcal{A}}_\Phi$ with $\alpha \in \Delta \cup \{\tilde{\alpha}\}$ we denote by $H(\Sigma, i)$ the wall of Σ orthogonal to $\tilde{\alpha}$ or σ_i, if $i = 0$ or $i > 0$, respectively. We write $H(w, i)$ instead of $H(w\overline{A_\circ}, i)$ for $w \in W_a$. The following theorem is the main result of this section. The reader is invited to verify the statement of the theorem in the case pictured in Figure 2.

Theorem 4.2. *The map ρ is a bijection from W_{Cat}^m to D_Φ^m. Moreover, for any $w \in W_{\mathrm{Cat}}^m$, $1 \le k \le m$ and $0 \le i \le \ell$, the point $\rho(w)$ lies on the wall $H(\Sigma_k^m, i)$ if and only if the wall $(ww_f^{-1})\, H(w_f, i)$ of $wA_\circ$ is of the form $H_{\alpha,k}$ and separates $wA_\circ$ from $A_\circ$.*

Proof. First we will check that ρ is injective. Indeed, if $\rho(w_1) = \rho(w_2)$, with $w_1, w_2 \in W_{\text{Cat}}^m$, then $w_2 w_1^{-1}$ fixes the origin, and hence $w_2 w_1^{-1} \in W$. Moreover, $w_2 w_1^{-1}$ carries $w_1 A_\circ$ to $w_2 A_\circ$, and both alcoves lie in the fundamental chamber of $\mathcal{A}_\Phi$. Since W acts simply transitively on the chambers of $\mathcal{A}_\Phi$, $w_2 w_1^{-1}$ must be the identity; that is, $w_1 = w_2$.

We next check that $\rho(w) \in D_\Phi^m$ for $w \in W_{\text{Cat}}^m$, and prove the second statement in the theorem. Clearly $\rho(w) \in \check{Q}$, so for the first claim it suffices to show that $\rho(w) \in p \overline{A_\circ}$. Let us write $w_f w^{-1} = t_\lambda u$, where $\lambda \in \check{Q}$, t_λ is the translation map in V which sends x to $x + \lambda$, and $u \in W$, so that $\rho(w) = \lambda$. Let $0 \le i \le \ell$, $1 \le k \le m$, and let H be the wall $(w w_f^{-1}) H(w_f, i)$ of $w A_\circ$. We need to show (i) that

$$(4.1) \qquad \begin{array}{ll} (\lambda, \tilde{\alpha}) \le mh + 1, & \text{if } i = 0, \\ (\lambda, \sigma_i) \ge 0, & \text{if } 1 \le i \le \ell, \end{array}$$

and (ii) that $H = H_{\alpha,k}$ for some $\alpha \in \Phi^+$ with H separating $w A_\circ$ from $A_\circ$ if and only if

$$(4.2) \qquad \begin{array}{ll} (\lambda, \tilde{\alpha}) = mh - m + k + 1, & \text{if } i = 0, \\ (\lambda, \sigma_i) = m - k, & \text{if } 1 \le i \le \ell. \end{array}$$

Let $H = H_{\alpha,r}$ for some $\alpha \in \Phi^+$ and nonnegative integer r. Observe first that H separates $w A_\circ$ from $A_\circ$ if and only if $H(w_f, i) = (w_f w^{-1}) H$ separates $w_f A_\circ$ from $(w_f w^{-1}) A_\circ = (t_\lambda u) A_\circ$ or, equivalently, if and only if

$$(4.3) \qquad \begin{array}{ll} (ux + \lambda, \tilde{\alpha}) > mh - m + 1, & \text{if } i = 0, \\ (ux + \lambda, \sigma_i) < m, & \text{if } 1 \le i \le \ell, \end{array}$$

for all $x \in A_\circ$. Next observe that $0 < (\beta, x) < 1$ for all $x \in A_\circ$ and $\beta \in \Phi^+$, and hence that $-1 < (\beta, ux) < 1$ holds for all such x and β. In view of Lemma 2.3, the fact that $t_\lambda u H_{\alpha,r} = w_f w^{-1} H = H(w_f, i)$ implies that

$$(4.4) \qquad H_{u\alpha, r + (u\alpha, \lambda)} = \begin{cases} H_{\tilde{\alpha}, mh-m+1}, & \text{if } i = 0, \\ H_{\sigma_i, m}, & \text{if } 1 \le i \le \ell. \end{cases}$$

