Face enumeration of order complexes and real-rootedness

KTH Combinatorics Seminar

May 2, 2024

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Supported by the Hellenic Foundation for Research and Innovation
HFRI - FM20 - 04537

Let

P = finite poset of rank $n-1$

$\Delta(P)$ = order complex of P

= {chains in P}

$c_k(P)$ = # k-element chains in P

= $f_{k-1}(\Delta(P))$.

Example. If

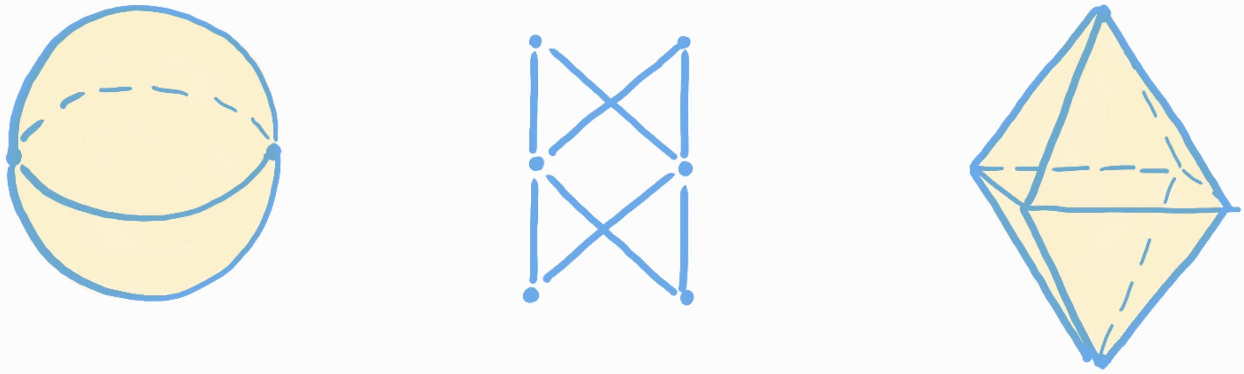
Γ = regular cell complex

P = face poset of Γ

then

$\Delta(P)$ = barycentric subdivision of Γ.

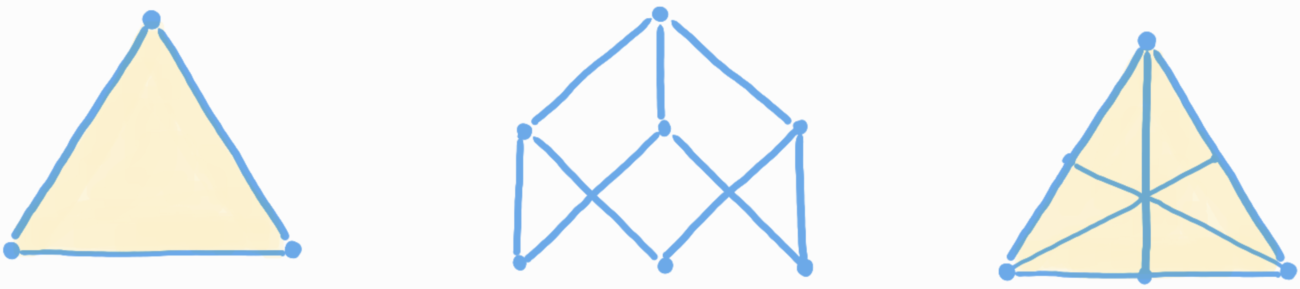
(a)



$\Gamma \qquad P \qquad \Delta(P)$

$$c_0(P) = 1, \quad c_1(P) = 6$$
$$c_2(P) = 12, \quad c_3(P) = 8$$

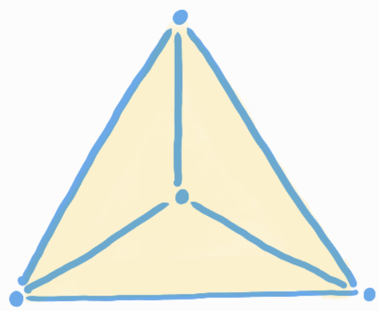
(b)



$\Gamma \qquad P \qquad \Delta(P)$

$$c_0(P) = 1, \quad c_1(P) = 7$$
$$c_2(P) = 12, \quad c_3(P) = 6$$

Remark. The order complex $\Delta(P)$ is *flag*, meaning that every clique in its one-skeleton is a face of $\Delta(P)$.



