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Note

On Hanlon's Eigenvalue Conjecture

R. M. ADIN*

Department of Mathematics and Computer Science, Bar-Ilan University, Ramat-Gan 52900, Israel

AND

C. A. ATHANASIADIS†

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

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P. Hanlon (*J. Combin. Theory Ser. A* **59** (1992), 218–239) has conjectured an explicit formula for the eigenvalues of certain combinatorial matrices related to the cohomology of nilpotent Lie algebras. Several special cases of this conjecture are now established.

1. INTRODUCTION

Let $G_k(a, b)$ be the disjoint union of the graphs $G_k(a, b; w)$, introduced in [H], for all possible w. Recall that if

$$M_k(x, y, \lambda) = \sum_{a,b,r} \mu_k(a, b; r)\, x^a y^b \lambda^r$$

is the generating function for the multiplicities $\mu_k(a, b; r)$ of r as an eigenvalue of $G_k(a, b)$, then Hanlon's remarkable conjecture may be stated as follows.

* Research supported in part by the Israel Science Foundation, administered by the Israel Academy of Sciences and Humanities, and by an internal research grant from Bar-Ilan University. E-mail: radin@bimacs.cs.biu.ac.il.

† E-mail: cathan@math.mit.edu.

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Conjecture (Hanlon [H]).

$$M_k(x, y, \lambda) = \prod_{i=0}^{k} (1 + x + y + \lambda^{i+1} xy).$$

Hanlon determined explicitly the eigenvalues of $G_k(a, b; w)$ under certain restrictions on the parameters, the so-called *stable case* [H, Theorem 2.5]. Since, for most values of a, b, and k, some values of w are not stable—the above conjecture remained unsettled for almost all cases.

The current paper contains a proof of Hanlon's conjecture in the following cases:

(i) $a = 1$, $b = 2$, arbitrary k (nonzero eigenvalues).

(ii) $a = 1$, arbitrary b and k (the zero eigenvalue).

Before we state precisely our results, let us recall some basic notation and background from [H]. Let $\mathscr{H}$ be the three-dimensional Heisenberg Lie algebra. As a complex vector space, $\mathscr{H}$ has a basis $\{e, f, x\}$ with Lie brackets

$$[e, f] = x, \qquad [e, x] = [f, x] = 0.$$

Now fix a nonnegative integer k and let $\mathscr{H}_k$ be the Lie algebra

$$\mathscr{H}_k = \mathscr{H} \otimes (\mathbf{C}[t]/(t^{k+1})),$$

with Lie bracket given by

$$[g \otimes p(t), h \otimes q(t)] = [g, h] \otimes p(t)\, q(t).$$

For $i \in [0, k] = \{0, 1, ..., k\}$, let e_i, f_i, and x_i denote $e \otimes t^i$, $f \otimes t^i$, and $x \otimes t^i$, respectively. These elements form the standard basis of $\mathscr{H}_k$, with the only nonzero brackets among them having the form

$$[e_i, f_j] = x_{i+j},$$

where $i + j \leqslant k$. Denote by E, F, and X the subspaces spanned by the e_i, f_i, and x_i, respectively. We then have

$$\mathscr{H}_k = E \oplus F \oplus X$$

and, also,

$$\Lambda \mathscr{H}_k = (\Lambda E) \wedge (\Lambda F) \wedge (\Lambda X),$$

where Λ stands for the exterior algebra. For a set of indices $I=\{i_1, i_2, ..., i_r\}$ with $0 \leqslant i_1 < i_2 < \cdots < i_r \leqslant k$ set, as usual, $e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_r}$ and, similarly, for f_I and x_I. The elements

$$e_A \wedge f_B \wedge x_C,$$

where A, B, and C range over all subsets of $[0, k]$, form a basis of $\Lambda \mathcal{H}_k$, which we call the *standard basis*. Denote by $V_k(a, b, c)$ the subspace $(\Lambda^a E) \wedge (\Lambda^b F) \wedge (\Lambda^c X)$ of $\Lambda \mathcal{H}_k$. Its standard basis consists of the elements $e_A \wedge f_B \wedge x_C$ satisfying $|A| = a$, $|B| = b$ and $|C| = c$.

