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NOTE

A Class of Labeled Posets and the Shi Arrangement of Hyperplanes

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We consider the class $\mathcal{P}_n$ of labeled posets on n elements which avoid certain three-element induced subposets. We show that the number of posets in $\mathcal{P}_n$ is $(n+1)^{n-1}$ by exploiting a bijection between $\mathcal{P}_n$ and the set of regions of the arrangement of hyperplanes in $\mathbb{R}^n$ of the form $x_i - x_j = 0$ or 1 for $1 \leqslant i < j \leqslant n$. It also follows that the number of posets in $\mathcal{P}_n$ with i pairs (a, b) such that $a < b$ is equal to the number of trees on $\{0, 1, ..., n\}$ with $\binom{n}{2} - i$ inversions. © 1997 Academic Press

1. THE RESULTS

Let $\mathcal{P}_n$ be the set of posets on $[n] := \{1, 2, ..., n\}$ which do not contain any of the three-element posets of Fig. 1, with $a < b < c$, as induced subposets. For any undefined terminology about posets we refer the reader to [8, Chap. 3]. The objective of this paper is to point out some surprising enumerative properties of $\mathcal{P}_n$. Our first theorem follows.

THEOREM 1.1. *The number of posets in* $\mathcal{P}_n$ *is* $(n+1)^{n-1}$.

We will prove this by a bijection between $\mathcal{P}_n$ and the set of *regions* of the arrangement of hyperplanes

$$x_i - x_j = 0 \qquad \text{for} \quad 1 \leqslant i < j \leqslant n,$$

$$x_i - x_j = 1 \qquad \text{for} \quad 1 \leqslant i < j \leqslant n$$

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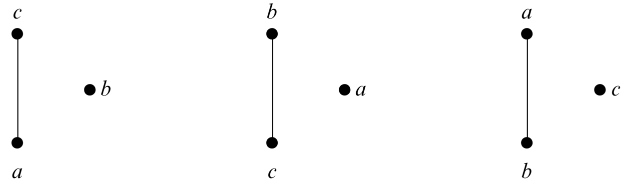

FIG. 1. The forbidden induced subposets.

in $\mathbb{R}^n$. This is the *Shi arrangement*, denoted by $\mathcal{S}_n$. It was first considered by Shi [7] who proved that it dissects $\mathbb{R}^n$ into $(n+1)^{n-1}$ regions. A number of simple proofs of Shi's result were found recently, see for example [1]. Our bijection combined with the work of Pak and Stanley [9, Sect. 5] (see also [10]) on the combinatorics of $\mathcal{S}_n$ implies a refinement of Theorem 1.1 which we describe next. Recall that an *inversion* of a tree T on the vertex set $\{0, 1, ..., n\}$ is a pair (i, j) with $1 \leqslant i < j \leqslant n$ such that the vertex j lies on the path in T from 0 to i. By a *relation* in a poset P we mean a pair of elements (a, b) of P such that $a <_P b$, where we denote by $\leqslant_P$ the partial order of P.

THEOREM 1.2. *For each $m = 0, 1, ..., \binom{n}{2}$, the number of posets in $\mathcal{P}_n$ with m relations equals the number of trees on $\{0, 1, ..., n\}$ with $\binom{n}{2} - m$ inversions.*

For the enumeration of labeled trees by the number of inversions see Mallows and Riordan [6], Gessel and Wang [3], or the summary of the known results given in [10, Section 3]. A conjectured algebraic interpretation of this statistic appears in Haiman [4, Section 2.3].

2. THE BIJECTION

Let R be a region of $\mathcal{S}_n$, i.e., a connected component of the space obtained from $\mathbb{R}^n$ by removing the hyperplanes of $\mathcal{S}_n$. For all $1 \leqslant i < j \leqslant n$, exactly one of the relations $x_i < x_j$, $0 < x_i - x_j < 1$, $x_i - x_j > 1$ holds on R. We define the poset $P = P_R$ on $[n]$ by requiring that for $1 \leqslant i < j \leqslant n$:

(i) $i <_P j$ if $x_i < x_j$ holds on R,

(ii) $i >_P j$ if $x_i - x_j > 1$ holds on R.