not flag

Definition. The f,h-polynomials of $\Delta(P)$ are defined as

$$f(\Delta(P),x) = \sum_{k=0}^{n} f_{k-1}(\Delta)\, x^k$$

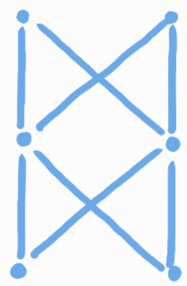
$$= \sum_{k=0}^{n} c_k(P)\, x^k$$

$$= \text{chain polynomial of } P$$

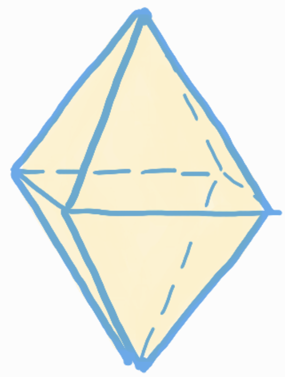
$$h(\Delta(P),x) = \sum_{k=0}^{n} f_{k-1}(\Delta)\, x^k (1-x)^{n-k}$$

$$= (1-x)^n f\left(\Delta, \frac{x}{1-x}\right).$$

Example. (a)



P

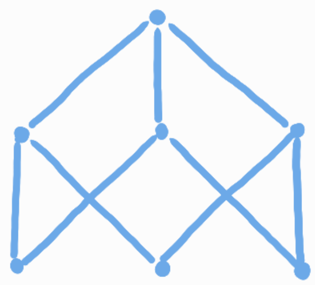


$\Delta(P)$

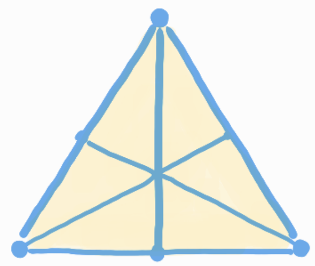
- $f(\Delta(P), x) = 1 + 6x + 12x^2 + 8x^3$
- $h(\Delta(P), x) = (1-x)^3 + 6x(1-x)^2 + 12x^2(1-x) + 8x^3$

$$= (1+x)^3$$

(b)