Let $\partial: \Lambda \mathcal{H}_k \to \Lambda \mathcal{H}_k$ be the *boundary map* defining the Koszul complex of $\mathcal{H}_k$. Thus, ∂ is a linear map defined on elements of the standard basis by

$$\partial(m_1 \wedge m_2 \wedge \cdots \wedge m_r)$$
$$= \sum_{1 \leqslant i < j \leqslant r} (-1)^{i+j-1} [m_i, m_j] m_1 \wedge \cdots \wedge \hat{m}_i \wedge \cdots \wedge \hat{m}_j \wedge \cdots \wedge m_r,$$

where $m \in \{e, f, x\}$. Then

$$\partial\partial = 0$$

and

$$H_*(\mathcal{H}_k) = \ker \partial / \operatorname{im} \partial$$

is the (Lie algebra) homology of $\mathcal{H}_k$. Define the *Laplacian operator* $L: \Lambda \mathcal{H}_k \to \Lambda \mathcal{H}_k$ to be

$$L = \partial\partial^* + \partial^*\partial,$$

where the adjoint of ∂ (the *coboundary map*) is taken with respect to the Hermitian form for which the standard basis of $\Lambda \mathcal{H}_k$ is orthonormal. Then (see [K]) $\ker L$ and $H_*(\mathcal{H}_k; \mathbf{C})$ are isomorphic as graded vector spaces, so that the (graded) multiplicity of zero as an eigenvalue of L gives the dimensions of homology groups of $\mathcal{H}_k$. The grading on $\Lambda \mathcal{H}_k$ is obtained by assigning degree 1 to each nonzero element of $\mathcal{H}_k$, so that nonzero elements of $V_k(a, b, c)$ have degree $a + b + c$. The maps ∂ and ∂^* do not preserve this grading (although L does), since

$$\partial: V_k(a, b, c) \to V_k(a-1, b-1, c+1)$$

and

$$\partial^*: V_k(a-1, b-1, c+1) \to V_k(a, b, c).$$

Each subspace $V_k(a, b, c)$ is therefore invariant under L. It is also clear that ∂ and ∂^* do preserve another grading of $\Lambda\mathscr{H}_k$, defined by assigning degree i to e_i, f_i, and x_i for each i. With this grading, an element $e_A \wedge f_B \wedge x_C$ of $\Lambda\mathscr{H}_k$ has degree $\|A\| + \|B\| + \|C\|$, where $\|S\|$ stands for the sum of the elements of S. This quantity is called the *weight* of the triple (A, B, C).

The adjacency matrix of the graph $G_k(a, b)$ (as defined in [H]) is the matrix representing the restriction of the Laplacian L to $V_k(a, b, 0)$ with respect to the standard basis. The adjacency matrix of the component $G_k(a, b; w)$ of $G_k(a, b)$ is the matrix representing the restriction of L to the homogeneous component of $V_k(a, b, 0)$ of total weight w. Note that $\partial^* = 0$ on $V_k(a, b, 0)$, and, hence, this restriction of L actually has the form $\partial^*\partial$, where

$$\partial\colon V_k(a, b, 0) \to V_k(a-1, b-1, 1)$$

and

$$\partial^*\colon V_k(a-1, b-1, 1) \to V_k(a, b, 0).$$

Let $\lfloor x \rfloor$ denote the largest integer not exceeding the real number x.

THEOREM 1. *For $a = 1$, $b = 2$, and arbitrary k, the nonzero eigenvalues of the Laplacian restricted to $V_k(a, b, 0)$ are the integers $1, ..., k+1$, each with multiplicity k:*

$$\mu_k(1, 2; r) = \begin{cases} k, & \textit{if} \quad 1 \leqslant r \leqslant k+1; \\ 0, & \textit{otherwise.} \end{cases}$$

Their distribution among the various weights w is

$$\mu_k(1, 2, w; r) = \begin{cases} 1, & \textit{if} \quad 1 \leqslant w \leqslant k,\ 1 \leqslant r \leqslant w+1,\ r \neq \lfloor (w+2)/2 \rfloor; \\ 1, & \textit{if} \quad k+1 \leqslant w \leqslant 2k,\ w-k \leqslant r \leqslant k+1,\ r \neq \lfloor (w+2)/2 \rfloor; \\ 0, & \textit{otherwise.} \end{cases}$$

THEOREM 2. *For $a = 1$ and arbitrary b and k, the multiplicity of the zero eigenvalue of the Laplacian restricted to $V_k(a, b, 0)$ is*

$$\mu_k(1, b; 0) = \binom{k+1}{1, b, k-b} = (k+1)\binom{k}{b}.$$

Its distribution among the various weights w is

$$\mu_k(1, b, w; 0) = \sum_{i=0}^{k} \# \left\{ (j_1, ..., j_b) \mid 1 \leqslant j_1 < \cdots < j_b \leqslant k, \sum_{t=1}^{b} j_t = w - i \right\}.$$

Sections 2 and 3 contain, respectively, proofs of the above two theorems.