To prove (4.1), suppose $i = 0$ first. It follows from (4.4) that either $u\alpha = \tilde{\alpha}$ and $r + (u\alpha, \lambda) = mh - m + 1$, or $u\alpha = -\tilde{\alpha}$ and $r + (u\alpha, \lambda) = -mh + m - 1$. Therefore $(\lambda, \tilde{\alpha}) = mh - m - r + 1$ or $mh - m + r + 1$. Thus to prove that $(\lambda, \tilde{\alpha}) \le mh + 1$ it suffices to show that we have $r \le m$ in the second case. Indeed, suppose that $(\lambda, \tilde{\alpha}) = mh - m + r + 1$. We may assume that $r \ge 1$. From the statement in (4.3) we conclude that $H = H_{\alpha,r}$ separates $w A_\circ$ from $A_\circ$. Since H is a facet of a region of Cat_Φ^m in the fundamental chamber of $\mathcal{A}_\Phi$ (Theorem 3.11), we must have $r \le m$. The case $1 \le i \le \ell$ is similar. From (4.4) we get that either $u\alpha = \sigma_i$ and $r + (u\alpha, \lambda) = m$, or $u\alpha = -\sigma_i$ and $r + (u\alpha, \lambda) = -m$. Therefore $(\lambda, \sigma_i) = m - r$ or $m + r$. If $(\lambda, \sigma_i) = m - r$ and $r \ge 1$, it follows again from the statement in (4.3) that $H = H_{\alpha,r}$ separates $w A_\circ$ from $A_\circ$, and hence $r \le m$, as before. Thus (4.1) holds, which means that $\rho(w) \in p \overline{A_\circ}$.

We now prove the second claim. Suppose first that $H = H_{\alpha,k}$ for some $\alpha \in \Phi^+$, and that H separates $w A_\circ$ from $A_\circ$. Then we have $k \ge 1$, and the analysis we just went through with r replaced by k shows that $(\lambda, \tilde{\alpha}) = mh - m - k + 1$ or $mh - m + k + 1$ if $i = 0$, and that $(\lambda, \sigma_i) = m - k$ or $m + k$ if $1 \le i \le \ell$. In view of (4.3) and the inequality $-1 < (ux, \beta) < 1$ for $x \in A_\circ$ and $\beta \in \Phi^+$, equation (4.2) follows. Conversely, suppose that (4.2) holds for some $1 \le k \le m$. Then (4.3) is

valid, and hence H separates $wA_\circ$ from $A_\circ$. Letting $H = H_{\alpha,r}$ for some $r \geq 1$, we deduce (4.4), as before, and that $(\lambda, \tilde{\alpha}) = mh - m - r + 1$ or $mh - m + r + 1$ if $i = 0$, while $(\lambda, \sigma_i) = m - r$ or $m + r$ if $1 \leq i \leq \ell$. It follows from (4.2) that $r = k$.

Finally, note that W_{Cat}^m and D_Φ^m are equinumerous by Theorem 1.1. Since the map $\rho : W_{\mathrm{Cat}}^m \to D_\Phi^m$ is injective, it has to be a bijection. $\square$

Corollary 4.3. *For any nonnegative integers $i_1, i_2, \ldots, i_m$ each of the quantities which appear in the statement of Corollary 3.12 is equal to the number of points in D_Φ^m which lie in i_k walls of Σ_k^m for all $1 \leq k \leq m$.*

Proof. Indeed, it follows from Theorem 4.2 that the number of $w \in W_{\mathrm{Cat}}^m$ such that i_k walls of $wA_\circ$ of the form $H_{\alpha,k}$ separate $wA_\circ$ from $A_\circ$ for $1 \leq k \leq m$ is equal to the number of points $x \in D_\Phi^m$ such that x lies in i_k walls of Σ_k^m for $1 \leq k \leq m$. $\square$

The following corollary implies Theorem 1.2.

Corollary 4.4. *For any nonnegative integer i each of the quantities which appear in the statement of Theorem 1.2 is equal to*

  (i) *the number of points in D_Φ^m which lie in i walls of $p\,\overline{A_\circ}$, and*
  (ii) *the number of $w \in W_{\mathrm{Cat}}^m$ such that i walls of $wA_\circ$ of the form $H_{\alpha,m}$ separate $wA_\circ$ from $A_\circ$.*

Proof. By Lemma 4.1, the number of orbits $x \in O^m(\Phi)$ with rank $r(x) = i$ is equal to the number of points in D_Φ^m which lie in i walls of $p\,\overline{A_\circ}$. The statement now follows by specializing Corollary 4.3 and recalling that $\Sigma_k^m = p\,\overline{A_\circ}$ for $k = m$. $\square$