In particular, i and j are incomparable in P if $0 < x_i - x_j < 1$ holds on R. It is easy to check that P is a well defined poset, that is, $x <_P y$ and $y <_P z$ imply $x <_P z$. For example, if R is the region of $\mathcal{S}_5$ defined by $x_3 > x_1 > x_5 > x_4 > x_2$ and $x_3 - x_4 > 1$, $x_1 - x_2 > 1$, $x_3 - x_5 < 1$, $x_1 - x_5 < 1$, and $x_1 - x_4 < 1$, then P_R is shown in Fig. 2.

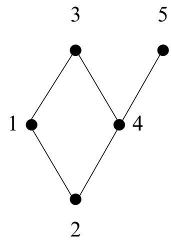

FIG. 2. A poset in $\mathscr{P}_5$.

Our main theorem can be stated as follows.

THEOREM 2.1. *The map* $R \mapsto P_R$ *defines a bijection between the set of regions of* $\mathscr{S}_n$ *and* $\mathscr{P}_n$.

Proof. First note that $P_R \in \mathscr{P}_n$. Indeed, if P_R contains the first poset of Fig. 1 as an induced subposet, then one is led to the contradiction $x_c > x_a > x_b > x_c$. The other two posets of Fig. 1 lead to similar contradictions.

The inverse map is as easy to describe. Given $P \in \mathscr{P}_n$, let R_P be the region in $\mathbb{R}^n$ for which $x_i < x_j$, $0 < x_i - x_j < 1$, or $x_i - x_j > 1$ holds if $i <_P j$, i and j are incomparable in P, or $i >_P j$, respectively, for all $1 \leqslant i < j \leqslant n$. We just have to show that this map is well defined, that is, R_P is nonempty.

First we show that the linear ordering of $x_1, x_2, ..., x_n$ imposed by the previous conditions is well defined. Consider the complete graph on the vertex set $\{x_1, x_2, ..., x_n\}$ with edges $x_i x_j$ oriented with the arrow pointing to the largest vertex, i.e., for $i < j$, $x_i \to x_j$ if $i <_P j$, and $x_j \to x_i$ otherwise. To check that this orientation is acyclic it suffices to check that there is no three-element cycle. Indeed, if $1 \leqslant i < j < k \leqslant n$, then the cycle $x_i \to x_j \to x_k \to x_i$ cannot occur, because otherwise $i <_P j <_P k$ but either $i >_P k$ or i and k are incomparable in P. The cycle $x_i \to x_k \to x_j \to x_i$ cannot occur because it either leads to contradictions similar to the previous one or it implies the existence of an induced subposet of P on $\{i, j, k\}$ as the first in Fig. 1. It follows that the orientation has the form $x_{w_j} \to x_{w_i}$, $1 \leqslant i < j \leqslant n$, for some permutation $w_1 w_2 \cdots w_n$ *of* $[n]$.

The region R_P is given by

$$x_{w_1} > x_{w_2} > \cdots > x_{w_n} \tag{1}$$

and also, for all $1 \leqslant i < j \leqslant n$ with $w_i < w_j$, the inequalities $x_{w_i} - x_{w_j} > 1$ or $x_{w_i} - x_{w_j} < 1$ if $w_i >_P w_j$ or w_i and w_j are incomparable in P, respectively. It is easy to see that R_P is nonempty if and only if the following, obviously necessary, condition is satisfied: whenever $x_a \geqslant x_b > x_c \geqslant x_d$ follows from

(1) with $b<c$, $a<d$, and $x_b - x_c > 1$ is chosen for the pair (b, c), then $x_a - x_d > 1$ is also chosen for the pair (a, d). Since either $a<c$ or $b<d$, it suffices to check the condition in the two special cases $c=d$ and $a=b$. For the first case, suppose that $x_a > x_b > x_c$ follows from (1) with $b<c$, $a<c$, and $x_b - x_c > 1$ is chosen for (b, c). Hence $b >_P c$. If $x_a - x_c < 1$ were chosen for (a, c) then a has to be incomparable to both b and c in P and P has the second forbidden poset in Fig. 1 as an induced subposet. The second case follows similarly from excluding the third kind of posets in Fig. 1. ∎

Theorem 2.1 and the result of Shi [7, Corollary 7.3.10], mentioned in the Introduction, imply Theorem 1.1. As in [9, Section 5], we let the *base region* R_0 of $\mathcal{S}_n$ be the region defined by the inequalities $x_1 > x_2 > \cdots > x_n$ and $x_1 - x_n < 1$. The bijection due to Pak and Stanley described in [9, Section 5] and one due to Kreweras [5] yield the following result for $\mathcal{S}_n$ (see [9, Theorem 5.1]).