2. NONZERO EIGENVALUES

In order to prove Theorem 1, let us first compute the eigenvalues of an interesting family of symmetric matrices. For a positive integer n, let

$$A_n = \mathrm{diag}(1, 2, ..., n) - T_n,$$

where T_n is the $n \times n$ "lower-right triangular" Hankel matrix defined by

$$T_n(i, j) = \begin{cases} 1, & \text{if } \quad i + j \geqslant n + 1; \\ 0, & \text{otherwise.} \end{cases}$$

For $n = 1, 2, 3, 4, 5$, A_n is

$$(0), \quad \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 2 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 2 & 0 & -1 & -1 \\ 0 & 0 & 2 & -1 & -1 \\ 0 & -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{pmatrix}.$$

LEMMA 3. *The eigenvalues of A_n are the elements of $\{0, 1, ..., n\} \setminus \{\lfloor (n+1)/2 \rfloor\}$, each with multiplicity one.*

Proof. To compute the determinant of $A_n - \lambda I$, add to the first row all the other rows. Now every entry in row 1 equals $-\lambda$, so factor out $-\lambda$ and add row 1 to the last row, which becomes $(0, ..., 0, n - \lambda)$. Expanding the determinant on its last row and then again on its first column, it follows that the eigenvalues of A_n are 0, n, and the eigenvalues of $A_{n-2} + I$. Induction on n now completes the proof. ∎

To compute the nonzero eigenvalues of the Laplacian for $a=1$, $b=2$ (and arbitrary k), let us use the following elementary fact from linear algebra: If $A\colon U\to W$ and $B\colon W\to U$ are linear transformations between finite dimensional vector spaces, then the nonzero eigenvalues of BA are the same as the nonzero eigenvalues of AB, including multiplicites. In other words,

$$\lambda^{\dim W}\operatorname{ch}(BA,\lambda)=\lambda^{\dim U}\operatorname{ch}(AB,\lambda),$$

where $\operatorname{ch}(T,\lambda)$ stands for the characteristic polynomial of T. Thus the eigenvalues we are looking for are the nonzero eigenvalues of the operator $\partial\partial^*\colon W\to W$, where

$$W=V_k(a-1,b-1,1)=V_k(0,1,1).$$

The standard basis for W consists of all $f_i\wedge x_j$ such that $i,j\in[0,k]$. Denote this basis element by (i,j), for brevity. The rule for describing the entries of the matrix of $\partial\partial^*$ with respect to this basis is as follows. The entry in row (i,j) and column (i',j') is 0 unless (i',j') can be obtained from (i,j) by the following procedure: Write $j=h''+i''$ with $h'',i''\in[0,k]$ but $i''\neq i$. Let j' be the sum of h'' and either i or i'', and let i' be the remaining one (i'' or i). Such a choice contributes (to the matrix entry) either 1 (when $i'=i$) or -1 (when $i'=i''$).

As was noted before, $\partial\partial^*$ preserves the weight $w=i+j$ of each basis element. Thus its matrix, with a suitable ordering of rows and columns, has block form with blocks indexed by weight. Take the row (and column) indices (i,j) of weight w in the following order:

$$\begin{array}{ll}(w,0),(w-1,1),\ldots,(0,w) & \text{if}\quad 0\leqslant w\leqslant k;\\ (k,w-k),(k-1,w-k+1),\ldots,(w-k,k) & \text{if}\quad k\leqslant w\leqslant 2k.\end{array}$$

The corresponding blocks are seen to be

$$B_{k,w}=\begin{cases}A_{w+1}, & \text{if}\quad 0\leqslant w\leqslant k;\\ A_{2k-w+1}+(w-k)I, & \text{if}\quad k\leqslant w\leqslant 2k,\end{cases}$$

where A_n are the matrices defined at the beginning of this section.