Remark 4.5. For $i = 0$, the number which appears in Theorem 1.2 is equal to

$$\prod_{i=1}^{\ell} \frac{e_i + (m-1)h + 1}{e_i + 1}.$$

Indeed, in view of the fact that $\check{Q}(\Phi) \subseteq Z(\Phi)$, the number of points in D_Φ lying in no wall of $p\,\overline{A_\circ}$ is equal to the cardinality of $\check{Q} \cap \Sigma_{m-1}^m$. By Lemma 2.2, the simplex Σ_{m-1}^m is a translate of $(p - h)\,\overline{A_\circ}$ by an element of the coweight lattice $Z(\Phi)$. It follows, for instance, from [5, Lemma 1] and the remark following this lemma that there exists a $w \in W_a$ such that $(p - h)\,\overline{A_\circ} = w(\Sigma_{m-1}^m)$. Thus $\check{Q} \cap \Sigma_{m-1}^m$ is equinumerous to $\check{Q} \cap (p - h)\,\overline{A_\circ}$. Our claim then follows from Corollary 4.4, Lemma 4.1 and Theorem 1.1, where m is replaced by $m - 1$.

5. THE CLASSICAL TYPES

In this section we compute the numbers which appear in Theorem 1.2 in the cases of the classical root systems, as follows.

Proposition 5.1. *Let Φ be a root system, W its Weyl group, and let T^m be as in Theorem 1.2. The number of W-orbits x of T^m with $r(x) = n - k$ is equal to*

$$\frac{1}{k}\binom{n-1}{k-1}\binom{mn}{k-1}, \qquad\qquad\qquad \text{if } \Phi = A_{n-1},$$

$$\binom{n}{k}\binom{mn}{k}, \qquad\qquad\qquad\qquad \text{if } \Phi = B_n \text{ or } C_n,$$

$$\binom{n}{k}\binom{mn-m}{k} + \binom{n-2}{k-2}\binom{mn-m+1}{k}, \qquad \text{if } \Phi = D_n.$$

Remark 5.2. The number of k-element antichains in Φ^+ has been computed for the exceptional root systems by V. Reiner and J. McCammond, as shown in the following table.

TABLE 1. The numbers $\mathrm{N}_\Phi(k)$ for the exceptional root systems.

k	0	1	2	3	4	5	6	7	8
$\Phi = G_2$	1	6	1						
$\Phi = F_4$	1	24	55	24	1				
$\Phi = E_6$	1	36	204	351	204	36	1		
$\Phi = E_7$	1	63	546	1470	1470	546	63	1	
$\Phi = E_8$	1	120	1540	6120	9518	6120	1540	120	1

In the remainder of this section we prove Proposition 5.1. We denote by $\mathrm{N}_\Phi^m(k)$ the number of W-orbits x of T^m with $r(x) = k$, which we simply write as $\mathrm{N}_\Phi(k)$ if $m = 1$.

5.1. The case $\Phi = A_{n-1}$. The coroot lattice $\check{Q}$ is the lattice of points in $\mathbb{Z}^n$ with zero coordinate sum, W is the symmetric group on n elements which acts on $\check{Q}$ by permuting coordinates, and $h = n$. If $Q_{m,n}$ is the set of points in $\mathbb{Z}^n_{mn+1}$ having zero coordinate sum, on which W also acts, then the natural map $\check{Q} \to Q_{m,n}$ induces an isomorphism of W-modules $\check{Q}/(mn + 1)\check{Q} \to Q_{m,n}$. Since n is relatively prime to $mn+1$, it follows that the set $Q_{m,n}$ is in bijection with the set $\mathbb{Z}^n_{mn+1}/(1, 1, \ldots, 1)$ of cosets of the cyclic subgroup of $\mathbb{Z}^n_{mn+1}$ generated by $(1, 1, \ldots, 1)$, and this bijection commutes with the natural action of W on the two sets. The stabilizer within W of such a coset, represented say by $a = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n_{mn+1}$, is generated by the transpositions in coordinates i and j for which $a_i = a_j$. It follows easily that $\mathrm{N}_\Phi^m(k)$ is equal to the number of W-orbits of such cosets for which $\{a_1, \ldots, a_n\}$ has $n - k$ elements, and hence to