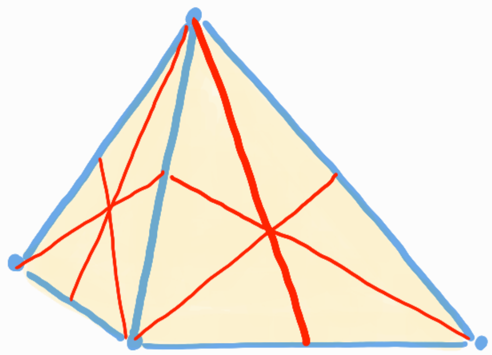
P



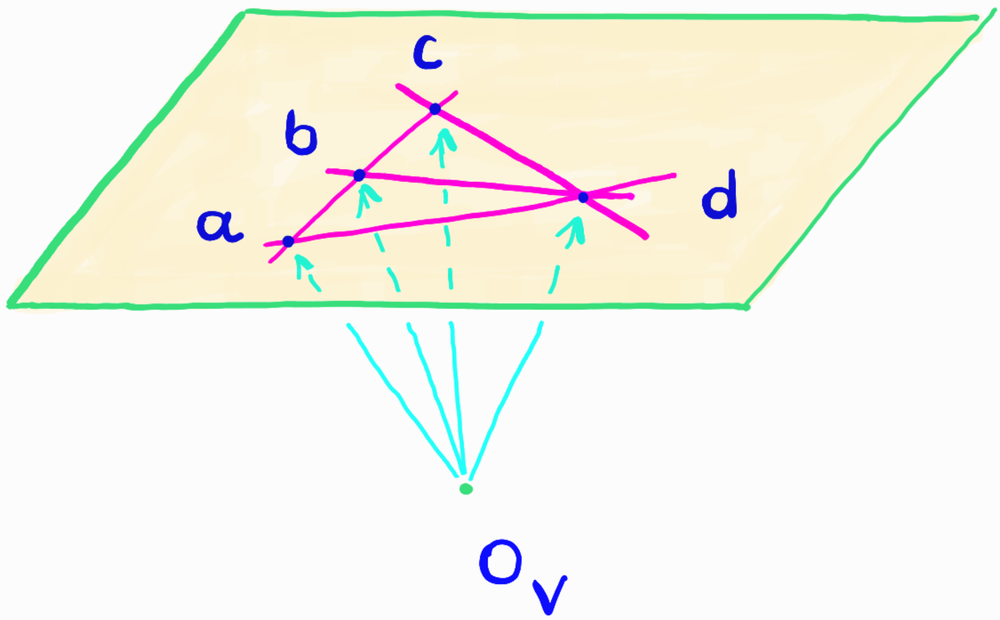
$\Delta(P)$

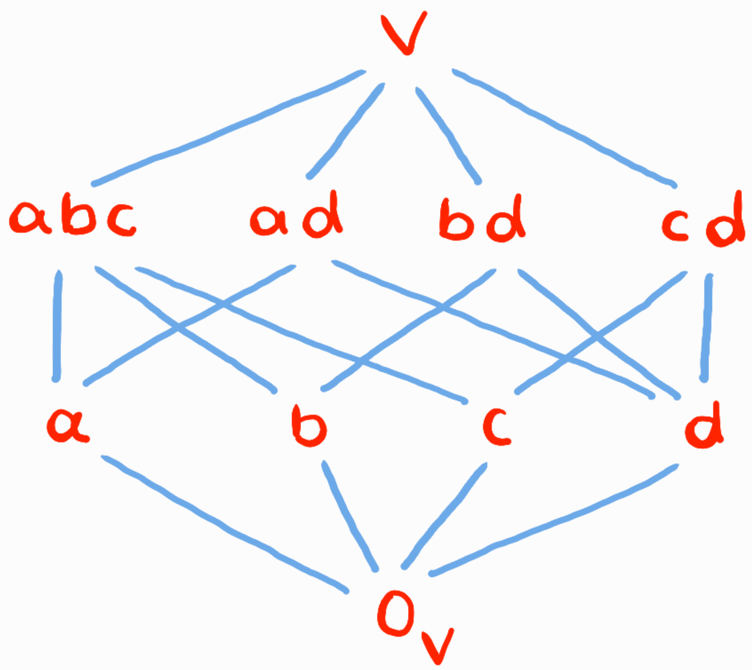
- $f(\Delta(P), x) = 1 + 7x + 12x^2 + 6x^3$
- $$\begin{aligned} h(\Delta(P), x) &= (1-x)^3 + 7x(1-x)^2 + \\ &\quad 12x^2(1-x) + 6x^3 \\ &= 1 + 4x + x^2. \end{aligned}$$

Conjecture (Brenti-Welker, 2008)
$f(\Delta(P), x)$ is real-rooted if P is the face lattice of a polytope.



Conjecture (A-Kalambogia Evangelinou) $f(\Delta(P), x)$ is real-rooted if P is a geometric lattice (the lattice of flats of a matroid).


c
b
d
a
0_V


V
abc
ad
bd
cd
a
b
c
d
0_V

Remark

(a) $f(\Delta(P), x)$ is real-rooted $\Leftrightarrow$ $h(\Delta(P), x)$ is real-rooted.

(b) For geometric lattices, even the unimodality of $h(\Delta(P), x)$ is open.

Even stronger conjectures make sense.

Conjecture (unpublished)