It now follows that the eigenvalues of $B_{k,w}$ are $\{0,1,\ldots,w+1\}\setminus\{\lfloor (w+2)/2\rfloor\}$ for $0\leqslant w\leqslant k$, and $\{w-k,w-k+1,\ldots,k+1\}\setminus\{\lfloor (w+2)/2\rfloor\}$ for $k\leqslant w\leqslant 2k$. Collecting the nonzeros among these numbers for $0\leqslant w\leqslant 2k$, one gets exactly the eigenvalues with uniform multiplicities claimed in Theorem 1. ∎

3. Multiplicity of the Zero Eigenvalue

It is clear that the special case $b=2$ of Theorem 2 may be proved using the arguments and computations of the previous section. However, the treatment of general values of b will proceed in a different route.

First note that, for an upper layer $V_k(a, b, 0)$,

$$\ker(L) = \ker(\partial^* \partial) = \ker(\partial).$$

Thus we are interested in the dimension of $\ker(\partial)$, where ∂ is restricted to $V_k(1, b, 0)$. Since the dimension of $V_k(1, b, 0)$ itself is easy to compute, it will suffice to find a (vector space) complement to $\ker(\partial)$ whose dimension is also easy to compute.

Lemma 4. $V_k(1, b, 0) = \ker(\partial) \oplus (f_0 \wedge V_k(1, b-1, 0))$.

Proof. For any index $i \in [0, k]$ and any set of indices $B \subseteq [0, k]$,

$$\partial(e_i \wedge f_B) = \sum_{j \in B} \pm f_{B \setminus \{j\}} \wedge x_{i+j},$$

where it is understood that $x_t = 0$ if $t > k$. Thus, for any index i_0 and set of indices S,

$$\partial\left(\sum_{i \in S} \pm e_{i_0+i} \wedge f_{S \setminus \{i\}}\right) = \sum_{i \in S} \sum_{j \in S \setminus \{i\}} \pm f_{S \setminus \{i,j\}} \wedge x_{i_0+i+j} = 0.$$

The summands in the double summation simply cancel in pairs; this fact is a close relative of the fundamental identity $\partial\partial = 0$ for the boundary map. In other words,

$$\sum_{i \in S} \pm e_{i_0+i} \wedge f_{S \setminus \{i\}} \in \ker(\partial) \qquad (\forall i_0 \in [0, k],\ S \subseteq [0, k]).$$

Taking now $S = \{0\} \cup B$ for $B \subseteq [1, k]$, it follows that

$$e_{i_0} \wedge f_B \in \ker(\partial) + (f_0 \wedge V_k(1, b-1, 0)),$$

where $b = |B|$. Thus

$$V_k(1, b, 0) = \ker(\partial) + (f_0 \wedge V_k(1, b-1, 0)).$$

In order to prove that this is a direct sum, consider a nonzero element $v \in f_0 \wedge V_k(1, b-1, 0)$. We shall prove that $\partial(v) \neq 0$. Indeed, let

$$v = \sum_{i, T} \alpha_{i,T} e_i \wedge f_0 \wedge f_T,$$

where the sets T do not contain the index 0 and at least one coefficient $\alpha_{i,T} \neq 0$. Let

$$i_0 = \min\{i \mid (\exists T)\ \alpha_{i,T} \neq 0\}.$$

For $i \in [0, k]$, let

$$X_i = x_i \wedge V_k(0, b-1, 0)$$

and let P_i be the projection from

$$V_k(0, b-1, 1) = \bigoplus_{i=0}^{k} X_i$$

onto its direct summand X_i. It is now clear that

$$P_{i_0}(\partial(v)) = \sum_{i_0, T} \alpha_{i_0, T} x_{i_0} \wedge f_T \neq 0,$$

so that indeed $\partial(v) \neq 0$.

Having proved that

$$V_k(1, b, 0) = \ker(\partial) \oplus (f_0 \wedge V_k(1, b-1, 0)),$$

it now follows that

$$\begin{aligned}\dim \ker(\partial) &= \dim V_k(1, b, 0) - \dim(f_0 \wedge V_k(1, b-1, 0)) \\ &= (k+1)\binom{k+1}{b} - (k+1)\binom{k}{b-1} = (k+1)\binom{k}{b},\end{aligned}$$

as claimed in Theorem 2. ∎

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