$$\frac{1}{mn + 1}\binom{n - 1}{n - k - 1}\binom{mn + 1}{n - k},$$

which is equivalent to the proposed formula. It is known that another set of coset representatives for $\mathbb{Z}^n_{mn+1}/(1, 1, \ldots, 1)$ is the set of *m-parking functions* of length n [23], meaning integer sequences $a = (a_1, a_2, \ldots, a_n)$ such that the unique increasing rearrangement $(b_1, b_2, \ldots, b_n)$ of a satisfies $0 \leq b_i \leq m(i - 1)$ for all i. Thus T^m is isomorphic as a W-module to the set of m-parking functions of length n, on which W acts. The orbits of this action are indexed by the increasing parking functions, in other words the increasing integer sequences $(a_1, a_2, \ldots, a_n)$ with $0 \leq a_i \leq m(i-1)$ for each i. In the case $m = 1$ it follows that $\mathrm{N}_\Phi(k)$ is equal to the number of lattice paths in $\mathbb{Z}^2$ with possible steps $(1, 0)$ and $(0, 1)$ which start at $(0, 0)$, end at (n, n), stay below the line $x = y$ and have $n - k - 1$ northeast corners. The latter is a standard combinatorial interpretation of the Narayana number $\frac{1}{k+1}\binom{n}{k}\binom{n-1}{k}$; see, for instance, [13, 24].

5.2. The case $\Phi = C_n$. We have $h = 2n$, $\check{Q} = \mathbb{Z}^n$, and W is the hyperoctahedral group which acts on $\check{Q}$ by changing signs and permuting coordinates. If $[-mn, mn] = \{-mn, \ldots, -1, 0, 1, \ldots, mn\}$, then it is clear that $\check{Q}/(2mn + 1)\check{Q}$ is isomorphic as a W-module to $\mathbb{Z}^n_{2mn+1}$, and hence to $[-mn, mn]^n$, on which W

acts. The orbits of this action are indexed by the increasing integer sequences $a = (a_1, a_2, \ldots, a_n)$ satisfying $0 \leq a_i \leq mn$. These sequences biject to lattice paths in $\mathbb{Z}^2$ which start at $(0,0)$ and end at (n, mn), where the path corresponding to a is the one which passes through the points $(i - 1, a_i)$ and (i, a_i) for $1 \leq i \leq n$. The stabilizer of a within W is minimally generated by the transpositions in coordinates i and $i + 1$ for those i with $a_i = a_{i+1}$ and the sign change in the first coordinate, if $a_1 = 0$. It follows that $\mathrm{N}_\Phi^m(k)$ is equal to the number of increasing sequences $(a_1, a_2, \ldots, a_n)$ with $0 \leq a_i \leq mn$ such that the set $\{a_1, \ldots, a_n\}$ has $n - k$ nonzero elements, and hence to the number of lattice paths in $\mathbb{Z}^2$ which start at $(0,0)$, end at (n, mn) and have $n - k$ northeast corners. The last statement implies that $\mathrm{N}_\Phi^m(k) = \binom{n}{n-k}\binom{mn}{n-k}$.

5.3. **The case $\Phi = B_n$.** We have $h = 2n$, $\check{Q}$ is the lattice of points in $\mathbb{Z}^n$ having even coordinate sum, and W is as in the type C_n case. By direct reasoning or by appealing to Lemma 4.1 we see that a set of representatives for the orbits of the W-action on $\check{Q}/(2mn + 1)\check{Q}$ is the set $O_{m,n}$ of (cosets of) increasing nonnegative integer sequences $a = (a_1, a_2, \ldots, a_n)$ such that $a_1 + a_2 + \cdots + a_n$ is even and

$$a_{n-1} + a_n \leq 2mn + 1.$$

Moreover, the stabilizer of a within W is minimally generated by the transpositions in coordinates i and $i + 1$ for those i with $a_i = a_{i+1}$, the sign change in the first coordinate, if $a_1 = 0$, and the element of W which exchanges the last two coordinates and changes their signs, if $a_{n-1} + a_n = 2mn + 1$. It follows that $\mathrm{N}_{B_n}^m(k)$ is equal to the number of sequences $a = (a_1, a_2, \ldots, a_n) \in O_{m,n}$ such that the set $\{a_1, \ldots, a_{n-1}, a_n, 2mn+1-a_n\}$ has $n-k+1$ nonzero elements. Let $O'_{m,n}$ be the set of increasing sequences $(a_1, a_2, \ldots, a_n)$ with $0 \leq a_i \leq mn$, considered in the type C_n case. The map $f : O'_{m,n} \rightarrow O_{m,n}$ which sends an element $(a_1, a_2, \ldots, a_n) \in O'_{m,n}$ to