$f(\Delta(P), x)$ is real-rooted if P is any rank-selected subposet of

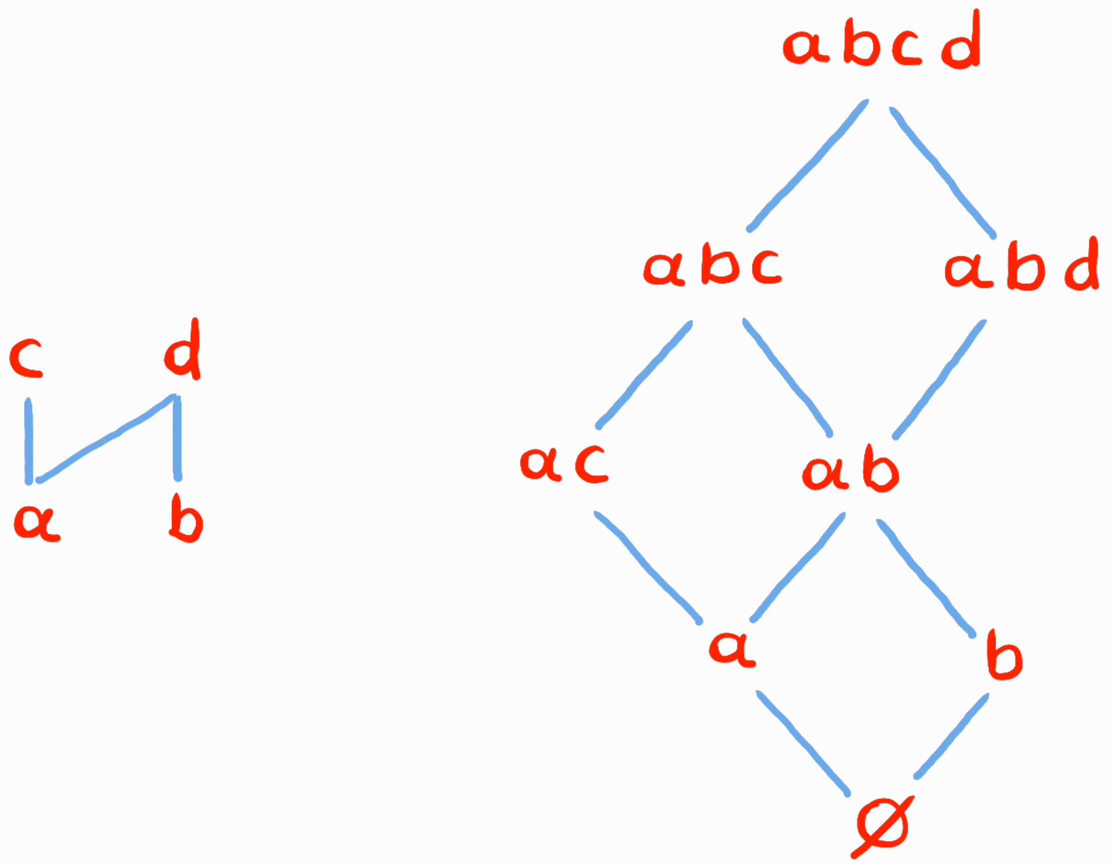
- the face lattice of a polytope, or
- a geometric lattice.

Question For which finite posets P is $f(\Delta(P), x)$ real-rooted?

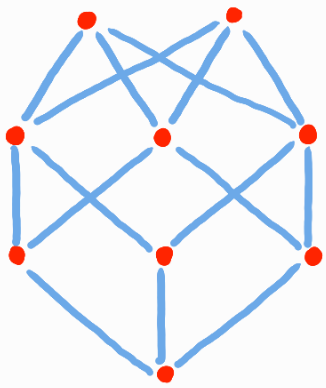
Question Is $f(\Delta(P), x)$ real-rooted for every doubly Cohen–Macaulay poset P?

Some answers:

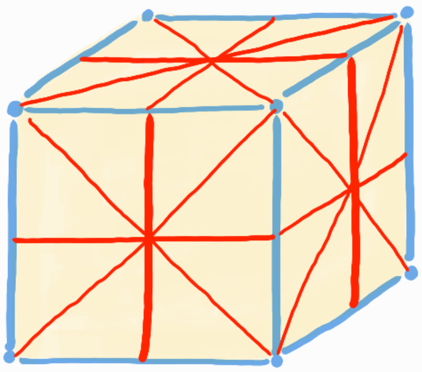
- No for distributive lattices (Stembridge, 2007)

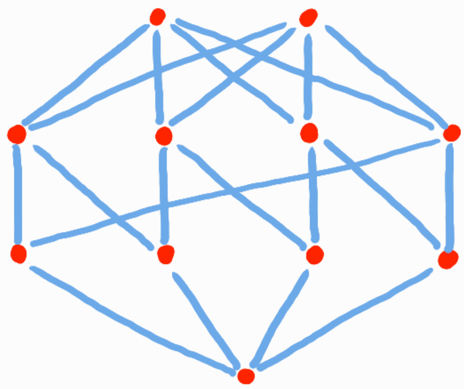


- Yes for Cohen-Macaulay simplicial posets (Brenti-Welker, 2008)



- Yes for face lattices of cubical polytopes (more generally, for shellable cubical posets, A 2021).





- Yes for the face lattices of Pyr Pyr(Q) and Prism(Q), if so for the face lattice of Q (AKE, 2023)

- Yes for subspace lattices and partition lattices of types A and B (A-KE, 2023).
- Yes for the lattices of flats of generalized paving matroids (Brändén - Saut Maia Leite)
- Yes for rank-selected subposets of Cohen-Macaulay simplicial posets (A-KE, 2023).