THEOREM 2.2 (Pak and Stanley). *For each $m = 0, 1, ..., \binom{n}{2}$, the number of regions R of $\mathcal{S}_n$ for which m hyperplanes of $\mathcal{S}_n$ separate R from R_0 is equal to the number of trees on $\{0, 1, ..., n\}$ with $\binom{n}{2} - m$ inversions.*

We can now derive Theorem 1.2.

Proof of Theorem 1.2. By construction, for $1 \leqslant i < j \leqslant n$, the map $R \mapsto P_R$ creates a relation $i <_P j$ in $P = P_R$ whenever $x_i - x_j = 0$ separates R from R_0 and a relation $i >_P j$ whenever $x_i - x_j = 1$ separates R from R_0. The result follows from Theorems 2.1 and 2.2. ∎

Remarks. 1. The proof of Theorem 2.2, outlined in [9, Section 5], combines two bijections. The one due to Pak and Stanley is between the regions of $\mathcal{S}_n$ and *parking functions* on the set $[n]$ and the one due to Kreweras is between parking functions on $[n]$ and trees on the vertex set $\{0, 1, ..., n\}$. It would be interesting to find a simpler and more direct proof of Theorem 2.2, or equivalently, of Theorem 1.2.

2. Paul Edelman [2] has defined a partial order on the set of regions R of a hyperplane arrangement $\mathcal{A}$ by inclusion of the sets of hyperplanes $S(R_0, R)$ which separate R from a fixed base region R_0. This order is usually called the *weak order* on the regions of $\mathcal{A}$. it follows from Theorem 2.1 that, for the Shi arrangement $\mathcal{S}_n$ and if R_0 is as above, the weak order is isomorphic to the set $\mathcal{P}_n$ partially ordered by inclusion of the sets of relations of the posets in $\mathcal{P}_n$. The minimal element R_0 corresponds to the antichain on $[n]$ and the maximal elements to the $n!$ linear orderings of $[n]$. The set $\mathcal{P}_n$ becomes a graded poset and the numbers in Theorem 1.2 are simply the cardinalities of various ranks.

REFERENCES

1. C. A. Athanasiadis and S. Linusson, A simple bijection for the regions of the Shi arrangement of hyperplanes, MSRI Preprint No. 1997-020, February, 1997.
2. P. H. Edelman, A partial order on the regions of $\mathbb{R}^n$ dissected by hyperplanes, *Trans. Amer. Math. Soc.* **283** (1984), 617–631.
3. I. Gessel and D.-L. Wang, Depth-first search as a combinatorial correspondence, *J. Combin. Theory Ser. A* **26** (1979), 308–313.
4. M. D. Haiman, Conjectures on the quotient ring by diagonal invariants, *J. Algebraic Combin.* **3** (1994), 17–76.
5. G. Kreweras, Une famille de polynômes ayant plusieurs propriétés énumératives, *Period. Math. Hungar.* **11** (1980), 309–320.
6. C. L. Mallows and J. Riordan, The inversion enumerator for labeled trees, *Bull. Amer. Math. Soc.* **74** (1968), 92–94.
7. J.-Y. Shi, "The Kazhdan–Lusztig Cells in Certain Affine Weyl Groups," Lecture Notes in Mathematics, Vol. 1179, Springer-Verlag, Berlin/Heidelberg/New York, 1986.
8. R. Stanley, "Enumerative Combinatorics," Vol. 1, Wadsworth & Brooks/Cole, Belmont, CA, 1992.
9. R. Stanley, Hyperplane arrangements, interval orders and trees, *Proc. Nat. Acad. Sci.* **93** (1996), 2620–2625.
10. R. Stanley, Hyperplane arrangements, parking functions and tree inversions, *in* "Mathematical Essays in Honor of Gian-Carlo Rota," Birkhäuser, Boston/Basel/Berlin, to appear.