$$\begin{cases} (a_1, \ldots, a_{n-1}, 2a_n - a_{n-1}), & \text{if } a_1 + \cdots + a_{n-2} \text{ is even,} \\ (a_1, \ldots, a_{n-1}, 2mn + 1 - 2a_n + a_{n-1}), & \text{if } a_1 + \cdots + a_{n-2} \text{ is odd,} \end{cases}$$

is a bijection having the property that $\{b_1, \ldots, b_{n-1}, b_n, 2mn + 1 - b_n\}$ has exactly one nonzero element more than $\{a_1, \ldots, a_n\}$, where $(b_1, \ldots, b_n) = f(a_1, \ldots, a_n)$; we leave it to the reader to check this. This shows that $\mathrm{N}_{B_n}^m(k) = \mathrm{N}_{C_n}^m(k)$.

5.4. **The case $\Phi = D_n$.** We have $h = 2n - 2$, where $n \geq 3$, $\check{Q}$ is as in the type B_n case, and W is the index two subgroup of the hyperoctahedral group on n elements which acts on $\check{Q}$ by changing an even number of signs and permuting coordinates. By direct reasoning or by appealing to Lemma 4.1, as in the previous case, we see that a set of representatives for the orbits of the W-action on $\check{Q}/(mh+1)\check{Q}$ is the set $O_{m,n}$ of (cosets of) integer sequences $a = (a_1, a_2, \ldots, a_n)$ such that $a_1 + a_2 + \cdots + a_n$ is even,

(5.1) $$0 \leq |a_1| \leq a_2 \leq \cdots \leq a_n$$

and

(5.2) $$a_{n-1} + a_n \leq 2m(n - 1) + 1.$$

The stabilizer of a within W is minimally generated by the transpositions in coordinates i and $i + 1$ for those i with $a_i = a_{i+1}$, the element of W which exchanges

the first two coordinates and switches their signs, if $a_1 + a_2 = 0$, and that which exchanges the last two coordinates and switches their signs, if equality holds in (5.2). It follows that $N_\Phi(k)$ is equal to the number of sequences $(a_1, a_2, \ldots, a_n) \in O_{m,n}$ such that the set $\{|a_1|, a_2, \ldots, a_{n-1}, a_n, 2m(n-1) + 1 - a_n\}$ has $n - k$ nonzero elements, if $a_1 = 0$ and $a_2 > 0$, and $n - k + 1$ nonzero elements otherwise. Let $O'_{m,n}$ be the set of integer sequences $(a_1, a_2, \ldots, a_n)$ satisfying (5.1) and $a_n \leq m(n-1)$. As in the type B_n case, the bijection $f : O'_{m,n} \to O_{m,n}$ which sends an element $(a_1, a_2, \ldots, a_n) \in O'_{m,n}$ to

$$
\begin{cases}
(a_1, \ldots, a_{n-1}, 2a_n - a_{n-1}), & \text{if } a_1 + \cdots + a_{n-2} \text{ is even,} \\
(a_1, \ldots, a_{n-1}, 2m(n-1) + 1 - 2a_n + a_{n-1}), & \text{if } a_1 + \cdots + a_{n-2} \text{ is odd,}
\end{cases}
$$

shows that $N_\Phi(k)$ is equal to the number of $(a_1, a_2, \ldots, a_n) \in O'_{m,n}$ such that $\{|a_1|, a_2, \ldots, a_n\}$ has $n - k - 1$ nonzero elements, if $a_1 = 0$ and $a_2 > 0$, and $n - k$ nonzero elements otherwise. The set of such sequences in $O'_{m,n}$ with $a_1 = a_2 = 0$ bijects to the set of lattice paths in $\mathbb{Z}^2$ starting at $(2, 0)$, ending at $(n, mn - m)$ and having $n - k$ northeast corners, which has cardinality $\binom{n-2}{n-k}\binom{mn-m}{n-k}$. Similar reasoning shows that there are $\binom{n-2}{n-k-2}\binom{mn-m}{n-k-1}$ such sequences in $O'_{m,n}$ with $a_1 = 0$ and $a_2 > 0$, and $2\binom{n-1}{n-k-1}\binom{mn-m}{n-k}$ having a_1 nonzero. Hence

$$
N_\Phi(k) = \binom{n-2}{k-2}\binom{mn-m}{n-k} + \binom{n-2}{k}\binom{mn-m}{n-k-1} + 2\binom{n-1}{k}\binom{mn-m}{n-k},
$$

which simplifies to the proposed formula.

Acknowledgements

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