- Yes for noncrossing partition lattices associated to finite Coxeter groups (A-KE, 2023).
- Yes for (3+1)-free posets (Stanley, 1998).

Basic method: Express $h(\Delta(P), x)$ as a nonnegative linear combination of real-rooted polynomials with nonnegative coefficients which have a common interleaver.

Recall the for real-rooted polynomials $p(x), q(x) \in \mathbb{R}[x]$ with roots

- $\cdots \leq \alpha_2 \leq \alpha_1 \leq 0$
- $\cdots \leq \beta_2 \leq \beta_1 \leq 0$

we say that $p(x)$ interlaces $q(x)$ if $\cdots \leq \alpha_2 \leq \beta_2 \leq \alpha_1 \leq \beta_1 \leq 0$ and write $p(x) \prec q(x)$.

Let

$$P = \text{simplicial poset of rank } n$$

$$f_{k-1}(P) = \#\text{ elements of rank } k$$

$$h(P,x) = \sum_{k=0}^{n} f_{k-1}(P)\, x^k (1-x)^{n-k}$$

$$= \sum_{k=0}^{n} h_k(P)\, x^k.$$

Note. If P is Cohen-Macaulay, then $h_k(P) \geq 0$ for every k.

Proposition (Brenti-Welker, 2008)

For every simplicial poset P of rank n

$$h(\Delta(P), x) = \sum_{k=0}^{n} h_k(P)\, p_{n,k}(x),$$

where

$$p_{n,k}(x) = \sum_{w \in \mathfrak{S}_{n+1} : w(1) = k+1} x^{\mathrm{des}(w)}$$

for $n \in \mathbb{N}$. Equivalently,

$$\frac{P_{n,k}(x)}{(1-x)^{n+1}} = \sum_{m\geq 0} m^k (1+m)^{n-k} x^m .$$

Note

$$\begin{pmatrix} P_{n,0}(x) \\ P_{n,1}(x) \\ \vdots \\ P_{n,n}(x) \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x & 1 & \cdots & 1 \\ x & x & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ x & x & & x \end{pmatrix} \begin{pmatrix} P_{n-1,0}(x) \\ P_{n-1,1}(x) \\ \vdots \\ P_{n-1,n-1}(x) \end{pmatrix}$$

and hence $P_{n,i}(x) < P_{n,j}(x)$ for $0 \leq i < j \leq n$.

Corollary For every Cohen-Macaulay simplicial poset P of rank n, $h(\Delta(P), x)$ is real-rooted and is interlaced by the Eulerian polynomial $A_n(x) := p_{n,0}(x)$.

Let

$$P = \text{cubical poset of rank } n+1$$

$$f_k(P) = \#\text{ elements of rank } k+1$$

$$\tilde{\chi}(P) = -1 + \sum_{k=0}^{n} (-1)^k f_k(P)$$

$$h(P,x) = \sum_{k=0}^{n+1} h_k(P)\, x^k$$

$$= \frac{1}{1+x} \left\{ 1 + \sum_{k=0}^{n} f_k(P)\, x^{k+1} \left(\frac{1-x}{2}\right)^{n-k} + (-1)^n \tilde{\chi}(P)\, x^{n+2} \right\}.$$

Proposition (A, 2023) For every cubical poset P of rank $n+1$,

$$h(\Delta(P), x) = \sum_{k=0}^{n+1} h_k(P)\, p^B_{n,k}(x),$$

where

$$\frac{p^B_{n,k}(x)}{(1-x)^{n+1}} = \begin{cases} \sum_{m\geq 0} (2m+1)^n x^m, & k=0 \\ \sum_{m\geq 0} 4m(2m-1)^{k-1}(2m+1)^{n-k} x^m, & 1\leq k\leq n \\ \sum_{m\geq 1} (2m-1)^n x^m, & k=n+1. \end{cases}$$

Note The $p^B_{n,k}(x)$ satisfy a recursion similar to the one satisfied by the $p_{n,k}(x)$.

Corollary For every shellable cubical poset P of rank $n+1$, $h(\Delta(P), x)$ is real-rooted and is interlaced by the Eulerian polynomial

$$B_n(x) := p^B_{n,0}(x).$$

Let

P = simplicial poset of rank n with minimum element $\hat{0}$

ρ = $\rho : P \to \{0, 1, \ldots, n\}$
= rank function of P

and for $S \subseteq \{1, 2, \ldots, n\}$ let

P_S = $\{x \in P : \rho(x) \in S\} \cup \{\hat{0}\}$
= S-rank selected subposet of P.

Proposition (A-KE 2023+) For every simplicial poset P of rank n and every $S \subseteq \{1, 2, \ldots, n\}$

$$h(\Delta(P_S), x) = \sum_{k=0}^{n} h_k(P)\, p_{n,k}^{n+1-S}(x),$$

where

$$p_{n,k}^{S}(x) = \sum_{\substack{w \in \mathfrak{S}_{n+1} \,:\, w(1) = k+1 \\ \mathrm{Des}(w) \subseteq S}} x^{\mathrm{des}(w)}.$$

Lemma For $S \subseteq \{1, 2, \dots, n\}$

$$P^{S}_{n,k}(x) = \begin{cases} \displaystyle\sum_{i=k}^{n-1} P^{S-1}_{n-1,i}(x), & 1 \notin S \\ \displaystyle x \sum_{i=0}^{k-1} P^{S-1}_{n-1,i}(x) + \\ \displaystyle\sum_{i=k}^{n-1} P^{S-1}_{n-1,i}(x), & 1 \in S. \end{cases}$$

As a result, $P^{S}_{n,i}(x) < P^{S}_{n,j}(x)$ for $0 \leq i < j \leq n$.

Corollary For every Cohen-Macaulay simplicial poset P of rank n and every $S \subseteq \{1, 2, \ldots, n\}$, $h(\Delta(P_S), x)$ is real-rooted and interlaced by

$$A_n^S(x) = \sum_{w \in \mathfrak{S}_n : \mathrm{Des}(w) \subseteq S} x^{\mathrm{des}(w)}.$$

Let

- Π_n = lattice of partitions of $\{1, 2, \ldots, n\}$
- $\mathcal{A}_n = \{1\} \times \{1, 1, 2\} \times \{1, 1, 1, 2, 2, 3\} \times \cdots \times \{1, 1, \ldots, 1, \ldots, n-2, n-2, n-1\}$

and for $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_{n-1}) \in \mathcal{A}_n$

- $\mathrm{des}(\sigma) = \#\, i \in \{1, 2, \ldots, n-2\} : \sigma_i \geqslant \sigma_{i+1}$.

Proposition (A-KE 2023+)

$$h(\Delta(\Pi_n), x) = \sum_{\sigma \in A_n} x^{\mathrm{des}(\sigma)}$$

is real-rooted for every $n \in \mathbb{N}$.

Let

- W = finite real reflection group of rank n
- NC_W = lattice of noncrossing partitions associated to W
- X = Coxeter type of W

Theorem (A - Douvropoulos - AK)

(a) $h(\Delta(NC_W), x)$ is a nonnegative linear combination of the polynomials $p_{n,k}(x)$ for $0 \le k \le n$ and hence real-rooted.

(b) Let $\mathcal{D}_n$ be the set of words $w = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n$ such that

$$1 \le |a_1|, a_2, \ldots, a_n \le n-1$$

and call $i \in \{1, 2, \ldots, n-1\}$ a descent if

- $|a_i| > a_{i+1}$ or
- $a_i = a_{i+1} > 0$.

Then, $h(\Delta(NC_W), x)$ is equal to

$$\begin{cases} \dfrac{1}{n+1} \displaystyle\sum_{w \in \{1,2,\ldots,n+1\}^n} x^{\operatorname{des}(w)}, & X = A_n \\ \displaystyle\sum_{w \in \{1,2,\ldots,n\}^n} x^{\operatorname{des}(w)}, & X = B_n \\ \displaystyle\sum_{w \in D_n} x^{\operatorname{des}(w)}, & X = D_n \end{cases}$$

(c) Let

$$S_r\left(\sum_{n\geq 0} a_n x^n\right) = \sum_{n\geq 0} a_{rn} x^n$$

be the zeroth r-Veronese operator. Then, $x^n h(\Delta(NC_W), 1/x)$ equals

$$\begin{cases} \dfrac{1}{n+1} S_{n+1}\left(x(1+x+\cdots+x^n)^{n+1}\right), & X = A_n \\ S_n\left(x(1+x+\cdots+x^{n-1})^{n+1}\right), & X = B_n \\ S_{n-1}\left((x+x^2)(1+x+\cdots+x^{n-2})^{n+1}\right), & X = D_n \end{cases}$$

As a result (Jochemko, 2021), $h(\Delta(NC_W), x)$ has a real-rooted symmetric decomposition with respect to n for every irreducible W.